Asymptotic behavior for nonlinear textiles with glued yarns

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Received xxx; | Revised xxx; | Accepted xxx

DOI: xxx/xxxx

RESEARCH ARTICLE

Asymptotic behavior for nonlinear textiles with glued yarns

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Abstract

This paper is dedicated to the homogenization and dimension reduction for a nonlinear elasticity problem of a textile structure. The structure is represented as a squared piece of cloth and modeled as a woven canvas made of long and thin fibers, crossing each other in a periodic pattern. The squared domain is partially clamped on the left and bottom. The fibers are assumed to be glued, condition that allows to extend the woven structure to a non perforate plate. The nonlinear elasticity problem is stated via minimization over the energy functional. The homogenization is made via the periodic unfolding method, with an additional dimension reduction. Since the existence of a minimum of the limit energy functional is not ensured, sufficient conditions for the existence are given by assuming that the material is homogeneous and isotropic.

KEYWORDS:
Homogenization, periodic unfolding method, dimension reduction, nonlinear elasticity, homogeneous and istotropic material, plates, beam structures

1 | INTRODUCTION

This work is dedicated to the homogenization and dimension reduction of a textile structure under large deformation, i.e. in a nonlinear elasticity context. Given the small parameters ε and r, the structure we take into consideration is a squared piece of cloth, consisting of long oscillating beams of length L, and with a square cross-section of width $2r$. The fibers cross each others in a 2ε periodic pattern and they are assumed to be glued when they are right above each others. The cloth is partially clamped on the left and bottom border, as shown in Figure 1 .

Assuming that $r \sim \varepsilon$, as ε goes to zero, the homogenization is done via unfolding method, an equivalent formulation of the two scale convergence first presented by Damlamian et al. in [6] and further developed in [8] and [7] in the frame of elasticity. For decomposition and limits for dimension reduction of plates and beams, we refer to [5,12,13,14]. The combination of homogenization and dimension reduction can be found in Chapter 11 of [7] and in [16]. As one can see in Table 1 , this work is a natural continuation of the investigations already done in the context of textile structures made of beams glued to each other, considered in [15,16] for the linear regime and in [18] for the Kármán regime. In all these cases, we will see that we can extend the glued fibers strucutre to a periodically perforated shell (for the homogenization, we refer to [9,10]). We would also like to refer to [19,20] concerning homogenization and dimension reduction for plates with large deformations.

The homogenization of this kind of structure in the context of large deformations leads to a better understanding of the limit model, and its effective asymptotic behavior, but also to its limitations in the actual approximation of a real textile. In this sense, we prove that the limit model is of the kind

$$\mathcal{J}(\mathcal{V}) \doteq \int_{\Omega} Q(\mathbf{II}(\mathcal{V}))\, dx - \int_{\Omega} f \cdot (\mathcal{V} - \mathbf{I}_d)\, dx.$$

Linear model ($\|e(u)\|^2 \sim \varepsilon^{5+\alpha}$) - glued fibers ($g = 0$ or $g \sim \varepsilon^{3+\alpha+\beta}$) [15,17]	Von Kármán model ($\|e(u)\|^2 \sim \varepsilon^5$) - glued fibers ($g = 0$ or $g \sim \varepsilon^{3+\beta}$) [18]	Nonlinear model ($\|(\nabla v)^T \nabla v - \mathbf{I}_3\|^2 \sim \varepsilon^3$) - glued fibers ($g = 0$ or $g \sim \varepsilon^{2+\beta}$)
Linear model ($\|e(u)\|^2 \sim \varepsilon^{5+\alpha}$) - strong contact ($g \sim \varepsilon^{3+\alpha}$) [17]	Von Kármán model ($\|e(u)\|^2 \sim \varepsilon^5$) - strong contact ($g \sim \varepsilon^3$)	Nonlinear model ($\|(\nabla v)^T \nabla v - \mathbf{I}_3\|^2 \sim \varepsilon^3$) - strong contact ($g \sim \varepsilon^2$)
Linear model ($\|e(u)\|^2 \sim \varepsilon^{5+\alpha}$) - loose contact ($g \sim \varepsilon^{2+\alpha}$) [11]	Von Kármán model ($\|e(u)\|^2 \sim \varepsilon^5$) - loose contact ($g \sim \varepsilon^2$)	Nonlinear model ($\|(\nabla v)^T \nabla v - \mathbf{I}_3\|^2 \sim \varepsilon^3$) - loose contact ($g \sim \varepsilon$)
Linear model ($\|e(u)\|^2 \sim \varepsilon^{5+\alpha}$) - very loose contact ($g \sim \varepsilon^{1+\alpha}$)	Von Kármán model ($\|e(u)\|^2 \sim \varepsilon^5$) - very loose contact ($g \sim \varepsilon$)	

TABLE 1 Let $\alpha, \beta > 0$, let v be a deformation, $u = v - I_d$ its associated displacement, and $e(u) \doteq \frac{1}{2}\big((\nabla u)^T + \nabla u\big)$. The table shows all possible cases according to the type of deformation (x-axis) and the allowed in-plane slide of the fibers (y-axis), which is modeled by a contact function g. The already studied cases are cited. On the top right, the case considered in this draft.

⟨tab0⟩

where Q is a quadratic form over the second fundamental form $\mathbf{II}$ of the middle-line displacement $\mathcal{V}$. We know that this model does not represent all the phenomena present in the nonlinear bending. However, we consider it a sufficiently good model to start with, and that can be extended to allow more configurations. As a comparison to the work done earlier in earlier in [19,20], this paper provides more details, also related to some symmetries and anti-symmetries w.r.t. some rotations and translations present in the plate structure, see Fig. 2 , 3 . Also, the algorithm to compute the effective bending properties is given, which have a practical effort for applied problems. On the other hand, the limit result that only developable surfaces are allowed is too strict. In this sense, conclusions are drawn and new suggestions are presented to fill this gap.

We start by giving some notation in Section 2, then in Section 3 we proceed to the construction of a woven canvas in the square $\Omega = (0, L)^2$, as shown in Figure 1 . The structure is assumed to be clamped on the left and bottom boundary. Similarly to

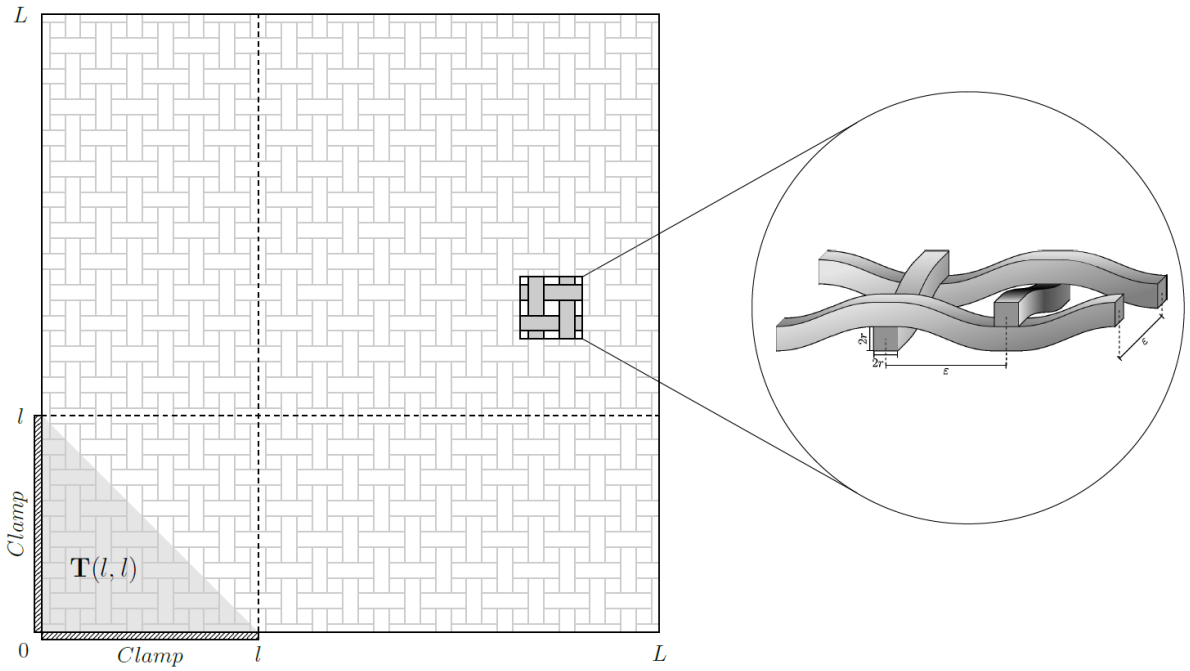

FIGURE 1 The 3D textile domain is made of long and thin fibers with squared cross section $2r$, which cross each others in a 2ε periodic pattern. A partial clamp is set on the left and bottom boundary of the domain.

⟨fig_uno⟩

what has been done in [15] and [18], the assumption of glued fibers allows to quickly convert the woven canvas into a periodically perforated plate S_ε^*, which will be the actual domain we will work on. Given a plate deformation $v \in H^1(S_\varepsilon^*)$, we decompose it as in [4] and derive the associated St.Venant tensor.

In Section 4, we proceed similarly as in [18] and state the nonlinear elasticity problem as a minimization over the energy functionals:

$$\inf_{v\in H^1(S_\varepsilon^*)} \mathcal{J}_\varepsilon(v) = \inf_{v\in H^1(S_\varepsilon^*)} \Big(\int_{S_\varepsilon^*} \widehat{W}_\varepsilon(\cdot, \nabla v)\, dx - \int_{S_\varepsilon^*} f_\varepsilon \cdot (v - I_d)\, dx \Big), \tag{1}$$

prob_0

where a_ε is the fourth order strain tensor describing the material law, and f_ε is the applied stress. Even though the existence of a minimizer for this problem is not ensured (the functionals $\mathcal{J}_\varepsilon$ are not coercive), we do not set more assumptions since we are merely interested on its solvability as $\varepsilon \to 0$.

However, in order to go to the limit we need to find a lower bound for the functionals, and this is done by choosing sufficient forces in Section 5 such that the resulting deformations are large:

$$\frac{1}{2}\|(\nabla v_\varepsilon)^T \nabla v_\varepsilon - \mathbf{I}_3\|_{L^2(S_\varepsilon^*)} \sim \varepsilon^{3/2}.$$

Then, compactness results ensure the weak convergence of the displacement fields. Section 6 is dedicated to the asymptotic behavior of the deformation fields and the form of the St.Venant tensor. To this scope we use two unfolding operators, related to each others: a 2D unfolding operator for the plate mid-line and a 3D global unfolding operator for the whole structure, where a dimension reduction is additionally applied. The limit behavior of the fields representing the mid-line of the plate $\mathcal{V}$ and their relation with the cross section rotation matrix $\mathbf{R}$ suggest a first characterization of the limit deformations, that is, they are developable surfaces.

Once the weak asymptotic behavior of the fields are found, we proceed in Section 7 to the definition of limit minimization problem:

$$\inf_{(\mathcal{V},\hat{v},\mathcal{Z})\in\mathbb{D}\times L^2(\Omega)} \mathcal{J}(\mathcal{V},\mathcal{Z},\hat{v}) = \inf_{(\mathcal{V},\hat{v},\mathcal{Z})\in\mathbb{D}\times L^2(\Omega)} \Big(\int_{\Omega\times\mathcal{Y}^*} Q(y)\big(\mathcal{E}(\mathcal{Z},\mathcal{V}) + \mathcal{E}_y(\hat{v})\big)\, dx'dy - |\mathcal{Y}^*| \int_\Omega f \cdot (\mathcal{V} - I_d)\, dx' \Big).$$

Even though, it is possible to show that the functional $\mathcal{J}$ is coercive, the existence of a minimizer is not ensured because of the presence of the shear fields $\mathcal{Z}$ which do not belong to a Hilbert space. For this reason, it is crucial to introduce a sufficient condition, (46), which ensures that our minimum is reached if $\mathcal{Z} = 0$. Even though such condition is not satisfied in general, it certainly holds for isotropic and homogeneous materials. With this condition, the existence of a minimizer in the limit is ensured and a full homogenization of the unfolded limit problem is done, giving the homogenizing coefficients and the final minimization problem over the homogenized energy functional:

At last, with a kind of Γ-convergence technique, we prove that such homogenized minimization problem is in fact the limit of the minimization problems (1) as $\varepsilon \to 0$.

The study of the limit behavior for the large deformations of a textile structure with glued fibers leads to some important outcomes, which we gather in the conclusive Section 8:

- Whenever the fibers are glued, a woven canvas can be extended to a periodically perforated plate, and treated accordingly;
- Assumption (46) is a sufficient condition for the existence of a minimizer for the limit functional. This assumption depends on the geometry of the periodic structure and the isotropy and homogeneity of the material, which may vary from case to case. For this purpose, the proof of Lemma 10 and Remark 3 furnish a sequence of steps to check such condition.
- The solution of the minimization problem for the homogenized energy functional is a developable surface.

As a final remark, we would like to mention that even thought the homogenized model that we present, which only allows developable surfaces in the limit, is in contrast with real experiments where folding appears (see e.g. the work done in [2]). This does not necessarily mean that it is incorrect, but rather incomplete: a forthcoming work, based on [3], will introduce penalizing terms in the right hand side, that vanish in the limit, so that the minimization problem has solution near zero, allowing more configurations since $\mathcal{Z}$ is not set to be zero.

2 | NOTATION

S2

Before getting started, we find it convenient to give in Table 2 the main definitions that will often appear throughout the work.

For the rest of the draft, the Einstein convention over repeated indexes will be used and, if not specified, the constants C, C_0 and C_1 do not depend on the parameter ε.

Symbol	Definition	Meaning
$L \in \mathbb{R}^+$	Constant	Lenght of the fibers.
$l < L \in \mathbb{R}^+$	Constant	Length of the partial clamp.
Ω	$\doteq (0, L)^2$	In-plane textile domain.
$\mathcal{Y}$	$\doteq (0, 2)^2$	In-plane reference cell.
$\varepsilon \in \mathbb{R}^+$	Small parameter	Distance between fibers.
$N_\varepsilon \in \mathbb{N}$	$\doteq \frac{L}{2\varepsilon}$	Number of 2ε-segments in L.
$n_\varepsilon \in \mathbb{N}$	$\doteq \frac{l}{2\varepsilon}$	Number of 2ε-segments in l.
$\mathcal{K}_\varepsilon$	$\doteq \{0, \dots, 2N_\varepsilon\}^2$	Set of nodes in $\overline{\Omega}$.
$\kappa \in [0, 1/3]$	Constant	Ratio between fiber's distance and their cross-section.
$r \in \mathbb{R}^+$	$\doteq \kappa\varepsilon$	Width of the fiber's cross section.
ω_κ	$\doteq (-\kappa, \kappa)^2$	Reference fiber's cross section.
$\omega_r = \omega_{\kappa\varepsilon}$	$\doteq (-r, r)^2 = (-\kappa\varepsilon, \kappa\varepsilon)^2$	Rescaled fiber's cross section.
$x \in \mathbb{R}^3$	$\doteq (x_1, x_2, x_3)$	Variables in the fixed reference frame.
$x' \in \mathbb{R}^2$	$\doteq (x_1, x_2)$	Variables in the fixed in-plane reference frame.
∂_i	$\doteq \frac{\partial}{\partial x_i}$	Partial derivative with respect to to x_i, for $i \in \{1, 2, 3\}$.
$\alpha, \beta \in \{1, 2\}$	Constant	Shorten notation for direction $\mathbf{e}_1$ and $\mathbf{e}_2$.
$a, b, c \in \{0, 1\}$	Constant	Shorten notation for the lines in the reference cell $\mathcal{Y}$.
$\mathbf{M}_3$	Group	Group of 3×3 matrices.
$\mathbf{S}_3$	Group	Group of 3×3 symmetric matrices.
I_d	Function	Identity map $I_d : x \in \mathbb{R}^3 \mapsto x \in \mathbb{R}^3$
$\mathbf{I}_3$	3×3 Matrix	Identity matrix
$\mathbf{I}$	2×2 Matrix	First fundamental form of a 3D surface
$\mathbf{II}$	2×2 Matrix	Second fundamental form of a 3D surface

TABLE 2

⟨tab1⟩

3 | THE TEXTILE STRUCTURE AND DECOMPOSITION OF THE DEFORMATION

⟨S3⟩ In this section we start by defining our textile structure, which consists of a woven canvas made of long, thin fibers crossing each other in a periodic pattern. Since the fibers are glued, we can see the structure as a perforated plate at first, and then extend the deformations on the perforated plate to deformations on a non-perforated one. For this last one, decomposition and St. Venant strain tensor are derived.

3.1 | Parameterization of a woven canvas made of glued fibers

We start by defining the 2-periodic function

$$\Phi(t) \doteq \begin{cases} -\kappa & \text{if } t \in [0, \kappa], \\ \kappa\Big(6\dfrac{(t-\kappa)^2}{(1-2\kappa)^2} - 4\dfrac{(t-\kappa)^3}{(1-2\kappa)^3} - 1\Big) & \text{if } t \in [\kappa, 1-\kappa], \\ \kappa & \text{if } t \in [1-\kappa, 1], \\ -\Phi(t-1) & \text{if } t \in [1, 2] \end{cases}$$

and we rescale it to a 2ε-periodic function setting $\Phi_\varepsilon(t) = \varepsilon\Phi\Big(\dfrac{t}{\varepsilon}\Big)$, which is piece-wise $\mathcal{C}^2(\mathbb{R})$ and overall $\mathcal{C}^1(\mathbb{R})$. By definition, such a function satisfies

$$\varepsilon^2\|\Phi_\varepsilon''\|_{L^\infty(\mathbb{R})} + \varepsilon\|\Phi_\varepsilon'\|_{L^\infty(\mathbb{R})} + \|\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C\varepsilon.$$

Let $P_r^{(1)}$ and $P_r^{(2)}$ be the straight reference rods in direction $\mathbf{e}_1$ and $\mathbf{e}_2$ defined by

$$P_r^{(1)} \doteq \left\{ x = (x_1, x_2, x_3) \in (0, L) \times (-r, r) \times \mathbb{R} \;\middle|\; x_3 \in (\Phi_\varepsilon(x_1) - r, \Phi_\varepsilon(x_1) + r) \right\},$$
$$P_r^{(2)} \doteq \left\{ x = (x_1, x_2, x_3) \in (-r, r) \times (0, L) \times \mathbb{R} \;\middle|\; x_3 \in (\Phi_\varepsilon(x_2) - r, \Phi_\varepsilon(x_2) + r) \right\}.$$

For each in plane direction, we define two sets of curved rods by

$$P_\varepsilon^{(1,q)} \doteq \left\{ x = (x_1, x_2, x_3) \in (0, L) \times (q\varepsilon - r, q\varepsilon + r) \times \mathbb{R} \;\middle|\; x_3 \in \left((-1)^{q+1}\Phi_\varepsilon(x_1) - r, (-1)^{q+1}\Phi_\varepsilon(x_1) + r\right) \right\},\ q \in \{0, \dots, 2N_\varepsilon\},$$
$$P_\varepsilon^{(2,p)} \doteq \left\{ x = (x_1, x_2, x_3) \in (p\varepsilon - r, p\varepsilon + r) \times (0, L) \times \mathbb{R} \;\middle|\; x_3 \in \left((-1)^{p}\Phi_\varepsilon(x_2) - r, (-1)^{p}\Phi_\varepsilon(x_2) + r\right) \right\},\ \ p \in \{0, \dots, 2N_\varepsilon\}.$$

Note that the quantities $(-1)^{q+1}$ and $(-1)^p$ denote the fact that the woven fibers are alternate, allowing crossing between them. The woven textile structure results to be

$$\mathcal{P}_\varepsilon \doteq \left(\bigcup_{q=0}^{2N_\varepsilon} \overline{P_\varepsilon^{(1,q)}} \cup \bigcup_{p=0}^{2N_\varepsilon} \overline{P_\varepsilon^{(2,p)}} \right) \cap \left(\Omega \times (-2r, 2r) \right),$$

where the restriction to $\Omega \times (-2r, 2r)$ is done to simplify the definition of the domain, since in fact $\mathcal{P}_\varepsilon \subset (-r, L+r)^2 \times (-2r, 2r)$. Now, since the fibers are glued, it is possible to partition the domain in small segments and fill the empty spaces as in Figure 2 . Doing so, we can extend our woven textile to the following structure:

$$S_\varepsilon^* \doteq \left(\mathcal{P}_\varepsilon \cap (\Omega \times \{0\}) \right) \times (-2r, 2r). \tag{2}$$

plate_holes

Note that this structure can be considered as a perforated plate, and that is why we name it S_ε^*, being * a notation for the holes.

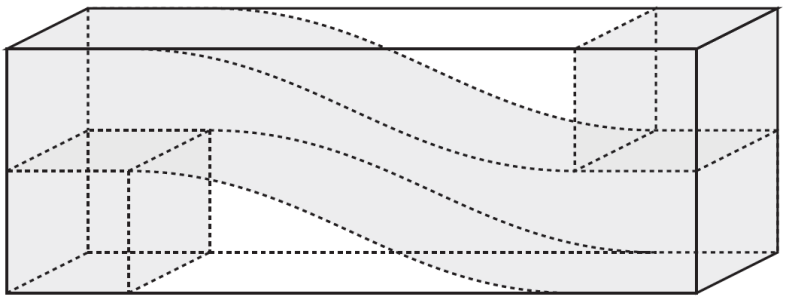

FIGURE 2 A small segment of the textile domain. In gray, the glued fibers belonging to $\mathcal{P}_\varepsilon$. It is possible to extend such structure to the whole parallelotope by filling the white spaces. Doing this for every segment, it will result in the perforated plate S_ε^*.

⟨fig_ext1⟩

3.2 | Boundary conditions and space of admissible deformations

Given the domain of our structure S_ε^*, we introduce the boundary conditions in the form of a partial clamp on the left and bottom boundary:

$$\boldsymbol{\gamma} = \left((0, l) \times \{0\} \right) \cup \left(\{0\} \times (0, l) \right), \qquad \boldsymbol{\Gamma}_r \doteq \boldsymbol{\gamma} \times (-2r, 2r). \tag{3}$$

BC

The space of admissible deformations of the whole textile structure is represented by the following set:

$$\mathbb{D}_\varepsilon^* \doteq \left\{ v \in H^1(S_\varepsilon^*)^3 \;\middle|\; v = I_d \ \text{ a.e. on } \ \partial S_\varepsilon^* \cap \boldsymbol{\Gamma}_r \right\}.$$

3.3 | From a woven canvas with glued fibers to a non-perforated plate

In juxtaposition to the perforated plate (2), we define the non-perforated one by

$$S_\varepsilon \doteq \Omega \times (-2r, 2r).$$

Accordingly, we define the space of admissible deformations including the boundary conditions by setting

$$\mathbb{D}_\varepsilon \doteq \left\{ v \in H^1(S_\varepsilon)^3 \;\middle|\; v = I_d \text{ a.e. on } \Gamma_r \right\}.$$

Below, we use a result proven in [18], which states that deformations defined over the woven canvas with glued fibers S_ε^* can be extended to deformations defined on the non-perforated plate S_ε.

Proposition 1 (Proposition 3.1 and Remark 3.2 in [18])**.** For every deformation $v \in H^1(S_\varepsilon^*)^3$, there exists $\widetilde{v} \in H^1(S_\varepsilon)^3$ such that

$$\widetilde{v}_{|S_\varepsilon^*} = v, \qquad \|dist(\widetilde{v}, SO(3))\|_{L^2(S_\varepsilon)} \le C \|dist(v, SO(3))\|_{L^2(S_\varepsilon^*)}.$$

Moreover, if $v \in \mathbb{D}_\varepsilon^*$ then $\widetilde{v} \in \mathbb{D}_\varepsilon$.

With the above proposition, we can now treat a deformation on the textile structure with glued yarns by using the results for the deformation of a plate (such as decomposition and associated strain tensor) extensively described in Section 3 of [4].

3.4 | Decomposition of a plate deformation

Let $v \in H^1(S_\varepsilon)^3$ be a plate deformation. Denote

$$\mathbf{D}(v) \doteq \|dist(\nabla v, SO(3))\|_{L^2(S_\varepsilon)} \qquad \forall v \in H^1(S_\varepsilon)^3.$$

From Theorem 3.4 in [4], there exists a constant $C(\Omega)$ depending on the mid-surface such that any $v \in H^1(S_\varepsilon)^3$ satisfying

$$\mathbf{D}(v) \le C(\Omega)\varepsilon^{3/2}, \tag{4}$$ R_cond

admits a decomposition of the kind:

$$v(x) = \mathcal{V}(x_1, x_2) + x_3 \mathbf{R}(x_1, x_2)\mathbf{e}_3 + \overline{v}(x) \qquad \text{a.e. } x \in S_\varepsilon, \tag{5}$$ EQ24New

where $\mathcal{V} \in H^1(\Omega)^3$, $\mathbf{R} \in H^1(\Omega; SO(3))$ and $\overline{v} \in H^1(S_\varepsilon)^3$. In particular, the quantity $\mathcal{V} + x_3\mathbf{R}\mathbf{e}_3$ is called elementary deformation, while the quantity $\overline{v}$ is called warping (or reminder term).

Given the above decomposition and the definition of boundary conditions (3), it follows that a plate deformation belongs to $\mathbb{D}_\varepsilon$ if and only if

$$\mathcal{V} = I_d \quad \text{and} \quad \mathbf{R} = \mathbf{I}_3 \qquad \text{a.e. on } \gamma, \qquad \overline{v} = 0 \quad \text{a.e. on } \Gamma_r. \tag{6}$$ EqBC

Again from Theorem 3.4 in [4], the above fields satisfy the following estimates ($\alpha \in \{1, 2\}$):

$$\begin{aligned} &\|\overline{v}\|_{L^2(S_\varepsilon)} + r\|\nabla\overline{v}\|_{L^2(S_\varepsilon)} \le Cr\mathbf{D}(v), \qquad &&\|\partial_\alpha \mathbf{R}\|_{L^2(\Omega)} \le \frac{C}{r^{3/2}}\mathbf{D}(v), \\ &\|\partial_\alpha \mathcal{V} - \mathbf{R}\mathbf{e}_\alpha\|_{L^2(\Omega)^3} \le \frac{C}{r^{1/2}}\mathbf{D}(v), \qquad &&\|\nabla v - \mathbf{R}\|_{L^2(S_\varepsilon)} \le C\mathbf{D}(v). \end{aligned} \tag{7}$$ E0

3.5 | Tensor of a plate deformation

ymgradbeam1⟩ Given the plate deformation $v \in H^1(S_\varepsilon)^3$ as in (5), we want to give an expression of the associated Green-St. Venant tensor

$$\frac{1}{2}\big((\nabla v)^T \nabla v - \mathbf{I}_3\big),$$

with which we will be later easier to go to the limit with.

First, we note that since $\mathbf{R}$ beongs to $SO(3)$, the following identity holds:

$$(\nabla v)^T \nabla v - \mathbf{I}_3 = \underbrace{(\nabla v - \mathbf{R})^T(\nabla v - \mathbf{R})}_{\mathbf{A}(v)^1} + \underbrace{\mathbf{R}^T(\nabla v - \mathbf{R}) + (\nabla v - \mathbf{R})^T\mathbf{R}}_{\mathbf{B}(v)}.$$

The matrix $\mathbf{B}(v)$ is the sum of two symmetric matrices

$$\mathbf{B}(v) = \mathbf{B}_{el}(v) + \mathbf{B}_w(\overline{v}),$$

[1] As we will see later, the term $\mathbf{A}(v)$ will weakly converge to zero in the limit and therefore does not need to be calculated explicitely.

where the first term collects the fields of the elementary displacemnt, it is defined by

$$\mathbf{B}_{el}(v) \doteq \begin{pmatrix} 2\big[(\partial_1\mathcal{V}-\mathbf{Re}_1)+z_3\partial_1\mathbf{Re}_3\big]\cdot\mathbf{Re}_1 & * & * \\ \big[(\partial_1\mathcal{V}-\mathbf{Re}_1)+z_3\partial_1\mathbf{Re}_3\big]\cdot\mathbf{Re}_2+\big[(\partial_2\mathcal{V}-\mathbf{Re}_2)+z_3\partial_2\mathbf{Re}_3\big]\cdot\mathbf{Re}_1 & 2\big[(\partial_2\mathcal{V}-\mathbf{Re}_2)+z_3\partial_2\mathbf{Re}_3\big]\cdot\mathbf{Re}_2 & * \\ \big[(\partial_1\mathcal{V}-\mathbf{Re}_1)+z_3\partial_1\mathbf{Re}_3\big]\cdot\mathbf{Re}_3 & \big[(\partial_2\mathcal{V}-\mathbf{Re}_2)+z_3\partial_2\mathbf{Re}_3\big]\cdot\mathbf{Re}_3 & 0 \end{pmatrix},$$

while the second term collects the fields coming of the warping and is defined by

$$\mathbf{B}_{w}(v) \doteq \begin{pmatrix} 2\partial_1\overline{v}\cdot\mathbf{Re}_1 & * & * \\ \partial_1\overline{v}\cdot\mathbf{Re}_2+\partial_2\overline{v}\cdot\mathbf{Re}_1 & 2\partial_2\overline{v}\cdot\mathbf{Re}_2 & * \\ \partial_1\overline{v}\cdot\mathbf{Re}_3+\partial_3\overline{v}\cdot\mathbf{Re}_1 & \partial_2\overline{v}\cdot\mathbf{Re}_3+\partial_3\overline{v}\cdot\mathbf{Re}_2 & 2\partial_3\overline{v}\cdot\mathbf{Re}_3 \end{pmatrix}.$$

In conclusion, we have

$$(\nabla v)^T\nabla v-\mathbf{I}_3=\mathbf{A}(v)+\mathbf{B}_{el}(v)+\mathbf{B}_{w}(v).$$

4 | THE NONLINEAR ELASTICITY PROBLEM

⟨S4⟩ As depicted in Figure 3 , we define the non-perforated and perforated two dimensional periodic cell by

$$Y \doteq (0,2)^2, \qquad Y^* = Y \setminus \Big(\bigcup_{a,b=1}^{2} [a+\kappa, a+1-\kappa]\times[b+\kappa, b+1-\kappa]\Big).$$

and the three dimensional ones by

$$\mathcal{Y} \doteq Y\times(-2\kappa,2\kappa), \qquad \mathcal{Y}^* \doteq Y^*\times(-2\kappa,2\kappa).$$

It is then clear that Ω, $\mathcal{S}_\varepsilon$ and $\mathcal{S}_\varepsilon^*$ consist of a paving of ε-sized cells of the kind Y, $\mathcal{Y}$ and $\mathcal{Y}^*$ respectively. We now set the total

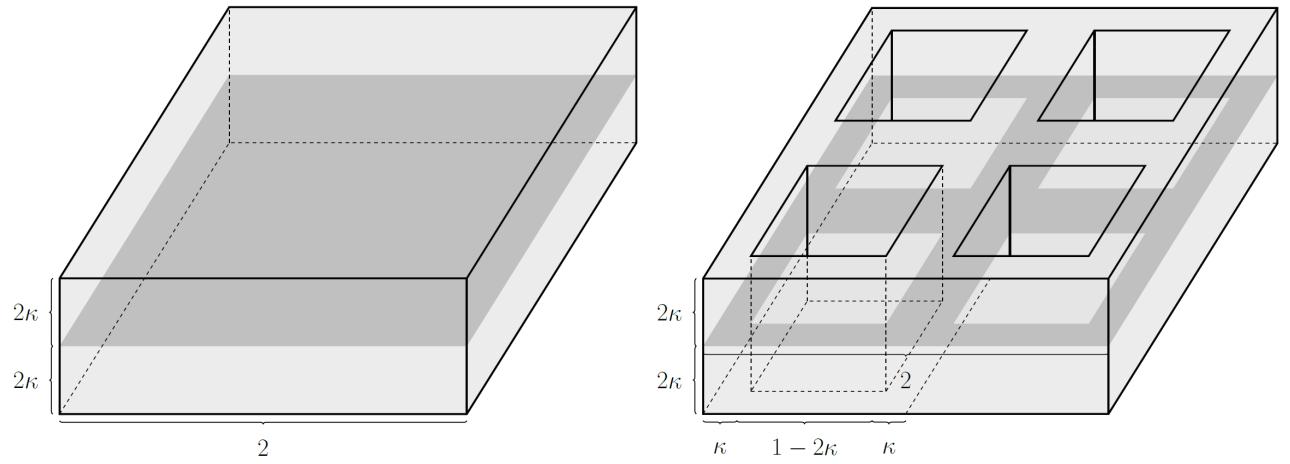

FIGURE 3 On the left, the 2-periodic reference cell $\mathcal{Y}$, in darker its restriction to the mid-surface Y. The non-perforated plate $\mathcal{S}_\varepsilon$ is a paving of ε cells of this kind. On the right, the 2-periodic perforated reference cell $\mathcal{Y}^*$, in darker its restriction to the mid-surface Y^*. The perforate plate $\mathcal{S}_\varepsilon^*$ is a paving of ε cells of this kind.

⟨fig_cells⟩

elastic energy to be the functional

$$\mathcal{J}_\varepsilon(v)=\int_{\mathcal{S}_\varepsilon^*}\widehat{W}_\varepsilon(\cdot,\nabla v)\,dx-\int_{\mathcal{S}_\varepsilon^*}f_\varepsilon\cdot(v-I_d)\,dx, \qquad \forall v\in\mathbb{D}_\varepsilon^*, \tag{8}$$

J

where the local energy density $\widehat{W}_\varepsilon$ is the map from $\mathcal{Y}^*\times\mathbf{M}_3\to\mathbb{R}^+\cup\{+\infty\}$ defined by

$$\widehat{W}_\varepsilon(\cdot,M)\doteq\widehat{W}\Big(\frac{\cdot}{\varepsilon},M\Big)\doteq\begin{cases} Q\Big(\frac{\cdot}{\varepsilon},\frac{1}{2}\big(M^TM-\mathbf{I}_3\big)\Big) & \text{if } \det(M)>0,\\ +\infty & \text{if } \det(M)\le 0.\end{cases}$$

In particular, Q is a positive quadratic form from $\mathcal{Y}^* \times \mathbf{S}_3 \to \mathbb{R}^+$ defined by

$$Q(y,S) \doteq a_{ijkl}(y)S_{ij}S_{kl}, \quad \text{for a.e. } y \in \mathcal{Y}^* \text{ and for all } S \in \mathbf{S}_3, \tag{9}$$ ineq_Q

where $a_{ijkl} \in L^\infty(\mathcal{Y}^*)$ are the coeffiecients of a fourth order tensor, which is periodic with respect of $\mathbf{e}_1$ and $\mathbf{e}_2$, symmetric ($a_{ijkl} = a_{jikl} = a_{klji}$) and positive definite. This last property implies that Q is coercive:

$$\exists C_0 > 0 \text{ such that } Q(y,S) \geq C_0|S|_F^2 \text{ for a.e. } y \in \mathcal{Y}^* \text{ and for all } S \in \mathbf{S}_3{}^2,$$

Hence, we have

$$\begin{aligned} &\exists C_0 > 0 \text{ such that } \forall M \in \mathbf{M}_3, \quad \text{such that} \quad \det(M) > 0, \\ &\widehat{W}(\cdot, M) \geq C_0|M^TM - \mathbf{I}_3|_F^2 \geq C_0 \mathrm{dist}^2\big(M, SO(3)\big) \qquad \text{a.e. in } \mathcal{Y}^*. \end{aligned} \tag{10}$$ ineq_W

Remark 1. An example of a local elastic energy satisfying such assumptions is given by St. Venant-Kirchhoff's material, for which (λ and μ are the Lamé constants)

$$\widehat{W}(M) \doteq \begin{cases} \frac{\lambda}{8}\big(\mathrm{tr}\big(M^TM - \mathbf{I}_3\big)\big)^2 + \frac{\mu}{4}\big(\mathrm{tr}\big(M^TM - \mathbf{I}_3\big)^2\big) & \text{if } \det(M) > 0, \\ +\infty & \text{if } \det(M) \leq 0. \end{cases}$$

Finally, the nonlinear elasticity problem for the textile reads as

$$m_\varepsilon \doteq \inf_{v \in \mathbb{D}_\varepsilon} \mathcal{J}_\varepsilon(v). \tag{11}$$ problem1

It is well known that the existence of a minimizer for $\mathcal{J}_\varepsilon$ in $\mathbb{D}_\varepsilon$ is still an open problem.

5 | A BOUND FOR THE DEFORMATION FIELDS

⟨S5⟩ In order to find the weak convergences of the fields involved in the nonlinear elasticity problem, that are, those that appear in the representation of tensor (see Subsection 3.5) and deformation (see (5)), we need to bound them.

We start by establishing their dependency on the quantity $\mathbf{D}(v)$.

Lemma 1. The fields $\mathbf{R}$ and $\mathcal{V}$ satisfy

$$\|\mathbf{R} - \mathbf{I}_3\|_{H^1(\Omega)} + \|\mathcal{V} - I_d\|_{H^1(\Omega)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\mathbf{D}(v). \tag{12}$$ est_fields1

Proof. Estimate $(12)_1$ is a direct consequence of estimate $(7)_2$, the boundary conditions (6) and the Poincaré inequality. Then, from $(12)_1$ and $(7)_3$ we have

$$\|\partial_\alpha \mathcal{V} - \mathbf{e}_\alpha\|_{L^2(\Omega)} \leq \|\partial_\alpha \mathcal{V} - \mathbf{R}\mathbf{e}_\alpha\|_{L^2(\Omega)} + \|(\mathbf{R} - \mathbf{I}_3)\mathbf{e}_\alpha\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\mathbf{D}(v),$$

thus $(12)_2$ follows from the above estimate, the boundary conditions (6) and the Poincaré inequality. □

5.1 | Assumption on the right hand side

⟨sec_forces⟩ Now, we want to give the relation between the quantity $\mathbf{D}(v)$ and the applied forces.

Without loss of generality, the seek for a minimizer in (11) implies that we can consider only deformations $v \in \mathbb{D}_\varepsilon^*$ such that $\mathcal{J}_\varepsilon(v) \leq \mathcal{J}_\varepsilon(I_d) = 0$. Hence, by the definition of (8) and property (10), we have

$$C_0\mathbf{D}(v)^2 \leq \int_{S_\varepsilon^*} \widehat{W}_\varepsilon(\cdot, \nabla v)\, dz \leq \left| \int_{S_\varepsilon^*} f_\varepsilon \cdot (v - I_d)\, dz \right|. \tag{13}$$ ineq

Let now f be in $H^1\big(\Omega \times (-2r, 2r)\big)^3$. Define the rescaled forces

$$f_\varepsilon(x) \doteq \varepsilon^\tau f(x_1, x_2) \quad \text{for a.e. } x = (x_1, x_2, x_3) \in \Omega \times (-2r, 2r). \tag{14}$$ forces_est

[2] We denote $|\cdot|_F$ the Frobenius norm of a 3×3 matrix.

By the decomposition in the previous subsection, we have that there exists a constant C_1 such that

$$\left|\int_{S_\varepsilon^*} f_\varepsilon \cdot (v - I_d)\, dz\right| \le \left|\int_{S_\varepsilon^*} f_\varepsilon \cdot \left(\left(\mathcal{V} - I_d\right) + x_3\left(\mathbf{R} - \mathbf{I}_3\right)\mathbf{e}_3 + \overline{v}_\varepsilon\right)\, dx\right|$$
$$\le C_1\sqrt{\varepsilon}\,\varepsilon^\tau \|f\|_{L^2(\Omega)}\left(\sqrt{\varepsilon}\|\mathcal{V} - I_d\|_{L^2(\Omega)} + \varepsilon\sqrt{\varepsilon}\|\mathbf{R} - \mathbf{I}_3\|_{L^2(\Omega)} + \|\overline{v}\|_{L^2(S_\varepsilon^v)}\right)$$

and thus from estimates (7), (12) and the force assumption (14), we get that

$$\mathbf{D}(v) \le \frac{C_1}{C_0}\varepsilon^{\tau-\frac{1}{2}}\|f\|_{L^2(\Omega)}. \tag{15}$$ GD_ineq

The constant does not depend on ε and τ.

5.2 | Fields bound and infimum for the minimization problem

In order to stay in the case of large deformations and under assumption (4)*, from estimate* (15) *we find convenient to choose* $\tau = 2$ *and have*

$$\mathbf{D}(v_\varepsilon) \le \frac{C_1}{C_0}\varepsilon^{3/2}\|f\|_{L^2(\Omega)}. \tag{16}$$ nowon_1

In addition, to ensure that (4) is satisfied and use decomposition (5) (see[4]), we further assume that

$$\|f\|_{L^2(\Omega)} \le \frac{C_0 C(\Omega)}{C_1}.$$

Applying (16) to estimates $(7)_1$ and (12), we can finally obtain a bound for the deformation fields

$$\|\mathcal{V}_\varepsilon - I_d\|_{H^1(\Omega)} + \|\mathbf{R}_\varepsilon - \mathbf{I}_3\|_{H^1(\Omega)} \le C, \qquad \|\overline{v}\|_{L^2(S_\varepsilon)} + \varepsilon\|\nabla\overline{v}\|_{L^2(S_\varepsilon)} \le C\varepsilon^{5/2}. \tag{17}$$ estimates

Moreover, assumption (16) implies that for every $v \in \mathbb{D}_\varepsilon$ satisfying $\mathcal{J}_\varepsilon(v) \le 0$, we get

$$\left|\int_{S_\varepsilon} f_\varepsilon \cdot (v - I_d)\, dz\right| \le C\varepsilon^3. \tag{18}$$ est_F

Hence,

$$0 \ge \mathcal{J}_\varepsilon(v) = \int_{S_\varepsilon} \widehat{W}_\varepsilon(\nabla v)\, dz - \int_{S_\varepsilon} f_\varepsilon \cdot (v - I_d)\, dz \ge -C\varepsilon^3.$$

As a consequence

$$-C\varepsilon^3 \le m_\varepsilon = \inf_{v\in\mathbb{D}_\varepsilon} \mathcal{J}_\varepsilon(v).$$

The constant is strictly positive and depends on $\|f\|_{L^2(\Omega)}$, but not on the parameter ε.

Given a minimizing sequence $\{v_\varepsilon^0\}_\varepsilon$ for the functional $\mathcal{J}_\varepsilon$ in $\mathbb{D}_\varepsilon$, the above estimate implies that we can set

$$m \doteq \liminf_{\varepsilon\to 0} \frac{m_\varepsilon}{\varepsilon^3} = \lim_{\varepsilon\to 0} \frac{\mathcal{J}_\varepsilon(v_\varepsilon^0)}{\varepsilon^3}.$$

Since $\mathcal{J}_\varepsilon(I_d) = 0$, without a loss of generality, we can assume that $\mathcal{J}_\varepsilon(v_\varepsilon^0) \le 0$.

6 | ASYMPTOTIC BEHAVIOR OF A SEQUENCE OF DEFORMATIONS

⟨S6⟩ In this section we give the asymptotic behavior of the fields of a sequence of deformations $\{v_\varepsilon\}_\varepsilon$ decomposed as in (5).

6.1 | Weak limit of the macroscopic fields

We first set the left and bottom limit boundary of Ω in the following way (see also Figure 1):

$$\mathbf{T}(l, l) \doteq \left\{\text{triangle whose vertexes are } (0,0),\ (0,l),\ (l,0)\right\}.$$

Then, we set the limit spaces

$$H^2_{\mathbf{T}}(\Omega)^3 \doteq \left\{\phi \in H^2(\Omega)^3 \,\middle|\, \phi = I_d \quad \text{a.e. in } \mathbf{T}(l,l)\right\},$$
$$H^1_{\mathbf{T}}(\Omega)^{3\times 3} \doteq \left\{\phi \in H^1(\Omega)^{3\times 3} \,\middle|\, \phi = \mathbf{I}_3 \text{ a.e. in } \mathbf{T}(l,l)\right\}$$

Below, we give the first weak convergences.

⟨61rev⟩ **Lemma 2.** There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\mathcal{V} \in H^2_{\mathbf{T}}(\Omega)^3$, $\mathbf{R} \in H^1_{\mathbf{T}}(\Omega; SO(3))$ such that

$$\begin{aligned} &\mathcal{V}_\varepsilon \to \mathcal{V} \quad \text{strongly in } H^1(\Omega)^3, \\ &\mathbf{R}_\varepsilon \rightharpoonup \mathbf{R} \quad \text{weakly in } H^1(\Omega)^{3\times 3} \text{ and strongly in } L^4(\Omega)^{3\times 3} \end{aligned} \tag{19}$$

EQ0

and $\mathcal{Z}_\alpha \in L^2(\Omega)^3$ such that

$$\frac{1}{\varepsilon}\left(\partial_\alpha \mathcal{V} - \mathbf{R}\mathbf{e}_\alpha\right) \rightharpoonup \mathcal{Z}_\alpha \quad \text{weakly in } L^2(\Omega)^3. \tag{20}$$

conv_Z

Moreover, the following limit identities hold:

$$\partial_1 \mathcal{V} = \mathbf{R}\mathbf{e}_1, \quad \partial_2 \mathcal{V} = \mathbf{R}\mathbf{e}_2 \quad \text{a.e. in } \Omega. \tag{21}$$

id

Proof. By estimates $(17)_1$, $(7)_3$ and Theorem 3.3 in[4], there exist $\mathcal{V} \in H^1(\Omega)^3$, $\mathbf{R} \in H^1(\Omega)^{3\times 3}$ and $\mathcal{Z}_\alpha \in L^2(\Omega)^3$ such that convergences (19)-(20) hold.

Now, we prove that $\mathbf{R} \in H^1(\Omega; SO(3))$. By convergence $(19)_2$ and the compact embedding of $H^1(\Omega)^{3\times 3}$ in $L^4(\Omega)^{3\times 3}$, we get that

$$\begin{aligned} &\mathbf{R}_\varepsilon \to \mathbf{R} \ \text{ strongly in } L^4(\Omega)^{3\times 3}, \\ &\mathbf{R}_\varepsilon^T \mathbf{R}_\varepsilon \to \mathbf{R}^T \mathbf{R} \ \text{ strongly in } L^2(\Omega)^{3\times 3}, \\ &\det(\mathbf{R}_\varepsilon) \to \det(\mathbf{R}) \ \text{ strongly in } L^{4/3}(\Omega). \end{aligned}$$

On the other hand, since $\mathbf{R}_\varepsilon \in SO(3)$ for every ε, we have that

$$\begin{aligned} &\mathbf{R}_\varepsilon^T \mathbf{R}_\varepsilon \to \mathbf{I}_3 \ \text{ strongly in } L^2(\Omega)^{3\times 3}, \\ &\det(\mathbf{R}_\varepsilon) \to 1 \ \text{ strongly in } L^{4/3}(\Omega). \end{aligned}$$

As a consequence, $\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$ and $\det(\mathbf{R}) = 1$ a.e. in Ω. Thus $\mathbf{R} \in H^1(\Omega; SO(3))$.

Equalities (21) follow from (20) and convergences (19). Moreover, (21) implies that $\nabla\mathcal{V} \in H^1(\Omega)^{3\times 2}$ and thus $\mathcal{V} \in H^2(\Omega)^3$.

It is left to prove that $\mathbf{R} \in H^1_{\mathbf{T}}(\Omega; SO(3))$ (and thus that $\mathcal{V} \in H^2_{\mathbf{T}}(\Omega)^3$). From the clamp conditions, we already know that in the limit, the following boundary conditions hold:

$$\mathcal{V} = I_d \quad \text{and} \quad \mathbf{R} = \mathbf{I}_3 \qquad \text{a.e. in } \boldsymbol{\gamma}. \tag{22}$$

lim_BC

Let λ be in $(0, l]$ and define the triangle $\mathbf{T}(\lambda, l)$ whose vertexes are $(0,0)$, $(\lambda, 0)$ and $(0, l)$. By integrating the quantity $\mathbf{R}$ on the domain $\mathbf{T}(\lambda, l)$ on the lines directed by the vector $\lambda\mathbf{e}_1 - \mathbf{e}_2$, using the limit identity (21) and the above limit boundary conditions (22), we obtain:

$$\begin{aligned} \int\limits_{\mathbf{T}(\lambda,1)} \mathbf{R}(z_1,z_2)(\lambda\mathbf{e}_1 - \mathbf{e}_2)\,dz_1dz_2 &= \int_0^l \int_0^{\lambda T} \mathbf{R}\Big(t, T - \frac{t}{\lambda}\Big)(\lambda\mathbf{e}_1 - \mathbf{e}_2)\,dtdT = \int_0^l \int_0^{\lambda T} \big(\lambda\partial_1\mathcal{V} - \partial_2\mathcal{V}\big)\Big(t, T - \frac{t}{\lambda}\Big)\,dtdT \\ = \int_0^l \int_0^{\lambda T} \lambda\partial_t\mathcal{V}\Big(t, T - \frac{t}{\lambda}\Big)\,dtdT &= \int_0^l \lambda\big(\mathcal{V}(\lambda T, 0) - \mathcal{V}(0,T)\big)\,dT = \int_0^l \lambda T(\lambda\mathbf{e}_1 - \mathbf{e}_2)\,dT = \int\limits_{\mathbf{T}(\lambda,l)} \mathbf{I}_3(\lambda\mathbf{e}_1 - \mathbf{e}_2)\,dz_1dz_2. \end{aligned}$$

As a consequence, we first get that $\mathbf{R}(\lambda\mathbf{e}_1 - \mathbf{e}_2)\cdot(\lambda\mathbf{e}_1 - \mathbf{e}_2) = |\lambda\mathbf{e}_1 - \mathbf{e}_2|^2$ a.e. in $\mathbf{T}(\lambda, l)$, from which we deduce that $\mathbf{R}(\lambda\mathbf{e}_1 - \mathbf{e}_2) = \lambda\mathbf{e}_1 - \mathbf{e}_2$ a.e. in $\mathbf{T}(\lambda, l)$. Similarly, we prove that $\mathbf{R}(\mathbf{e}_1 - \lambda\mathbf{e}_2) = \mathbf{e}_1 - \lambda\mathbf{e}_2$ a.e. in $\mathbf{T}(l, \lambda)$ for all $\lambda \in (0, l]$.

Now, let a be in $(0, l)$. Consider the triangle $\mathbf{T}(a, a)$ whose vertexes are $(0,0)$, $(a, 0)$ and $(0, a)$. Clearly, we have $\mathbf{T}(a,a) \subset \mathbf{T}(a,l)$ and $\mathbf{T}(a,a) \subset \mathbf{T}(l,a)$. So in $T(a,a)$ we have

$$\mathbf{R}(\mathbf{e}_1 - a\mathbf{e}_2) = \mathbf{e}_1 - a\mathbf{e}_2 \qquad \text{and} \qquad \mathbf{R}(a\mathbf{e}_1 - \mathbf{e}_2) = a\mathbf{e}_1 - \mathbf{e}_2 \qquad \text{a.e. in } \mathbf{T}(a,a).$$

Thus

$$\mathbf{R}\mathbf{e}_\alpha = \mathbf{e}_\alpha, \quad \alpha \in \{1,2\}, \quad \text{a.e. in } \mathbf{T}(a,a).$$

Hence $\mathbf{R} = \mathbf{I}_3$ a.e. in $\mathbf{T}(a,a)$. As a is any element in $(0,l)$, we have $\mathbf{R} = \mathbf{I}_3$ a.e. in $\mathbf{T}(l,l)$. □

From the regularity of $\mathbf{R}$, some more identities hold in the limit.

⟨cor1⟩ **Corollary 1.** There exist two anti-symmetric matrices $\mathbf{A}^{(1)}, \mathbf{A}^{(2)} \in L^2(\Omega)^{3\times 3}$ such that

$$\partial_1 \mathbf{R} = \mathbf{R}\,\mathbf{A}^{(1)} \quad \text{and} \quad \partial_2 \mathbf{R} = \mathbf{R}\,\mathbf{A}^{(2)}. \tag{23}$$ id_2

Moreover, we have that

$$\mathbf{A}^{(1)}\,\mathbf{e}_2 = \mathbf{A}^{(2)}\,\mathbf{e}_1. \tag{24}$$ id_3

Proof. Equalities (23) directly follow from the fact that $\mathbf{R} \in H^1(\Omega; SO(3))$, while (24) is an easy consequence of (21) and (23). □

Since $\mathbf{R}$ depends on $\mathcal{V}$ via identity (23) and the map $\mathcal{V} \mapsto \mathbf{R}$ is one-to-one, from now on we will denote such dependency by $\mathbf{R}(\mathcal{V})$. This allows to define the following limit set:

$$\mathbb{D} \doteq \left\{ \mathcal{V} \in H^2_{\mathbf{T}}(\Omega)^3 \;\middle|\; \exists! \mathbf{R} \in H^1_{\mathbf{T}}(\Omega; SO(3)) \text{ such that } \partial_\alpha \mathcal{V} = \mathbf{R}\mathbf{e}_\alpha \right\}.$$

Now, for every deformation of the plate $\mathcal{V} \in \mathbb{D}$, the principal directions of the tangential plane ant any point $x' \in \Omega$ are defined by $\langle \partial_1 \mathcal{V}, \partial_2 \mathcal{V} \rangle = \langle \mathbf{R}(\mathcal{V})\mathbf{e}_1, \mathbf{R}(\mathcal{V})\mathbf{e}_2 \rangle$ and the known orthogonal vector to the surface by $\mathbf{n} \doteq \mathbf{R}(\mathcal{V})\mathbf{e}_3$. In this sense, the associated second fundamental form is defined by (we denote the entry terms with the usual notation $\mathbf{L}, \mathbf{M}, \mathbf{N} \in L^2(\Omega)$):

$$\mathbf{II} = \begin{pmatrix} \mathbf{L} & * \\ \mathbf{M} & \mathbf{N} \end{pmatrix} \doteq \begin{pmatrix} \mathbf{R}\mathbf{e}_3 \cdot \partial_1 \mathbf{R}\mathbf{e}_1 & * \\ \mathbf{R}\mathbf{e}_3 \cdot \partial_1 \mathbf{R}\mathbf{e}_2 & \mathbf{R}\mathbf{e}_3 \cdot \partial_2 \mathbf{R}\mathbf{e}_2 \end{pmatrix}, \quad \text{a.e. } x' \in \Omega. \tag{25}$$ second_form

Note that we have

$$\mathbf{M}(\mathcal{V}) = \mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_1 \mathbf{R}(\mathcal{V})\mathbf{e}_2 = \mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{12} \mathcal{V} = \mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{21} \mathcal{V} = \mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_2 \mathbf{R}(\mathcal{V})\mathbf{e}_1$$

With this notation, identities (23)-(24) lead to the following characterization of the anti-symmetric matrices in Corollary 1:

$$\mathbf{A}^{(1)} = \begin{pmatrix} 0 & 0 & -\mathbf{L} \\ 0 & 0 & -\mathbf{M} \\ \mathbf{L} & \mathbf{M} & 0 \end{pmatrix}, \qquad \mathbf{A}^{(2)} = \begin{pmatrix} 0 & 0 & -\mathbf{M} \\ 0 & 0 & -\mathbf{N} \\ \mathbf{M} & \mathbf{N} & 0 \end{pmatrix} \tag{26}$$ matrices

We can now get our first important conclusion.

Corollary 2. The admissible deformations of $\mathbb{D}$ transforms the plate's mid-surface $\Omega \times \{0\}$ into a developable surface.

Proof. Since the principal curvatures k_1 and k_2 are the roots of the second order equation $\det(\mathbf{II} - k\mathbf{I}) = 0$, it follows that the Gaussian curvature $K = k_1 k_2$ is equal to zero if and only if such equation has no constant term, which is satisfied if and only if $\det(\mathbf{II}) = 0$, or equivalently, if and only if $\mathbf{LN} - \mathbf{M}^2 = 0$. But this is true due to Lemma 9 in Appendix. □

Remark 2. Note that since the mid-surface of the plate can be transformed into a developable surface, it follows that the nonlinear homogenized plate has a very few amount of possible configurations. In particular, this leads to the impossibility to cover half a sphere.

6.2 | Weak limit of the unfolded macroscopic fields

We can now proceed to the unfolding of the macroscopic fields of the plate deformations. Since these fields lie on the mid-surface of the plate, which is the domain Ω, we start by defining the following unfolding operator.

Definition 1. (In-plane unfolding operator) For every measurable function φ in $L^2(\Omega)$, one defines the measurable function $\mathcal{T}_\varepsilon(\varphi)$ in $L^2(\Omega \times \mathcal{Y})$ by

$$\mathcal{T}_\varepsilon(\varphi)(z', y') = \varphi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon y'\right) \quad \text{for a.e. } (z', y') \in \Omega \times Y^*.$$

Its properties are the typical ones and can be found in [7]. Among them, we recall that

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\Omega\times\mathcal{Y})} \le C\|\varphi\|_{L^2(\Omega)}, \qquad \forall \varphi \in L^2(\Omega).$$

We have the following.

⟨61rev-0⟩ **Lemma 3.** There exist $\widehat{\mathbf{R}} \in L^2(\Omega; H^1_{per,0}(Y^*))^{3\times 3}$ such that

$$\begin{aligned} &\mathcal{T}_\varepsilon\big(\mathcal{V}_\varepsilon\big) \to \mathcal{V} \ \text{ strongly in } L^2(\Omega; H^1(Y^*))^3, \\ &\mathcal{T}_\varepsilon\big(\nabla\mathcal{V}_\varepsilon\big) \to \nabla\mathcal{V} \ \text{ strongly in } L^2(\Omega\times Y^*)^{3\times 2} \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\mathcal{T}_\varepsilon\big(\mathbf{R}_\varepsilon\big) \rightharpoonup \mathbf{R}(\mathcal{V}) \ \text{ strongly in } L^2(\Omega; H^1(Y^*))^{3\times 3}, \\ &\mathcal{T}_\varepsilon\big(\nabla\mathbf{R}_\varepsilon\big) \rightharpoonup \nabla\mathbf{R}(\mathcal{V}) + \nabla_y\widehat{\mathbf{R}} \ \text{ weakly in } L^2(\Omega\times Y^*)^{3\times 3}. \end{aligned} \tag{28}$$

Moreover, there exists $\widehat{\mathfrak{u}} \in L^2(\Omega; H^1_{per,0}(Y^*))^3$ such that

$$\mathcal{T}_\varepsilon\big(\partial_\alpha\mathcal{V}_\varepsilon - \mathbf{R}_\varepsilon\mathbf{e}_\alpha\big) \rightharpoonup \mathcal{Z}_\alpha + \partial_{y_\alpha}\widehat{\mathfrak{u}} - \widehat{\mathbf{R}}\mathbf{e}_\alpha \ \text{ weakly in } L^2(\Omega; H^1(Y^*))^3. \tag{29}$$

In addition, there exist an anti-symmetric matrix $\widehat{\mathbf{A}} \in L^2(\Omega; H^1_{per,0}(Y^*)^{3\times 3}$ such that

$$\widehat{\mathbf{R}} = \mathbf{R}(\mathcal{V})\,\widehat{\mathbf{A}} \qquad \text{a.e. in } \Omega\times Y^*. \tag{30}$$

Proof. Convergences (27) are a consequence of $(19)_1$ and Proposition 1.39 in [7], while (28) are a consequence of $(19)_2$ and Corollary 1.37 in [7]. Convergence (29) is a consequence of (20), (28) and Lemma 11.11 of [7].

By the strong convergence (20) and Proposition 1.9 of [7], we get

$$\mathcal{T}_\varepsilon\big(\mathbf{R}_\varepsilon\big) \to \mathbf{R}(\mathcal{V}) \quad \text{ strongly in } L^2(\Omega\times Y^*)^{3\times 3}. \tag{31}$$

Now we prove (30). By the fact that for every ε we have $\mathbf{R}_\varepsilon \in SO(3)$ a.e. in Ω, it follows that $(\partial_1\mathbf{R}_\varepsilon)^T\mathbf{R}_\varepsilon + \mathbf{R}_\varepsilon^T\partial_1\mathbf{R}_\varepsilon = 0$ a.e. in Ω. Hence, from convergences (28), (31) and Corollary 1.18 of [7], in the limit we obtain

$$\big(\partial_1\mathbf{R}(\mathcal{V}) + \partial_{y_1}\widehat{\mathbf{R}}\big)^T\mathbf{R}(\mathcal{V}) + \mathbf{R}(\mathcal{V})^T\big(\partial_1\mathbf{R}(\mathcal{V}) + \partial_{y_1}\widehat{\mathbf{R}}\big) = 0 \quad \text{a.e. in } \Omega\times Y^*,$$

which together with identity (23) implies that

$$\big(\partial_{y_1}\widehat{\mathbf{R}}\big)^T\mathbf{R}(\mathcal{V}) + \mathbf{R}(\mathcal{V})^T\big(\partial_{y_1}\widehat{\mathbf{R}}\big) = 0 \quad \text{a.e. in } \Omega\times Y^*.$$

Hence, identity (30) follows since $\widehat{\mathbf{R}} \in L^2(\Omega; H^1_{per,0}(Y^*))$. □

6.3 | Weak limit of the unfolded warping

We can now proceed to the unfolding of the warping. Before doing so, we need to define a second unfolding operator, which allows to unfold the complete three dimensional textile structure S^*_ε.

Definition 2 (Global unfolding operator)**.** For every measurable function Φ on S^*_r, one defines the measurable functions $\Pi_\varepsilon(\Phi)$ on $\Omega\times\mathcal{Y}^*$ by

$$\Pi_\varepsilon(\Phi)(z', y) = \Phi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big] + \varepsilon y\Big) \quad \text{for a.e. } (z', y) \in \Omega\times\mathcal{Y}^*.$$

Note that this unfolding operator changes the convergence rate, since a dimension reduction is additionally applied. We have

$$\|\Pi_\varepsilon(\phi)\|_{L^2(\Omega\times\mathcal{Y}^*)} \le \frac{C}{\sqrt{\varepsilon}}\|\phi\|_{L^2(S^*_\varepsilon)}, \qquad \forall\phi \in L^2(S^*_\varepsilon). \tag{32}$$

Moreover, we note that such operator is related to the previous one and due to the Trace Theorem, the convergences of the macroscopic fields in Lemma 3 unfolded via $\mathcal{T}_\varepsilon$ will be the same if unfolded via Π_ε.

Below, we show the asymptotic behavior of the unfolded warping components.

⟨warp⟩ **Lemma 4.** There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\overline{v} \in L^2(\Omega; H^1_{per}(\mathcal{Y}^*))^3$ such that the following convergence holds:

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon(\overline{v}_\varepsilon) \rightharpoonup \overline{v} \ \text{ weakly in } \quad L^2(\Omega; H^1(\mathcal{Y}^*))^3. \tag{33}$$

Proof. The proof follows from estimates $(17)_2$, the change of convergence rate (32) for Π_ε and Theorem 1.36 in [7]. □

As a consequence of convergences (31), (33) and Corollary 1.18 of[7], we have (recall the definition of $\mathbf{B}_w$ in Subsection 3.5)

$$\frac{1}{2\varepsilon}\Pi_\varepsilon\big(\mathbf{B}_w(\overline{v}_\varepsilon)\big) \rightharpoonup \mathcal{E}_y\big(\mathbf{R}(\mathcal{V})^T\overline{v}\big) \quad \text{weakly in} \quad L^2(\Omega\times\mathcal{Y}^*)^{3\times3}, \tag{34}$$

strain_warp

where for every $\phi \in H^1(\mathcal{Y}^*)^3$, the quantity $\mathcal{E}_y$is the symmetric gradient defined by

$$\mathcal{E}_y(\phi) = \begin{pmatrix} \partial_{y_1}\phi\cdot\mathbf{e}_1 & * & * \\ \frac{1}{2}\big(\partial_{y_1}\phi\cdot\mathbf{e}_2+\partial_{y_2}\phi\cdot\mathbf{e}_1\big) & \partial_{y_2}\phi\cdot\mathbf{e}_2 & * \\ \frac{1}{2}\big(\partial_{y_1}\phi\cdot\mathbf{e}_3+\partial_{y_3}\phi\cdot\mathbf{e}_1\big) & \frac{1}{2}\big(\partial_{y_2}\phi\cdot\mathbf{e}_3+\partial_{y_3}\phi\cdot\mathbf{e}_2\big) & \partial_{y_3}\phi\cdot\mathbf{e}_3 \end{pmatrix}.$$

6.4 | Limit Green-St.Venant tensor of the plate deformation

Note that the tensor admits a weak limit in the form of a weak convergent subsequence. Indeed, from (10), (13), (18) and (32), we derive that

$$\Big\|\frac{1}{\varepsilon}\Pi_\varepsilon\big((\nabla v_\varepsilon)^T\nabla v_\varepsilon-\mathbf{I}_3\big)\Big\|_{L^2(\Omega\times\mathcal{Y}^*)}\le C.$$

We now give a characterization of the tensor in the limit. From the computation in Subsection 3.5, we know that

$$\frac{1}{2\varepsilon}\Pi_\varepsilon\big((\nabla v_\varepsilon)^T\nabla v_\varepsilon-\mathbf{I}_3\big)=\frac{1}{2\varepsilon}\Pi_\varepsilon\big(\mathbf{A}(v_\varepsilon)+\mathbf{B}_{el}(v_\varepsilon)+\mathbf{B}_w(v_\varepsilon)\big).$$

We start with the weak convergence of the term $\mathbf{A}$.

⟨lem66⟩ **Lemma 5.** We have

$$\frac{1}{2\varepsilon}\Pi_\varepsilon\big(\mathbf{A}(v_\varepsilon)\big)\rightharpoonup 0 \qquad \text{weakly in } \ L^2(\Omega\times\mathcal{Y}^*)^{3\times3}. \tag{35}$$

eq-limA

Proof. Step 1. We show that the convergence of the LHS of (35) *holds strongly in the space* $L^1(\Omega\times\mathcal{Y}^*)^{3\times3}$.
From estimate $(7)_4$ and the change of convergence rate of the unfolding operator (32), we have that

$$\|\Pi_\varepsilon(\nabla v_\varepsilon-\mathbf{R}_\varepsilon)\|_{L^2(\Omega\times\mathcal{Y}^*)}\le\frac{C}{\sqrt{\varepsilon}}\|\nabla v_\varepsilon-\mathbf{R}_\varepsilon\|_{L^2(S_\varepsilon)}\le C\varepsilon.$$

Hence

$$\frac{1}{\sqrt{2\varepsilon}}\Pi_\varepsilon\big(\nabla v_\varepsilon-\mathbf{R}_\varepsilon\big)\to 0 \quad \text{strongly in } \ L^2(\Omega\times\mathcal{Y}^*)^{3\times3}$$

and thus

$$\frac{1}{\varepsilon}\Pi_\varepsilon\big((\nabla v_\varepsilon-\mathbf{R}_\varepsilon)^T(\nabla v_\varepsilon-\mathbf{R}_\varepsilon)\big)=\frac{1}{\sqrt{2\varepsilon}}\Pi_\varepsilon\big((\nabla v_\varepsilon-\mathbf{R}_\varepsilon)^T\big)\frac{1}{\sqrt{2\varepsilon}}\Pi_\varepsilon\big(\nabla v_\varepsilon-\mathbf{R}_\varepsilon\big)\to 0 \quad \text{strongly in } L^1(\Omega\times\mathcal{Y}^*)^{3\times3}.$$

Step 2. We prove the lemma statement.
From Step 1, the sequence $\Pi_\varepsilon(\mathbf{A}(v_\varepsilon))$ converges strongly to zero in $L^1(\Omega\times\mathcal{Y}^*)^{3\times3}$. Moreover, it is bounded in $L^2(\Omega\times\mathcal{Y}^*)^{3\times3}$. Indeed, from (10), (13) and (18), we have

$$C_0\mathbf{D}(v_\varepsilon)^2\le C_0\int_{S_\varepsilon}|(\nabla v_\varepsilon)^T\nabla v_\varepsilon-\mathbf{I}_3|_F^2\,dz\le\int_{S_\varepsilon}\widehat{W}_\varepsilon(\cdot,\nabla v_\varepsilon)\,dz\le\Big|\int_{S_\varepsilon}f_\varepsilon\cdot(v_\varepsilon-I_d)\,dz\Big|\le C\varepsilon^3.$$

So, using inequality (10) we get

$$\|(\nabla v_\varepsilon)^T\nabla v_\varepsilon-\mathbf{I}_3\|_{L^2(S_\varepsilon)}\le C\varepsilon^{3/2}.$$

Thus

$$\begin{aligned}\frac{1}{\varepsilon}\Big\|\Pi_\varepsilon\big((\nabla v_\varepsilon-\mathbf{R}_\varepsilon)^T(\nabla v_\varepsilon-\mathbf{R}_\varepsilon)\big)\Big\|_{L^2(\Omega\times\mathcal{Y}^*)}&\le\frac{C}{\varepsilon\sqrt{\varepsilon}}\|(\nabla v_\varepsilon-\mathbf{R}_\varepsilon)^T(\nabla v_\varepsilon-\mathbf{R}_\varepsilon)\|_{L^2(S_\varepsilon)}\\&\le\frac{C}{\varepsilon\sqrt{\varepsilon}}\|(\nabla v_\varepsilon)^T\nabla v_\varepsilon-\mathbf{I}_3\|_{L^2(S_\varepsilon)}+\frac{2C}{\varepsilon\sqrt{\varepsilon}}\|\nabla v_\varepsilon-\mathbf{R}_\varepsilon\|_{L^2(S_\varepsilon)}\le C.\end{aligned}$$

The boundedness in L^2 and the strong convergence in L^1 imply the weak convergence in L^2. The proof of (35) is complete. □

Now we proceed to the weak convergence of the terms $\mathbf{B}_{el}+\mathbf{B}_w$.

⟨lem67⟩ **Lemma 6.** One has

$$\frac{1}{2\varepsilon}\Pi_\varepsilon\big(\mathbf{B}_{el}(v_\varepsilon)+\mathbf{B}_w(v_\varepsilon)\big) \rightharpoonup \mathcal{E}(\mathcal{Z},\mathcal{V})+\mathcal{E}_y(\widehat{v}) \quad \text{weakly in } L^2(\Omega\times\mathcal{Y}^*)^{3\times 3},$$

where the second quantity is the symmetric gradient of a function $\widehat{v}\in L^2(\Omega;H^1_{per}(\mathcal{Y}^*))^3$, while the first quantity is the symmetric matrix

$$\mathcal{E}(\mathcal{Z},\mathcal{V}) \doteq \begin{pmatrix} \mathcal{Z}_{11}-y_3\mathbf{L}(\mathcal{V}) & * & 0\\ \mathcal{Z}_{12}-y_3\mathbf{M}(\mathcal{V}) & \mathcal{Z}_{22}-y_3\mathbf{N}(\mathcal{V}) & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{36}$$ ER

where $\mathbf{L},\mathbf{M},\mathbf{N}\in L^2(\Omega)$ are defined in (25), and where

$$\mathcal{Z}_{\alpha\beta} \doteq \frac{1}{2}\big(\mathcal{Z}_\alpha\cdot\partial_\beta\mathcal{V}+\mathcal{Z}_\beta\cdot\partial_\alpha\mathcal{V}\big), \qquad (\alpha,\beta)\in\{1,2\}^2. \tag{37}$$ zetas

Proof. Step 1. The weak convergences.
From Lemma 3, convergence (31) and Corollary 1.18 of[7], together with convergence (34), we get that

$$\frac{1}{2\varepsilon}\Pi_\varepsilon\big(\mathbf{B}_{el}(v_\varepsilon)+\mathbf{B}_w(v_\varepsilon)\big) \rightharpoonup \mathbf{E}+\widehat{\mathbf{E}}+\mathcal{E}_y(\mathbf{R}^T\overline{v}) \quad \text{weakly in } L^2(\Omega\times\mathcal{Y}^*)^{3\times 3},$$

where the first quantity $\mathbf{E}$ is the symmetric matrix that gathers only the macroscopic spaces and is defined by

$$\mathbf{E} \doteq \begin{pmatrix} \big(\mathcal{Z}_1+y_3\partial_1\mathbf{Re}_3\big)\cdot\mathbf{Re}_1 & * & *\\ \frac{1}{2}\Big[\big(\mathcal{Z}_1+y_3\partial_1\mathbf{Re}_3\big)\cdot\mathbf{Re}_2+\big(\mathcal{Z}_2+y_3\partial_2\mathbf{Re}_3\big)\cdot\mathbf{Re}_1\Big] & \big(\mathcal{Z}_2+y_3\partial_2\mathbf{Re}_3\big)\cdot\mathbf{Re}_2 & *\\ \frac{1}{2}\big(\mathcal{Z}_1+y_3\partial_1\mathbf{Re}_3\big)\cdot\mathbf{Re}_3 & \frac{1}{2}\big(\mathcal{Z}_2+y_3\partial_2\mathbf{Re}_3\big)\cdot\mathbf{Re}_3 & 0\end{pmatrix} \tag{38}$$ macro_s

and the second quantity $\widehat{\mathbf{E}}$ is the symmetric matrix that gathers the macroscopic-microscopic fields and is defined by

$$\widehat{\mathbf{E}} \doteq \begin{pmatrix} \big(\partial_{y_1}\widehat{\mathfrak{u}}-\widehat{\mathbf{Re}}_1+y_3\partial_{y_1}\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_1 & * & *\\ \frac{1}{2}\Big[\big(\partial_{y_1}\widehat{\mathfrak{u}}-\widehat{\mathbf{Re}}_1+y_3\partial_{y_1}\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_2+\big(\partial_{y_2}\widehat{\mathfrak{u}}-\widehat{\mathbf{Re}}_2+y_3\partial_{y_2}\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_1\Big] & \big(\partial_{y_2}\widehat{\mathfrak{u}}-\widehat{\mathbf{Re}}_2+y_3\partial_{y_2}\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_2 & *\\ \frac{1}{2}\big(\partial_{y_1}\widehat{\mathfrak{u}}-\widehat{\mathbf{Re}}_1+y_3\partial_{y_1}\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_3 & \frac{1}{2}\big(\partial_{y_2}\widehat{\mathfrak{u}}-\widehat{\mathbf{Re}}_2+y_3\partial_{y_2}\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_3 & 0\end{pmatrix}. \tag{39}$$ micro_s

Step 2. Simplification of the limit form of the tensor.
Setting (37) and taking into account the limit identities (23), we have that

$$\partial_1\mathbf{Re}_3\cdot\mathbf{Re}_2=\partial_2\mathbf{Re}_3\cdot\mathbf{Re}_1 \qquad \text{and} \qquad \partial_1\mathbf{Re}_3\cdot\mathbf{Re}_3=\partial_2\mathbf{Re}_3\cdot\mathbf{Re}_3=0.$$

At the same time, the fact that $\mathbf{R}\in SO(3)$ implies that $\mathbf{Re}_3\cdot\mathbf{Re}_1=0$ and $\mathbf{Re}_3\cdot\mathbf{Re}_2=0$, thus that ($\alpha\in\{1,2\}$)

$$\partial_1\mathbf{Re}_3\cdot\mathbf{Re}_1=-\mathbf{Re}_3\cdot\partial_1\mathbf{Re}_1 \qquad \text{and} \qquad \partial_\alpha\mathbf{Re}_3\cdot\mathbf{Re}_2=-\mathbf{Re}_3\cdot\partial_\alpha\mathbf{Re}_2$$

Applying these changes to the quantity (38) and using the notation in (25), we get that

$$\mathbf{E}=\mathcal{E}(\mathcal{Z},\mathcal{V})+\begin{pmatrix} 0 & 0 & *\\ 0 & 0 & *\\ \frac{1}{2}\mathcal{Z}_1\cdot\mathbf{Re}_3 & \frac{1}{2}\mathcal{Z}_2\cdot\mathbf{Re}_3 & 0\end{pmatrix}.$$

Now, note that the quantity (39) can be rewritten as

$$\widehat{\mathbf{E}}=\begin{pmatrix} \partial_{y_1}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_1 & * & *\\ \frac{1}{2}\Big[\partial_{y_1}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_2+\partial_{y_2}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_1\Big] & \partial_{y_2}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_2 & *\\ \frac{1}{2}\partial_{y_1}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_1 & \frac{1}{2}\partial_{y_2}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3\big)\cdot\mathbf{Re}_2 & 0\end{pmatrix}-\begin{pmatrix} \widehat{\mathbf{Re}}_1\cdot\mathbf{Re}_1 & * & *\\ \frac{1}{2}\big(\widehat{\mathbf{Re}}_1\cdot\mathbf{Re}_2+\widehat{\mathbf{Re}}_2\cdot\mathbf{Re}_1\big) & \widehat{\mathbf{Re}}_2\cdot\mathbf{Re}_2 & *\\ \frac{1}{2}\widehat{\mathbf{Re}}_1\cdot\mathbf{Re}_3 & \frac{1}{2}\widehat{\mathbf{Re}}_2\cdot\mathbf{Re}_3 & 0\end{pmatrix}$$

Define the displacement $\widehat{v}\in L^2(\Omega;H^1_{per}(\mathcal{Y}^*))^3$ by setting

$$\widehat{v} \doteq \mathbf{R}^T\big(\mathfrak{u}+y_3\widehat{\mathbf{Re}}_3+\overline{v}\big)+y_3\big((\mathcal{Z}_1\cdot\mathbf{Re}_3)\mathbf{e}_1+(\mathcal{Z}_1\cdot\mathbf{Re}_3)\mathbf{e}_2\big). \tag{40}$$ v

Straightforward computations give

$$\partial_{y_1}\widehat{v}=\mathbf{R}^T\partial_{y_1}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3+\overline{v}\big), \qquad \partial_{y_2}\widehat{v}=\mathbf{R}^T\partial_{y_2}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{Re}}_3+\overline{v}\big), \qquad \partial_{y_3}\widehat{v}=\mathbf{R}^T\widehat{\mathbf{Re}}_3+\partial_{y_3}\overline{v}+(\mathcal{Z}_1\cdot\mathbf{Re}_3)\mathbf{e}_1+(\mathcal{Z}_1\cdot\mathbf{Re}_3)\mathbf{e}_2.$$

and therefore we obtain

$$
\begin{aligned}
&\mathbf{E}+\widehat{\mathbf{E}}+\mathcal{E}_y(\mathbf{R}^T\overline{v})\\
&=\mathcal{E}(\mathcal{Z},\mathcal{V})-\begin{pmatrix} \widehat{\mathbf{R}}\mathbf{e}_1\cdot\mathbf{R}\mathbf{e}_1 & * & *\\ \frac{1}{2}\big(\widehat{\mathbf{R}}\mathbf{e}_1\cdot\mathbf{R}\mathbf{e}_2+\widehat{\mathbf{R}}\mathbf{e}_2\cdot\mathbf{R}\mathbf{e}_1\big) & \widehat{\mathbf{R}}\mathbf{e}_2\cdot\mathbf{R}\mathbf{e}_2 & *\\ \frac{1}{2}\widehat{\mathbf{R}}\mathbf{e}_1\cdot\mathbf{R}\mathbf{e}_3 & \frac{1}{2}\widehat{\mathbf{R}}\mathbf{e}_2\cdot\mathbf{R}\mathbf{e}_3 & 0\end{pmatrix}\\
&+\begin{pmatrix} \partial_{y_1}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{R}}\mathbf{e}_3\big)\cdot\mathbf{R}\mathbf{e}_1 & * & *\\ \frac{1}{2}\Big[\partial_{y_1}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{R}}\mathbf{e}_3\big)\cdot\mathbf{R}\mathbf{e}_2+\partial_{y_2}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{R}}\mathbf{e}_3\big)\cdot\mathbf{R}\mathbf{e}_1\Big] & \partial_{y_2}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{R}}\mathbf{e}_3\big)\cdot\mathbf{R}\mathbf{e}_2 & *\\ \frac{1}{2}\partial_{y_1}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{R}}\mathbf{e}_3\big)\cdot\mathbf{R}\mathbf{e}_1 & \frac{1}{2}\partial_{y_2}\big(\widehat{\mathfrak{u}}+y_3\widehat{\mathbf{R}}\mathbf{e}_3\big)\cdot\mathbf{R}\mathbf{e}_2 & 0\end{pmatrix}+\begin{pmatrix} 0 & 0 & *\\ 0 & 0 & *\\ \frac{1}{2}\mathcal{Z}_1\cdot\mathbf{R}\mathbf{e}_3 & \frac{1}{2}\mathcal{Z}_2\cdot\mathbf{R}\mathbf{e}_3 & 0\end{pmatrix}+\mathcal{E}_y(\mathbf{R}^T\overline{v})\\
&=\mathcal{E}(\mathcal{Z},\mathcal{V})-\begin{pmatrix} \widehat{\mathbf{R}}\mathbf{e}_1\cdot\mathbf{R}\mathbf{e}_1 & * & *\\ \frac{1}{2}\big(\widehat{\mathbf{R}}\mathbf{e}_1\cdot\mathbf{R}\mathbf{e}_2+\widehat{\mathbf{R}}\mathbf{e}_2\cdot\mathbf{R}\mathbf{e}_1\big) & \widehat{\mathbf{R}}\mathbf{e}_2\cdot\mathbf{R}\mathbf{e}_2 & *\\ \frac{1}{2}\big(\widehat{\mathbf{R}}\mathbf{e}_1\cdot\mathbf{R}\mathbf{e}_3+\widehat{\mathbf{R}}\mathbf{e}_3\cdot\mathbf{R}\mathbf{e}_1\big) & \frac{1}{2}\widehat{\mathbf{R}}\mathbf{e}_2\cdot\mathbf{R}\mathbf{e}_3 & \widehat{\mathbf{R}}\mathbf{e}_3\cdot\mathbf{R}\mathbf{e}_3\end{pmatrix}+\mathcal{E}_y(\widehat{v}).
\end{aligned}
$$

The lemma statement follows by the above equality and identity (30). □

Finally, from Lemmas 5 and 6, we get that

$$\frac{1}{2\varepsilon}\Pi_\varepsilon\big((\nabla v_\varepsilon)^T\nabla v_\varepsilon-\mathbf{I}_3\big)\rightharpoonup\mathcal{E}(\mathcal{Z},\mathcal{V})+\mathcal{E}_y(\widehat{v})\quad\text{weakly in } L^2(\Omega\times\mathcal{Y}^*),$$

where $\mathcal{E}$ and $\widehat{v}$ are defined in (36) and (40) respectively.

7 | THE LIMIT MINIMIZATION PROBLEM AND HOMOGENIZATION

⟨S7⟩ In this section we go to the limit with problem (11) and give a characterization of the minimization problem for the limit energy functionals. Then, we give the sufficient conditions to ensure that such problem admits a minimum. At last, we prove that such minimum is in fact the limit of the minimum for the energy functionals as $\varepsilon\to 0$.

7.1 | Limit of the total elastic energy functional

In order to go to the limit for $\varepsilon\to 0$ and find the limit form of the total energy functional, the limiting results of Section 6 and the properties of the periodic unfolding will be needed. Denote the space

$$\mathbf{H}^1_{per}(\mathcal{Y}^*)\doteq\Big\{\mathbf{c}\in H^1_{per}(\mathcal{Y}^*)^3\;\Big|\;\int_{\mathcal{Y}^*}\mathbf{c}\,dy=0\Big\}$$

and we equip it with the semi-norm

$$\|\mathbf{c}\|_{\mathbf{H}^1_{per}(\mathcal{Y}^*)}=\sqrt{\int_{\mathcal{Y}^*}|\mathcal{E}_y(\mathbf{c})|^2_F dy}.$$

Note that this semi-norm is in fact a norm equivalent to the usual norm of $H^1_{per}(\mathcal{Y}^*)^3$.

Now, let $(\mathcal{V},\mathcal{Z},\widehat{v})\in\mathbb{D}\times L^2(\Omega)^3\times L^2(\Omega;\mathbf{H}^1_{per}(\mathcal{Y}^*))$ and define

$$\mathcal{J}(\mathcal{V},\mathcal{Z},\widehat{v})\doteq\int_{\Omega\times\mathcal{Y}^*}Q(y)\big(\mathcal{E}(\mathcal{Z},\mathcal{V})+\mathcal{E}_y(\widehat{v})\big)\,dx'dy-|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}-I_d)\,dx'.\tag{41}$$ J_lim

In the following, we show that the above defined quantity is an admissible limit energy functional.

⟨prop2⟩ **Proposition 2.** Let $\{v^0_\varepsilon\}_\varepsilon\in\mathbb{D}_\varepsilon$ be a minimizing sequence of admissible deformations satisfying

$$\liminf_{\varepsilon\to0}\frac{m_\varepsilon}{\varepsilon^3}=\lim_{\varepsilon\to0}\frac{\mathcal{J}_\varepsilon(v^0_\varepsilon)}{\varepsilon^3}.$$

We have

$$\mathcal{J}(\mathcal{V}^0,\mathcal{Z}^0,\widehat{v}^0)\le\liminf_{\varepsilon\to0}\frac{m_\varepsilon}{\varepsilon^3}=\lim_{\varepsilon\to0}\frac{\mathcal{J}_\varepsilon(v^0_\varepsilon)}{\varepsilon^3}.\tag{42}$$ J_prop

where $(\mathcal{V}^0, \mathcal{Z}^0, \hat{v}^0) \in \mathbb{D} \times L^2(\Omega)^3 \times L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*))$ is given by Lemmas 2 and 6.

Proof. As a consequence of the weak convergences in Lemmas 3,4,5 and 6, together with the weak lower semi-continuity of the quadratic form of the functional $\mathcal{J}$, we get that

$$\sum_{\alpha=1}^{2} \int_{\Omega\times\mathcal{Y}^*} Q(y)\big(\mathcal{E}(\mathcal{Z}^0, \mathcal{V}^0) + \mathcal{E}_Y(\hat{v}^0)\big)\,dx'dy \le \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{S^*_\varepsilon} Q\Big(\Big\{ \frac{z}{\varepsilon} \Big\}, \frac{1}{2}\big((\nabla v^0_\varepsilon)^T \nabla v^0_\varepsilon - \mathbf{I}_3\big) \Big)\,dx. \tag{43}$$ EQ-105

On the other hand, the product between strong and weak convergences leads to

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{S_\varepsilon} f_\varepsilon \cdot (v^0_\varepsilon - I_d)\,dx \to \int_{\Omega\times\mathcal{Y}^*} f \cdot (\mathcal{V}^0 - I_d)\,dx'dy = |\mathcal{Y}^*| \int_{\Omega} f \cdot (\mathcal{V}^0 - I_d)\,dx'. \tag{44}$$ EQ-106

Hence, the statement follows from (43) and (44) □

7.2 | Homogenization of the limit energy functional

Let $\mathbf{A} \in L^\infty(\mathcal{Y}^*)^{6\times 6}$ be the matrix associated to the positive quadratic form Q. We rewrite the limit energy functional $\mathcal{J}$ defined by (41) in the matrix representation and with the explicit values of the second fundamental form given in (25) by setting

$$\mathcal{J}(\mathcal{V}, \mathcal{Z}, \hat{v}) = \int_{\Omega\times\mathcal{Y}^*} \mathbf{A} \left[\begin{pmatrix} \mathcal{Z}_{11} - y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{11}\mathcal{V} \\ \mathcal{Z}_{12} - y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{12}\mathcal{V} \\ \mathcal{Z}_{22} - y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{22}\mathcal{V} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\hat{v}) \\ e_{12,y}(\hat{v}) \\ e_{22,y}(\hat{v}) \\ e_{33,y}(\hat{v}) \\ e_{13,y}(\hat{v}) \\ e_{23,y}(\hat{v}) \end{pmatrix} \right] \cdot \left[\begin{pmatrix} \mathcal{Z}_{11} - y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{11}\mathcal{V} \\ \mathcal{Z}_{12} - y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{12}\mathcal{V} \\ \mathcal{Z}_{22} - y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{22}\mathcal{V} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\hat{v}) \\ e_{12,y}(\hat{v}) \\ e_{22,y}(\hat{v}) \\ e_{33,y}(\hat{v}) \\ e_{13,y}(\hat{v}) \\ e_{23,y}(\hat{v}) \end{pmatrix} \right] dx'dy - |\mathcal{Y}^*| \int_{\Omega} f \cdot (\mathcal{V} - I_d)\,dx'. \tag{45}$$ J_matrix

Now, note that the existence of a minimum for the functional $\mathcal{J}(\mathcal{V}, \mathcal{Z}, \hat{v})$ cannot be ensured, since $\mathbb{D} \times L^2(\Omega)^3 \times L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*))$ is not a subset of a Hilbert space. For this reason, before proceeding to the actual homogenization we need to introduce an assumption so that the minimum of the functional does not depend on $\mathcal{Z} \in L^2(\Omega)^3$.

From now on, we assume that the quadratic form Q introduced by (9) *is such that*

$$\mathcal{J}(\mathcal{V}, 0, \hat{v}) \le \mathcal{J}(\mathcal{V}, \mathcal{Z}, \hat{v}), \qquad \forall (\mathcal{V}, \mathcal{Z}, \hat{v}) \in \mathbb{D} \times L^2(\Omega)^3 \times L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*)). \tag{46}$$ condition

It is important to note that this condition is not satisfied in general. Nonetheless, we will give in Appendix examples of materials, for which the condition (46) *is satisfied.*

In order to find the minimum for the functional $\mathcal{J}(\mathcal{V}, 0, \hat{v})$ with respect to $\hat{v} \in L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*))$, we consider the associated Euler-Lagrange equation and get that the minimum is reached with the solution to the following problem:

$$\begin{cases} \text{Find } \hat{\mathbf{c}} \in L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*)) \text{ such that:} \\ \displaystyle\int_{\Omega\times\mathcal{Y}^*} \mathbf{A} \left[\begin{pmatrix} -y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{11}\mathcal{V} \\ -y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{12}\mathcal{V} \\ -y_3\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{22}\mathcal{V} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\hat{\mathbf{c}}) \\ e_{12,y}(\hat{\mathbf{c}}) \\ e_{22,y}(\hat{\mathbf{c}}) \\ e_{33,y}(\hat{\mathbf{c}}) \\ e_{13,y}(\hat{\mathbf{c}}) \\ e_{23,y}(\hat{\mathbf{c}}) \end{pmatrix} \right] \cdot \begin{pmatrix} e_{11,y}(\hat{w}) \\ e_{12,y}(\hat{w}) \\ e_{22,y}(\hat{w}) \\ e_{33,y}(\hat{w}) \\ e_{13,y}(\hat{w}) \\ e_{23,y}(\hat{w}) \end{pmatrix} dx'dy = 0, \qquad \forall \hat{w} \in L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*)). \end{cases}$$

We introduce the first three vectors of the canonical basis of $\mathbb{R}^6$ and denote them by:

$$\mathbf{V}_{11} \doteq \mathbf{e}_1, \qquad \mathbf{V}_{12} = \mathbf{V}_{21} \doteq \mathbf{e}_2, \qquad \mathbf{V}_{22} \doteq \mathbf{e}_3$$

and denote the 6-entries vector representing the symmetric gradient by

$$\overrightarrow{e_y} = \begin{pmatrix} e_{11,y} & e_{12,y} & e_{22,y} & e_{33,y} & e_{13,y} & e_{23,y} \end{pmatrix}^T$$

With this notation, the correctors are the solutions to the following cell problems:

$$\begin{cases} \text{Find } \widehat{\chi}_{\alpha\beta} \in \mathbf{H}^1_{per}(\mathcal{Y}^*) \text{ such that} \\ \displaystyle\int_{\mathcal{Y}^*} \mathbf{A}\left(y_3\mathbf{V}_{\alpha\beta} + \overrightarrow{e_y}\big(\widehat{\chi}_{\alpha\beta}\big)\right) \cdot \overrightarrow{e_y}(\widehat{\mathbf{d}})\,dy = 0, \qquad \forall \widehat{\mathbf{d}} \in \mathbf{H}^1_{per}(\mathcal{Y}^*), \qquad \forall (\alpha,\beta) \in \{1,2\}^2. \end{cases} \tag{47}$$ cell_sols

Hence, we obtain that the problem

$$\text{Find } \widehat{\mathbf{c}} \in L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*)) \text{ such that}$$
$$\mathcal{J}(\mathcal{V}, 0, \widehat{\mathbf{c}}) = \min_{\widehat{v} \in L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*))} \mathcal{J}(\mathcal{V}, 0, \widehat{v}), \qquad \forall \mathcal{V} \in \mathbb{D}$$

admits the solution

$$\widehat{\mathbf{c}} = -\sum_{\alpha,\beta=1}^{2} \big(\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{\alpha\beta}\mathcal{V}\big)\widehat{\chi}_{\alpha\beta}. \tag{48}$$ EQ62

With the above assumption, the limit minimization problem reduces to the following:

$$\inf_{\mathcal{V}\in\mathbb{D}} \mathcal{J}(\mathcal{V}, 0, \widehat{\mathbf{c}}), \tag{49}$$ min_1

where $\widehat{\mathbf{c}} \in L^2(\Omega; \mathbf{H}^1_{per}(\mathcal{Y}^*))$ is given by (48) and where $\mathbb{D}$ is a closed subset of a Hilbert space.

Before proceeding to the existence of a minimum for the above problem (49), we continue the homogenization process and obtain the homogenized energy functional, with which it will be easier to deal. Namely, replacing (48) in (45) and integrating over the reference cell $\mathcal{Y}^*$, we first obtain the homogenizing coefficients: $((\alpha, \beta) \in \{1, 2\}^2)$

$$\mathbf{c}^{hom}_{\alpha\beta\alpha'\beta'} \doteq \frac{1}{|\mathcal{Y}^*|}\int_{\mathcal{Y}^*} \mathbf{A}(y)\left(-y_3\mathbf{V}_{\alpha\beta} + \overrightarrow{e_y}\big(\widehat{\chi}_{\alpha\beta}\big)(y)\right) \cdot \left(-y_3\mathbf{V}_{\alpha'\beta'} + \overrightarrow{e_y}\big(\widehat{\chi}_{\alpha'\beta'}\big)(y)\right)\, dy.$$

The homogenized energy functional $\mathcal{J}$ can therefore be rewritten in the following homogenized form:

$$\mathcal{J}_0(\mathcal{V}) \doteq \mathcal{J}\big(\mathcal{V}, 0, \widehat{\mathbf{c}}\big) = \int_\Omega \mathbf{c}^{hom}_{\alpha\beta\alpha'\beta'}\big(\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{\alpha\beta}\mathcal{V}\big)\big(\mathbf{R}(\mathcal{V})\mathbf{e}_3 \cdot \partial_{\alpha'\beta'}\mathcal{V}\big)\, dx' - |\mathcal{Y}^*| \int_\Omega f \cdot (\mathcal{V} - I_d)\, dx'. \tag{50}$$ J_0

Finally, from the minimization problem (49) can be rewritten in the following form:

$$\inf_{\mathcal{V}\in\mathbb{D}} \mathcal{J}_0(\mathcal{V}).$$

7.3 | Existence of a minimum in $\mathbb{D}$ for the homogenized limit functional $\mathcal{J}_0$

For every $\zeta \in \mathbb{R}^3$, set

$$\mathcal{E}(\zeta) = \begin{pmatrix} -y_3\zeta_1 & * & 0 \\ -y_3\zeta_3 & -y_3\zeta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In order to prove that the functional $\mathcal{J}_0$ admits a minimum on $\mathbb{D}$, a couple of preparatory lemmas are needed.

lem8 **Lemma 7.** Let $(\zeta, \widehat{w})$ be in $\mathbb{R}^3 \times \mathbf{H}^1_{per}(\mathcal{Y}^*)$. If

$$\mathcal{E}(\zeta) + \mathcal{E}_y(\widehat{w}) = 0 \quad \text{a.e. in } \mathcal{Y}^*, \tag{51}$$ EQ811

then $\zeta = 0$ and $\widehat{w} = 0$.

Moreover, there exists two constant $C_0, C_1 > 0$ such that for every $(\zeta, \widehat{w}) \in \mathbb{R}^3 \times \mathbf{H}^1_{per}(\mathcal{Y}^*)$

$$C_0\big(|\zeta|^2 + \|\widehat{w}\|^2_{\mathbf{H}^1(\mathcal{Y}^*)}\big) \le \|\mathcal{E}(\zeta) + \mathcal{E}_y(\widehat{w})\|^2_{L^2(\mathcal{Y}^*)} \le C_1\big(|\zeta|^2 + \|\widehat{w}\|^2_{\mathbf{H}^1(\mathcal{Y}^*)}\big). \tag{52}$$ EQ812

Proof. The proof is done in the same way as for Lemma 6.5 in [18], but in a weaker formulation. Define the map

$$\boldsymbol{\tau}(y) \doteq \begin{pmatrix} -y_1y_3\zeta_1 - y_2y_3\zeta_3 \\ -y_1y_3\zeta_3 - y_2y_3\zeta_2 \\ -\frac{y_1^2}{2}\zeta_1 - \frac{y_2^2}{2}\zeta_2 - y_1y_2\zeta_3 \end{pmatrix},$$

which by construction satisfies $\mathcal{E}_y(\boldsymbol{\tau}) = \mathcal{E}(\zeta)$. Under assumption (51) we get $\mathcal{E}_y(\boldsymbol{\tau} + \widehat{w}) = 0$. Hence, $\boldsymbol{\tau} + \widehat{w}$ is a rigid motion and there exist $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^3$ such that $\boldsymbol{\tau}(y) + \widehat{w}(y) = \mathbf{a} + \mathbf{b} \wedge y$, which leads to

$$\widehat{w}(y) = \begin{pmatrix} \mathbf{a}_1 + y_3\mathbf{b}_2 - y_2\mathbf{b}_3 \\ \mathbf{a}_2 - y_3\mathbf{b}_1 + y_1\mathbf{b}_3 \\ \mathbf{a}_3 + y_2\mathbf{b}_1 - y_1\mathbf{b}_2 \end{pmatrix} + \begin{pmatrix} y_1y_3\zeta_1 + y_2y_3\zeta_3 \\ y_1y_3\zeta_3 + y_2y_3\zeta_2 \\ \frac{y_1^2}{2}\zeta_1 + \frac{y_2^2}{2}\zeta_2 + y_1y_2\zeta_3 \end{pmatrix}$$

By the 2-periodicity in direction $\mathbf{e}_1$ ($\widehat{w}(0,y_2,y_3) = \widehat{w}(2,y_2,y_3)$) and direction $\mathbf{e}_2$ ($\widehat{w}(y_1,0,y_3) = \widehat{w}(y_1,2,y_3)$), we obtain that $\zeta_1 = \zeta_2 = 0$ and that

$$\begin{cases} \mathbf{b}_3 \pm y_3\zeta_3 = 0, \\ \mathbf{b}_2 - y_2\zeta_3 = 0, \\ \mathbf{b}_1 + y_1\zeta_3 = 0, \end{cases}$$

which immediately implies that $\mathbf{b} = 0$ and $\zeta_3 = 0$. Finally, from the fact that $\widehat{w} \in \mathbf{H}^1_{per}(\mathcal{Y}^*)$ we get $\mathbf{a} = 0$ and thus that $\widehat{w} = 0$. Concerning inequalities (52), we have that the right hand side is obvious, while the left hand side follows from a contradiction argumentation. □

em_positive⟩ **Lemma 8.** The quadratic form associated to $\mathcal{J}_0$ is positive definite.

Proof. We will use a contradiction argumentation. Let ζ be a not null 2×2 symmetric matrix such that

$$\mathbf{c}^{hom}_{\alpha\beta\alpha'\beta'}\,\zeta_{\alpha\beta}\zeta_{\alpha'\beta'} \le 0.$$

From the definition of $\mathbf{c}^{hom}$, we proceed backwards with the homogenization steps and obtain that

$$0 \ge \mathbf{c}^{hom}_{\alpha\beta\alpha'\beta'}\,\zeta_{\alpha\beta}\zeta_{\alpha'\beta'} = \int_{\mathcal{Y}^*} \mathbf{A}\big[\mathcal{E}(\zeta) + \mathcal{E}_y(\widehat{\mathbf{h}})\big]\cdot\big[\mathcal{E}(\zeta) + \mathcal{E}_y(\widehat{\mathbf{h}})\big]\,dy, \qquad \text{where } \mathbf{h} = -\sum_{\alpha,\beta=1}^{2}\zeta_{\alpha\beta}\widehat{\chi}_{\alpha\beta}.$$

From the coercivity of the matrix $\mathbf{A}$, we immediately get that the above inequality leads to $\mathcal{E}(\zeta) + \mathcal{E}_y(\widehat{\mathbf{h}}) = 0$, thus Lemma 7 implies that ζ is a null matrix, which is a contradiction. □

We are ready to prove the main results of this subsection.

Theorem 1. The functional $\mathcal{J}_0$ admits a minimum on $\mathbb{D}$.

Proof. Set

$$m \doteq \inf_{\mathcal{V}\in\mathbb{D}} J_0(\mathcal{V}), \qquad m \in [-\infty, 0].$$

Step 1. We show that $m \in (-\infty, 0]$.
We assume that $\mathcal{J}_0(\mathcal{V}) \le 0$ for every $\mathcal{V} \in \mathbb{D}$. This, together with the coercivity of $\mathcal{J}_0$, implies that

$$\|\mathbf{R}(\mathcal{V})\mathbf{e}_3\cdot\partial_1\mathbf{R}(\mathcal{V})\mathbf{e}_1\|^2_{L^2(\Omega)} + 2\|\mathbf{R}(\mathcal{V})\mathbf{e}_3\cdot\partial_1\mathbf{R}(\mathcal{V})\mathbf{e}_2\|^2_{L^2(\Omega)} + \|\mathbf{R}(\mathcal{V})\mathbf{e}_3\cdot\partial_2\mathbf{R}(\mathcal{V})\mathbf{e}_2\|^2_{L^2(\Omega)} \le C|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}-I_d)\,dz'.$$

From the definition of anti-symmetric matrices $\mathbf{A}^{(1)}(\mathcal{V})$ and $\mathbf{A}^{(2)}(\mathcal{V})$ from Corollary 1, we can rewrite the above estimate in the following way:

$$\|\mathbf{A}^{(1)}_{31}(\mathcal{V})\|^2_{L^2(\Omega)} + \|\mathbf{A}^{(1)}_{32}(\mathcal{V})\|^2_{L^2(\Omega)} + \|\mathbf{A}^{(2)}_{32}(\mathcal{V})\|^2_{L^2(\Omega)} \le C|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}-I_d)\,dz'.$$

Now, from the definition of Frobenius norm and the matrix representation (26), we get

$$\begin{aligned}\sum_{\alpha=1}^{2}\|\mathbf{A}^{(\alpha)}(\mathcal{V})\|^2_{L^2(\Omega)^{3\times 3}} = \sum_{\alpha=1}^{2}\sum_{i,j=1}^{3}\int_\Omega |\mathbf{A}^{(\alpha)}_{ij}(x')|^2\,dx' &= 2\big(\|\mathbf{A}^{(1)}_{31}(\mathcal{V})\|^2_{L^2(\Omega)} + 2\|\mathbf{A}^{(1)}_{32}(\mathcal{V})\|^2_{L^2(\Omega)} + \|\mathbf{A}^{(2)}_{32}(\mathcal{V})\|^2_{L^2(\Omega)}\big)\\ &\le 2|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}-I_d)\,dx'.\end{aligned}$$

From the fact that $\mathbf{R}(\mathcal{V}) \in SO(3)$ and thus $\|\mathbf{R}(\mathcal{V})\|_{L^\infty(\Omega)} \le \sqrt{3}$, the above inequality and the fact that $\mathcal{V} \in H^2_{\mathbf{T}}(\Omega)$ implies that

$$\|\mathcal{V}-\mathbf{I}_d\|^2_{H^2(\Omega)} \le C\sum_{\alpha,\beta=1}^{2}\|\partial_{\alpha\beta}\mathcal{V}\|^2_{L^2(\Omega)} \le C\big(\|\partial_1\mathbf{R}(\mathcal{V})\|^2_{L^2(\Omega)} + \|\partial_2\mathbf{R}(\mathcal{V})\|^2_{L^2(\Omega)}\big) \le C|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}-I_d)\,dz'.$$

Now, we have

$$|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}-I_d)\,dz' \le C\|f\|_{L^2(\Omega)}\|\mathcal{V}-\mathbf{I}_d\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}\|\mathcal{V}-\mathbf{I}_d\|_{H^2(\Omega)}$$

from which we derive a bound for $\|\mathcal{V}-\mathbf{I}_d\|_{H^2(\Omega)}$. Thus

$$\forall \mathcal{V}\in\mathbb{D} \quad \text{such that} \quad \mathcal{J}_0(\mathcal{V})\le 0 \quad \text{we have} \quad \|\mathcal{V}-\mathbf{I}_d\|_{H^2(\Omega)}\le C\|f\|_{L^2(\Omega)}.$$

As a consequence, we can give a lower bound for $\mathcal{J}_0$.

Step 2. We show that m is a minimum.
Let $\{\mathcal{V}_n^0\}_n\in\mathbb{D}$ be a minimizing sequence satisfying $\mathcal{J}_0(\mathcal{V}_n^0)\le\mathcal{J}_0(\mathbf{I}_d)=0$ and such that

$$m=\inf_{\mathcal{V}\in\mathbb{D}}\mathcal{J}_0(\mathcal{V})=\lim_{n\to+\infty}\mathcal{J}_0(\mathcal{V}_n^0). \tag{53}$$ m

From Step 1, there exists a constant $C>0$ independent on n and such that

$$\|\mathcal{V}_n^0-\mathbf{I}_d\|_{H^2(\Omega)^3}\le C\|f\|_{L^2(\Omega)}.$$

By the fact that $\mathbb{D}$ is a closed subset of a Hilbert space, there exist a subsequence of $\{n\}$, still denoted $\{n\}$, such that

$$\mathcal{V}_n^0\rightharpoonup\mathcal{V}^0 \quad \text{weakly in } \mathbb{D}. \tag{54}$$ conv_int

In particular. we have

$$\mathbf{A}^{(\alpha)}(\mathcal{V}_n^0)\rightharpoonup\mathbf{A}^{(\alpha)}(\mathcal{V}^0) \qquad \text{weakly in } \ L^2(\Omega)^{3\times 3}.$$

Now, from the positive definiteness of the quadratic form of the homogenized limit functional (50) in Lemma 8 we know that the quadratic part of $\mathcal{J}_0$ is weak lower semi-continuous. Hence, this fact and the weak convergence (54) imply that

$$\begin{aligned}\mathcal{J}_0(\mathcal{V}^0)&=\int_\Omega \mathbf{c}^{hom}_{\alpha\beta\alpha'\beta'}\mathbf{A}^{(\alpha)}_{3\beta}(\mathcal{V}^0)\,\mathbf{A}^{(\alpha')}_{3\beta'}(\mathcal{V}^0)\,dx'-|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}^0-I_d)\,dx'\\&\le\liminf_{n\to+\infty}\int_\Omega \mathbf{c}^{hom}_{\alpha\beta\alpha'\beta'}\mathbf{A}^{(\alpha)}_{3\beta}(\mathcal{V}_n^0)\,\mathbf{A}^{(\alpha')}_{3\beta'}(\mathcal{V}_n^0)\,dx'-\lim_{n\to+\infty}|\mathcal{Y}^*|\int_\Omega f\cdot(\mathcal{V}_n^0-I_d)\,dx'\le\lim_{n\to+\infty}\mathcal{J}_0(\mathcal{V}_n^0)=m.\end{aligned}$$

On the other hand, from equality (53), we have that

$$\mathcal{J}_0(\mathcal{V}^0)\le m\le\mathcal{J}_0(\mathcal{V}), \qquad \text{for every } \ \mathcal{V}\in\mathbb{D},$$

thus the infimum m is in fact a minimum. □

7.4 | The minimum of $\mathcal{J}_0$ is the limit of the energy functionals.

Set

$$m\doteq\min_{\mathcal{V}\in\mathbb{D}}\mathcal{J}_0(\mathcal{V}). \tag{55}$$ minimum

It is left to show that m coincides with the limit of the energy functionals as $\varepsilon\to 0$.

it-theorem⟩? **Theorem 2.** Suppose that f_ε is defined as in Subsection 5.1, and let $\mathcal{J}_0$ be the energy functional defined in (50). For every minimizing sequence $\{v_\varepsilon\}_\varepsilon$ for the functionals $\mathcal{J}_\varepsilon$ we have

$$\lim_{\varepsilon\to0}\frac{\mathcal{J}_\varepsilon(v_\varepsilon)}{\varepsilon^3}=\lim_{\varepsilon\to0}\frac{m_\varepsilon}{\varepsilon^3}=m=\min_{\mathcal{V}\in\mathbb{D}}\mathcal{J}_0(\mathcal{V}).$$

Proof. Step 1. In this step, we prove that for every minimizing sequence $\{v_\varepsilon\}_\varepsilon$ for the functionals $\mathcal{J}_\varepsilon$ we have

$$\lim_{\varepsilon\to0}\frac{\mathcal{J}_\varepsilon(v_\varepsilon)}{\varepsilon^3}=\liminf_{\varepsilon\to0}\frac{m_\varepsilon}{\varepsilon^3}\ge m.$$

In order to prove that, let $\{v_\varepsilon^0\}_\varepsilon$ be the minimizing sequence for the functionals $\mathcal{J}_\varepsilon$ introduced in Proposition 2. From (42), the definition of the limit energy functional $\mathcal{J}_0$ and due to assumption (46), we have

$$\lim_{\varepsilon\to0}\frac{\mathcal{J}_\varepsilon(v_\varepsilon^0)}{\varepsilon^3}=\liminf_{\varepsilon\to0}\frac{m_\varepsilon}{\varepsilon^3}\ge\mathcal{J}(\mathcal{V}^0,\mathcal{Z}^0,\hat{v}^0)\ge\mathcal{J}(\mathcal{V}^0,0,\hat{v}^0)\ge\min_{\hat{v}\in L^2(\Omega;\mathbf{H}^1_{per,0}(\mathcal{Y}^*))}\mathcal{J}(\mathcal{V}^0,0,\hat{v})=\mathcal{J}(\mathcal{V}^0,0,\hat{\mathbf{c}})=\mathcal{J}_0(\mathcal{V}^0)\ge m.$$

Step 2. In this step, we prove that for every $\mathcal{W}\in\mathbb{D}$, we have

$$\limsup_{\varepsilon\to0}\frac{m_\varepsilon}{\varepsilon^3}\le\mathcal{J}_0(\mathcal{W}). \tag{56}$$ limsup_eq

Let $(\mathcal{W}, \widehat{w})$ be in $\mathbb{D}\times L^2(\Omega;\mathbf{H}^1_{per}(\mathcal{Y}^*))$ and let $\{(\mathcal{W}_\varepsilon, \mathbf{R}_\varepsilon, \widehat{w}_\varepsilon)\}\in W^{2,\infty}(\Omega)^3\times W^{1,\infty}(\Omega)^{3\times 3}\times W^{1,\infty}(S_\varepsilon)$ be the sequence of fields given in Lemma 11 in Appendix that approximates $(\mathcal{W}, \mathbf{R}(\mathcal{W}), \widehat{w})$. Without loss of generality, we can choose

$$\widehat{w} = W\widehat{W}, \qquad \text{with} \quad W\in C_c^\infty(\Omega), \quad \text{and} \quad \widehat{W}\in \mathbf{H}^1_{per}(\mathcal{Y}^*)\cap W^{1,\infty}(\mathcal{Y}^*)^3.$$

In this sense, we can set

$$\widehat{w}_\varepsilon\Big(x_1, x_2, \frac{x}{\varepsilon}\Big) \doteq W(x')\widehat{W}\Big(\frac{x}{\varepsilon}\Big) \qquad \text{a.e. } x\in S^*_\varepsilon.$$

We define the following sequence of deformations:

$$w_\varepsilon\Big(x', \frac{x}{\varepsilon}\Big) \doteq \mathcal{W}_\varepsilon(x') + x_3\mathbf{R}_\varepsilon(x')\mathbf{e}_3 + \varepsilon^2 W(x')\widehat{W}\Big(\frac{x}{\varepsilon}\Big) \qquad \text{a.e. } x\in S^*_\varepsilon.$$

Straightforward calculations lead to that, from the properties of the global unfolding operator, we obtain that

$$\nabla w_\varepsilon - \mathbf{R}_\varepsilon = \Big(\partial_1\mathcal{W}_\varepsilon - \mathbf{R}_\varepsilon\mathbf{e}_1 + x_3\partial_1\mathbf{R}_\varepsilon\mathbf{e}_3 + \varepsilon W\partial_{y_1}\widehat{W} + \varepsilon^2\partial_1 W\widehat{W}\Big|\ \partial_2\mathcal{W}_\varepsilon - \mathbf{R}_\varepsilon\mathbf{e}_2 + x_3\partial_2\mathbf{R}_\varepsilon\mathbf{e}_3 + \varepsilon W\partial_{y_2}\widehat{W} + \varepsilon^2\partial_2 W\widehat{W}\Big|\ \varepsilon W\partial_{y_3}\widehat{W}\Big). \tag{57}$$ lim_1

So, we have

$$\|\nabla w_\varepsilon - \mathbf{R}_\varepsilon\|_{L^\infty(\Omega\times\mathcal{Y}^*)} \le \sum_{\alpha=1}^{2}\big(\|\partial_\alpha\mathcal{W}_\varepsilon - \mathbf{R}_\varepsilon\mathbf{e}_\alpha\|_{L^\infty(\Omega)} + \varepsilon\kappa\|\partial_\alpha\mathbf{R}_\varepsilon\mathbf{e}_3\|_{L^\infty(\Omega)}\big) + \varepsilon\|W\nabla_y\widehat{W}\|_{L^\infty(\Omega\times\mathcal{Y}^*)} + \varepsilon^2\|\nabla W\widehat{W}\|_{L^\infty(\Omega\times\mathcal{Y}^*)}.$$

Now, from the estimates in Lemma 11 and the fact that $\kappa\le 1/3$, we get that

$$\|\nabla w_\varepsilon - \mathbf{R}_\varepsilon\|_{L^\infty(S^*_\varepsilon)} \le \frac{5}{48} + 2\varepsilon\big(\|W\|_{W^{1,\infty}(\Omega)}\|\widehat{W}\|_{W^{1,\infty}(\mathcal{Y}^*)}\big). \tag{58}$$ est_int

So, from the fact that for every $M\in\mathbf{M}_3$, one has $\det(M) = M\mathbf{e}_1\cdot(M\mathbf{e}_2\wedge M\mathbf{e}_3)$, the distributive properties of the dot and cross product and the fact that $|\det(M)|\le\|M\mathbf{e}_1\|_2\|M\mathbf{e}_2\|_2\|M\mathbf{e}_3\|$, where $\|\cdot\|_2$ is the Euclidean norm, we first get that

$$\begin{aligned}
&|\det(\nabla w_\varepsilon) - \det(\mathbf{R}_\varepsilon)|\\
&\le \|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_1\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_2\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_3\|_2 + \|\mathbf{R}_\varepsilon\mathbf{e}_1\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_2\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_3\|_2\\
&\quad + \|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_1\|_2\|\mathbf{R}_\varepsilon\mathbf{e}_2\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_3\|_2 + \|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_1\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_2\|_2\|\mathbf{R}_\varepsilon\mathbf{e}_3\|_2\\
&\quad + \|\mathbf{R}_\varepsilon\mathbf{e}_1\|_2\|\mathbf{R}_\varepsilon\mathbf{e}_2\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_3\|_2 + \|\mathbf{R}_\varepsilon\mathbf{e}_1\|_2\|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_2\|_2\|\mathbf{R}_\varepsilon\mathbf{e}_3\|_2 + \|(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\mathbf{e}_1\|_2\|\mathbf{R}_\varepsilon\mathbf{e}_2\|_2\|\mathbf{R}_\varepsilon\mathbf{e}_3\|_2.
\end{aligned}$$

Then, from the above estimate and (58) and the fact that $\|\mathbf{R}_\varepsilon\mathbf{e}_i\|_2 = 1$ a.e. in S^*, we obtain

$$|(\det(\nabla w_\varepsilon)-1)(x)| = |\det(\nabla w_\varepsilon)-\det(\mathbf{R}_\varepsilon)(x)| \le \Big(\frac{5}{48}\Big)^3 + 3\sqrt{3}\Big(\frac{5}{48}\Big)^2 + 9\Big(\frac{5}{48}\Big) + C\varepsilon\big(\|W\|_{W^{1,\infty}(\Omega)}\|\widehat{W}\|_{W^{1,\infty}(\mathcal{Y}^*)}\big), \text{ a.e. in } S^*_\varepsilon$$

from which it follows that if ε is sufficiently small, we have $|\det(\nabla w_\varepsilon) - 1| < 1$ a.e. in S^*_ε, thus that

$$\det(\nabla w_\varepsilon) > 0 \qquad \text{a.e. in } S^*_\varepsilon.$$

Therefore, from the estimates and convergences in Lemma 11 and identity (57), we derive that

$$\|\Pi_\varepsilon\big(\nabla w_\varepsilon - \mathbf{R}_\varepsilon\big)\|_{L^2(\Omega\times\mathcal{Y}^*)^{3\times 3}} \le C\varepsilon$$

Since, for any ε, the identity

$$(\nabla w_\varepsilon)^T\nabla w_\varepsilon - \mathbf{I}_3 = (\nabla w_\varepsilon - \mathbf{R}_\varepsilon)^T(\nabla w_\varepsilon - \mathbf{R}_\varepsilon) + \mathbf{R}_\varepsilon^T(\nabla w_\varepsilon - \mathbf{R}_\varepsilon) + (\nabla w_\varepsilon - \mathbf{R}_\varepsilon)^T\mathbf{R}_\varepsilon + (\mathbf{R}_\varepsilon - \mathbf{R}\mathbf{R}(\mathcal{W}))^T\mathbf{R}_\varepsilon + \mathbf{R}^T(\mathbf{R}_\varepsilon - \mathbf{R}\mathbf{R}(\mathcal{W})), \tag{59}$$ id_lim

holds, the above estimate implies that

$$\|\Pi_\varepsilon\big((\nabla w_\varepsilon)^T\nabla w_\varepsilon - \mathbf{I}_3\big)\|_{L^2(\Omega\times\mathcal{Y}^*)^{3\times 3}} \le C\varepsilon.$$

Applying the unfolding operator to identity (59), the convergences in Lemma 11 lead to

$$\begin{aligned}
\frac{1}{2\varepsilon}\Pi_\varepsilon\big((\nabla w_\varepsilon - \mathbf{R}_\varepsilon)^T(\nabla w_\varepsilon - \mathbf{R}_\varepsilon)\big) &\to 0 \quad \text{strongly in} \quad L^4(\Omega\times\mathcal{Y}^*)^3,\\
\frac{1}{2\varepsilon}\Pi_\varepsilon\big(\mathbf{R}_\varepsilon - \mathbf{R}(\mathcal{W})\big) &\to 0 \quad \text{strongly in} \quad L^2(\Omega\times\mathcal{Y}^*)^3,\\
\frac{1}{2\varepsilon}\Pi_\varepsilon\big(\mathbf{R}_\varepsilon^T(\nabla w_\varepsilon - \mathbf{R}_\varepsilon) + (\nabla w_\varepsilon - \mathbf{R}_\varepsilon)^T\mathbf{R}_\varepsilon\big) &\to \begin{pmatrix} -y_3\mathbf{R}(\mathcal{W})\mathbf{e}_3\cdot\partial_1\mathbf{R}(\mathcal{W})\mathbf{e}_1 & * & *\\ -y_3\mathbf{R}(\mathcal{W})\mathbf{e}_3\cdot\partial_1\mathbf{R}(\mathcal{W})\mathbf{e}_2 & -y_3\mathbf{R}(\mathcal{W})\mathbf{e}_3\cdot\partial_2\mathbf{R}(\mathcal{W})\mathbf{e}_2 & *\\ 0 & 0 & 0\end{pmatrix} + \mathcal{E}_y(\widehat{w})\\
&= \mathcal{E}\big(0, \mathbf{R}(\mathcal{W})\big) + \mathcal{E}_y(\widehat{w}) \quad \text{strongly in} \quad L^2(\Omega\times\mathcal{Y}^*)^{3\times 3}.
\end{aligned}$$

As a consequence, we get that

$$\frac{1}{2\varepsilon}\Pi_\varepsilon\left((\nabla w_\varepsilon)^T\nabla w_\varepsilon - \mathbf{I}_3\right) \to \mathcal{E}\big(0,\mathbf{R}(\mathcal{W})\big) + \mathcal{E}_y(\widehat{w}) \quad \text{ strongly in } \quad L^2(\Omega\times\mathcal{Y}^*)^{3\times 3}.$$

The above convergence, together with the strong convergence of the forces, implies that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(\mathcal{W}_\varepsilon,\mathbf{R}_\varepsilon,\widehat{w}_\varepsilon) &= \lim_{\varepsilon\to 0}\int\limits_{\Omega\times\mathcal{Y}^*}\Pi_\varepsilon\Big(\mathcal{Q}\big(y,(\nabla w_\varepsilon)^T\nabla w_\varepsilon - \mathbf{I}_3\big)\Big)\,dx'dy - \int\limits_\Omega f_\varepsilon\cdot(\mathcal{W}_\varepsilon - I_d)\,dx'\\ &= \int\limits_{\Omega\times\mathcal{Y}^*}\mathcal{Q}\big(y,\mathcal{E}\big(0,\mathbf{R}(\mathcal{W})\big)+\mathcal{E}_y(\widehat{w})\big) - |\mathcal{Y}^*|\int\limits_\Omega f\cdot(\mathcal{W}-I_d)\,dx'.\end{aligned}$$

This allows to reach the following inequality:

$$\limsup_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^3} \le \lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(\mathcal{W}_\varepsilon,\mathbf{R}_\varepsilon,\widehat{w}_\varepsilon) = \mathcal{J}(\mathcal{W},0,\widehat{w}), \qquad \forall\mathcal{W}\in\mathbb{D} \text{ and } \widehat{w}=\mathcal{W}\widehat{\mathcal{W}}\in C_c^\infty(\Omega;\mathbf{H}^1_{per}(\mathcal{Y}^*)\cap W^{1,\infty}(\mathcal{Y}^*))^3.$$

Since the space $C_c^\infty(\Omega;\mathbf{H}^1_{per}(\mathcal{Y}^*)\cap W^{1,\infty}(\mathcal{Y}^*)^3)$ is dense in $L^2(\Omega;\mathbf{H}^1_{per}(\mathcal{Y}^*))$, the above inequality holds for every $\widehat{w}$ in $L^2(\Omega;\mathbf{H}^1_{per}(\mathcal{Y}^*))$. Thus, it holds especially for $\widehat{\mathbf{c}}$ and therefore

$$\limsup_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^3} \le \mathcal{J}(\mathcal{W},0,\widehat{\mathbf{c}}) = \mathcal{J}_0(\mathcal{W}), \qquad \forall\mathcal{W}\in\mathbb{D}.$$

The proof of (56) is complete.

Step 3. Combining the results of Step 1 and Step 2, we obtain that for $\mathcal{V}^0\in\mathbb{D}$ limit of a minimizing sequence, we have

$$m \le \mathcal{J}_0(\mathcal{V}^0) = \liminf_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^3} \le \limsup_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^3} \le \min_{\mathcal{W}\in\mathbb{D}}\mathcal{J}_0(\mathcal{W}) = m.$$

As a consequence, the minimum m defined in (55) is the limit of the energy functionals. □

8 | CONCLUSIONS

⟨S8⟩ In this last section, we draw the conclusions for the homogenization of a textile structure with glued fibers subject to nonlinear deformations:

(i) Since the fibers are glued, the textile can be extended to a periodically perforated plate, and treated accordingly. This simplification has been already done in [18], where a von-Kármán regime was taken into account.

(ii) The existence of a minimum for the limit energy functional (41) is ensured only if the minimum is reached for $\mathcal{Z}=0$, hence if the condition on the material behavior (46) holds. This sufficient condition depends on the geometry of the periodic structure and the isotropy and homogeneity of the material, which may vary from case to case, as shown in Remark 3. In this sense, even though conditions for this particular textile has been given explicitly, no general formulation could be made, nor it was possible to affirm that condition (46) is necessary.

(iii) Any limit deformation $\mathcal{V}\in\mathbb{D}$ is a developable surface. This also implies that the solution of the minimization problem for the homogenized energy functional is a developable surface. This triviality of the solutions is due to the limit identity (21) highlights the limitations of the homogenization of the nonlinear model if compared with reality, where folds appear. This is due to the fact that in reality, ε is not really zero and thus (21) is not an identity, but rather a controlled distance. A forthcoming work, based on [3], will introduce penalizing terms in the right hand side, that vanish in the limit:

$$\begin{aligned}\mathcal{J}_\varepsilon^{ext}(v_\varepsilon) = &\int\limits_{S_\varepsilon^*}Q\Big(\frac{\cdot}{\varepsilon},\frac{1}{2}\big(\mathbf{A}(v_\varepsilon)+\mathbf{B}_{el}(v_\varepsilon)+\mathbf{B}_w(v_\varepsilon)\big)\Big)\,dx - \int\limits_{S_\varepsilon^*}f_\varepsilon\cdot(\mathcal{V}_\varepsilon - I_d)\,dx\\ &+\varepsilon\|\partial_1\mathbf{R}_\varepsilon\mathbf{e}_2+\partial_2\mathbf{R}_\varepsilon\mathbf{e}_1\|_{L^2(\Omega)^3}+\varepsilon^3\|\partial_1\mathcal{V}_\varepsilon\cdot\mathbf{R}_\varepsilon\mathbf{e}_2-\partial_2\mathcal{V}_\varepsilon\cdot\mathbf{R}_\varepsilon\mathbf{e}_1\|_{L^2(\Omega)}, \qquad \forall v_\varepsilon\in\mathbb{D}_\varepsilon^*.\end{aligned}$$

In this sense, the minimization problem has solution near zero, allowing more configurations since $\mathcal{Z}$ is not set to be zero. We can then reconstruct the solution after the homogenization.

9 | APPENDIX

Here in Appendix we collect some additional results.

9.1 | Additional regularity of the anti-symmetric matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$

⟨lem_abc⟩ **Lemma 9.** Let $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ be the matrices defined in (26) with entries $\mathbf{L}, \mathbf{M}, \mathbf{N} \in L^2(\Omega)$. We have that

$$\mathbf{A}^{(2)}\mathbf{A}^{(1)} = \mathbf{A}^{(1)}\mathbf{A}^{(2)} \qquad \text{and} \qquad \partial_1\mathbf{A}^{(2)} = \partial_2\mathbf{A}^{(1)} \qquad \text{in} \quad H^{-1}(\Omega)^{3\times 3}$$

and

$$\mathbf{M}^2 = \mathbf{L}\mathbf{N}, \qquad \partial_1\mathbf{M} = \partial_2\mathbf{L}, \qquad \partial_1\mathbf{N} = \partial_2\mathbf{M}.$$

Proof. Since Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary, we extend $\mathcal{V}$ in a field, still denoted $\mathcal{V}$, belonging to $H^2(\mathbb{R}^2)^3$ and such that $\nabla\mathcal{V} \in L^\infty(\mathbb{R}^2)^{3\times 2}$ (see Theorem 3.15 and Lemma 3.16 in [1]). Then, we regularize the field $\mathcal{V}$ in oder to obtain a sequence $\{\mathcal{V}_n\}_n$ in $C^\infty(\overline{\Omega})^3$ such that

$$\begin{aligned} &\mathcal{V}_n \rightarrow \mathcal{V} \quad \text{strongly in} \;\; H^2(\Omega)^3, \\ &\nabla\mathcal{V}_n \rightharpoonup \nabla\mathcal{V} \quad \text{weakly* in} \;\; L^\infty(\Omega)^{3\times 2}. \end{aligned} \tag{60}$$ EQ65

Let us define the sequence of matrix fields $\{\mathbf{R}_n\}_n$ by setting

$$\mathbf{R}_n = \left(\partial_1\mathcal{V}_n \;\; \partial_2\mathcal{V}_n \;\; \partial_1\mathcal{V}_n \wedge \partial_2\mathcal{V}_n\right).$$

By construction $\mathbf{R}_n$ belongs to $C^\infty(\overline{\Omega})^{3\times 3}$ and is uniformly bounded in $H^1(\Omega)^{3\times 3} \cap L^\infty(\Omega)^{3\times 3}$. From $(60)_1$ we have ($\alpha \in \{1,2\}$)

$$\mathbf{R}_n\mathbf{e}_\alpha \rightarrow \mathbf{R}\mathbf{e}_\alpha \quad \text{strongly in } H^1(\Omega)^3.$$

Then, taking into account the above convergence and (60), we easily check that

$$\mathbf{R}_n\mathbf{e}_3 \rightharpoonup \mathbf{R}\mathbf{e}_3 \quad \text{weakly in } H^1(\Omega)^3, \qquad \mathbf{R}_n \rightarrow \mathbf{R} \quad \text{weakly* in } L^\infty(\Omega)^{3\times 3}.$$

Now,we define the sequences of matrices $\{\mathbf{A}_n^{(\alpha)}\}_n$ such that ($\alpha \in \{1,2\}$)

$$\mathbf{A}_n^{(\alpha)} \doteq \mathbf{R}^T\partial_\alpha\mathbf{R}_n.$$

One has $\mathbf{A}_n^{(\alpha)} \in H^1(\Omega)^{3\times 3}$ and

$$\mathbf{A}_n^{(\alpha)} \rightharpoonup \mathbf{A}^{(\alpha)} \quad \text{weakly in } L^2(\Omega)^{3\times 3}.$$

Note that

$$\begin{aligned} \partial_{12}\mathbf{R}_n = \partial_1\big(\partial_2\mathbf{R}_n\big) = \partial_1\big(\mathbf{R}\mathbf{A}_n^{(2)}\big) = (\partial_1\mathbf{R})\mathbf{A}_n^{(2)} + \mathbf{R}\partial_1\mathbf{A}_n^{(2)} = \mathbf{R}\big(\mathbf{A}^{(1)}\mathbf{A}_n^{(2)} + \partial_1\mathbf{A}_n^{(2)}\big), \\ \partial_{21}\mathbf{R}_n = \partial_2\big(\partial_1\mathbf{R}_n\big) = \partial_2\big(\mathbf{R}\mathbf{A}_n^{(1)}\big) = (\partial_2\mathbf{R})\mathbf{A}_n^{(1)} + \mathbf{R}\partial_2\mathbf{A}_n^{(1)} = \mathbf{R}\big(\mathbf{A}^{(2)}\mathbf{A}_n^{(1)} + \partial_2\mathbf{A}_n^{(1)}\big), \end{aligned}$$

which leads to

$$\mathbf{A}^{(1)}\mathbf{A}_n^{(2)} + \partial_1\mathbf{A}_n^{(2)} = \mathbf{A}^{(2)}\mathbf{A}_n^{(1)} + \partial_2\mathbf{A}_n^{(1)}.$$

By the fact that

$$\begin{aligned} \mathbf{A}^{(1)}\mathbf{A}_n^{(2)} \rightharpoonup \mathbf{A}^{(1)}\mathbf{A}^{(2)} \quad \text{weakly in } L^1(\Omega)^{3\times 3}, \\ \mathbf{A}^{(2)}\mathbf{A}_n^{(1)} \rightharpoonup \mathbf{A}^{(2)}\mathbf{A}^{(1)} \quad \text{weakly in } L^1(\Omega)^{3\times 3}, \end{aligned}$$

we pass to the limit in the space of distributions $\big(\mathcal{D}(\Omega)'\big)^{3\times 3}$ and we obtain

$$\mathbf{A}^{(1)}\mathbf{A}^{(2)} + \partial_1\mathbf{A}^{(2)} = \mathbf{A}^{(2)}\mathbf{A}^{(1)} + \partial_2\mathbf{A}^{(1)} \quad \text{a.e in } \big(\mathcal{D}(\Omega)'\big)^{3\times 3}.$$

Finally, a simple computation shows that

$$\mathbf{A}^{(1)}\mathbf{A}^{(2)} = \begin{pmatrix} -\mathbf{L}\mathbf{M} & -\mathbf{L}\mathbf{N} & 0 \\ -\mathbf{M}^2 & -\mathbf{M}\mathbf{N} & 0 \\ 0 & 0 & -\mathbf{M}(\mathbf{L}+\mathbf{N}) \end{pmatrix}, \qquad \mathbf{A}^{(2)}\mathbf{A}^{(1)} = \begin{pmatrix} -\mathbf{L}\mathbf{M} & -\mathbf{M}^2 & 0 \\ -\mathbf{L}\mathbf{N} & -\mathbf{M}\mathbf{N} & 0 \\ 0 & 0 & -\mathbf{M}(\mathbf{L}+\mathbf{N}) \end{pmatrix}$$

Hence,

$$\begin{pmatrix} 0 & 0 & -(\partial_2\mathbf{L} - \partial_1\mathbf{M}) \\ 0 & 0 & -(\partial_2\mathbf{M} - \partial_1\mathbf{N}) \\ \partial_2\mathbf{L} - \partial_1\mathbf{M} & \partial_2\mathbf{M} - \partial_1\mathbf{N} & 0 \end{pmatrix} = \partial_2\mathbf{A}^{(1)} - \partial_1\mathbf{A}^{(2)} = \mathbf{A}^{(1)}\mathbf{A}^{(2)} - \mathbf{A}^{(2)}\mathbf{A}^{(1)} = \begin{pmatrix} 0 & \mathbf{M}^2 - \mathbf{L}\mathbf{N} & 0 \\ -(\mathbf{M}^2 - \mathbf{L}\mathbf{N}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that $\mathbf{M}^2 = \mathbf{LN}$, that $\partial_2 \mathbf{L} = \partial_1 \mathbf{M}$ and that $\partial_2 \mathbf{M} = \partial_1 \mathbf{N}$. □

9.2 | Examples of materials for which condition (46) is satisfied

In order to satisfy condition (46), we first need to understand the geometry of our periodic structure and spot the transformations for which it is invariant.

If we extend by periodicity the reference cell

$$\mathcal{Y}^*_{per} \doteq \text{Interior}\Big(\bigcup_{(p,q)\in\mathbb{Z}^2} \big(2p\mathbf{e}_1 + 2q\mathbf{e}_2 + \overline{\mathcal{Y}^*}\big)\Big),$$

w note that it is invariant under the following $\mathbb{R}^3 \to \mathbb{R}^3$ maps:

$$\begin{aligned}
&\text{C1:} \quad y \mapsto (1 + y_1, y_2, -y_3),\\
&\text{C2:} \quad y \mapsto (y_1, 1 + y_2, -y_3),\\
&\text{C3:} \quad y \mapsto (1 + y_1, 1 + y_2, y_3) \ \text{ and } \ y \mapsto (y_1, y_2, -y_3).
\end{aligned}$$

In the next Lemma, we prove that

⟨lem9⟩ **Lemma 10.** Let $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbf{H}^1_{per}(\mathcal{Y}^*)$. Set

$$\mathcal{E}(\mathbf{a}, \mathbf{b}) \doteq \begin{pmatrix} \mathbf{b}_1 + y_3\mathbf{a}_1 & * & * \\ \mathbf{b}_2 + y_3\mathbf{a}_2 & \mathbf{b}_3 + y_3\mathbf{a}_3 & * \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{E}_y(\mathbf{c}) \doteq \begin{pmatrix} e_{11,y}(\mathbf{c}) & * & * \\ e_{12,y}(\mathbf{c}) & e_{22,y}(\mathbf{c}) & * \\ e_{13,y}(\mathbf{c}) & e_{23,y}(\mathbf{c}) & e_{33,y}(\mathbf{c}) \end{pmatrix}.$$

Define the quadratic form

$$Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) \doteq \int_{\mathcal{Y}^*} \mathbf{A} \left[\begin{pmatrix} \mathbf{b}_1 + y_3\mathbf{a}_1 \\ \mathbf{b}_2 + y_3\mathbf{a}_2 \\ \mathbf{b}_3 + y_3\mathbf{a}_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\mathbf{c}) \\ e_{12,y}(\mathbf{c}) \\ e_{22,y}(\mathbf{c}) \\ e_{33,y}(\mathbf{c}) \\ e_{13,y}(\mathbf{c}) \\ e_{23,y}(\mathbf{c}) \end{pmatrix} \right] \cdot \left[\begin{pmatrix} \mathbf{b}_1 + y_3\mathbf{a}_1 \\ \mathbf{b}_2 + y_3\mathbf{a}_2 \\ \mathbf{b}_3 + y_3\mathbf{a}_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\mathbf{c}) \\ e_{12,y}(\mathbf{c}) \\ e_{22,y}(\mathbf{c}) \\ e_{33,y}(\mathbf{c}) \\ e_{13,y}(\mathbf{c}) \\ e_{23,y}(\mathbf{c}) \end{pmatrix} \right] dy, \tag{61}$$ quad

where the matrix $\mathbf{A} \in L^\infty(\mathcal{Y}^*_{per})^{6\times 6}$ is symmetric and positive-definite.

If $\mathbf{A}$ is invariant under transformation C1, that is

$$\mathbf{A}(1 + y_1, y_2, -y_3) = \mathbf{A}(y) \ \text{ for a.e. } \ y \in \mathcal{Y}^*_{per}, \tag{62}$$ C1_inv

then we have

$$\min_{\mathbf{c}\in\mathbf{H}^1_{per}(\mathcal{Y}^*)} Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) = Q_a(\mathbf{a}) + Q_b(\mathbf{b})$$

where Q_a and Q_b are positive-definite quadratic forms in $\mathbb{R}^3$.

Proof. We first solve the following minimization problem

$$\min_{\mathbf{c}\in\mathbf{H}^1_{per}(\mathcal{Y}^*)} Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) = Q(\mathbf{a}, \mathbf{b}, \mathring{\mathbf{c}}) = \mathcal{Q}(\mathbf{a}, \mathbf{b}).$$

So, the displacement $\mathring{\mathbf{c}} \in \mathbf{H}^1_{per}(\mathcal{Y}^*)^3$ is the solution of the following variational problem:

$$\begin{cases} \text{Find } \mathring{\mathbf{c}} \in \mathbf{H}^1_{per}(\mathcal{Y}^*) \text{ such that} \\ \displaystyle\int_{\mathcal{Y}^*} \mathbf{A} \left[\begin{pmatrix} \mathbf{b}_1 + y_3\mathbf{a}_1 \\ \mathbf{b}_2 + y_3\mathbf{a}_2 \\ \mathbf{b}_3 + y_3\mathbf{a}_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\mathring{\mathbf{c}}) \\ e_{12,y}(\mathring{\mathbf{c}}) \\ e_{22,y}(\mathring{\mathbf{c}}) \\ e_{33,y}(\mathring{\mathbf{c}}) \\ e_{13,y}(\mathring{\mathbf{c}}) \\ e_{23,y}(\mathring{\mathbf{c}}) \end{pmatrix} \right] \cdot \begin{pmatrix} e_{11,y}(\mathbf{d}) \\ e_{12,y}(\mathbf{d}) \\ e_{22,y}(\mathbf{d}) \\ e_{33,y}(\mathbf{d}) \\ e_{13,y}(\mathbf{d}) \\ e_{23,y}(\mathbf{d}) \end{pmatrix} dy = 0, \qquad \forall \mathbf{d} \in \mathbf{H}^1_{per}(\mathcal{Y}^*). \end{cases}$$

We introduce the 6 correctors as solutions of the following cell problems (δ_{ij} is the Kronecker symbol):

$$\begin{cases} \text{Find } (\widehat{\chi}_1^a, \widehat{\chi}_2^a, \widehat{\chi}_3^a, \widehat{\chi}_1^b, \widehat{\chi}_2^b, \widehat{\chi}_3^b) \in \mathbf{H}^1_{per}(\mathcal{Y}^*)^6 \text{ such that} \\ \displaystyle\int_{\mathcal{Y}^*} \mathbf{A}\left[\begin{pmatrix} y_3\delta_{1i} \\ y_3\delta_{2i} \\ y_3\delta_{3i} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\widehat{\chi}_i^a) \\ e_{12,y}(\widehat{\chi}_i^a) \\ e_{22,y}(\widehat{\chi}_i^a) \\ e_{33,y}(\widehat{\chi}_i^a) \\ e_{13,y}(\widehat{\chi}_i^a) \\ e_{23,y}(\widehat{\chi}_i^a) \end{pmatrix}\right] \cdot \begin{pmatrix} e_{11,y}(\mathbf{d}) \\ e_{12,y}(\mathbf{d}) \\ e_{22,y}(\mathbf{d}) \\ e_{33,y}(\mathbf{d}) \\ e_{13,y}(\mathbf{d}) \\ e_{23,y}(\mathbf{d}) \end{pmatrix} dy = 0, \qquad \forall \mathbf{d} \in \mathbf{H}^1_{per}(\mathcal{Y}^*), \quad \forall i \in \{1,2,3\}, \\ \displaystyle\int_{\mathcal{Y}^*} \mathbf{A}\left[\begin{pmatrix} y_3\delta_{1i} \\ y_3\delta_{2i} \\ y_3\delta_{3i} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\widehat{\chi}_i^b) \\ e_{12,y}(\widehat{\chi}_i^b) \\ e_{22,y}(\widehat{\chi}_i^b) \\ e_{13,y}(\widehat{\chi}_i^b) \\ e_{23,y}(\widehat{\chi}_i^b) \\ e_{33,y}(\widehat{\chi}_i^b) \end{pmatrix}\right] \cdot \begin{pmatrix} e_{11,y}(\mathbf{d}) \\ e_{12,y}(\mathbf{d}) \\ e_{22,y}(\mathbf{d}) \\ e_{13,y}(\mathbf{d}) \\ e_{23,y}(\mathbf{d}) \\ e_{33,y}(\mathbf{d}) \end{pmatrix} dy = 0, \qquad \forall \mathbf{d} \in \mathbf{H}^1_{per}(\mathcal{Y}^*), \quad \forall i \in \{1,2,3\}. \end{cases} \tag{63}$$ cell_sols

Hence, we obtain that $\mathring{\mathbf{c}}$ depends linearly on the vectors $\mathbf{a}$, $\mathbf{b}$ and the correctors

$$\mathring{\mathbf{c}} \doteq \sum_{i=1}^{3} \mathbf{a}_i \widehat{\chi}_i^a + \sum_{i=1}^{3} \mathbf{b}_i \widehat{\chi}_i^b. \tag{64}$$ c_found

Now, set

$$\mathbf{A} \doteq \begin{pmatrix} \mathbf{A}_1 & \mathbf{O}_{42} \\ \mathbf{O}_{24} & \mathbf{A}_2 \end{pmatrix}, \tag{65}$$ exp_A

where $\mathbf{A}_1 \in L^\infty(\mathcal{Y}^*_{per})^{4\times4}$, $\mathbf{A}_2 \in L^\infty(\mathcal{Y}^*_{per})^{2\times2}$, $\mathbf{O}_{42} \in L^\infty(\mathcal{Y}^*_{per})^{4\times2}$ with zero entries and $\mathbf{O}_{24} \in L^\infty(\mathcal{Y}^*_{per})^{2\times4}$, also with zero entries. Replacing (64) in the quadratic form (61), we can rewrite it in the following compact form:

$$\mathcal{Q}(\mathbf{a},\mathbf{b}) = \int_{\mathcal{Y}^*} \mathbf{A}_1(\overline{\mathbf{b}} + y_3\overline{\mathbf{a}}) \cdot (\overline{\mathbf{b}} + y_3\overline{\mathbf{a}}) + \mathbf{A}_1\left[E_y(\widehat{\chi}^a)\overline{\mathbf{a}} + E_y(\widehat{\chi}^b)\overline{\mathbf{b}}\right] \cdot (\overline{\mathbf{b}} + y_3\overline{\mathbf{a}})\, dy, \tag{66}$$ compact_Q

where

$$\overline{\mathbf{a}} \doteq \begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix}, \ \overline{\mathbf{b}} \doteq \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \in \mathbb{R}^4, \qquad \text{and} \qquad E_y(\widehat{\chi}) \doteq \begin{pmatrix} e_{11,y}(\widehat{\chi}_1) & e_{11,y}(\widehat{\chi}_2) & e_{11,y}(\widehat{\chi}_3) \\ e_{12,y}(\widehat{\chi}_1) & e_{12,y}(\widehat{\chi}_2) & e_{12,y}(\widehat{\chi}_3) \\ e_{22,y}(\widehat{\chi}_1) & e_{22,y}(\widehat{\chi}_2) & e_{22,y}(\widehat{\chi}_3) \\ e_{33,y}(\widehat{\chi}_1) & e_{33,y}(\widehat{\chi}_2) & e_{33,y}(\widehat{\chi}_3) \end{pmatrix} \in L^2(\mathcal{Y}^*)^{4\times3}.$$

We can rewrite (66) as

$$\begin{aligned} \mathcal{Q}(\mathbf{a},\mathbf{b}) =& \Big(\int_{\mathcal{Y}^*} \left[y_3^2\mathbf{A}_1 + y_3\mathbf{A}_1E_y(\widehat{\chi}^a)\right]\overline{\mathbf{a}} \cdot \overline{\mathbf{a}}\, dy + \int_{\mathcal{Y}^*} \left[\mathbf{A}_1 + \mathbf{A}_1E_y(\widehat{\chi}^b)\right]\overline{\mathbf{b}} \cdot \overline{\mathbf{b}}\, dy \Big) \\ &+ 2\Big(\int_{\mathcal{Y}^*} y_3\mathbf{A}_1\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}\, dy + \int_{\mathcal{Y}^*} A_1\Big[\big(E_y(\widehat{\chi}^a)\,\overline{\mathbf{a}}\big) \cdot \overline{\mathbf{b}} + y_3\big(E_y(\widehat{\chi}^b)\overline{\mathbf{b}}\big) \cdot \overline{\mathbf{a}}\Big]\, dy \Big). \end{aligned} \tag{67}$$ par

To prove the lemma statement it is left to show that if $\mathbf{A}$ is invariant with respect to transformation C1, then the term in the second parenthesis is identically equal to zero.

Hence, consider the transformation

$$\phi \in H^1_{per}(\mathcal{Y}^*)^3 \mapsto \widetilde{\phi} \in H^1_{per}(\mathcal{Y}^*)^3, \qquad \widetilde{\phi}(y) = \phi_1(\widetilde{y})\mathbf{e}_1 + \phi_2(\widetilde{y})\mathbf{e}_2 - \phi_3(\widetilde{y})\mathbf{e}_3 \quad \text{for a.e. } \widetilde{y} \doteq (1+y_1)\mathbf{e}_1 + y_2\mathbf{e}_2 - y_3\mathbf{e}_3 \in \mathcal{Y}^*.$$

For any $\phi \in H^1_{per}(\mathcal{Y}^*)^3$ we have

$$\begin{pmatrix} e_{11,y}(\widetilde{\phi}))(y) \\ e_{12,y}(\widetilde{\phi})(y) \\ e_{22,y}(\widetilde{\phi})(y) \\ e_{33,y}(\widetilde{\phi})(y) \\ e_{13,y}(\widetilde{\phi})(y) \\ e_{23,y}(\widetilde{\phi})(y) \end{pmatrix} = \begin{pmatrix} e_{11,y}(\phi)(\widetilde{y}) \\ e_{12,y}(\phi)(\widetilde{y}) \\ e_{22,y}(\phi)(\widetilde{y}) \\ e_{33,y}(\phi)(\widetilde{y}) \\ -e_{13,y}(\phi)(\widetilde{y}) \\ -e_{23,y}(\phi)(\widetilde{y}) \end{pmatrix}.$$

Let $\mathbf{d}$ be in $\mathbf{H}^1_{per}(\mathcal{Y}^*)$. Choosing $\widetilde{\mathbf{d}} \in \mathbf{H}^1_{per}(\mathcal{Y}^*)$ as test-function in equation $(63)_1$, we obtain

$$\int_{\mathcal{Y}^*} \mathbf{A}(y) \left[y_3 \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\widehat{\chi}^a_i)(y) \\ e_{12,y}(\widehat{\chi}^a_i)(y) \\ e_{22,y}(\widehat{\chi}^a_i)(y) \\ e_{33,y}(\widehat{\chi}^a_i)(y) \\ e_{13,y}(\widehat{\chi}^a_i)(y) \\ e_{23,y}(\widehat{\chi}^a_i)(y) \end{pmatrix} \right] \cdot \begin{pmatrix} e_{11,y}(\widetilde{\mathbf{d}})(y) \\ e_{12,y}(\widetilde{\mathbf{d}})(y) \\ e_{22,y}(\widetilde{\mathbf{d}})(y) \\ e_{33,y}(\widetilde{\mathbf{d}})(y) \\ e_{13,y}(\widetilde{\mathbf{d}})(y) \\ e_{23,y}(\widetilde{\mathbf{d}})(y) \end{pmatrix} dy = 0.$$

Now, we use the change of variable $\widetilde{y}$, this gives

$$\int_{\mathcal{Y}^*} \mathbf{A}(\widetilde{y}) \left[-y_3 \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\widehat{\chi}^a_i)(\widetilde{y}) \\ e_{12,y}(\widehat{\chi}^a_i)(\widetilde{y}) \\ e_{22,y}(\widehat{\chi}^a_i)(\widetilde{y}) \\ e_{33,y}(\widehat{\chi}^a_i)(\widetilde{y}) \\ e_{13,y}(\widehat{\chi}^a_i)(\widetilde{y}) \\ e_{23,y}(\widehat{\chi}^a_i)(\widetilde{y}) \end{pmatrix} \right] \cdot \begin{pmatrix} e_{11,y}(\widetilde{\mathbf{d}})(\widetilde{y}) \\ e_{12,y}(\widetilde{\mathbf{d}})(\widetilde{y}) \\ e_{22,y}(\widetilde{\mathbf{d}})(\widetilde{y}) \\ e_{33,y}(\widetilde{\mathbf{d}})(\widetilde{y}) \\ e_{13,y}(\widetilde{\mathbf{d}})(\widetilde{y}) \\ e_{23,y}(\widetilde{\mathbf{d}})(\widetilde{y}) \end{pmatrix} dy = 0,$$

which can be rewritten as (since $\mathbf{A}$ satisfies (62))

$$\int_{\mathcal{Y}^*} \mathbf{A}(y) \left[-y_3 \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ e_{12,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ e_{22,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ e_{33,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ -e_{13,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ -e_{23,y}(\widetilde{\widehat{\chi}^a_i})(y) \end{pmatrix} \right] \cdot \begin{pmatrix} e_{11,y}(\mathbf{d})(y) \\ e_{12,y}(\mathbf{d})(y) \\ e_{22,y}(\mathbf{d})(y) \\ e_{33,y}(\mathbf{d})(y) \\ -e_{13,y}(\mathbf{d})(y) \\ -e_{23,y}(\mathbf{d})(y) \end{pmatrix} dy = 0.$$

Due to the expression (65) of the matrix $\mathbf{A}$, we obtain

$$\int_{\mathcal{Y}^*} \mathbf{A}(y) \left[-y_3 \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e_{11,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ e_{12,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ e_{22,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ e_{33,y}(\widetilde{\widehat{\chi}^a_i})(y) \\ e_{13,y}(\widehat{\chi}^a_i)(y) \\ e_{23,y}(\widehat{\chi}^a_i)(y) \end{pmatrix} \right] \cdot \begin{pmatrix} e_{11,y}(\mathbf{d})(y) \\ e_{12,y}(\mathbf{d})(y) \\ e_{22,y}(\mathbf{d})(y) \\ e_{33,y}(\mathbf{d})(y) \\ e_{13,y}(\mathbf{d})(y) \\ e_{23,y}(\mathbf{d})(y) \end{pmatrix} dy = 0.$$

This gives

$$\widetilde{\widehat{\chi}^a_i} = -\widehat{\chi}^a_i, \qquad \text{a.e. } y \in \mathcal{Y}^*, \qquad i \in \{1,2,3\}.$$

Proceeding in a similar way we prove that

$$\widetilde{\widehat{\chi}^b_i} = \widehat{\chi}^b_i, \qquad \text{a.e. } y \in \mathcal{Y}^*, \qquad i \in \{1,2,3\}.$$

Finally, a change of variable and property (62) imply that

$$\int_{\mathcal{Y}^*} y_3 \mathbf{A}_1(y)\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}\, dy = \int_{\mathcal{Y}^*} -y_3 \mathbf{A}_1(\widetilde{y})\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}\, dy = -\int_{\mathcal{Y}^*} y_3 \mathbf{A}_1(y)\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}\, dy,$$

$$\int_{\mathcal{Y}^*} \mathbf{A}_1(y) E_y\big(\widehat{\chi}^a(y)\big)\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}\, dy = \int_{\mathcal{Y}^*} \mathbf{A}_1(\widetilde{y}) E_y\big(\widehat{\chi}^a(\widetilde{y})\big)\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}\, dy = -\int_{\mathcal{Y}^*} \mathbf{A}_1(y) E_y\big(\widehat{\chi}^a(y)\big)\overline{\mathbf{a}} \cdot \overline{\mathbf{b}}\, dy,$$

$$\int_{\mathcal{Y}^*} y_3 \mathbf{A}_1(y) E_y\big(\widehat{\chi}^b(y)\big)\overline{\mathbf{b}} \cdot \overline{\mathbf{a}}\, dy = \int_{\mathcal{Y}^*} -y_3 \mathbf{A}_1(\widetilde{y}) E_y\big(\widehat{\chi}^b(\widetilde{y})\big)\overline{\mathbf{b}} \cdot \overline{\mathbf{a}}\, dy = -\int_{\mathcal{Y}^*} \mathbf{A}_1(y) E_y\big(\widehat{\chi}^b(y)\big)\overline{\mathbf{b}} \cdot \overline{\mathbf{a}}\, dy$$

and therefore the above three integrals are equal to zero. As a consequence, the second parenthesis in (67) is equal to zero and the statement follows by setting

$$\mathcal{Q}_a(\mathbf{a}) \doteq \int_{\mathcal{Y}^*} \left[y_3^2 \mathbf{A}_1 + y_3 \mathbf{A}_1 E_y(\widehat{\chi}^a)\right] \overline{\mathbf{a}} \cdot \overline{\mathbf{a}}\, dy, \qquad \mathcal{Q}_b(\mathbf{b}) \doteq \int_{\mathcal{Y}^*} \left[\mathbf{A}_1 + \mathbf{A}_1 E_y(\widehat{\chi}^b)\right] \overline{\mathbf{b}} \cdot \overline{\mathbf{b}}\, dy,$$

which are positive-definite quadratic forms. □

⟨rem4⟩ *Remark 3.* With an identical argumentation, it is possible to prove that the same result holds if instead of transformation C1, the matrix $\mathbf{A}$ is invariant under transformation C2 or C3. In particular, the invariance under such transformations is ensured whenever a material is homogeneous and isotropic.

However, two more considerations are worth mentioning:

- A sufficient condition for the lemma statement to hold is the absence of mixed terms in the quadratic form (66), i.e., that the second parenthesis in (67) is equal to zero. In general, given a periodic reference cell and a quadratic form defined on it, one need to verify manually such condition, looking for invariant transformation in the cell geometry and giving the consequent assumptions on the material.
- Since the condition is not necessary, it is possible that the quadratic form (66) admits its minimum for $\mathbf{b} = 0$ even if the second parenthesis in (67) is not zero.

9.3 | Strong convergence of the test functions

We recall an equivalent formulation of Lemma C in [3], where estimates (68) hold with a stricter bound. Note that we can assume one fourth of the bounds given in the Lemma's original formulation since, in that proof, they have been chosen arbitrarily small.

⟨lem11⟩ **Lemma 11.** Let $\mathcal{W}$ be in $\mathbb{D}$. There exists a sequence of pairs $\{(\mathcal{W}_\varepsilon, \mathbf{R}_\varepsilon)\}_\varepsilon$ in $W^{2,\infty}(\Omega)^3 \times W^{1,\infty}(\Omega)^{3\times 3}$ such that

$$\mathcal{W}_\varepsilon = I_d, \quad \mathbf{R}_\varepsilon = \mathbf{I}_3, \quad \text{a.e. in } \boldsymbol{\gamma}.$$

The following convergences hold ($\alpha \in \{1,2\}$):

$$\begin{aligned}
&\mathcal{W}_\varepsilon \to \mathcal{W} \quad \text{strongly in} \quad H^2(\Omega)^3,\\
&\mathbf{R}_\varepsilon \to \mathbf{R}(\mathcal{W}) \quad \text{strongly in} \quad H^1(\Omega)^{3\times 3},\\
&\frac{1}{\varepsilon}\left(\mathbf{R}_\varepsilon - \mathbf{R}\right) \to 0 \quad \text{strongly in} \quad L^2(\Omega)^3,\\
&\frac{1}{\varepsilon}\left(\partial_\alpha \mathcal{W}_\varepsilon - \mathbf{R}_\varepsilon \mathbf{e}_\alpha\right) \to 0 \quad \text{strongly in} \quad L^2(\Omega)^3.
\end{aligned}$$

Moreover, the following estimates hold ($\alpha \in \{1,2\}$):

$$\|\mathrm{dist}(\mathbf{R}_\varepsilon, SO(3))\|_{L^\infty(\Omega)} \le \frac{1}{32}, \qquad \|\partial_\alpha \mathcal{W}_\varepsilon - \mathbf{R}_\varepsilon \mathbf{e}_\alpha\|_{L^\infty(\Omega)} \le \frac{1}{32}, \qquad \|\mathbf{R}_\varepsilon\|_{W^{1,\infty}(\Omega)} \le \frac{1}{16\varepsilon}. \tag{68}$$ EQ98

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How to cite this article: R. Falconi, G. Griso, and J. Orlik (2023), Asymptotic behavior for nonlinear textiles with glued yarns, *European Journal of Mechanics - A/Solids*, *2023;xxx:xxx.*