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Mourre theory for analytically fibered operators revisited

F. Nier ^{*}
C. Gérard [†]

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Abstract

About 25 years ago our article “Mourre theory for analytically fibered operators” was published in J. of Functional Analysis. This article proposed a general construction of a conjugate operator for a wide class of self-adjoint analytically fibered hamiltonians, provided that one accepts a more accurate notion of threshold. It is only recently that Olivier Poisson mentioned us a problem with the statement that $H_0 \in \mathcal{C}^\infty(A_I)$. Actually even $H_0 \in \mathcal{C}^2(A_I)$ or $H_0 \in \mathcal{C}^{1+0}(A_I)$, which is crucial for the full application of Mourre theory, is problematic with our initial construction. However the statement and the construction can be modified in order to make work all the theory. This is explained here.

Keywords: Mourre theory, stratification of subanalytic sets and maps, unitary connections

MSC2020: 32B20, 35P25, 47A40, 53B35, 81U99.

1 Introduction

In [GeNi] we introduced the notion of an analytically fibered operator

$$H_0 = \int_M^\oplus H_0(k) dv(k)$$

where:

- (i) M is a real analytic manifold and $dv(k)$ stands for a volume density which can be written $dv(k) = e^{2V(k)} dk_1 \dots dk_{\dim M}$ in any coordinate chart with $V(k)$ real analytic;
- (ii) \mathcal{H}' is a Hilbert space, $\mathcal{L}(\mathcal{H}')$ and $\mathcal{L}^\infty(\mathcal{H}')$ respectively denote the spaces of bounded and compact operators in \mathcal{H}' ;
- (iii) for all $k \in M$, $(H_0(k), D(H_0(k)))$ is a self-adjoint operator in \mathcal{H}' such that $M \ni k \mapsto (H_0(k) + i)^{-1} \in \mathcal{L}^\infty(\mathcal{H}')$ is real analytic (actually this can be replaced by $(H_0(k) + i)^{-1}$ is a real analytic section of $\mathcal{L}^\infty(\mathcal{F})$ where \mathcal{F} is a real analytic Hilbert bundle $p_{\mathcal{F}} : \mathcal{F} \rightarrow M$ with fiber $\mathcal{F}_k \sim \mathcal{H}'$);

^{*}LAGA, Université de Paris XIII, 99 avenue J.B. Clément, F-93430 Villetaneuse, France. nier@math.univ-paris13.fr

[†]Université Paris-Saclay, UMR-CNRS 8628, Laboratoire mathématiques d'Orsay, F-91405 Orsay, christian.gerard@universite-paris-saclay.fr

- (iv) when $\Sigma = \{(k, \lambda) \in M \times \mathbb{R}, \lambda \in \text{Spec}(H_0(k))\}$ is the analytic characteristic variety in $M \times \mathbb{R}$, the map $p_{\Sigma \rightarrow \mathbb{R}} : \Sigma \rightarrow \mathbb{R}$ given by $p_{\Sigma \rightarrow \mathbb{R}}(k, \lambda) = \lambda = p_{\mathbb{R}}(k, \lambda)$ is proper. We distinguish by notations the natural projection $p_{\mathbb{R}} : M \times \mathbb{R} \rightarrow \mathbb{R}$ from its restriction to Σ , $p_{\Sigma \rightarrow \mathbb{R}} = p_{\mathbb{R}}|_{\Sigma}$.

After noticing that the analytic characteristic variety can be partitionned in the semi-analytic sets according to the multiplicity μ of the eigenvalue λ of $H_0(k)$, $\Sigma = \bigsqcup_{\mu} \Sigma_{\mu}$, the general theory of subanalytic sets (see [Hardt, Hiro, DeLe]) tells us that there exists a (locally finite) Whitney stratification \mathcal{S} of Σ compatible with $p_{\Sigma \rightarrow \mathbb{R}}$, which means here:

- $\Sigma = \bigsqcup_{S \in \mathcal{S}} S$, where any $S \in \mathcal{S}$ is an analytic connected manifold embedded in $\mathbb{R} \times M$;
- $\overline{S} \cap S' \neq \emptyset$ implies $S' = S$ or $S' \subset \partial S \stackrel{\text{def}}{=} \overline{S} \setminus S$;
- for all $S \in \mathcal{S}$ there exists μ_S such that $S \subset \Sigma_{\mu_S}$;
- for all $S \in \mathcal{S}$ either $p_{\Sigma \rightarrow \mathbb{R}}(S)$ is an open interval and $dp_{\Sigma \rightarrow \mathbb{R}}|_{TS}$ does not vanish (rank of $dp_{\Sigma \rightarrow \mathbb{R}}|_{TS}$ is constantly equal to 1) or $p_{\Sigma \rightarrow \mathbb{R}}(S) = \{1pt\}$ (rank of $dp_{\Sigma \rightarrow \mathbb{R}}|_{TS}$ is constantly equal to 0).

We refer the reader to the founding articles [Hardt][Hiro] and to [Loja][BiMi] for a panoramic or historical presentation. Another reference concerned with the analysis of subanalytic Lipschitz functions is [DeLe]. The application of these general ideas to our case is given in [GeNi]. We called the locally finite set $\tau = \bigsqcup_{\text{rank } dp_{\Sigma \rightarrow \mathbb{R}}|_{TS} = 0} p_{\Sigma \rightarrow \mathbb{R}}(S)$ in \mathbb{R} , the set of *thresholds* for H_0 . Our main statement in [GeNi] was

“Theorem” 3.1 in [GeNi] 1. *There exists a discrete set τ determined by H_0 so that for any interval $I \Subset \mathbb{R} \setminus \tau$, there exists an operator A_I essentially self-adjoint on $D(A_I) = \mathcal{C}_{comp}^{\infty}(M; \mathcal{F})$ satisfying the following properties:*

- i) For all $\chi \in \mathcal{C}_{comp}^{\infty}(I)$, there exists a constant $c_{\chi} > 0$ so that

$$\chi(H_0)[H_0, iA_I]\chi(H_0) \geq c_{\chi}\chi(H_0)^2. \quad (1)$$

- ii) The multi-commutators $\text{ad}_{A_I}^k(H_0)$ are bounded for all $k \in \mathbb{N}$.

- iii) The operator A_I is a first order differential operator in k whose coefficients belong to $\mathcal{C}^{\infty}(M; \mathcal{L}(\mathcal{F}))$ and there exists $\chi \in \mathcal{C}_{comp}^{\infty}(\mathbb{R} \setminus \tau)$ such that $A_I = \chi(H_0)A_I = A_I\chi(H_0)$.

As mentionned above, the second statement ii) is problematic with the construction of [GeNi]. It was actually not carefully checked in [GeNi]. However the general idea works after some modifications. Below is the proper statement.

Theorem 1.1. *For any interval $I \Subset \mathbb{R} \setminus \tau$, there exists an operator \tilde{A}_I essentially self-adjoint on $D(\tilde{A}_I) = \mathcal{C}_{comp}^{\infty}(M; \mathcal{F})$ which satisfies the following properties:*

- i) For all $\lambda \in I$ there exists $\delta_{\lambda} > 0$ such that

$$1_{[\lambda - \delta_{\lambda}, \lambda + \delta_{\lambda}]}(H_0)[H_0, i\tilde{A}_I]1_{[\lambda - \delta_{\lambda}, \lambda + \delta_{\lambda}]}(H_0) \geq \frac{1}{2}1_{[\lambda - \delta_{\lambda}, \lambda + \delta_{\lambda}]}(H_0).$$

- ii) The multi-commutators $\text{ad}_{\tilde{A}_I}^k(H_0)$ are bounded for all $k \in \mathbb{N}$ and $H_0 \in \mathcal{C}^{\infty}(\tilde{A}_I)$.

iii) The operator \tilde{A}_I is a first order differential operator in k whose coefficients belong to $\mathcal{C}^\infty(M; \mathcal{L}(\mathcal{F}))$ and there exists $\chi \in \mathcal{C}_{comp}^\infty(\mathbb{R} \setminus \tau)$ such that $\tilde{A}_I = \chi(H_0)\tilde{A}_I = \tilde{A}_I\chi(H_0)$.

The construction of the new conjugate operator \tilde{A}_I slightly differs from the one of A_I in order to ensure the condition ii). The price to pay is about Mourre's inequality i) which is now only local (with a uniform constant which is here $\frac{1}{2}$). But we recall that such a local inequality is sufficient for developing all the theory. Additional comments are given in Section 5.1 after the construction of \tilde{A}_I is done.

2 Preliminary remarks and notations

The notion of vector (Hilbert) bundles although pointed out and later used in [GeNi2] was not accurately enough considered in [GeNi]. We specify its use now. The local analysis of [GeNi], putting the stress of locally scalar differential operators, was essentially correct. But gluing all the pieces together, where the dimensions of vector subbundles change, must be done carefully by taking into account the stratified structure of Σ . Elementary remarks and notations are collected here, while we recall the local initial construction of the operator A_I . Its correct variation \tilde{A}_I will be done later.

2.1 Open covering and partition of unity on Σ

The main idea of [GeNi] which still works is that the construction and the analysis can be localized on the characteristic variety Σ . Because $p_{\Sigma \rightarrow \mathbb{R}} : \Sigma \rightarrow \mathbb{R}$ is proper, $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}) \subset \Sigma$ is compact and an open covering $\bigcup_{i \in \mathcal{I}} \Omega_i$ of $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I})$ can always be assumed finite. A partition of unity $\sum_{i \in \mathcal{I}} g_i(\lambda, k)^2 \equiv 1$ over a neighborhood of $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I})$, subordinate to $\bigcup_{i \in \mathcal{I}} \Omega_i$, is made with $g_i \in \mathcal{C}_{comp}^\infty(\Omega_i)$.

Like in [GeNi], the open sets Ω_i , $i \in \mathcal{I}$, will be cylindrical above M , with $\Omega_m = \bigcup_{n=1}^{N_m} \omega_m \times J_{m,n} = \bigcup_{m=1}^{N_m} \Omega_{m,n}$, ω_m open set of M and $J_{m,n}$ open interval. The finite family \mathcal{I} can then be either $\mathcal{I} = \{1, \dots, m_I\}$ or $\mathcal{I} = L(I) \stackrel{def}{=} \{(m, n) \in \mathbb{N}^2, 1 \leq n \leq N_m, 1 \leq m \leq m_I\}$ for a more local presentation on the energy axis. We use the convention that ω is an open set of M and Ω is an open set in $M \times \mathbb{R}$.

It is convenient to introduce another interval \tilde{I} such that

$$I \Subset \tilde{I} \Subset \mathbb{R} \setminus \tau$$

and all the open intervals $J_{m,n}$, $1 \leq n \leq N_m$, $1 \leq m \leq m_I$, will satisfy $J_{m,n} \Subset \tilde{I}$. With I we shall associate the finite (\mathcal{I} is locally finite and $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I})$ is compact) set of strata

$$\mathcal{S}_I = \left\{ S \in \mathcal{I}, \bar{S} \cap p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}) \neq \emptyset \right\}.$$

The assumption $\bar{I} \subset \mathbb{R} \setminus \tau$ implies

$$\forall S \in \mathcal{S}_I, \text{rank } dp_{\Sigma \rightarrow \mathbb{R}}|_{TS} \equiv 1.$$

Proposition 2.1. *Let I, \tilde{I} , be two open intervals such that $I \Subset \tilde{I} \Subset \mathbb{R} \setminus \tau$. There exists open sets $\omega_m \subset M$, for $1 \leq m \leq m_I$, and open intervals $J_{m,n} \Subset \tilde{I}$ for $(m, n) \in L(I) = \{(m, n) \in \mathbb{N}^2, 1 \leq n \leq N_m, 1 \leq m \leq m_I\}$ such that:*

(1) setting $\Omega_{m,n} = \omega_m \times J_{m,n}$, $\bigcup_{(m,n) \in L(I)} \Omega_{m,n}$ is an open covering of $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I})$;

(2) $J_{m,n} \cap J_{m,n'} = \emptyset$ for $n \neq n'$, $1 \leq n, n' \leq N_m$;

(3) for any $(m, n) \in L(I)$ there exists a unique stratum $S_{m,n} \in \mathcal{S}_I$ (rank $dp_{\Sigma \rightarrow \mathbb{R}}|_{TS_{m,n}} \equiv 1$) and $\Omega_{m,n} \cap \Sigma$ is a (finite) union of $\Omega_{m,n} \cap S'$, $S' \in \mathcal{S}_I$, $S_{m,n}$ included in every S' (i.e. $S' = S_{m,n}$ or $S_{m,n} \subset \partial S' = \overline{S'} \setminus S'$);

(4) $1_{J_{m,n}}(H_0(k))$ has a constant rank $\mu_{m,n}$ for $k \in \omega_m$ and there exists another interval $J'_{m,n} \Subset J_{m,n}$ such that $1_{J_{m,n}}(H_0(k)) = 1_{J'_{m,n}}(H_0(k))$ for all $k \in \omega_m$;

(5) for all $(m, n) \in L(I)$ there exists a real analytic section $W_{m,n}$ of $\mathcal{L}(\mathbb{C}^{\mu_{m,n}}; \mathcal{F}|_{\omega_m})$ with $W_{m,n}(k) : \mathbb{C}^{\mu_{m,n}} \rightarrow \text{Ran } 1_{J_{m,n}}(H_0(k))$ unitary for all $k \in \omega_m$ so that

$$W_{m,n}(k)^* H_0(k) 1_{J_{m,n}}(H_0(k)) W_{m,n}(k) = \tilde{H}_{m,n}(k)$$

is a real analytic $\mu_{m,n} \times \mu_{m,n}$ matrix with respect to $k \in \omega_{m,n}$;

(6) there exists a real analytic vector field $X_{m,n}$ on $\omega_{m,n}$ such that

$$\begin{aligned} \forall k \in \omega_{m,n}, \quad X_{m,n} \tilde{H}_{m,n}(k) &\geq \frac{1}{2} \text{Id}_{\mathbb{C}^{\mu_{m,n}}}, \\ [W_{m,n} \circ X_{m,n} \circ W_{m,n}^*] H_0(k) 1_{J_{m,n}}(H_0(k)) &\geq \frac{1}{2} 1_{J_{m,n}}(H_0(k)); \end{aligned}$$

(7) if $\Omega_{m,n} \cap \Omega_{m',n'} \cap \Sigma \neq \emptyset$ and $\mu_{m,n} \geq \mu_{m',n'}$ then

$$\begin{aligned} \forall k \in \omega_m \cap \omega_{m'}, \quad 1_{J_{m,n}}(H_0(k)) 1_{J_{m',n'}}(H_0(k)) &= 1_{J_{m',n'}}(H_0(k)), \\ \text{and } \begin{cases} S_{m,n} \subset \partial S_{m',n'} \\ \text{or } S_{m',n'} \subset \partial S_{m,n} \text{ and } \mu_{m,n} = \mu_{m',n'} \\ \text{or } S_{m',n'} = S_{m,n} \text{ and } \mu_{m,n} = \mu_{m',n'}. \end{cases} \end{aligned}$$

Proof. The construction of the open covering will be done in several steps. Actually because $K_I = p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}))$ is compact in M it suffices to consider carefully the situation in a neighborhood $\omega_0 \Subset M$ of any $k_0 \in p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}))$. We use the index $m = 0$ for this local analysis and in the different steps the open set $\omega_0 \ni k_0$ will be reduced in order to fulfill all the properties. So let us fix $k_0 \in p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}))$. We start with three remarks:

- The set $\text{Spec}(H_0(k_0)) \cap \bar{I}$ is finite and is written $\{\lambda_n, 1 \leq n \leq N_0\}$.
- With $m = 0$, the properties 3 (uniqueness is imposed by $(k_0, \lambda_n) \in S_{0,n}$), 4, 5, 6 hold in any open neighborhood $\omega' \subset \omega_0$ of k_0 . So these properties are hereditary after reducing the neighborhood ω_0 .
- Locally the real analytic vector bundle $\mathcal{F}|_{\omega_0}$, for ω_0 small enough, is transformed by a unitary transform $U_0(k) : \mathcal{F}_k \rightarrow \mathcal{H}'$, real analytic with respect to $k \in \omega_0$, into the trivial bundle $\omega_0 \times \mathcal{H}'$. So working with the trivial real analytic vector bundle $\omega_0 \times \mathcal{H}'$ is not a restriction here.

For a fixed $\lambda_n \in \bar{I} \cap \text{Spec}(H_0(k_0))$, there is a unique stratum $S_{0,n} \in \mathcal{S}_I$ and a unique $\mu_{0,n} \in \mathbb{N}$ such that $(k_0, \lambda_n) \in S_{0,n} \subset \Sigma_{\mu_{0,n}}$. Consider the set $\mathcal{S}_{0,n} = \{S' \in \mathcal{S}, (k_0, \lambda_n) \in \overline{S'}\} \subset \mathcal{S}_I$. Because the stratification \mathcal{S} is locally finite, there exists an open neighborhood $\omega_{0,n}^1 \times J_{0,n}^1$ of (k_0, λ_n) such that $\omega_{0,n}^1 \times J_{0,n}^1 \cap \overline{S'} \neq \emptyset$ is equivalent to $S' \in \mathcal{S}_{0,n}$ (with $\mathcal{S}_{0,n} \subset \mathcal{S}_I$). Meanwhile $(k_0, \lambda_n) \in S_{0,n}$ gives $S_{0,n} \cap \overline{S'} \neq \emptyset$ and therefore $S_{0,n} = S'$ or $S_{0,n} \subset \partial S'$ with

$\dim S_{0,n} < \dim S'$ in this case. This is the local in energy version of the property (3) which is hereditary for another neighborhood $\omega_{0,n}^2 \times J_{0,n}^2 \subset \omega_{0,n}^1 \times J_{0,n}^1$. In particular we may assume $J_{0,n}^2 \Subset \tilde{I}$ for every $1 \leq n \leq N_0$ and $J_{0,n}^2 \cap J_{0,n'}^2 = \emptyset$ for $n \neq n'$, $1 \leq n, n' \leq N_0$. From now the intervals $J_{0,n} = J_{0,n}^2$, $1 \leq n \leq N_0$ are fixed and the properties (2) and (3) as well as $J_{0,n} \Subset \tilde{I}$ are granted.

Again for n fixed there exists a path in $\gamma_{0,n}$ in \mathbb{C} around $J_{0,n}$ such that

$$1_{\{\lambda_n\}}(H_0(k_0)) = 1_{J_{0,n}}(H_0(k_0)) = \frac{1}{2i\pi} \int_{\gamma_{0,n}} \frac{dz}{z - H_0(k_0)}.$$

With the last formula the spectral projector

$$\pi_{0,n}(k) = \frac{1}{2i\pi} \int_{\gamma_{0,n}} \frac{dz}{z - H_0(k)}$$

is analytic in an open neighborhood $\omega_{0,n}^2 \subset \omega_{0,n}^1$ of k_0 . Additionally choosing $\omega_{0,n}^2$ small enough implies $\|\pi_{0,n}(k) - \pi_{0,n}(k_0)\|_{\mathcal{L}(\mathcal{H}^{\prime})} < 1$ and Nagy's formula (see e.g. [MaSo][NeSo])

$$W_{0,n}(k) = (1 - (P_2 - P_1))^{-1/2} [P_2 P_1 - (1 - P_2)(1 - P_1)] \quad \text{with } P_1 = \pi_{0,n}(k_0), P_2 = \pi_{0,n}(k),$$

provides a unitary operator $W_{0,n}(k)$ in \mathcal{H}^{\prime} , which depends analytically on $k \in \omega_{0,n}$ and such that $W_{0,n}(k)\pi_{0,n}(k_0) = \pi_{0,n}(k)$. Therefore

$$H_0(k)\pi_{0,n}(k) = \pi_{0,n}(k)H_0(k)\pi_{0,n}(k) = \frac{1}{2i\pi} \int_{\gamma_0} \frac{z dz}{z - H_0(k)}$$

is unitarily equivalent to $\tilde{H}_{0,n}(k) = \pi_{0,n}(k_0)W_{0,n}^*(k)H_0(k)W_{0,n}(k)\pi_{0,n}(k_0) \in \mathcal{L}(\mathbb{C}^{\mu_{0,n}})$ after identifying $\text{Ran}\pi_{0,n}(k_0)$ with $\mathbb{C}^{\mu_{0,n}}$. Again this property, which is the condition (5) is hereditary for any neighborhood $\omega_{0,n}^3 \times J_{0,n}$ of (k_0, λ_n) with $\omega_{0,n}^3 \subset \omega_{0,n}^2$.

The local version of condition (4) in $J_{0,n}$ is almost done: the rank of $1_{J_{0,n}}(H_0(k))$ is constantly equal to $\mu_{0,n}$ for $k \in \omega_{0,n}^3 \subset \omega_{0,n}^3$. The function $\max \text{Spec}(H_0(k)) \cap J_{0,n}$ and $\min \text{Spec}(H_0(k))$ are continuous functions of $k \in \omega_{0,n}^3$, equal to $\lambda_{0,n}$ when $k = k_0$ belonging to the open interval $J_{0,n}$. Hence the neighborhood $\omega_{0,n}^3$ of k_0 can be chosen small enough so that $1_{J_{0,n}}(H_0(k)) = 1_{J'_{0,n}}(H_0(k))$ with $J'_{0,n} \Subset J_{0,n}$ and this will be true for any neighborhood of k_0 $\omega_{0,n}^4 \subset \omega_{0,n}^3$.

The local problem is thus reduced to the case of a self-adjoint matrix $\tilde{H}_{0,n}(k) = \tilde{H}_{0,n}(k)^* \in M_{\mu_{0,n}}(\mathbb{C})$ which is real analytic with respect to $k \in \omega_{0,n}^4$.

Because $\text{Tr}[\tilde{H}_{0,n}(k)] = \text{Tr}[H_0(k)1_{J_{0,n}}(H_0(k))]$ is analytic on $\omega_{0,n}^4$, the set

$$\tilde{S}_{0,n} = \left\{ (k, \frac{1}{\mu_{0,n}} \text{Tr}[\tilde{H}_{0,n}(k)]), k \in \omega_{0,n}^4 \right\}$$

is an analytic submanifold of $\omega_{0,n}^4 \times J_{0,n}$ diffeomorphic to $\omega_{0,n}^4$ via the projection $p_M : \omega_{0,n}^4 \times J_{0,n} \rightarrow \omega_{0,n}^4$. Since $S_{0,n} \cap (\omega_{0,n}^4 \times J_{0,n})$ is a real analytic manifold contained in $\tilde{S}_{0,n}$, $p_M(S_{0,n} \cap (\omega_{0,n}^4 \times J_{0,n}))$ is a real analytic submanifold of $\omega_{0,n}^4$. For $\omega_{0,n}^4$ small enough, there exists real analytic coordinates $(k', k'') \in \mathbb{R}^{\dim S_{0,n}} \times \mathbb{R}^{\dim M - \dim S_{0,n}}$ around k_0 such that $S_{0,n} \cap (\omega_{0,n}^4 \times J_{0,n}) = \left\{ ((k', 0), \frac{1}{\mu_{0,n}} \text{Tr}[\tilde{H}_{0,n}(k', 0)]), (k', 0) \in \omega_{0,n}^4 \right\}$. Our assumption that $\text{rank } dp_{\Sigma \rightarrow \mathbb{R}}|_{TS_{0,n}} = 1$ implies that the image of $\frac{\partial}{\partial \lambda}$ by the projection from $T(M \times \mathbb{R})|_{S_{0,n}}$ to $TS_{0,n}$ is a non vanishing real analytic vector field $X_{0,n} = \sum_{j=1}^{\dim S_{0,n}} a_j(k', 0) \frac{\partial}{\partial k_j}$, with $k' = (k_1, \dots, k_{\dim S_{0,n}})$, such that $X_{0,n}\tilde{H}_{0,n}(k', 0) = \text{Id}_{\mathbb{C}^{\mu_{0,n}}}$. Because (k', k'') are coordinates on $\omega_{0,n}^4$, this vector field is a real analytic vector field on $\omega_{0,n}^4$ and $X_{0,n}\tilde{H}_{0,n}(k)$ is a real

analytic self-adjoint matrix. By continuity the open set $\omega_{0,n}^4$ can be reduced such that $X_{0,n}\tilde{H}_{0,n}(k) \geq \frac{1}{2}\text{Id}_{\mathbb{C}^{\mu_{0,n}}}$. This proves the condition 6 and the vector field $X_{0,n}$ can be taken unchanged, except the restriction, for any neighborhood of k_0 contained in $\omega_{0,n}^4$.

Finally for any open neighborhood

$$\omega_{k_0} \subset \bigcap_{n=1}^{N_0} \omega_{0,n}^4$$

of k_0 the inclusion $J_{0,n} \Subset \tilde{I}$ and the conditions (2),(3),(4),(5),(6) are satisfied by the family

$$(\Omega_{0,n} = \omega_0 \times J_{0,n}, S_{0,n}, \mu_{0,n}, W_{0,n}, \tilde{H}_{0,n}, X_{0,n})_{1 \leq n \leq N_0} \quad \text{with} \quad \omega_0 = \omega_{k_0}.$$

Without condition (7), we could conclude with (1) by extracting a finite covering out of $K_I \subset \bigcup_{k_0 \in K_I} \omega_{k_0}$ of the compact set $K_I = p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}))$.

The condition (7) requires an additional specification of the open neighborhood ω_{k_0} of k_0 . It is a global property of the extracted finite covering $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}) \subset \bigcup_{(m,n) \in L(I)} \Omega_{m,n}$ because it is about all the possible intersections. It is actually obtained by a connectedness argument after choosing a specific family of local connected neighborhoods $(\omega_{k_0})_{k_0 \in K_I}$ with all intersections connected. This is possible on a manifold. Let us fix any riemannian metric γ on M . Then:

- any geodesically convex set is arcwise connected and the intersection of geodesically convex sets is geodesically convex;
- for any $k_0 \in M$ there exists $\varepsilon_{\gamma, k_0} > 0$ such that for all $\varepsilon \leq \varepsilon_{\gamma, k_0}$ the geodesic ball $B_\gamma(k_0, \varepsilon)$ is geodesically convex (see [Lee]-Exercise 6-4)

Take for any $k_0 \in K_I = p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}))$

$$\omega_{k_0} = B_\gamma(k_0, \varepsilon_0) \quad , \quad 2\varepsilon_0 < \varepsilon_{\gamma, k_0}$$

$$B_\gamma(k_0, 2\varepsilon_0) \subset \bigcap_{n=1}^{N_0} \omega_{0,n}^4$$

and let $K_I \subset \bigcup_{m=1}^{m_I} \omega_m$ be the finite extracted covering with $\omega_m = B_\gamma(k_m, \varepsilon_m)$. The covering (1) is given by $\Omega_{m,n} = \omega_m \times J_{m,n}$. And $\Omega_{m,n} \cap \Omega_{m',n'} \cap \Sigma \neq \emptyset$ implies $B_\gamma(k_m, \varepsilon_m) \cap B_\gamma(k_{m'}, \varepsilon_{m'}) \neq \emptyset$ and therefore $d_\gamma(k_m, k_{m'}) < \varepsilon_m + \varepsilon_{m'} \leq 2\max(\varepsilon_m, \varepsilon_{m'})$. By symmetry we can focus on the case $\max(\varepsilon_m, \varepsilon_{m'}) = \varepsilon_m$ for which $k_{m'} \in B_\gamma(k_m, 2\varepsilon_m) \subset \omega_m^4$. On the connected set $B_\gamma(k_m, 2\varepsilon_m) \cap \omega_{m'}^4 \subset \omega_m^4 \cap \omega_{m'}^4$, the spectral projector $1_{J_{m,n}}(H_0(k))1_{J_{m',n'}}(H_0(k))$ is real analytic with respect to k with a constant rank $\leq \min(\mu_{m,n}, \mu_{m',n'})$. Because this set contains $k_{m'}$ for which $J_{m,n} \cap J_{m',n'} \cap \text{Spec}(H_0(k_{m'})) = \{\lambda_{m',n'}\}$ with the multiplicity $\mu_{m',n'}$,

$$\mu_{m',n'} \leq \mu_{m,n} \quad \text{and} \quad 1_{J_{m,n}}(H_0(k))1_{J_{m',n'}}(H_0(k)) = 1_{J_{m',n'}}(H_0(k))$$

must hold for all $k \in B_\gamma(k_m, 2\varepsilon_m) \cap B_\gamma(k_{m'}, \varepsilon_{m'}) \supset \omega_m \cap \omega_{m'}$. Because in this case $(k_{m'}, \lambda_{m',n'}) \in S_{m',n'}$ we get

$$S_{m',n'} \cap \omega_{m,n}^4 \times J_{m,n} \supset S_{m',n'} \cap B(k_m, 2\varepsilon_m) \times J_{m,n} \supset \{(k_{m'}, \lambda_{m',n'})\} \neq \emptyset$$

and the property (3) implies $S_{m,n} \subset \partial S_{m',n'}$ or $(S_{m',n'} = S_{m,n}$ and $\mu_{m,n} = \mu_{m',n'})$. By symmetry with $\varepsilon_{m'} = \max(\varepsilon_m, \varepsilon_{m'})$, the equality $\mu_{m,n} = \mu_{m',n'}$ is compatible with $S_{m',n'} \in \partial S_{m,n}$. Although the ordering of ε_m and $\varepsilon_{m'}$ does not appear in the final statement, the condition (3) says that $\varepsilon_m = \varepsilon_{m'}$ is possible only when $S_{m,n} = S_{m',n'}$. \square

Once the open covering $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}) \in \cup_{(m,n) \in L(I)} \Omega_{m,n}$ is chosen according to Proposition 2.1 with $\Omega_{m,n} = \omega_m \times J_{m,n}$, several objects can be associated with it. Take a partition of unity

$$\sum_{m=1}^{M_I} g_m^2(k) \equiv 1 \text{ on a neighborhood of } K_I = p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I})). \quad (2)$$

Because $\text{Spec}(H_0(k)) \cap \bar{I} \subset \bigsqcup_{n=1}^{N_m} J_{m,n}$ for $k \in \omega_m$ and all $m \in \{1, \dots, m_I\}$, we obtain:

$$\sum_{(m,n) \in L(I)} g_m^2(k) 1_{J_{m,n}}(\lambda) \equiv 1 \text{ on a neighborhood of } p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}).$$

With $(m,n) \in L(I)$ we associated the interval $J'_{m,n} \in J_{m,n}$ in Proposition 2.1-(4) such that $1_{J_{m,n}}(H_0(k)) = 1_{J'_{m,n}}(H_0(k))$ for all $k \in \omega_m$. By taking $\chi_{m,n} \in \mathcal{C}_{\text{comp}}^\infty(J_{m,n})$ with $\chi_{m,n} \equiv 1$ on a neighborhood of $\bar{J}'_{m,n}$, we obtain the smooth partition of unity

$$\sum_{(m,n) \in L(I)} g_m(k)^2 \chi_{m,n}^2(\lambda) \equiv 1 \text{ on a neighborhood of } p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}), \quad (3)$$

$$\text{with } \forall k \in \omega_m, \quad g_m(k) 1_{J_{m,n}}(H_0(k)) = g_m(k) \chi_{m,n}(H_0(k)). \quad (4)$$

Definition 2.2. *With the open covering of Proposition 2.1, the set of pairs $(m,n) \in L(I)$ is naturally endowed with an asymmetric relation \triangleleft (asymmetric means that $x \triangleleft y$ and $y \triangleleft x$ never happens). Actually $(m,n) \triangleleft (m',n')$ is defined by:*

$$(i) \quad (\omega_m \times J_{m,n}) \cap (\omega_{m'} \times J_{m',n'}) \cap \Sigma \neq \emptyset;$$

$$(ii) \quad \dim S_{m,n} < \dim S_{m',n'}.$$

Lemma 2.3. *For $(m,n), (m',n') \in L(I)$ the relation $(m,n) \triangleleft (m',n')$ implies*

$$(1) \quad m \neq m' \text{ and } S_{m,n} \subset \partial S_{m',n'};$$

$$(2) \quad \text{for all } k \in \omega_m \cap \omega_{m'}, \text{ all the operators } \chi_{m,n}(H_0(k)) \chi_{m',n'}(H_0(k)), 1_{J_{m,n}}(H_0(k)) 1_{J_{m',n'}}(H_0(k)), 1_{J_{m',n'}}(H_0(k)) \text{ and } \chi_{m',n'}(H_0(k)) \text{ are equal.}$$

The result (2) still holds when the assumption $(m,n) \triangleleft (m',n')$ is replaced by the more general condition $(\Omega_{m,n} \cap \Omega_{m',n'} \cap \Sigma \neq \emptyset \text{ and } \overline{S_{m,n}} \subset \overline{S_{m',n'}})$.

Proof. If $m = m'$ with $\Omega_{m,n} \cap \Omega_{m,n'} \neq \emptyset$ then $J_{m,n} \cap J_{m,n'} \neq \emptyset$, where Proposition 2.1-(2) implies $n = n'$ and Proposition 2.1-(3) implies $S_{m,n} = S_{m',n'}$ in contradiction with Definition 2.2-ii). In the second statement the equality $1_{J_{m,n}}(H_0(k)) 1_{J_{m',n'}}(H_0(k)) = 1_{J_{m',n'}}(H_0(k))$ is given by the classification of Proposition 2.1-(7) where $\dim S_{m,n} < \dim S_{m',n'}$ eliminates the cases $\mu_{m,n} < \mu_{m',n'}$ and $S_{m,n} = S_{m',n'}$. The choice of the cut-off $\chi_{m,n}$ was done such that $\chi_{m,n}(H_0(k)) = 1_{J_{m,n}}(H_0(k))$ for all $k \in \omega_m$ and all $(m,n) \in L(I)$.

The more general condition includes the additional case $S_{m,n} = S_{m',n'}$ where all the operators of (2) are equal to $1_{J_{m,n}}(H_0(k))$. \square

The classification of Proposition 2.1-(7) says that $\Omega_{m,n} \cap \Omega_{m',n'} \cap \Sigma \neq \emptyset$ is classified into the three mutually exclusive cases

$$S_{m,n} \triangleleft S_{m',n'} \quad \text{or} \quad S_{m',n'} \triangleleft S_{m,n} \quad \text{or} \quad S_{m,n} = S_{m',n'},$$

while the last case does not mean $(m,n) = (m',n')$. It can be summarized as

$$(\Omega_{m,n} \cap \Omega_{m',n'} \cap \Sigma \neq \emptyset) \Rightarrow \overline{S_{m,n}} \subset \overline{S_{m',n'}} \text{ or } \overline{S_{m',n'}} \subset \overline{S_{m,n}}$$

while we recall that $\bar{S} \subset \bar{S}'$, which means $S = S'$ or $S \subset \partial S' = \bar{S}' \setminus S'$, is an order relation on \mathcal{S} . This leads to the following useful result.

Proposition 2.4. Any family $(m_\ell, n_\ell)_{1 \leq \ell \leq L}$ of $L(I)$ such that $\bigcap_{\ell=1}^L \Omega_{m_\ell, n_\ell} \cap \Sigma \neq \emptyset$ can be ordered such that $\overline{S_{m_\ell, n_\ell}} \subset \overline{S_{m_{\ell+1}, n_{\ell+1}}}$.

With such an order, all the operators $\chi_{m_\ell, n_\ell}(H_0(k))\chi_{m_{\ell'}, n_{\ell'}}(H_0(k))$, $1_{J_{m_\ell, n_\ell}}(H_0(k))1_{J_{m_{\ell'}, n_{\ell'}}}(H_0(k))$, $1_{J_{m_{\ell'}, n_{\ell'}}}(H_0(k))$ and $\chi_{m_{\ell'}, n_{\ell'}}(H_0(k))$ are equal for $k \in \omega_{m_\ell} \cap \omega_{m_{\ell'}}$ when $1 \leq \ell \leq \ell' \leq L$. In particular all the operators $\chi_{m_\ell, n_\ell}(H_0(k))\chi_{m_L, n_L}(H_0(k))$, $1_{J_{m_\ell, n_\ell}}(H_0(k))1_{J_{m_L, n_L}}(H_0(k))$, $1_{J_{m_L, n_L}}(H_0(k))$ and $\chi_{m_L, n_L}(H_0(k))$ are equal for $k \in \bigcap_{\ell'=1}^L \omega_{m_{\ell'}}$ for all $\ell \in \{1, \dots, L\}$.

2.2 Unitary connections

A connection ∇ , on a Hilbert bundle $p_{\mathcal{F}} : \mathcal{F} \rightarrow M$ is a \mathbb{R} (or \mathbb{C})-linear map from $\mathcal{C}^\infty(M; \mathcal{F})$ to $\mathcal{C}^\infty(M; T^*M \otimes \mathcal{F})$ with the Leibniz rule

$$\forall f \in \mathcal{C}^\infty(M; \mathbb{R}), \forall s \in \mathcal{C}^\infty(M; \mathcal{F}), \quad \nabla(fs) = (df)s + f(\nabla s)$$

or in terms of covariant derivatives

$$\forall X \in \mathcal{C}^\infty(M; TX), \forall f \in \mathcal{C}^\infty(M; \mathbb{R}), \forall s \in \mathcal{C}^\infty(M; \mathcal{F}), \quad \nabla_X(fs) = (Xf)s + f(\nabla_X)s.$$

The curvature $R^\nabla \in$ is given by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

and it defines an element of $\mathcal{C}^\infty(M; \Lambda^2 T^*M \otimes \text{End}(\mathcal{F}))$, where in all our cases $\dim \mathcal{F}_k < \infty$ or $\dim \mathcal{F}_k = \infty$, $\text{End}(\mathcal{F}) = \mathcal{L}(\mathcal{F})$.

In a local chart open set $\omega \subset M$, there exists a unitary map $U(k)$ in \mathcal{H}' with depends analytically on $k \in \omega$ such that

$$u(k) \mapsto e^{-V(k)}U(k)u(k)$$

defines a unitary map from $L^2(\omega, dk; \mathcal{H}')$ to $\mathcal{F}|_\omega$ endowed with the metric $\langle \cdot, \cdot \rangle_{\mathcal{H}'} e^{2V(k)}$. Hence on $\mathcal{F}|_\omega$ there always exists a flat unitary connection given by

$$\nabla^\omega = e^{-V(k)}U(k) \left[\sum_{i=1}^{\dim M} \frac{\partial}{\partial k_i} dk_i \right] e^{V(k)}U(k)^{-1} = U(k) \left[\sum_{i=1}^{\dim M} \frac{\partial}{\partial k_i} dk_i \right] U(k)^{-1} + dV \otimes \text{Id}_{\mathcal{F}} \quad (5)$$

Here flat means $R^{\nabla^\omega} \equiv 0$ while the unitarity of the Hilbert bundle with fiber \mathcal{H}' and the metric $\langle \cdot, \cdot \rangle_{\mathcal{H}'} e^{2V(k)}$ means

$$X \left[\langle s, s' \rangle_{\mathcal{H}'} e^{2V(k)} \right] = \langle \nabla_X^\omega s, s' \rangle_{\mathcal{H}'} e^{2V(k)} + \langle s, \nabla_X^\omega s' \rangle_{\mathcal{H}'} e^{2V(k)}.$$

In particular when $X = \sum_{i=1}^{\dim M} X_i \partial_{k_i} \in \mathcal{C}^\infty(\omega; TM)$, $\nabla_X^\omega + \frac{1}{2} \text{div} X$, with $\text{div} X = \sum_{i=1}^{\dim M} \partial_{k_i} X_i$, is an antisymmetric operator on $\mathcal{C}_{comp}^\infty(\omega; \mathcal{F})$.

Let $M = \bigcup_{i \in \mathcal{I}} \omega_i$ be a locally finite open covering of M , with $\overline{\omega_i}$ compact, and let $\sum_{i \in \mathcal{I}} \chi_i^2$, $\chi_i \in \mathcal{C}_{comp}^\infty(\omega_i)$ be a subordinate partition of unity with $\sum_{i \in \mathcal{I}} \chi_i^2 \equiv 1$. When $(\nabla_i)_{i \in \mathcal{I}}$ is a family of unitary connections on $\mathcal{F}|_{\omega_i}$ then

$$\nabla^M = \sum_{i \in \mathcal{I}} \chi_i \nabla_i \chi_i \quad (6)$$

is a unitary connection on \mathcal{F} .

A connection ∇^M on $p_{\mathcal{F}} : \mathcal{F} \rightarrow M$, allows to define a connection still denoted by ∇^M on $p_{\mathcal{L}(\mathcal{F})} : \mathcal{L}(\mathcal{F}) \rightarrow M$ via

$$(\nabla_X^M A)s = \nabla_X^M (As) - A(\nabla_X^M s)$$

which ensures the Leibnitz rule $\nabla^M(As) = (\nabla^M A)s + A(\nabla^M s)$. The unitarity of ∇^M implies $(\nabla_X^M A)^* = -\nabla_X^M(A^*) + \lambda_X A^*$ where the factor $\lambda_X A^*$ is due to the differentiation of the volume form $d\nu(k)$ in the integration by part.

Remember also that the difference between two connections on $p_{\mathcal{F}} : \mathcal{F} \rightarrow M$ is given by $\nabla^2 - \nabla^1 = L \in \mathcal{C}^\infty(M; T^*M \otimes \mathcal{L}(\mathcal{F}))$. This holds true when \mathcal{F} is finite dimensional and when $\dim \mathcal{F} = \infty$ the considered connections ∇^2 are the ones for which the property holds for a given fixed connection ∇^1 , e.g. $\nabla^1 = \nabla^M = \sum_{i \in \mathcal{I}} \chi_i \nabla^i \chi_i$. A typical example is when ∇^2 is given by the same formula as $\nabla^1 = \nabla^M$ but with a different partition of unity $\sum_{i \in \mathcal{I}} \chi_{2,i}^2 \equiv 1$ subordinate to $M = \bigcup_{i \in \mathcal{I}} \omega_i$. The induced connections on $p_{\mathcal{L}(\mathcal{F})} : \mathcal{L}(\mathcal{F}) \rightarrow M$ satisfy $(\nabla_X^2 - \nabla_X^1)A = [L_X, A]$. In particular, the factor $e^{2V(k)}$ in the metric $\langle \cdot, \cdot \rangle_{\mathcal{H}'} e^{2V(k)}$ can be forgotten while considering the connection ∇^ω induced on $p_{\mathcal{L}(\mathcal{F})} : \mathcal{L}(\mathcal{F}) \rightarrow \omega$ with ∇^ω given by (5) because $L_X = (dV(X))\text{Id}_{\mathcal{H}'}$. When both ∇^2 and ∇^1 are unitary connections, $L_X^* = (\nabla_X^2 - \nabla_X^1)^* = -(\nabla_X^2 - \nabla_X^1) = -L_X$ and L_X is anti-symmetric.

In a local coordinate system (k_1, \dots, k_d) in $\omega \subset M$, the following relations

$$\left[\nabla_{\frac{\partial}{\partial k_i}}^2, \nabla_{\frac{\partial}{\partial k_j}}^1 \right] = R^\nabla \left(\frac{\partial}{\partial k_i}, \frac{\partial}{\partial k_j} \right) \in \mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F})), \quad (7)$$

$$\nabla_{\frac{\partial}{\partial k_i}}^2 - \nabla_{\frac{\partial}{\partial k_i}}^1 \in \mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F})), \quad (8)$$

ensure that the following objects are well defined.

Definition 2.5. *Let ω be an open set of M . A differential operator B of order μ with coefficients in $\mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ is a local operator which can be written in any local coordinate system*

$$B = \sum_{|\alpha| \leq \mu} B_\alpha(k) \nabla_{\frac{\partial}{\partial k_1}}^{\alpha_1} \dots \nabla_{\frac{\partial}{\partial k_d}}^{\alpha_d}. \quad (9)$$

It is initially defined with the domain $D(B) = \mathcal{C}_{comp}^\infty(\omega; \mathcal{F})$ and it is said symmetric according to the general definition

$$\forall \varphi, \psi \in \mathcal{C}_{comp}^\infty(\omega; \mathcal{F}), \quad \langle \varphi, B\psi \rangle_{L^2(M; \mathcal{F})} = \langle B\varphi, \psi \rangle_{L^2(M; \mathcal{F})}.$$

A first order differential operator B with $\mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ coefficients, is said to have a scalar principal part if there exists $X \in \mathcal{C}^\infty(\omega; TM)$ and $B_0 \in \mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ such that

$$B = \nabla_X + B_0. \quad (10)$$

In (9) the covariant derivatives do not commute but (7) ensures that the form (9) is preserved, after modifying the coefficients B_α , when the order is changed. The relation (8) ensures that the general forms (9) and (10) are not changed when the connection ∇ is changed.

We will focus on first order differential operators with $\mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ coefficients where ω and the fiber bundle will be specified further.

When ω is a local chart open set with $\mathcal{F}|_\omega = e^{-V(\cdot)} U(\cdot) L^2(\omega; \mathcal{H}')$ and ∇^ω defined by (5), a first order differential operator can be written

$$B = e^{-V(k)} U(k) \left[\sum_{i=1}^d \underbrace{B_{1,i}(k) X_i(k)}_{=\tilde{B}_{1,i}(k)} \frac{\partial}{\partial k_i} + B_0(k) \right] e^{V(k)} U(k)^{-1}$$

with $B_{1,i}, B_0, \tilde{B}_{1,i} \in \mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{H}'))$ and $X = \sum_{i=1}^d X_i(k) \frac{\partial}{\partial k_i} \in \mathcal{C}^\infty(\omega; TM)$. It has a scalar principal part when all the $B_{1,i}$'s equal $\text{Id}_{\mathcal{H}'}$.

The compositions BA or AB with $B = B_1 \nabla_X^\omega + B_0$, $B_1, B_0 \in \mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ and $A \in \mathcal{C}^\infty(\omega; \mathcal{L})$ equal

$$BA = B_1 A \nabla_X^\omega + (B_1 (\nabla_X^\omega A)) + B_0 A \quad \text{and} \quad AB = AB_1 \nabla_X^\omega + AB_0,$$

and the commutator $[B, A]$ belongs to $\mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ when $[B_1, A] = 0$, which is the case if B has a scalar principal part.

When $[B_1^1, B_1^2] = 0$, the commutator (on $\mathcal{C}^\infty(\omega; \mathcal{F})$) $[B^1, B^2]$ with $B^k = B_1^k \nabla_{X^k}^\omega + B_0^k$ is a first order differential operators with $\mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ -coefficients:

$$\begin{aligned} [B^1, B^2] &= (B_1^1 (\nabla_{X^1}^\omega B_1^2)) \nabla_{X^2}^\omega - (B_1^2 (\nabla_{X^2}^\omega B_1^1)) \nabla_{X^1}^\omega + (B_1^1 B_1^2) \nabla_{[X^1, X^2]}^\omega \\ &\quad + [B_0^1, B_0^2] + (B_1^1 \nabla_{X^1}^\omega B_0^2) - (B_1^2 \nabla_{X^2}^\omega B_0^1) + B_1^1 B_1^2 \mathcal{R}^{\nabla^\omega}(X^1, X^2), \end{aligned}$$

and this is not true in general when $[B_1^1, B_1^2] \neq 0$.

When ∇^ω is a unitary connection the formal adjoint of $B = B_1 \nabla_X^\omega + B_0$ equals

$$B^* = -\nabla_X^\omega B_1^* - (\text{div } X) B_1^* + B_0^* = -B_1^* \nabla_X^\omega + B_0^* - (\nabla_X^\omega B_1^* + (\text{div } X) B_1^*).$$

First order differential operators and their adjoint are fully understood by their local description with a unitary connection ∇^ω and we can start with (5). This will be applied with $\omega = \omega_m$, $1 \leq m \leq M_I$. In order to use the localization on Σ in the variables (k, λ) , it is convenient to associate a new unitary connection with a finite orthogonal decomposition of the fiber bundle $p_{\mathcal{F}} : \mathcal{F} \rightarrow \omega$. Namely assume that $(\pi_n(k))_{0 \leq n \leq N}$ are orthogonal projections in the fiber \mathcal{F}_k , with $\sum_{n=0}^N \pi_n(k) = \text{Id}_{\mathcal{F}_k}$, which depend analytically on k (actually \mathcal{C}^∞ -regularity is only used here). The latter means that $\tilde{\pi}_n(k) = U(k)^{-1} \pi_n(k) U(k)$ is an orthogonal projection in \mathcal{H}' which is analytic with respect to $k \in \omega$, while we know

$$\pi_n(k)^2 = \pi_n(k) = \pi_n(k)^* \quad , \quad \sum_{n=0}^N \pi_n(k) = \text{Id}_{\mathcal{F}_k}.$$

When ∇^ω is a unitary connection on $p_{\mathcal{F}} : \mathcal{F} \rightarrow \omega$, endowed with the metric $\langle \cdot, \cdot \rangle_{\mathcal{H}'} e^{2V(k)}$, this induces a natural connection $\nabla^{\omega, n}$ on $p_{\pi_n \mathcal{F}} : \pi_n \mathcal{F} \rightarrow \omega$ simply given by

$$\nabla^{\omega, n} = \pi_n \circ \nabla^\omega \circ \pi_n = (\nabla^\omega - (\nabla^\omega \pi_n)) \Big|_{\mathcal{C}^\infty(\omega; \pi_n \mathcal{F})} = (\nabla - [(\nabla^\omega \pi_n), \pi_n]) \Big|_{\mathcal{C}^\infty(\omega; \pi_n \mathcal{F})},$$

and sometimes called the *adiabatic connection*. The last equality is actually due to

$$\nabla_X^\omega (\pi_n) = \nabla_X^\omega (\pi_n^2) = (\nabla_X^\omega \pi_n) \pi_n + \pi_n (\nabla_X^\omega \pi_n)$$

which implies

$$\pi_n (\nabla_X^\omega \pi_n) \pi_n = 0.$$

Then

$$\nabla^{\omega, \pi} = \oplus_{n=0}^N \nabla^{\omega, n} = \oplus_{n=0}^N \pi_n \nabla^\omega \pi_n = \nabla^\omega - \sum_{n=1}^N (\nabla^\omega \pi_n) \pi_n$$

is a new unitary connexion, $\nabla^{\omega, \pi}$, such that for any block diagonal $A \in \mathcal{C}^\infty(\omega; \mathcal{L}(\mathcal{F}))$, $A = \sum_{n=1}^N \pi_n A_n \pi_n$,

$$\nabla^{\omega, \pi} A = \sum_{n=0}^N \pi_n \nabla^\omega A_n \pi_n = \sum_{n=0}^N \pi_n \nabla^{\omega, n} A_n \pi_n, \quad (11)$$

and $\nabla_X^{\omega, \pi} A$ remains block diagonal for any $X \in \mathcal{C}^\infty(\omega; TM)$.

Actually any finite collection of unitary connections $(\nabla^{\omega, n})_{0 \leq n \leq N}$ on the fiber bundles

$p_{\pi_n \mathcal{F}} : \pi_n \mathcal{F} \rightarrow \omega$, endowed with the metric $\langle \cdot, \cdot \rangle_{\mathcal{H}'} e^{2V(k)}$, gives rise to a unitary connection $\nabla^{\omega, \pi} = \sum_{n=1}^N \pi_n \nabla^{\omega, n} \pi_n$ on $p_{\mathcal{F}} : \mathcal{F} \rightarrow \omega$ with the same property (11). Such a unitary connection $\nabla^{\omega, \pi}$ or alternatively $\nabla^{\omega} = \nabla^{\omega, \pi}$ is characterized by

$$\forall n \in \{1, \dots, N\}, \quad \nabla^{\omega} \pi_n = 0 \quad \text{in} \quad \mathcal{C}^{\infty}(\omega; \mathcal{L}(\mathcal{F}))$$

if we notice $\pi_0 = \text{Id}_{\mathcal{F}} - \sum_{n=1}^N \pi_n$.

A first version of the operator A_I , essentially as presented in [GeNi] is done as follows. We use the open covering $\cup_{(m,n) \in L(I)} \Omega_{m,n}$, $\Omega_{m,n} = \omega_m \times J_{m,n}$, of $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I})$ given by Proposition 2.1. Above the open set ω_m , $1 \leq m \leq M_I$, use the decomposition $\text{Id}_{\mathcal{F}_k} = \oplus_{n=1}^{N_m} 1_{J_{m,n}}(H_0(k)) \oplus 1_{J_{m,0}}(H_0(k)) = \oplus_{n=0}^{N_m} \pi_{m,n}(k)$, with $J_{m,0} = \mathbb{R} \setminus \cup_{n=1}^{N_m} J_{m,n}$ and

$$\pi_{m,n}(k) = 1_{J_{m,n}}(H_0(k)), \quad 0 \leq n \leq N_m, \quad 1 \leq m \leq M_I. \quad (12)$$

Consider the unitary connection $\nabla^{\omega_m, \pi} = \sum_{n=0}^{N_m} \pi_{m,n} \nabla^{m,n} \pi_{m,n}$, with $\nabla^{m,n}$ a unitary connection defined on $p_{\pi_{m,n} \mathcal{F}} : \pi_{m,n} \mathcal{F} \rightarrow \omega_m$ by

- for $n = 0$, $\nabla^{m,0} = \pi_{m,0} \nabla^{\omega_m} \pi_{m,0}$ with ∇^{ω_m} defined by (5) with $\omega = \omega_m$,
- for $n \in \{1, \dots, N_m\}$, $\nabla_X^{m,n} = W_{m,n} X W_{m,n}^* + dV(X)$ where $W_{m,n}(k)$ is the unitary map from $\mathbb{C}^{\mu_{m,n}} \sim \pi_{m,n} \mathcal{F}_{k_0}$ to $\pi_{m,n} \mathcal{F}_k$ introduced in Proposition 2.1-(5).

We consider the operator

$$A_m = i \sum_{n=0}^{N_m} g_m(k) \left[\nabla_{X_{m,n}}^{m,n} + \frac{1}{2} (\text{div} X_{m,n}) \pi_{m,n} \right] g_m(k) \quad (13)$$

$$= i \sum_{n=1}^{N_m} g_m(k) \chi_{m,n}(H_0(k)) \left[\nabla_{X_{m,n}}^{\omega_m, \pi} + \frac{1}{2} (\text{div} X_{m,n}) \right] \chi_{m,n}(H_0(k)) g_m(k) \quad (14)$$

where $X_{m,0} = 0$ and $X_{m,n}$ for $1 \leq n \leq N_m$ is the real analytic vector field introduced in Proposition 2.1-6) such that

$$\forall k \in \omega_m, \quad \nabla_{X_{m,n}}^{m,n} (H_0(k) 1_{J_{m,n}}(k)) \geq \frac{1}{2} 1_{J_{m,n}}(H_0(k)). \quad (15)$$

The right-hand side of (13) shows that A_m is a first order differential operator with $\mathcal{C}_{\text{comp}}^{\infty}(\omega_m; \mathcal{L}(\mathcal{F}))$ coefficients according to the general setting of Definition 2.5. Because $\nabla^{\omega_m, \pi}$ is a unitary connection, A_m is symmetric on $\mathcal{C}_{\text{comp}}^{\infty}(M; \mathcal{F})$ while the commutator $[H_0, iA_m]$ satisfies

$$[H_0, iA_m] = g_m(k) \left[\sum_{n=1}^{N_m} \nabla_{X_{m,n}}^{m,n} (H_0(k) 1_{J_{m,n}}(H_0(k))) \right] g_m(k) \geq \frac{1}{2} g_m^2(k) 1_{\bar{I}}(H_0(k)).$$

Clearly the operator $[H_0, iA_m] = \sum_{n=1}^{N_m} \pi_{m,n} B_{m,n} \pi_{m,n}$ with $B_{m,n} \in \mathcal{C}_{\text{comp}}^{\infty}(\omega_m; \mathcal{L}(\pi_{m,n} \mathcal{F}))$, $\text{supp} B_{m,n} \subset \text{supp} g_m$, and iterated commutators $\text{ad}_{iA_m}^k$ preserve this class.

By summing $A_I = \sum_{m=1}^{M_I} A_m$ we obtain

$$[H_0, iA_I] \geq \frac{1}{2} \sum_{m=1}^{M_I} g_m(k)^2 1_{\bar{I}}(H_0(k)) = \frac{1}{2} 1_{\bar{I}}(H_0).$$

So everything seems to be done here, and this is where we definitely omitted an argument in [GeNi]. Actually when $\omega_{m_1} \cap \omega_{m_2} \neq \emptyset$ or more precisely when $g_{m_1} g_{m_2} \neq 0$, the block diagonal decomposition $\oplus_{n=0}^{N_m} 1_{J_{m,n}}(H_0(k))$ do not coincide for $m = m_1$ and $m = m_2$.

While considering the iterated commutator $[A_{m_j}, [A_{m_1}, H_0]]$, $j = 1, 2$, we must check that $[A_{m_1}, H_0]$ has the local structure $\sum_{n=1}^{N_m} \pi_{m,n} B_{m,n} \pi_{m,n}$, $B_{m,n} \in \mathcal{C}_{comp}^\infty(\omega_m; \mathcal{L}(\pi_{m,n} \mathcal{F}))$, both for $m = m_1$ and $m = m_2$. And this is not true in general. This can be corrected by modifying the unitary connections $\nabla^{\omega_m, \pi}$. But the additional terms deteriorate Mourre's inequality which can be finally recovered only on a small energy interval $[\lambda - \varepsilon, \lambda + \varepsilon]$, $\lambda \in \bar{I}$ and $\varepsilon > 0$ small enough. This is done in the following sections.

3 Modified unitary connection

Here we change the unitary connection $\nabla^{\omega_m, \pi} = \sum_{n=0}^{N_m} \pi_{m,n} \nabla^{\omega_m} \pi_{m,n}$ into a new unitary connection $\tilde{\nabla}^{\omega_m, \pi} = \sum_{n=0}^{N_m} \pi_{m,n} \tilde{\nabla}^{m,n} \pi_{m,n}$ whose the construction depends on the partition of unity (2)(3)(4). Remember the definition (12) of $\pi_{m,n}$. Actually it suffices to specify the new connection $\tilde{\nabla}^{m,n}$ on the fiber bundle $p_{\pi_{m,n} \mathcal{F}} : \pi_{m,n} \mathcal{F} \rightarrow \omega_m$, for every fixed pair (m, n) , but all the possible non empty intersections $\omega_\alpha = \bigcap_{m' \in \alpha} \omega_{m'}$, for $m \in \alpha \subset \{1, \dots, M_I\}$, $\emptyset \neq \omega_\alpha \subset \omega_m$, must be considered. The new operator \tilde{A}_I defined like $A_I = \sum_{m=1}^{M_I} A_m$ with this new unitary connection will satisfy for all pairs $(m', n') \in L(I)$,

$$\text{ad}_{\tilde{A}_I} B_{m', n'} \in \mathcal{C}_{comp}^\infty(\omega_{m'}; \mathcal{L}(\pi_{m', n'} \mathcal{F})) \quad \text{with} \quad \text{supp ad}_{\tilde{A}_I} B_{m', n'} \subset \text{supp } g_{m'} \quad (16)$$

for any $B_{m', n'} \in \mathcal{C}_{comp}^\infty(\omega_{m'}; \mathcal{L}(\pi_{m', n'} \mathcal{F}))$, with $\text{supp } B_{m', n'} \subset \text{supp } g_{m'} \subseteq \omega_{m'}$, and we will deduce

$$\forall k \in \mathbb{N}, \quad \text{ad}_{\tilde{A}_I}^k H_0 \in \mathcal{L}(L^2(M; \mathcal{F})).$$

For the moment we still work with first order differential operators with $\mathcal{C}^\infty(M; \mathcal{L}(\mathcal{F}))$ coefficients (see Definition 2.5), acting on $\mathcal{C}_{comp}^\infty(M; \mathcal{F})$. The question of their self-adjoint realizations as well as the functional analytic meaning of commutators will be discussed later.

By expanding (16) with

$$\tilde{A}_I = i \sum_{(m,n) \in L(I)} g_m(k) \left(\tilde{\nabla}_{X_{m,n}}^{m,n} + \frac{(\text{div } X_{m,n})}{2} \pi_{m,n} \right) g_m(k) \quad (\text{remember } X_{m,0} = 0)$$

the commutator equals

$$\begin{aligned} [i \tilde{A}_I, B_{m', n'}] &= - \sum_{(m,n) \in L(I)} g_m(k) \left[\left(\tilde{\nabla}_{X_{m,n}}^{m,n} + \frac{(\text{div } X_{m,n})}{2} \pi_{m,n} \right), B_{m', n'} \right] g_m(k) \\ &= - \sum_{\substack{\Omega_{m,n} \cap \Omega_{m', n'} \cap \Sigma \neq \emptyset \\ (m,n) \in L(I)}} g_m(k) \left[\left(\tilde{\nabla}_{X_{m,n}}^{m,n} + \frac{(\text{div } X_{m,n})}{2} \pi_{m,n} \right), \pi_{m', n'} B_{m', n'} \pi_{m', n'} \right] g_m(k). \end{aligned}$$

for any $(m', n') \in L(I)$, where we recall $\Omega_{m,n} = \omega_m \times J_{m,n}$ for $(m, n) \in L(I)$ as in Proposition 2.1 with $\text{supp } g_m \subset \omega_m$ and $1_{J_{m,n}}(H_0(k)) = \pi_{m,n}(k)$ for $k \in \omega_m$. In the above sum, expand the commutator $[\tilde{\nabla}_{X_{m,n}}^{m,n}, \pi_{m', n'} B_{m', n'} \pi_{m', n'}]$ as:

$$[\tilde{\nabla}_{X_{m,n}}^{m,n}, \pi_{m', n'}] B_{m', n'} \pi_{m', n'} + \pi_{m', n'} [\tilde{\nabla}_{X_{m,n}}^{m,n}, B_{m', n'}] \pi_{m', n'} + \pi_{m', n'} B_{m', n'} [\tilde{\nabla}_{X_{m,n}}^{m,n}, \pi_{m', n'}]$$

while, as operator valued differential operators on $\omega_m \cap \omega_{m'}$,

$$\begin{aligned} [\tilde{\nabla}_{X_{m,n}}^{m,n}, \pi_{m', n'}] &= \pi_{m,n} \tilde{\nabla}_{X_{m,n}}^{m,n} \pi_{m,n} \pi_{m', n'} - \pi_{m', n'} \pi_{m,n} \tilde{\nabla}_{X_{m,n}}^{m,n} \pi_{m,n} \\ &= \left[\tilde{\nabla}_{X_{m,n}}^{m,n}, \pi_{m,n} \pi_{m', n'} \right] = \tilde{\nabla}_{X_{m,n}}^{m,n} (\pi_{m,n} \pi_{m', n'}). \end{aligned}$$

Therefore the condition (16) is satisfied when

$$\tilde{\nabla}^{m,n}(\pi_{m,n}\pi_{m',n'})=0 \quad \text{above } \text{supp } g_m \cap \text{supp } g_{m'}, \quad (17)$$

for all pairs (m,n) and (m',n') in $L(I)$ such that $\Omega_{m,n} \cap \Omega_{m',n'} \cap \Sigma \neq \emptyset$. This can be discussed by using the order introduced in Proposition 2.4 after the classification of cases in Proposition 2.1-(7) and Definition 2.2.

Proposition 3.1. *Let I and \bar{I} be fixed intervals such that $I \subseteq \bar{I} \subset \mathbb{R} \setminus \tau$ and let the partition of unity given by (2)(3)(4) after Proposition 2.1 be fixed.*

The unitary connections $(\tilde{\nabla}^{m,n})_{(m,n) \in L(I)}$ on the vector bundles $p_{\pi_{m,n}\mathcal{F}} : \pi_{m,n}\mathcal{F} \rightarrow \omega_m$ can be defined such that (17) holds true for all pairs (m,n) and (m',n') in $L(I)$. These connections satisfy the additional property that when $(k_0, \lambda_0) \in \Omega_{m,n} \cap p_{\Sigma \rightarrow \mathbb{R}}(\bar{I}) \subset \Sigma$ and the set $\{(m',n') \in L(I), (k_0, \lambda_0) \in \Omega_{m',n'}\} = \{(m_\ell, n_\ell), 1 \leq \ell \leq L\}$ is ordered like in Proposition 2.4, there exists an open neighborhood \mathcal{V}_{k_0} of k_0 such that

$$\pi_{m_L, n_L} \circ \tilde{\nabla}^{m,n} \circ \pi_{m_L, n_L} = \pi_{m_L, n_L} \circ \nabla^{m,n} \circ \pi_{m_L, n_L} \quad \text{on } \mathcal{V}_{k_0}.$$

The proof will be done in several steps. It can be done for any fixed pair $(m,n) \in L(I)$. We firstly construct a unitary connection ∇^α on the restricted vector bundle $p_{\pi_{m,n}\mathcal{F}} : \pi_{m,n}\mathcal{F} \rightarrow \omega_\alpha$ where $m \in \alpha \subset \{1, \dots, M_{I_1}\}$ and $\omega_\alpha = \bigcap_{m' \in \alpha} \omega_{m'} \neq \emptyset$. Then we use a partition of unity in order to glue all these connections ∇^α .

Lemma 3.2. *Let $\mathcal{P}_\mathcal{G} = \{\pi_i, i \in \mathcal{I}\}$ be a finite non empty set of self-adjoint projectors $\pi_i^2 = \pi_i = \pi_i^*$ in a unital commutative \mathcal{C}^* -algebra \mathcal{A} . For $\gamma \subset \mathcal{I}$ set $\pi_\gamma = \prod_{i \in \gamma} \pi_i$ with the convention $\pi_\emptyset = 1_{\mathcal{A}}$. Assume that $\pi \in \mathcal{P}_\mathcal{G}$ implies $1 - \pi \in \mathcal{P}_\mathcal{G}$. Then the \mathcal{C}^* -algebra generated by $\mathcal{P}_\mathcal{G}$, $\mathcal{C}^*(\mathcal{P}_\mathcal{G})$, is finite dimensional and a basis is given by the family, Γ , made of $\pi_\gamma \neq 0$ such that for all $i \in \mathcal{I}$, $\pi_i \pi_\gamma = 0$ when $i \notin \gamma$. In particular for any $i \in \mathcal{I}$, π_i admits a unique decomposition $\pi_i = \sum_{\gamma \in \Gamma} c_{i,\gamma} \pi_\gamma$ with $c_{i,\gamma} \in \{0, 1\}$.*

Proof. The commutative \mathcal{C}^* -algebra generated by $\mathcal{P}_\mathcal{G}$ is the set of linear combinations

$$\sum_{\gamma \subset \mathcal{I}} c_\gamma \pi_\gamma, \quad c_\gamma \in \mathbb{C},$$

where all the π_γ 's are self-adjoint projectors.

For any $\gamma \subset \mathcal{I}$ and any $i \in \mathcal{I}$ we can write

$$\pi_\gamma = \pi_i \pi_\gamma + (1_{\mathcal{A}} - \pi_i) \pi_\gamma = \pi_i \pi_\gamma + \pi_{i'} \pi_\gamma$$

with $\pi_{i'} = 1_{\mathcal{A}} - \pi_i \in \mathcal{P}_\mathcal{G}$. We deduce that any π_γ , $\gamma \subset \mathcal{I}$, can be written as a finite sum of $\pi_{\gamma'}$, $\gamma' \subset \mathcal{I}$, such that $\pi_i \pi_{\gamma'} = 0$ for all $i \in \mathcal{I} \setminus \gamma'$.

Our family Γ , is thus a linear generating family of $\mathcal{C}^*(\mathcal{P}_\mathcal{G})$. It is linearly independent because $\sum_{\gamma \in \mathcal{F}} c_\gamma \pi_\gamma = 0$ implies $c_{\gamma_1} \pi_{\gamma_1} = \pi_{\gamma_1} [\sum_{\gamma \in \mathcal{F}} c_\gamma \pi_\gamma] = 0$ and therefore $c_{\gamma_1} = 0$ for all $\gamma_1 \in \Gamma$. \square

We now construct the connection ∇^α on $p_{\pi_{m,n}\mathcal{F}} : \pi_{m,n}\mathcal{F} \rightarrow \omega_\alpha$ when $\omega_\alpha = \bigcap_{m' \in \alpha} \omega_{m'} \neq \emptyset$ for $m \in \alpha \subset \{1, \dots, M_I\}$.

Proposition 3.3. *Fix $\alpha \subset \{1, \dots, M_I\}$ and $(m,n) \in L(I)$. Assume $m \in \alpha \subset \{1, \dots, M_I\}$ with $\omega_\alpha = \bigcap_{m' \in \alpha} \omega_{m'} \neq \emptyset$. There exists a linearly independent finite family $(\pi_\gamma)_{\gamma \in \Gamma_\alpha}$ of spectral projections $\pi_\gamma(k) = 1_{J_\gamma}(H_0(k))$, $J_\gamma \subset J_{m,n}$, which are analytic with respect to $k \in \omega_\alpha$, and such that*

- $\pi_\gamma \pi_{\gamma'} = 0$ if $\gamma \neq \gamma'$, $\gamma, \gamma' \in \Gamma_\alpha$,
- for all pairs (m', n') , $m' \in \alpha$, $0 \leq n' \leq N_{m'}$, the spectral projection $\pi_{m,n} \pi_{m',n'}$ admits a unique decomposition

$$\pi_{m,n} \pi_{m',n'} = \sum_{\gamma \in \Gamma_\alpha} c_{m',n',\gamma} \pi_\gamma \quad , \quad c_{m',n',\gamma} \in \{0, 1\} .$$

The unitary connection $\nabla^\alpha = \sum_{\gamma \in \Gamma_\alpha} \pi_\gamma \circ \nabla^{m,n} \circ \pi_\gamma$ on $p_{\pi_{m,n}\mathcal{F}} : \pi_{m,n}\mathcal{F} \rightarrow \omega_\alpha$ then satisfies

$$\forall m' \in \alpha, \forall n' \in \{0, \dots, N_{k_{m'}}\}, \quad \nabla^\alpha(\pi_{m,n} \pi_{m',n'}) = 0 \quad \text{on } \omega_\alpha .$$

If $J_\gamma \cap \bar{I} \neq \emptyset$ for $\gamma \in \Gamma_\alpha$, then $\{(m', n'), m' \in \alpha, 0 \leq n' \leq N_{m'}, J_\gamma \subset J_{m',n'}\} \subset L(I)$ can be ordered like in Proposition 2.4 and written $\{(m_\ell, n_\ell), 1 \leq \ell \leq L_\gamma\}$ with

$$J_\gamma = \bigcap_{\ell=1}^{L_\gamma} (J_{m,n} \cap J_{m_\ell, n_\ell}) = J_{m,n} \cap J_{m_{L_\gamma}, n_{L_\gamma}} \quad , \quad \pi_\gamma = \pi_{m,n} \pi_{m_{L_\gamma}, n_{L_\gamma}} .$$

Proof. Let us work in the unital commutative \mathcal{C}^* -algebra, \mathcal{A} , of continuous functions of $f(k, H_0(k))$, $k \in \omega_\alpha$, such that $\pi_{m,n}(k)f(k, H_0(k)) = f(k, H_0(k))$ with the unit $\pi_{m,n}$.

For any $m' \in \alpha$, we know the fiberwise identity $\bigoplus_{n'=0}^{N_{k_{m'}}} \pi_{m',n'}(k) = \text{Id}_{\mathcal{F}_k}$ for $k \in \omega_\alpha \subset \omega_{m'}$ and

$$\sum_{n'=0}^{N_{k_{m'}}} \pi_{m,n} \pi_{m',n'} = \pi_{m,n} = 1_{\mathcal{A}} .$$

By setting $\mathcal{I}_0 = \{(m', n'), m' \in \alpha, 0 \leq n' \leq N_{k_{m'}}\}$ and $\mathcal{P}_{\mathcal{I}_0} = \{\pi_{m,n} \pi_{m',n'}, (m', n') \in \mathcal{I}_0\}$ and $\mathcal{P}_{\mathcal{I}} = \mathcal{P}_{\mathcal{I}_0} \cup \{1_{\mathcal{A}} - \pi_{m,n} \pi_{m',n'}, (m', n') \in \mathcal{I}_0\}$, $\mathcal{I} = \mathcal{I}_0 \times \{0, 1\}$, we deduce

$$\mathcal{C}^*(\mathcal{P}_{\mathcal{I}_0}) = \mathcal{C}^*(\mathcal{P}_{\mathcal{I}}) .$$

The family $\mathcal{P}_{\mathcal{I}}$ is stable by $\pi \rightarrow 1_{\mathcal{A}} - \pi$ and Lemma 3.2 can be applied.

By construction the basis $(\pi_\gamma)_{\gamma \in \Gamma_\alpha}$ of $\mathcal{C}^*(\mathcal{P}_{\mathcal{I}_0}) = \mathcal{C}^*(\mathcal{P}_{\mathcal{I}})$, is made of products

$$\prod_{(m',n') \in \gamma_0} (\pi_{m,n} \pi_{m',n'}) \circ \prod_{(m',n') \in \gamma_1} (\pi_{m,n} - \pi_{m,n} \pi_{m',n'}) \quad (18)$$

for $\gamma_0, \gamma_1 \subset \mathcal{I}_0$. They can be written $1_{J_\gamma}(H_0(k))$ with $J_\gamma \subset J_{m,n}$ and they are all analytic with respect to $k \in \omega_\alpha$.

The property of the unitary connection ∇^α is due to $\pi_\gamma(\pi_{m,n} \pi_{m',n'}) \in \{\pi_\gamma, 0\}$ for all $\gamma \in \Gamma_\alpha$, $m' \in \alpha$ and all $n' \in \{0, \dots, N_{k_{m'}}\}$.

When $J_\gamma \cap \bar{I} \neq \emptyset$ with $\pi_\gamma = 1_{J_\gamma}(H_0(k))$, the pairs (m', n') involved in the first factor of (18) must satisfy $n' \geq 1$ while in the second factor the pairs must be $(m', 0)$. With $\pi_{m,n} - \pi_{m,n} \pi_{m',0} = \pi_{m,n} \sum_{n'=1}^{N_{m'}} \pi_{m',n'}$, this implies that π_γ is a positive linear combination of products of $\pi_{m',n'}$, $m' \in \alpha$, $(m', n') \in L(I)$. But the two properties of the basis $(\pi_\gamma)_{\gamma \in \Gamma_\alpha}$ says that it must be the product of the all the π_{m_ℓ, n_ℓ} , $1 \leq \ell \leq L_\gamma$ with $m_\ell \in \alpha$, $(m_\ell, n_\ell) \in L(I)$ and $J_\gamma \subset J_{m_\ell, n_\ell}$. For such pairs (m', n') we have

$$\forall k \in \omega_\alpha, \quad 0 \neq \pi_\gamma(k) \leq \pi_{m',n'}(k),$$

and the intersection $\bigcap_{\ell=1}^{L_\gamma} \Omega_{m_\ell} \cap \Sigma \supset (\omega_\alpha \times J_\gamma) \cap \Sigma \neq \emptyset$. By ordering the set $\{(m_\ell, n_\ell), 1 \leq \ell \leq L_\gamma\}$ like in Proposition 2.4 we get $J_\gamma = J_{m_{L_\gamma}, n_{L_\gamma}}$ and $\pi_\gamma(k) = \pi_{m_{L_\gamma}, n_{L_\gamma}}(k)$ for $k \in \omega_\alpha$. \square

Proof of Proposition 3.1. Fix $m \in \{1, \dots, M_I\}$. The set $K = \bigcup_{m' \neq m} \text{supp}(g_m \cap \text{supp } g_{m'})$ is a compact subset of ω_m . For any $\alpha \subset \{1, \dots, M_I\}$ such that $m \in \alpha$, $\#\alpha \geq 2$, set

$$\tilde{\omega}_\alpha = \omega_\alpha \setminus \bigcup_{m' \notin \alpha} \text{supp } g_{m'} = \bigcap_{m' \in \alpha} \omega_{m'} \setminus \left(\bigcup_{m' \notin \alpha} \text{supp } g_{m'} \right).$$

These are open sets and they make an open covering of K ,

$$K \subset \bigcup_{\substack{\alpha \subset \{1, \dots, M_I\} \\ m \in \alpha \\ \#\alpha \geq 2}} \tilde{\omega}_\alpha.$$

Therefore there exists a partition of unity

$$\sum_{m \in \alpha \subset \{1, \dots, M_I\}} \Theta_\alpha^2 \equiv 1 \quad \text{on } \omega_m$$

with

- $\Theta_\alpha \in \mathcal{C}_{\text{comp}}^\infty(\tilde{\omega}_\alpha)$ when $\#\alpha \geq 2$,
- $\Theta_{\{m\}} \in \mathcal{C}^\infty(\omega_m)$ and $\text{supp } \Theta_{\{m\}} \subset \omega_m \setminus K$.

The unitary connection $\tilde{\nabla}$ on $p_{\pi_{m,n}\mathcal{F}} : \pi_{m,n}\mathcal{F} \rightarrow \omega_m$ is defined as

$$\tilde{\nabla}^{m,n} = \sum_{m \in \alpha \subset \{1, \dots, M_I\}} \Theta_\alpha \nabla^\alpha \Theta_\alpha$$

where ∇^α is the connection above $\omega_\alpha \supset \tilde{\omega}_\alpha$ introduced in Proposition 3.3. For any $m' \in \{1, \dots, M_I\}$ such that $\text{supp } g_m \cap \text{supp } g_{m'} \neq \emptyset$ we get

$$\tilde{\nabla}^{m,n} = \sum_{\{m, m'\} \subset \alpha \subset \{1, \dots, M_I\}} \Theta_\alpha \nabla^\alpha \Theta_\alpha$$

with $\sum_{\{m, m'\} \subset \alpha \subset \{1, \dots, M_I\}} \Theta_\alpha^2 \equiv 1$

on a neighborhood of $\mathcal{V}_{m, m'}$ of $\text{supp } g_m \cap \text{supp } g_{m'}$. This implies for any $n' \in \{1, \dots, N_{m'}\}$,

$$\tilde{\nabla}^{m,n}(\pi_{m,n}\pi_{m',n'}) = \sum_{\{m, m'\} \subset \alpha \subset \{1, \dots, M_I\}} \Theta_\alpha^2 \nabla^\alpha(\pi_{m,n}\pi_{m',n'}|_{\omega_\alpha}) \quad \text{on } \mathcal{V}_{m, m'},$$

where Proposition 3.3 says $\nabla^\alpha(\pi_{m,n}\pi_{m',n'}|_{\omega_\alpha}) = 0$ when $m' \in \alpha$.

Therefore (17) holds true for all pairs (m, n) and (m', n') in $L(I)$.

For the second property we again fix $m \in \{1, \dots, M_I\}$, $(k_0, \lambda_0) \in \Omega_{m,n} \cap p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}) \subset \Sigma$, and consider the set

$$\{(m', n') \in L(I), (k_0, \lambda_0) \in \Omega_{m', n'}\} = \{(m_\ell, n_\ell), 1 \leq \ell \leq L\}.$$

The intersection $\bigcap_{\ell=1}^L \Omega_{m_\ell, n_\ell} \cap \Sigma \ni (k_0, \lambda_0)$ and the family $((m_\ell, n_\ell))_{1 \leq \ell \leq L}$ can be ordered according to Proposition 2.4.

Define

$$\alpha_0 = \{m_\ell, 1 \leq \ell \leq L\}$$

where obviously $m \in \alpha_0$ and $k_0 \in \omega_{\alpha_0}$. A stronger property says that $k_0 \in \omega_\alpha$ for $m \in \alpha \subset \{1, \dots, M_I\}$, implies $\alpha \subset \alpha_0$: Actually if $k_0 \in \omega_{m'}$, then the condition $(k_0, \lambda_0) \in \Sigma \cap (\omega_{m'} \times \bar{I})$ implies the existence of $(m', n') \in L(I)$ such that $\lambda_0 \in \mathcal{J}_{m', n'}$. Thus $k_0 \in \omega_{m'}$ implies $m' \in \alpha_0$

and therefore $k_0 \notin \omega_\alpha$ for $\alpha \neq \alpha_0$. Because $\text{supp } \Theta_\alpha \subseteq \omega_\alpha$ for $m \in \alpha \neq \alpha_0$, $\mathcal{V}_{k_0} = \omega_{\alpha_0} \setminus \cup_{m \in \alpha \neq \alpha_0} \text{supp } \Theta_\alpha$ is an open neighborhood of k_0 . We now write

$$\tilde{\nabla}^{m,n} = \sum_{m \in \alpha \subset \alpha_0} \Theta_\alpha \nabla^\alpha \Theta_\alpha \quad \text{on } \mathcal{V}_{k_0},$$

and $\sum_{m \in \alpha \subset \alpha_0} \Theta_\alpha^2 \equiv 1 \quad \text{on } \mathcal{V}_{k_0}.$

With the notations of Proposition 3.3, for $m \in \alpha \subset \alpha_0$ there exists $\gamma \in \Gamma_\alpha$ such that \mathcal{J}_γ contains $\lambda_0 \in \text{Spec}(H_0(k_0)) \cap \bar{I}$. For such a γ , Proposition 3.3 provides a pair $(m_{\alpha,\gamma}, n_{\alpha,\gamma}) \in L(I)$, $m_{\alpha,\gamma} \in \alpha$ such that $\pi_\gamma = \pi_{m,n} \pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}}$, $(k_0, \lambda_0) \in \Omega_{m_{\alpha,\gamma}, n_{\alpha,\gamma}}$, and $\pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}} \circ \nabla^\alpha \circ \pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}} = \pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}} \circ \nabla^{m,n} \circ \pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}}$ on ω_α and therefore on $\mathcal{V}_{k_0} \subset \omega_{\alpha_0} \subset \omega_\alpha$. Because $(k_0, \lambda_0) \in \Omega_{m_{\alpha,\gamma}, n_{\alpha,\gamma}}$, there exist $\ell \in \{1, \dots, L\}$ such that $(m_{\alpha,\gamma}, n_{\alpha,\gamma}) = (m_\ell, n_\ell)$ and the ordering of Proposition 2.4 says $\pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}} \pi_{m_L, n_L} = \pi_{m_\ell, n_\ell} \pi_{m_L, n_L} = \pi_{m_L, n_L}$ and

$$\begin{aligned} \pi_{m_L, n_L} \circ \tilde{\nabla}^{m,n} \circ \pi_{m_L, n_L} \Big|_{\mathcal{V}_{k_0}} &= \sum_{m \in \alpha \subset \alpha_0} \Theta_\alpha \pi_{m_L, n_L} \pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}} \nabla^\alpha \pi_{m_{\alpha,\gamma}, n_{\alpha,\gamma}} \pi_{m_L, n_L} \Theta_\alpha \Big|_{\mathcal{V}_{k_0}} \\ &= \pi_{m_L, n_L} \circ \left[\sum_{m \in \alpha \subset \alpha_0} \Theta_\alpha \nabla^{m,n} \Theta_\alpha \right] \circ \pi_{m_L, n_L} \Big|_{\mathcal{V}_{k_0}} \\ &= \pi_{m_L, n_L} \circ \nabla^{m,n} \circ \pi_{m_L, n_L} \Big|_{\mathcal{V}_{k_0}}. \end{aligned}$$

□

4 The operator \tilde{A}_I

We specify here the construction and properties of \tilde{A}_I and finish the proof of Theorem 1.1. As mentioned in the beginning of Section 3 the operator \tilde{A}_I is given as a first-order differential operator with $\mathcal{C}_{comp}^\infty(M; \mathcal{L}(\mathcal{F}))$ -coefficients by

$$\tilde{A}_I = i \sum_{m=1}^{M_I} \sum_{n=1}^{N_m} g_m(k) \left(\tilde{\nabla}_{X_{m,n}}^{m,n} + \frac{(\text{div } X_{m,n})}{2} \pi_{m,n} \right) g_m(k)$$

where

- $\sum_{m=1}^{M_I} g_m^2(k) \equiv 1$ in a neighborhood of $p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}))$, is the partition of unity subordinate to $p_M(\Sigma \cap p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\bar{I}_1)) \cup \cup_{m=1}^{M_I} \omega_m$, introduced in (2)(3)(4) after Proposition 2.1;
- $X_{m,n}$ is a vector field such that (15) holds, of which the local construction was given in Proposition 2.1-(6);
- $\tilde{\nabla}^{m,n}$ is the modified unitary connection given by Proposition 3.1.

By construction $\cup_{(m,n) \in L(I)}^{M_I} \mathcal{J}_{m,n} \subseteq \tilde{I}$ and there exists $\chi \in \mathcal{C}_{comp}^\infty(\tilde{I}; \mathbb{R})$ such that

$$\tilde{A}_I \chi(H_0) = \tilde{A}_I = \chi(H_0) \tilde{A}_I \quad \text{on } \mathcal{C}_{comp}^\infty(M; \mathcal{F}). \quad (19)$$

As a differential operator we get

$$[H_0, iA] = \sum_{(m,n) \in L(I)} g_m(k) \pi_{m,n}(k) (\tilde{\nabla}_{X_{m,n}}^{m,n} H_0(k)) \pi_{m,n}(k) g_m(k) = \sum_{(m',n') \in L(I)} B_{m',n'}(k)$$

with $B_{m',n'} \in \mathcal{C}_{comp}^\infty(\omega_{m'}; \mathcal{L}(\pi_{m,n} \mathcal{F}))$ with $\text{supp } B_{m',n'} \subset \text{supp } g_{m'}$. But because the connection $\tilde{\nabla}^{m,n}$ satisfies (17) we deduce that for all $j \in \mathbb{N}$,

$$\text{ad}_{i\tilde{A}_I}^j [H_0, i\tilde{A}_I] \in \mathcal{L}(L^2(M; \mathcal{F})).$$

If \tilde{A}_I is essentially self-adjoint on $\mathcal{C}_{comp}^\infty(M; \mathcal{F})$ and that its self-adjoint extension, still denoted by \tilde{A}_I , satisfies $e^{it\tilde{A}_I}D(H_0) \subset D(H_0)$, for all $t \in \mathbb{R}$, then $H_0 \in \mathcal{C}^\infty(i\tilde{A}_I)$.

The second statement is a consequence of (19) and it suffices to check the essential self-adjointness of \tilde{A}_I . Because $p_M(\Sigma \cap p_{\Sigma \rightarrow \mathbb{R}}^{-1} \text{supp } \chi)$ is compact with χ chosen like in (19), there exists a smooth domain ω with $\bar{\omega}$ compact in M such that the coefficients of \tilde{A}_I belong to $\mathcal{C}_{comp}^\infty(\omega; \mathcal{L}(\mathcal{F}))$. Take a finite covering $\bar{\omega} \subset \bigcup_{n=1}^{n_{max}} \omega_n$ of coordinate open charts ω_n . When $\sum_{n=1}^{n_{max}} \theta_n^2(k) \equiv 1$ in a neighborhood of $\bar{\omega}$ with $\theta_n \in \mathcal{C}_{comp}^\infty(\omega_n; \mathbb{R})$ consider the operator

$$-\Delta_\omega = - \sum_{n=1}^{n_{max}} \theta_n \Delta_{\omega_n} \theta_n$$

$$\text{with } \Delta_{\omega_n} = \sum_{i=1}^{\dim M} (\nabla_{\frac{\partial}{\partial k_i}}^M)^2, \quad ,$$

where $(k_1, \dots, k_{\dim M})$ are coordinates on ω_n and ∇^M is the unitary connection defined by (6). In a neighborhood of $\bar{\omega}$, $-\Delta_\omega$ is an elliptic second order operator with a scalar principal part. Moreover there exists a constant $C_\omega > 0$ such that $C_\omega - \Delta_\omega$ is non negative on $\mathcal{C}_{comp}^\infty(\omega; \mathcal{L}(\mathcal{F}))$. The standard elliptic regularity on the smooth domain $\bar{\omega}$ (see e.g. [ChPi]) says that the Friedrichs extension of $C_\omega - \Delta_\omega$, which is the Dirichlet realization denoted by $\mathcal{N} = C_\omega - \Delta_\omega^D$, is self-adjoint with domain $H^2(\bar{\omega}; \mathcal{F}) \cap H_0^1(\omega; \mathcal{F})$ and that $\mathcal{C}_{comp}^\infty(\omega; \mathcal{F})$ is a core for $\mathcal{N} = (C_\omega - \Delta_\omega^D)$.

The differential operator \tilde{A}_I acts on $\mathcal{C}_{comp}^\infty(\omega; \mathcal{F})$ with

$$\|\tilde{A}_I \varphi\| \leq C \|\mathcal{N} \varphi\|$$

$$|\langle \tilde{A}_I \varphi, \mathcal{N} \varphi \rangle - \langle \mathcal{N} \varphi, \tilde{A}_I \varphi \rangle| \leq C \|\mathcal{N}^{1/2} \varphi\|^2$$

for all $\varphi \in \mathcal{C}_{comp}^\infty(\omega; \mathcal{F})$. The second inequality is due to the fact that \mathcal{N} is a second order differential operator with a scalar principal part so that $[\mathcal{N}, \tilde{A}_I]$ is a second order differential operator with $\mathcal{C}_{comp}^\infty(\omega; \mathcal{L}(\mathcal{F}))$ coefficients. Nelson's commutator theorem (see e.g. [ReSi]) tells us that \tilde{A}_I is essentially self-adjoint on $\mathcal{C}_{comp}^\infty(\omega; \mathcal{F})$ and therefore on $\mathcal{C}_{comp}^\infty(M; \mathcal{F})$.

The final point is about the local Mourre inequality (1). Let $\lambda_0 \in \bar{I}$ for any $k_0 \in M$ such that $(k_0, \lambda_0) \in \Sigma$, Proposition 3.1 provides a neighborhood \mathcal{V}_0 of k_0 and a pair $(m_L, n_L) = (m_L(k_0), n_L(k_0))$ such that $(k_0, \lambda_0) \in \Omega_{m_L, n_L}$

$$\forall (m, n) \in L(I), \quad \pi_{m_L, n_L} \circ \tilde{\nabla}^{m, n} \circ \pi_{m_L, n_L} = \pi_{m_L, n_L} \circ \nabla^{m, n} \pi_{m_L, n_L},$$

while $\pi_{m_L, n_L} \pi_{m, n} \in \{\pi_{m_L, n_L}, 0\}$.

For $k \in \mathcal{V}_{k_0}$ we deduce

$$\begin{aligned} \pi_{m_L, n_L} [H_0, i\tilde{A}_I] \pi_{m_L, n_L} &= \pi_{m_L, n_L} [H_0, - \sum_{(m, n) \in L(I)} g_m(m) \pi_{m_L, n_L} \tilde{\nabla}_{X_{m, n}}^{m, n} \pi_{m_L, n_L} g_m(k)] \pi_{m_L, n_L} \\ &= \pi_{m_L, n_L} [H_0, - \sum_{(m, n) \in L(I)} g_m(k) \nabla^{m, n} g_m(k)] \pi_{m_L, n_L} \\ &\geq \frac{1}{2} \pi_{m_L, n_L} \left(\sum_{(m, n) \in L(I)} g_m(k)^2 \pi_{m, n} \right) \pi_{m_L, n_L} \\ &\geq \frac{1}{2} \pi_{m_L, n_L} = \frac{1}{2} 1_{J_{m_L, n_L}}(H_0(k)). \end{aligned}$$

Now $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\{\lambda_0\}) \cap \Sigma$ is compact and $\bigcup_{(k_0, \lambda_0) \in \Sigma} \mathcal{V}_{k_0} \times J_{m_L(k_0), n_L(k_0)}$ is an open covering of it. We can find $N_{\lambda_0} \in \mathbb{N}$ such that $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(\{\lambda_0\}) \cap \Sigma \subset \bigcup_{v=1}^{N_{\lambda_0}} \mathcal{V}_{k_v} \times J_{m_L(k_v), n_L(k_v)}$. Therefore we can take

$\delta > 0$ such that $p_{\Sigma \rightarrow \mathbb{R}}^{-1}([\lambda_0 - \delta, \lambda_0 + \delta]) \cap \Sigma \subset \bigcup_{\nu=1}^{N_{\lambda_0}} \mathcal{V}_{k_\nu} \times \mathbf{J}_{m_{L(k_\nu)}, n_{L(k_\nu)}}$ and we obtain

$$1_{[\lambda_0 - \delta, \lambda_0 + \delta]}(H_0)[H_0, i\tilde{A}_I]1_{[\lambda_0 - \delta, \lambda_0 + \delta]}(H_0) \geq \frac{1}{2}1_{[\lambda_0 - \delta, \lambda_0 + \delta]}(H_0),$$

which is (1).

5 Comments and examples

The changes between the initial and the new versions requires several comments in particular to convince the reader that it does not change the applications which were made e.g. in [GeNi2]. Some simple examples in dimension 2 are then given in order to illustrate the multistep construction of \tilde{A}_I .

5.1 Comments

- a) The explicit form of the conjugate operator \tilde{A}_I is not important but it is important for the applications that it remains a first-order differential operator with $\mathcal{C}^\infty(M; \mathcal{L}(\mathcal{F}))$ -coefficients. The main difference between “Theorem” 3.1 in [GeNi] and the new version Theorem 1.1 is actually the local form (1) of Mourre’s inequality around the energy λ . But this does not make any problem because Mourre theory is essentially local in energy (see [Mou1][Mou2][ABG]).
- b) The local in energy version of Mourre inequality is actually the stable version when perturbations $H_0 + V$ are considered. From this point of view Theorem 3.3 of [GeNi] which was a litteral application of the general Proposition 7.5.6 in [ABG], remains valid after replacing A_I by its new version \tilde{A}_I . Note that a regularity $H_0 \in C^{1+\varepsilon}(A)$ with $\varepsilon > 0$, or $\varepsilon = 0$ interpreted as a Dini-type continuity (see [ABG]), is required. This was definitely incorrect in our initial version with $A = A_I$ and this works now with $A = \tilde{A}_I$.
- c) In all the examples presented in [GeNi] and in particular for the case of periodic Schrödinger operators developed in [GeNi2], the Hilbert bundle $p_{\mathcal{F}} : \mathcal{F} \rightarrow M$ is actually a trivial one $\mathcal{F} = M \times \mathcal{H}'$, with $M = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. The initial connection ∇^M can be taken as the trivial one and this is probably the reason why we were not very careful about the use of connections in [GeNi]. However the now more accurate and correct version, shows that it is important to clarify this point especially when finite rank subbundles $p_{\pi_{m,n}\mathcal{F}} : \pi_{m,n}\mathcal{F} \rightarrow \omega_m$ are considered.
- d) In [GeNi2] with $\mathcal{F} = \mathbb{T}^d \times \mathcal{H}'$ the perturbation V that we considered were $\mathcal{L}(\mathcal{H}')$ -valued pseudo-differential operator $V(D_k)$ of order $-\mu$, $\mu > 0$, with a scalar principal part. The negative order $-\mu < 0$ implies that $V(D_k)$ is a relatively compact perturbation of H_0 while the scalar principal part implies $V(D_k) \in \mathcal{C}^{1+\mu}(\tilde{A}_I)$ and $H_0 + V(D_k) \in \mathcal{C}^{1+\mu}(A_{U_1, I_2})$. All the arguments developed in [GeNi2], concerned with the absence of singular continuous spectrum, the virial theorem or propagation estimates, work after replacing A_I by \tilde{A}_I . About the asymptotic completeness of the wave operators, which is a consequence of Proposition 7.5.6 in [ABG] or Theorem 3.3 in [GeNi] when $\mu > 1$, we recall that it is still an open question in the long range case $\mu > 0$, as it is discussed in [GeNi2].

- e) About the absence of singular continuous spectrum for the free operator H_0 away from the set of thresholds τ , a conjugate operator is not necessary and we refer the reader to [Kuc] for a spectral measure argument for it.
- f) Theorem 1.1 and its proof were given in a global form, with \tilde{A}_I constructed for the whole energy intervals $I \in \tilde{I} \in \mathbb{R} \setminus \tau$. This was done in order to stick as much as possible to the initial version. This imposes the treatment of possibly several energy intervals $J_{m,n}$, $1 \leq n \leq N_m$ above ω_m . Actually the construction can be made simpler if one works from the beginning in $I =]\lambda - \delta, \lambda + \delta[$ and $\tilde{I} =]\lambda - 2\delta, \lambda + 2\delta[$ with $\delta > 0$ small enough so that for every $m \in \{1, \dots, M_I\}$, $N_m = 1$. This simplifies the connections $\nabla^{\omega_m, \pi} = \pi_{m,1} \nabla^{\omega_m} \pi_{m,1}$ on $p_{\pi_{m,1}, \mathcal{F}} : \pi_{m,1} \mathcal{F} \rightarrow \omega_m$ with a single term and only one vector field $X_{m,1}$ above every ω_m is used in the construction of \tilde{A}_I . However while gluing the operators $\nabla_{X_{m,1}}^{\omega_m, 1}$ with the partition of unity $\sum_{m=1}^{M_I} g_m^2(k) \equiv 1$ in a neighborhood of $p_M(p_{\Sigma \rightarrow \mathbb{R}}^{-1}([\lambda - \delta, \lambda + \delta]) \cap \Sigma)$, the rank of the various projections $\pi_{m,1}$ changes and the modification of the connections $\nabla^{m,1}$ into $\tilde{\nabla}^{m,1}$ is still necessary in order to ensure $H_0 \in \mathcal{C}^\infty(\tilde{A}_I)$. This is probably a simpler way for visualizing the whole construction, of which the generalization relies on the simple Lemma 3.2, and this is how it will be illustrated in the examples below.
- g) Generically crossings of eigenvalues of a matricial self-adjoint operators occur along codimension 2 strata. When $\dim M = 2$, singular strata with $\dim S < M$ are thus points and these singular points are removed while working in the energy intervals $I \in \mathbb{R} \setminus \tau$. So when $\dim M = 2$ and generically, $p_{\Sigma \rightarrow \mathbb{R}}^{-1}(I)$ is made of disconnected strata of dimension 2 locally diffeomorphic to M , and H_0 is locally (on Σ) a scalar operator. The analysis is much simpler. Such two dimensional problems were considered in [FeWe] and appear in the modelling of graphene.
- h) The assumption that the mapping $p_{\Sigma \rightarrow \mathbb{R}} : \Sigma \rightarrow \mathbb{R}$ is proper is used in a fundamental way, in order to make a stratification which is compatible with the projection $p_{\Sigma \rightarrow \mathbb{R}}$. In some examples where a finite stratification is obviously given it can be forgotten and the construction of a conjugate operator \tilde{A}_I can be done without this assumption, especially when the differential of $p_{S \rightarrow \mathbb{R}}$, the restriction of $p_{\Sigma \rightarrow \mathbb{R}}$ to every stratum S , is uniformly away from 0. Then the construction relies on the local understanding of incidence of strata with the possible changes of the rank of the associated projectors. Another use $p_{\Sigma \rightarrow \mathbb{R}} : \Sigma \rightarrow \mathbb{R}$ being proper, is for the construction of a self-adjoint operator for Nelson's commutator method, but this can be easy for some non compact manifold M (e.g. $M = \mathbb{R}^d$). Such a simple example is discussed below.
- i) One may wonder about the mixing of the stratification of subanalytic sets, which makes sense in the analytic category with proper analytic mapping, with \mathcal{C}^∞ partition of unity. It works for our construction of \tilde{A}_I . A question is whether the incidence of strata, as well as the associated spectral projectors, could be fully handled within the subanalytic category with the help of Lojasiewicz inequalities.

5.2 Examples

We give two examples with $M = \mathbb{R}^2$ and $\mathcal{H}' = \mathbb{R}^2$ with pictures in order to visualize in simple cases our general construction and to make more explicit the problem set by the

initial construction of A_I . Although these examples are non generic according to comment **g**), the two dimensional case, $M = \mathbb{R}^2$, allows simple pictures. Other examples dimension 3 with $M = \mathbb{R}^3$ or $M = \mathbb{T}^3$, motivated by the study of Maxwell equations were recently considered in [Poi]. In this text a more accurately designed conjugate operator taking into account the more specific structure of the operator allows to consider the Limiti Absorption Principle at the threshold energies.

First example: Consider

$$H_0(k_1, k_2) = \begin{pmatrix} k_1^2 + k_2^2 + k_2 + k_1 & k_1 k_2 \\ k_1 k_2 & k_1^2 + k_2^2 + k_2 - k_1 \end{pmatrix} = k_1^2 + k_2^2 + k_2 \text{Id}_{\mathbb{R}^2} + k_1 \begin{pmatrix} 1 & k_2 \\ k_2 & -1 \end{pmatrix} \quad (k_1, k_2) \in \mathbb{R}^2$$

The characteristic variety is given by

$$\Sigma = \left\{ k_1^2 + k_2^2 + k_2 \pm k_1 \sqrt{1 + k_2^2}, (k_1, k_2) \in \mathbb{R}^2 \right\}$$

and is partitionned into 15 strata given by

$$\lambda = k_1^2 + k_2^2 + k_2 \pm k_1 \sqrt{1 + k_2^2}$$

with the following choices

- $\pm k_1 > 0, \pm(k_2 + 1/2) > 0, \dim S = 2$, multiplicity 1, in blue or red on Figure 1;
- $k_1 = 0, \pm(k_2 + 1/2) > 0, \dim S = 1$, multiplicity 2, green on Figure 1;
- $\pm k_1 > 0, k_2 = -1/2, \dim S = 1$, multiplicity 1, black on Figure 1;
- $k_1 = 0, k_2 = -1/2, \dim S = 0$, multiplicity 2, black on Figure 1.

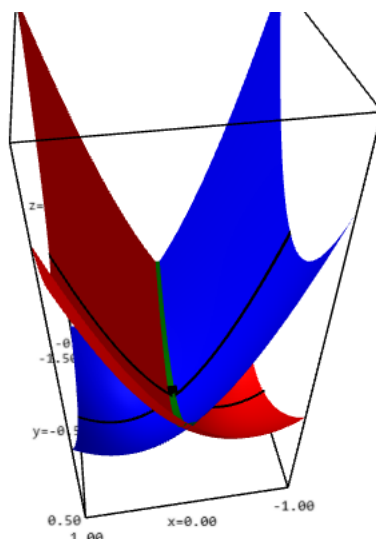


Figure 1: Stratification of Σ

There is one threshold at energy $-\frac{1}{4}$ corresponding to $(k_1 = 0, k_2 = -1/2, \lambda = -\frac{1}{4})$. While working in the energy set $\tilde{I} \Subset]-\frac{1}{4}, +\infty[$ the number of strata can be reduced to 5 the red, blue ones ($\dim S = 2$, multiplicity 1) and the green one ($\dim S = 1$, multiplicity 2) on

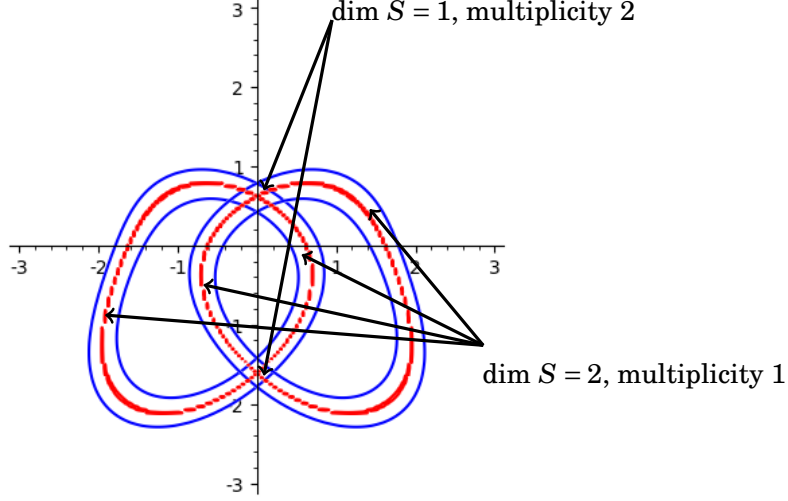


Figure 2: Level sets $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, $\lambda_0 = 1$, $\delta = 0.4$ (blue), $\delta = 0.04$ (red).

Figure 1.

The situation is better seen by considering the level sets in \mathbb{R}^2 around an energy λ_0 (in Figure 2 $\lambda_0 = 1$). Let us look more carefully around a point along the k_2 -axis where λ_0 is an eigenvalue with multiplicity 2. Three points k_{m_0} , $\dim S_{k_{m_0}} = 1$, k_{m_1} and k_{m_2} with $\dim S_{k_{m_1}} = \dim S_{k_{m_2}} = 2$, the boundaries of open set ω_{m_j} (dashed lines) and of $\text{supp } g_{m_j}$ (black lines) are represented for $j = 0, 1, 2$ in Figure 3.

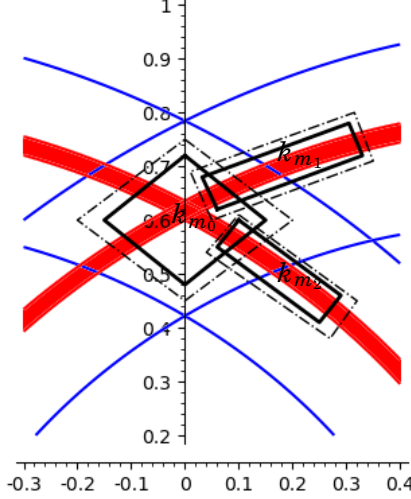


Figure 3: Open sets ω_m (dashed lines) and $\text{supp } g_m$ (black lines) around the points k_m . Level sets $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, $\lambda_0 = 1$, $\delta = 0.4$ (blue), $\delta = 0.04$ (red).

In Figure 3 the projectors $\pi_{m_j,1}$ have the minimal rank 1 for $j = 1, 2$ and the connection $\nabla^{m_j,1}$ need not to be modified above ω_{m_j} . It is not the case for m_0 where $\text{rank } \pi_{m_0,1} = 2$ while $\text{supp } g_{m_0}$ intersect $\text{supp } g_{m_1}$ and $\text{supp } g_{m_2}$. The connection $\tilde{\nabla}^{m_0,1}$ differs from the trivial connection $\nabla^{m_0,1}$, $\nabla_X^{m_0,1} = X \otimes \text{Id}_{\mathbb{R}^2}$. Above a neighborhood of $\text{supp } g_{m_j} \cap \text{supp } g_{m_0}$ for $j = 1, 2$ (black areas in Figure 4), it equals $\tilde{\nabla}^{m_0,1} = \pi_{m_j,1} \nabla^{m_0,1} \pi_{m_j,1} + (\text{Id}_{\mathbb{R}^2} - \pi_{m_j,1}) \nabla^{m_0,1} (\text{Id}_{\mathbb{R}^2} - \pi_{m_j,1})$.

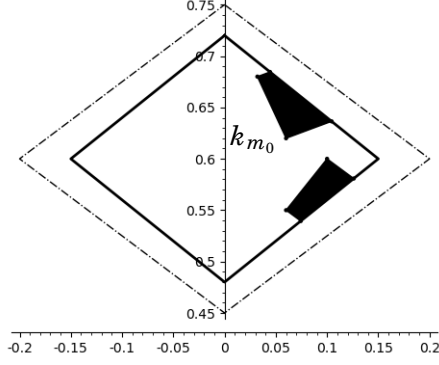


Figure 4: Localization in ω_{m_0} (dashed limits) of $\text{supp } g_{m_0} \cap \text{supp } g_{m_j}$ (in black), $j = 1, 2$, around which the connection must be modified.

Second example: This is a simplified version of the first one, for which we will explicitly compare the two operators A_I and \tilde{A}_I , with

$$H_0(k_1, k_2) = \begin{pmatrix} k_2 + k_1 & k_1 k_2 \\ k_1 k_2 & k_2 - k_1 \end{pmatrix} = k_2 \text{Id}_{\mathbb{R}^2} + k_1 \begin{pmatrix} 1 & k_2 \\ k_2 & -1 \end{pmatrix}.$$

Here the characteristic variety is

$$\Sigma = \left\{ k_2 \pm k_1 \sqrt{1 + k_2^2}, (k_1, k_2) \in \mathbb{R}^2 \right\}.$$

It does not fulfil the condition that $p_{\Sigma \rightarrow \mathbb{R}} : \Sigma \rightarrow \mathbb{R}$ is proper. But as mentioned in Comment **h**) above, it is not a problem because there is an obvious finite stratification of Σ compatible with $p_{\Sigma \rightarrow \mathbb{R}} : \Sigma \rightarrow \mathbb{R}$. It is given by the 5 strata given by

$$\lambda = \lambda_{\pm}(k_1, k_2) = k_2 \pm k_1 \sqrt{1 + k_2^2}$$

with the following choices:

- $\pm k_1 > 0$, $\dim S_{\pm, \pm} = 2$, multiplicity 1, in blue or red on Figure 5;
- $k_1 = 0$, $\dim S_0 = 1$, multiplicity 2, in green on Figure 5.

There is no threshold because $\partial_{k_2} k_2 = 1$ along $p_M S_0 = \{(0, k_2), k_2 \in \mathbb{R}\}$ while $\nabla \lambda_{\pm} = \begin{pmatrix} \pm \sqrt{1 + k_2^2} \\ 1 \pm \frac{k_2}{\sqrt{1 + k_2^2}} k_1 \end{pmatrix}$ satisfies $|\nabla \lambda_{\pm}| \geq 1$ on $p_M(S_{\pm, \pm}) = \{(k_1, k_2), \pm k_1 > 0\}$. In particular $\frac{\nabla \lambda_{\pm}}{|\nabla \lambda_{\pm}|^2} \cdot \nabla$ is a smooth vector field on $p_M(S_{\pm, \pm})$ such that $\frac{\nabla \lambda_{\pm}}{|\nabla \lambda_{\pm}|^2} \cdot \nabla \lambda_{\pm} = 1$. Notice also that

$$\partial_{k_2} H_0 = \begin{pmatrix} 1 & k_1 \\ k_1 & 1 \end{pmatrix} \geq \frac{1}{2} \text{Id}_{\mathbb{R}^2} \quad \text{for } |k_1| \leq \frac{1}{2}.$$

The spectral projectors associated with the eigenvalues $\lambda_{\pm}(k_1, k_2)$ for $k_1 \neq 0$ are given by

$$\pi_+ = \frac{1}{2\sqrt{1 + k_2^2}} \begin{pmatrix} 1 + \sqrt{1 + k_2^2} & k_2 \\ k_2 & -1 + \sqrt{1 + k_2^2} \end{pmatrix}, \quad \pi_- = \frac{1}{2\sqrt{1 + k_2^2}} \begin{pmatrix} \sqrt{1 + k_2^2} - 1 & -k_2 \\ -k_2 & \sqrt{1 + k_2^2} + 1 \end{pmatrix}.$$

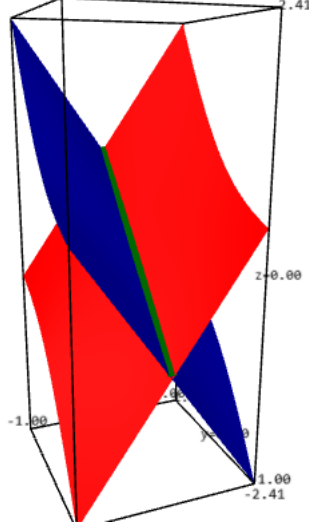


Figure 5: Stratification of Σ in the second example.

Take $g_0 \in \mathcal{C}_{comp}^\infty(]-1/2, 1/2[; \mathbb{R})$, $g_- \in \mathcal{C}^\infty(]-\infty, 1/4[; \mathbb{R})$ and $g_+ \in \mathcal{C}^\infty(]1/4, +\infty[; \mathbb{R})$ such that $g_0^2 + g_-^2 + g_+^2 \equiv 1$ on \mathbb{R} . The operator A_I of [GeNi] is then given by

$$-iA_I = g_0(k_1) \frac{\partial}{\partial k_2} g_0(k_1) + \sum_{\varepsilon_1, \varepsilon_2 \in \{+, -\}} g_{\varepsilon_1}(k_1) \pi_{\varepsilon_2} \left[\frac{\nabla \lambda_{\varepsilon_2}}{|\nabla \lambda_{\varepsilon_2}|^2} \cdot \nabla + \frac{1}{2} \operatorname{div} \left(\frac{\nabla \lambda_{\varepsilon_2}}{|\nabla \lambda_{\varepsilon_2}|^2} \right) \right] \pi_{\varepsilon_2} g_{\varepsilon_1}(k_1).$$

Then $[H_0, iA_I]$ equals

$$g_0^2(k_1) \begin{pmatrix} 1 & k_1 \\ k_1 & 1 \end{pmatrix} + g_+^2(k_1) + g_-^2(k_1) = \operatorname{Id}_{\mathbb{R}^2} + g_0^2(k_1) k_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Because $\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi_\pm \right] \neq 0$, the double commutator $[[H_0, iA_I], iA_I]$ is a first-order differential operator with a non vanishing principal part for $k_1 \in \operatorname{supp} g_0 \cap \operatorname{supp} g_\pm$ and it is not bounded.

For \tilde{A}_I we change the trivial connection in $\{(k_1, k_2), |k_1| < 1/2\}$: Take $\theta_0 \in \mathcal{C}_{comp}^\infty(]-1/4, 1/4[; \mathbb{R})$ and $\theta_1 \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$, $\operatorname{supp} \theta_1 \subset \mathbb{R} \setminus [-\frac{1}{8}, \frac{1}{8}]$ and $\theta_0^2 + \theta_1^2 \equiv 1$ on \mathbb{R} . Then replace $\frac{\partial}{\partial k_2}$ by

$$\tilde{\nabla}_{\frac{\partial}{\partial k_2}} = \theta_0(k_1) \frac{\partial}{\partial k_2} \theta_0(k_1) + \theta_1(k_1) [\pi_+ \frac{\partial}{\partial k_2} \pi_+ + \pi_- \frac{\partial}{\partial k_2} \pi_-] \theta_1(k_1)$$

and take

$$-i\tilde{A}_I = g_0(k_1) \tilde{\nabla}_{\frac{\partial}{\partial k_2}} g_0(k_1) + \sum_{\varepsilon_1, \varepsilon_2 \in \{+, -\}} g_{\varepsilon_1}(k_1) \pi_{\varepsilon_2} \left[\frac{\nabla \lambda_{\varepsilon_2}}{|\nabla \lambda_{\varepsilon_2}|^2} \cdot \nabla + \frac{1}{2} \operatorname{div} \left(\frac{\nabla \lambda_{\varepsilon_2}}{|\nabla \lambda_{\varepsilon_2}|^2} \right) \right] \pi_{\varepsilon_2} g_{\varepsilon_1}(k_1).$$

Computing $\tilde{\nabla}_{\frac{\partial}{\partial k_2}} H_0$ gives

$$\begin{aligned} \tilde{\nabla}_{\frac{\partial}{\partial k_2}} H_0 &= \theta_0^2(k_1) \begin{pmatrix} 1 & k_1 \\ k_1 & 1 \end{pmatrix} + \theta_1^2(k_1) \left[\frac{\partial \lambda_+}{\partial k_2} \pi_+ + \frac{\partial \lambda_-}{\partial k_2} \pi_- \right] \\ &= \operatorname{Id}_{\mathbb{R}^2} + \theta_0^2(k_1) k_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \theta_1^2(k_1) \frac{k_2 k_1}{\sqrt{1 + k_2^2}} (\pi_+ - \pi_-) \end{aligned}$$

and the commutator $[H_0, i\tilde{A}_I]$ equals now

$$\text{Id}_{\mathbb{R}^2} + k_1\theta_0^2(k_1)g_0^2(k_1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \theta_1^2(k_1)g_0^2(k_1)\frac{k_2k_1}{\sqrt{1+k_2^2}}(\pi_+ - \pi_-).$$

Because $\theta_0g_0 \equiv 0$ on $\text{supp } g_{\pm}$ and $\pi_+ - \pi_-$ commutes with π_{\pm} , the double commutator $[[H_0, i\tilde{A}_I], i\tilde{A}_I]$ is now a zeroth order operator. It is bounded and the iterated commutator can be checked to be bounded: Actually all the non diagonal terms with respect to the decomposition $\text{Id}_{\mathbb{R}^2} = \pi_+ + \pi_-$, generated after every commutation with $i\tilde{A}_I$, are supported on $\text{supp } \theta_0$ and therefore vanish on $\text{supp } g_{\pm}$.

Here the Mourre estimate can be written globally because

$$\theta_1^2(k_1)g_0^2(k_1)\frac{k_2k_1}{\sqrt{1+k_2^2}}(\pi_+ - \pi_-) \geq -\frac{1}{2}\theta_1^2(k_1)\text{Id}_{\mathbb{R}^2}$$

and

$$k_1g_0^2(k_1)\theta_0^2(k_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq -\frac{1}{2}\theta_0^2(k_1)\text{Id}_{\mathbb{R}^2}.$$

We obtain

$$[H_0, i\tilde{A}_I] \geq \frac{1}{2}\text{Id}_{\mathbb{R}^2}.$$

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