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Sharing the cost of hazardous transportation networks and the Priority Shapley value*

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Abstract

We consider the cost sharing issue resulting from the maintenance of a hazardous waste transportation network represented by a sink tree. The participating agents are located on the nodes of the network and must transport their waste to the sink through costly network portions. We introduce the Liability rule, which is inspired by the principles applied by the courts to settle cost-allocation disputes in the context of hazardous waste. We provide an axiomatic characterization of this rule. Furthermore, we show that the Liability rule coincides with the Priority Shapley value, a new allocation rule on an appropriate class of multi-choice games arising from hazardous waste transportation problems. Finally, we also axiomatize the Priority Shapley value on the full domain of multi-choice games.

Keywords: Hazardous waste, transportation network, Liability rule, Priority Shapley value, multi-choice games.
JEL Codes: C71, Q53, R42.

1. Introduction

1.1. Maintaining a hazardous waste transportation network

Worldwide, industrial activities generate an estimated 400 million tons of hazardous waste each year.[1] Much of this waste is transported from the point of generation to facilities dedicated to its treatment and disposal. Any accident that occurs during the transportation of hazardous materials has enormous consequences because of the damage it can cause to human health and the environment. It is therefore necessary to ensure that the transportation network is maintained in a manner that minimizes the risk of accidents. One way to do this is to apply the *polluter pays principle*, one of the key tenets of the European Union's environmental policy. This principle implies that polluters should bear not only the costs of their pollution, but also the costs of measures to prevent it. This could involve charging waste producers based on the amount of waste

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[1] https://www.theworldcounts.com/challenges/use-of-chemicals/hazardous-waste-production.

they generate and using these funds to cover the costs of maintaining and operating the hazardous waste transportation network.

In this article, we consider a set of agents (countries, companies, etc.) involved in a hazardous waste transportation network. This network is modeled by a directed tree with a unique global sink (also called a sink tree). The nodes represent the agents involved in the network and the sink is a treatment facility. Each agent has a certain amount of waste that needs to be transported from its location to the sink. There are costs associated with properly maintaining the network. The amount of waste passing through each portion of the network determines the cost of maintaining this portion. From this, the total cost of maintaining the network can be calculated and then allocated to the parties involved. Therefore, a central regulator must determine the extent to which the parties are liable. In addition to the volume of waste an agent must transport, this task can also naturally depend on several elements: (a) How far is this agent from the treatment facility? (b) What is the level of traffic on the road its waste travels on? (c) Do other agents have a high level of waste to transport? We introduce an allocation rule which takes these characteristics of the problem into account as well as other important facets of a hazardous waste transportation/maintenance network.

Indeed, the management of hazardous waste is particularly regulated, for instance by the *Comprehensive Environmental Response, Compensation, and Liability Act* (CERCLA or Superfund Act). Our work is in line with the rich literature in tort law based on this regulation, which seeks to impose liability on potentially responsible parties for hazardous waste management. According to Erkut et al. (2007), this risk assessment is the most important aspect of hazardous waste management. It is inseparable from the allocation of costs resulting from the management of hazardous waste (transportation, cleanup, environmental damage, product loss, etc.). Although this literature deals primarily with liability and cost trade-offs in the event of an accident, we draw extensively on it in our transportation network maintenance framework. To do so, we assimilate an agent's risk to the network, or equivalently, its liability, to the amount of waste it must transport. In this manner, the mechanisms and principles that environmental regulations use to allocate liability help us determine how to share the total maintenance cost of the network.

One of the key features of CERCLA is that it imposes joint and several liability on those who contaminate the environment, referred to as *potentially responsible parties* (PRPs). CERCLA Section 107(a) recognizes transporters of hazardous waste as one of four classes of PRPs. The imposition of joint and several liability means that one of the PRPs must bear the entire cost of the network, regardless of the liability of other parties. This regime ensures that costs are paid in a timely and efficient manner without shifting the burden to innocent third parties.

CERCLA Section 113(f)(1) provides that the PRP that has borne the cost of the network may bring an action in federal court (the central regulator) against other PRPs. This is a legal process known as CERCLA allocation. The purpose of an allocation is to seek contribution from the other PRPs for a share of the costs based on their degree of responsibility. Contribution is the right to recover from joint parties when, and to the extent that, the party has paid more than its proportionate share. These contribution claims may be based on appropriate equitable factors.

Although it is not defined which equitable factors courts should consider in the allocation process, two sets of criteria are often used: the *Gore factors* and the *Torres factors*. The Gore factors are: (1) the ability of the parties to demonstrate that their contribution to the discharge, release, or disposal of hazardous waste can be distinguished; (2) the amount of hazardous waste involved; (3) the degree of toxicity of the hazardous waste involved; (4) the degree of involvement

of the parties in the production, treatment, transportation, or disposal of the hazardous waste; (5) the degree of care exercised by the parties with respect to the hazardous waste involved, taking into account the characteristics of such hazardous waste; and (6) the degree of cooperation between the parties and the local government to prevent harm to public health or the environment. Meanwhile, the Torres factors are: (1) the extent to which the costs relate to waste for which each party is responsible; (2) the degree of fault of each party; (3) the extent to which the party has benefited from the disposal of the waste; and (4) the ability to pay its share of the costs. In this article, we rely in particular on Gore factors 2, 4 and 6 and on Torres factors 1 and 4 as explained hereafter.

As explained by Hall et al. (1994), cost allocation among responsible parties is exacerbated by the presence of *orphan shares*, which are the shares left by absent or insolvent PRPs that must be allocated among the other parties. These orphan shares are often substantial. This clearly argues for the provision of high transportation network maintenance costs through the implementation of installment payments as an insurance scheme.[2] A novelty of our approach is to construct such a provisioning system using an allocation rule that implements these features. Specifically, the share of the network costs incurred by an agent is divided into as many levels as the number of waste units transported by that agent, and a provisioning principle is established in which the payment associated with a given waste shipment already includes a portion of the costs incurred by the higher waste levels. We call the resulting allocation rule the *Liability rule*. This allocation rule is also based on two other determining principles. First, agents are not held liable for the costs of the portions of the network that their waste do not use. This clearly echoes the aforementioned element (a), the Gore factor (4) and the Torres factor (1). Second, an agent is not held liable for the cost increases resulting from the fact that other agents expose the network to greater risk levels than the risk level of this agent. This principle is related to element (c) and the Gore factors (2) and (4).

Our first main result is an axiomatic characterization of the Liability rule by invoking axioms inspired by principles in tort law. According to Pidot and Ratliff (2018), the cost allocation under CERCLA is characterized by two methods. The first and most common method follows from the maxim that "equality is equity" and provides that the liable parties will end by paying equal shares. The second and increasingly used method relies on the comparative impact of the involved parties. Our axioms can be considered as mixtures between these two methods as we will emphasize in the body of the article. In fact some axioms impose equal payoffs for some agents across two different interrelated waste transportation problems while distinguishing agents according to their importance in each of these problems.

Our second main result proves that the Liability rule coincides with a natural extension of the *Shapley value* (Shapley, 1953) on an appropriate class of multi-choice games constructed from hazardous waste transportation problems. Multi-choice games extend the classical model of cooperative games with transferable utility by allowing the possibility that agents cooperate at various intensities within coalitions. In this class of games, it is possible to distinguish the different levels of cooperation/activity of an agent when assessing its overall performance in a game. As for our

[2]Insurance-based incentives in hazardous waste management under CERCLA are analyzed in Kehne (1986). The author determines the conditions that are needed for effective insurance-based incentives to arise and the circumstances that allow insurance-based incentives to effectively promote deterrence. He then argues that insurance-based incentives have the potential to significantly improve the control of environmental risks, provided significant statutory changes of CERCLA. The main change would be to limit generators' and transporters' liability to damages attributable to wastes that they have generated or transported. This clearly brings to mind the principles underlying the Liability rule that we consider in this article.

Liability rule, an allocation rule for multi-choice games specifies a payoff for each agent and each cooperation level of this agent. We introduce the *Priority Shapley value* for multi-choice games as the average of specific marginal vectors in a spirit similar to the construction of the Shapley value in the case of a classic cooperative game. We only consider the orderings of the pairs of agents/activity levels that are consistent in the sense that an agent can activate its given activity level only if all other agents have activated either all lower activity levels or their maximal activity level. Then, the marginal contribution of the pair agent/activity level is computed and shared evenly among this activity level and the its lower levels. On the full domain of multi-choice games, we provide two axiomatic characterizations of the Priority Shapley value. The axiom set includes adaptations of some axioms invoked in the characterization of the Liability rule as well as axioms specifically designed for multi-choice games based on well known principles in cooperative game theory.

1.2. Related literature

Our approach can be connected to several strands of literature.

The closest article to ours is probably Techer (2023) in which the same cost sharing model on a hazardous waste transportation network is studied but through another allocation rule, called the *Responsability rule*. This allocation rule essentially differs from our Liability rule in that a given level of waste of an agent cannot be impacted by the fact that this agent may expose the network to greater risk levels. We show that the Liability rule and the Responsability rule admit comparable axiomatic characterizations, i.e. characterizations in which the axiom set differs with respect to two axioms only. Moreover, we provide another characterization of the Liability rule by adapting the axiom set used by Techer (2023) to characterize the Responsability rule.

As mentioned in the previous section, we also contribute to the literature on multi-choice games in which a couple of extensions of the classical Shapley value have been proposed. Beyond the two axiomatic characterizations of the Priority Shapley value, we also compare this allocation rule to the two closest allocation rules for multi-choice games provided by Derks and Peters (1993) and Lowing and Techer (2022). In particular, Techer (2023) proves that the Responsability value coincides with the multi-choice Shapley value introduced in Lowing and Techer (2022) on the class of multi-choice games arising from hazardous waste transportation networks. This offers an additional element of comparison with our Liability rule.

Our approach can as well be related to the literature on cleaning polluted rivers in which each agent is characterized by a segment of a river and a cleaning cost for this segment. The polluted river can be linear (Ni and Wang, 2007; Alcade-Unzu et al., 2015) or modelled by a sink tree (Dong et al., 2012; van den Brink et al., 2018). The goal here is to share among the agents the total cleaning cost, possibly by taking into account that the agents located upstream have a responsibility to the agents located downstream. The main difference with our article is that a cleaning cost is limited to one segment (and so is directly attributable to one agent) whereas our hazardous waste travels through the network, which involves at each portion all the several agents using that portion. Furthermore, the literature on cleaning polluted rivers looks for allocations specifying a cost share for each agent while we determine finer allocations that distinguish a cost sub-share for each waste level of each agent. Put differently, only two levels of cooperation are possible in the problems of cleaning a polluted river: full cooperation or the absence of cooperation.

Our work also shares some similarities with the literature on minimum cost spanning trees. Both approaches seek to share the connection cost to a specific node but with two significant differences. At the level of the model, the first differences are that our cost functions can vary with the shipped

quantities and that we do not include any preliminary optimization step during which the optimal (with minimum total cost) spanning tree is determined from an initially complete network. Our sink tree is given. At the level of the allocation rule, we specify the cost share associated with each waste level of each agent whereas the literature on minimum cost spanning trees only determines coarser allocations specifying a unique cost share for each agent. Furthermore, allocations for minimum cost spanning trees can depend on the cost of the edges of the network that are not part of a minimum cost spanning tree. As in our work, the links between minimum cost spanning tree problems and cooperative games are studied. For instance, Kar (2002) characterizes the Shapley value of the cooperative games arising from minimum cost spanning tree problems.

Finally, our approach can be related to the literature on cost allocation in collaborative transportation, which has grown rapidly in recent years, as highlighted in Guajardo and Rönnqvist (2016). This literature often uses the tools of cooperative game theory. For example, Cruijssen et al. (2010) use the Shapley value to allocate cost savings in cooperative transportation problems, while other game theoretic allocation rules are computed in Flisberg et al. (2015). The two articles propose applications on real data (on a big logistic service provider for the grocery industry in the Netherlands and on transportation of forest fuels in Sweden) but do not offer axiomatic studies. Our main contribution to this literature is to combine the risk assessment, which is the primary ingredient that distinguishes hazardous waste transportation problems from other transportation problems, and the cost allocation issue. On the one hand, the aforementioned article do not deal with the transport of hazardous materials. On the other hand, Guo and Luo (2022) provides an overview of recent research on risk assessment in hazardous materials transportation, but do not address the issue of cost allocation.

1.3. Outline of the article

The rest of the article is organized as follows. Section 2 provides some preliminaries. Section 3 introduces and motivates the model. The Liability rule and its axiomatic characterizations are presented in section 4. Section 5 draws a parallel between the Liability rule and the Priority Shapley value for multichoice games. The Priority Shapley value is axiomatically characterized in section 6. Section 7 provides concluding remarks. All proofs are relegated to the Appendix in section 8.

2. Preliminaries

2.1. Notation

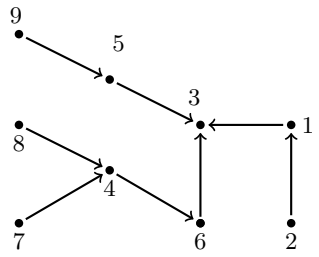
For a finite set A, the symbol $|A|$ denotes its cardinality. For any element $i \in A$, henceforth the singleton $\{i\}$ is denoted by i. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $\overrightarrow{x}$ denotes the vector $(x, \dots, x) \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, $x \wedge y$ stands for $(\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\}) \in \mathbb{R}^n$. For each $i \in \{1, \dots, n\}$, $e^i = (0, \dots, 0, 1, 0, \dots, 0)$ is the vector whose components are all zero, except the one on component i that equals 1.

2.2. Sink trees

A **directed graph** is a pair $g = (N, E)$ where N is a finite set of nodes (representing the set of agents) and $E \subseteq N \times N$ is a binary irreflexive relation. Each element (i, i') of E represents a directed link from i to i'. A **(directed) path** from i to i' in g is an ordered sequence of distinct nodes $(i_1, \dots, i_q)$ such that $i_1 = i$, $i_q = i'$ and, for each $t = 1, \dots, q-1$, $(i_t, i_{t+1}) \in E$. A directed

path $(i_1, \ldots, i_q)$ from i to i' in g induces a **(directed) cycle** in g if $(i', i) \in E$. A directed graph is **acyclic** if it contains no cycle. A directed graph is a **sink tree** if it is acyclic and satisfies the following two conditions: there is exactly one node i_0 such that, for each $i \in N \setminus i_0$, $(i_0, i) \notin E$, and, there is exactly one directed path from i to i_0. The node i_0 is the sink of the tree. Thereafter, only sink trees are considered. Given a sink tree $g = (N, E)$ and $i \in N$, denote by $P_g(i)$ the set of agents i' such that $(i', i) \in E$. Each such a node i' is a (direct) **predecessor** of i in g. If $P_g(i)$ is empty, i is a **leaf** of g. Reciprocally, each node $i' \in N$ such that $(i, i') \in E$ is a direct **successor** of i in g. By definition of a sink tree, each $i \in N \setminus i_0$ has a unique successor in g and i_0 has no successor in g. Next, denote by $D_g(i)$ the set of nodes different from i and belonging the unique path from i to i_0. This corresponds to the set of nodes located **downstream** of i along the path towards the seek i_0. By definition of a sink tree, for each $i \in N \setminus i_0$, $i_0 \in D_g(i)$, and $D_g(i_0) = \emptyset$. Finally, let $U_g(i)$ be the set of nodes i', different from i, such that i belongs to the unique path from i' to i_0. This set corresponds to the set of nodes located **upstream** of i in g. Of course, for each node $i \in N$, $P_g(i) \subseteq U_g(i)$ and $U_g(i_0) = N \setminus i_0$. For any non-empty set of nodes $S \subseteq N$, we use the following notation: $P_g[S] = \cup_{i \in S} P_g(i) \cup i$, $D_g[S] = \cup_{i \in S} D_g(i) \cup i$, and $U_g[S] = \cup_{i \in S} U_g(i) \cup i$.

Example 1. Consider the agent set $N = \{1, \ldots, 9\}$ and the sink tree $g = (N, E)$ represented by the figure below. Node 3 is the sink of the tree. The set of leaves is $\{2, 7, 8, 9\}$. For instance, the set of agents located downstream of 4 is $D_g(4) = \{6, 3\}$, and the set of agents located upstream of agent 6 is $U_g(6) = \{4, 8, 7\}$.



□

3. The model

Consider a **transportation network** connecting a finite set of agents N (directly or indirectly) of size $n \in \mathbb{N}$ to a special node d called the delivery node. Such transportation network structure is modeled by a sink tree $g = (N \cup d, E)$, where d is the sink of the tree. Each element $(i, i') \in E$ is called a portion of the network from i to i'. Since g is a sink tree, for each agent $i \in N$, there is exactly one portion starting from i to some other i'. When no confusion arises, we simply denote such a portion (i, i') by i. Each agent $i \in N$ is endowed with a given amount of hazardous waste $w_i \in \mathbb{N}$ to be treated by a treatment facility. The treatment facility is located at the delivery node d. Each agent is able to ship any amount $j \in \{0, \ldots, w_i\}$ of waste. Element $s = (s_i)_{i \in N} \in \prod_{i \in N} \{0, \ldots, w_i\}$, is referred to a **waste profile** that indicates the amount shipped by each agent. The profile of maximal amount of waste is $w = (w_i)_{i \in N}$ and $\max(w)$ stands for the greatest value in w. Given N, w, and any amount of waste $j \leq \max(w)$, $Q^w(j) \subseteq N$ denotes the subset of agents able to ship a quantity j of waste, i.e., $Q^w(j) = \{i \in N : j \leq w_i\}$. For any two amounts of waste j, j' such that $j \geq j'$, one obviously has $Q^w(j) \subseteq Q^w(j')$. Furthermore, for any

waste amount $j \leq \max(w)$, define the j-**waste profile** $w \wedge \overrightarrow{j}$ in which each agent i either ships the waste amount j if $i \in Q^w(j)$, or w_i otherwise.

Shipping any amount of waste through a portion carries risks and so generates a cost. This cost is considered as the cost of maintaining and operating this portion. For each portion $i \in N$, let $C_i : \mathbb{N} \to \mathbb{R}_+$ be the cost function of this portion. For each $i \in N$ the cost function C_i is non-negative and non-decreasing over $\mathbb{N}$ and $C_i(0) = 0$. The null cost function C^0 is such that $C^0(x) = 0$ whatever $x \in \mathbb{N}_+$. The interpretation is as follows. For each portion $i \in N$, the cost function C_i only depends on the total amount of waste passing through this portion, i.e., it only depends on the sum of the waste shipped by all agents located upstream i including agent i's own waste. Therefore, the maintenance cost resulting from the transport of waste profile s through the portion i is given by

$$C_i\Big(\sum_{i' \in U_g[i]} s_{i'}\Big).$$

Hence, the **total cost of maintaining and operating the network** is then given by

$$\sum_{i \in N} C_i\Big(\sum_{i' \in U_g[i]} w_{i'}\Big). \tag{1}$$

Denote by $C = (C_i)_{i \in N}$ a **cost profile**. For each $i' \in N$, define the cost profile $\lambda^{C,i'}$ such that $\lambda^{C,i'}_{i'} = C_{i'}$ and $\lambda^{C,i'}_{i} = C^0$ for each other $i \in N \setminus i'$. Observe that the cost profile C can be decomposed as:

$$C = \sum_{i' \in N} \lambda^{C,i'}. \tag{2}$$

For the rest of the article, it is useful to consider the following decomposition of $\lambda^{C,i'}$. For each $k \in \{1, \ldots, \max(w)\}$, define the cost function $\lambda^{C,i',k}$ as:

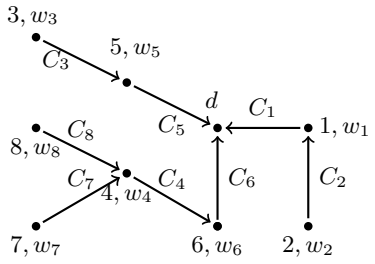
$$\forall s \in \prod_{i \in N} \{0, \ldots, w_i\}, \quad \lambda^{C,i',k}(s) = \lambda^{C,i'}\Big(\sum_{i \in U_g[i']} s_i \wedge k\Big) - \lambda^{C,i'}\Big(\sum_{i \in U_g[i']} s_i \wedge (k-1)\Big).$$

It follows that

$$\lambda^{C,i'} = \sum_{k=1}^{\max(w)} \lambda^{C,i',k} \text{ and so } C = \sum_{i' \in N} \sum_{k=1}^{\max(w)} \lambda^{C,i',k}. \tag{3}$$

A **hazardous waste transportation problem** P on a fixed agent set N and delivery node d is a triplet (g, w, C) where g is a sink tree over $N \cup d$, w is a profile of maximal amount of waste and C is a cost profile. Let $\mathfrak{P}$ be the set of all hazardous waste transportation problems on $N \cup d$.

Example 2. The following figure represents a hazardous waste transportation problem (g, w, C).



□

The question that arises is how to allocate the total maintenance and operation cost of the hazardous waste transportation network. To that end, one introduces a solution concept for the domain $\mathfrak{P}$. An **allocation rule** is a function on $\mathfrak{P}$ that assigns a payoff (or a cost share) $f_{(i,j)}(g,w,C) \in \mathbb{R}$ to each waste amount $0 < j \leq w_i$ of each agent $i \in N$. The accumulated payoff $\sum_{j\in\{1,\dots,w_i\}} f_{(i,j)}(g,w,C)$ is what agent i pays for the maintenance of the network. As explained in the introduction, this total payoff is split in w_i sub-payoffs, one for each waste unit transported by agent i, in order to set up an installment payment system. The objective is to ensure that the network can be maintained, possibly by (partially) anticipating the costs that high waste levels will generate. Alternatively, one could imagine that the transport of the hazardous waste is carried out in several shipments (w_i representing the number of shipments associated with agent i) and that the maintenance of the network requires a payment $f_{(i,j)}(g,w,C)$ from agent i for each shipment j.

4. Axiomatic study and the Liability rule

This section conducts an axiomatic study leading to a characterization of a new rule called the Liability rule. The principles behind this set of axioms are inspired by the liability regimes for the transportation of hazardous waste. Finally, the Liability rule is discussed in relation to the Responsibility rule, recently introduced by Techer (2023). From this discussion, new comparable characterizations of the Responsibility rule and the Liability rule are provided.

4.1. Axiomatic characterization of the Liability rule

The classical axiom of Efficiency provides a reference point specifying the total maintenance cost to be allocated.

Axiom 1 (**Efficiency).** For each $(g,w,C) \in \mathfrak{P}$, it holds that

$$\sum_{i\in N}\sum_{j=1}^{w_i} f_{(i,j)}(g,w,C) = \sum_{i\in N} C_i\Big(\sum_{i'\in U_g[i]} w_{i'}\Big).$$

The axiom simply recalls that all the maintenance costs of the network must be covered and provisioned by the participating agents. Since the agents cooperate to take charge of the network total maintenance cost decided by the central regulator, the axiom of Efficiency is an explicit transposition of the Gore factor 6.

The second axiom proposes a first way to limit the responsibility of an agent. It is based on the risk that an agent puts on the network, measured by the quantity of waste it must transport.

Axiom 2 (**Independence of other higher waste amounts).** For each $(g,w,C) \in \mathfrak{P}$, and each pair of agents $\{i,i'\} \subseteq N$ such that $w_i < w_{i'}$, it holds that

$$\forall j \in \{1,\dots,w_i\}, \quad f_{(i,j)}(g,w,C) = f_{(i,j)}(g,w-e^{i'},C).$$

This axiom imposes that the liability of that agent can be affected by comparable risks but should not be affected if other agents expose the network to greater risk levels. In this sense, this is a limited liability principle. In describing the consequences of a change for an agent in the amount of waste shipped by another agent, this axiom reflects the Gore factors 2 and 4. The axiom of Independence of other higher waste amounts is similar but weaker than the axiom of Independence of higher waste amounts in Techer (2023), which requires that the cost paid by an agent for a given amount of waste does not depend on any greater amount of waste (including the possible agent's own greater waste levels). This last axiom is questionable if it is considered necessary to provision the costs of the transportation network because it does not allow to take into account the cost of the remaining higher waste levels of an agent in its installment payments.

Remark 1. Consider any $(g, w, C) \in \mathfrak{P}$, any $k \leq \max(w)$ and any $i \in N$ such that $k \geq w_i$. A repeated application of Independence of other higher waste amounts yields the following equalities:

$$\forall j \in \{1, \dots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(i,j)}(g, w \wedge \overrightarrow{k}, C).$$

The third axiom offers a second way to limit the responsibility of an agent, this time based on the location of this agent.

Axiom 3 (Path consistency). For each $i \in N$ and each pair $\{(g, w, C), (g, w, C')\} \subseteq \mathfrak{P}$ such that $C'_\ell = C_\ell$, for each $\ell \in D_g[i]$, it holds that

$$\forall j \in \{1, \dots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(i,j)}(g, w, C').$$

The axiom of Path consistency is borrowed from Techer (2023). It states that agents are jointly and severally liable for the portions that their waste use but not beyond that. An agent is not liable for the other portions and hence it should not pay for the risk created on the portions that its waste does not use. This is consistent with Gore factor 4 and Torres factor 1. Our axiom is also clearly related to the polluter pays principle: if the transport of waste is likely to cause pollution, then the potential polluters should pay the cost. The axiom of Path consistency is similar in spirit to the Independence of upstream costs invoked in Ni and Wang (2007) and Alcade-Unzu et al. (2015) in the context of polluted river problems, which imposes that the total cost paid by a region be independent of the waste located upstream from this region. The principle underlying Path consistency is also close to other axioms in the literature on cost sharing problems in networks as underlined by Techer (2023).

The next axiom says something on the nature of the installment payments of the agents with the highest risk. More specifically, it requires the principle of equality, according to which a decrease in this risk should affect equally all remaining installment payments.

Axiom 4 (Intra balanced contributions for highest contributors). For each $(g, w, C) \in \mathfrak{P}$, each $i \in N$ such that $w_i = \max(w)$, and each pair of waste amounts $\{j, j'\} \subseteq \{1, \dots, w_i\}$ for i, it holds that

$$f_{(i,j)}(g, w, C) - f_{(i,j)}(g, w - e^i, C) = f_{(i,j')}(g, w, C) - f_{(i,j')}(g, w - e^i, C),$$

with the convention that, for each $i \in N$, $f_{(i,w_i)}(g, w - e^i, C) = 0$.

This kind of principle of balanced contributions has been extensively used in axiomatizations since Myerson (1977), and in particular in multi-choice games (see Klijn et al., 1999, among others). However, Myerson (1977) uses this principle to compare the payoffs of two different agents, not to compare the payoffs of a given agent's levels of waste, as in our axiom.

Remark 2. Intra balanced contributions for highest contributors can be rewritten as follows: for each $(g, w, C) \in \mathfrak{P}$, each $i \in N$ such that $w_i = \max(w)$, it holds that

$$\forall j \in \{1, \dots, w_i - 1\}, \quad f_{(i,j)}(g, w, C) = f_{(i,w_i)}(g, w, C) + f_{(i,j)}(g, w - e^i, C).$$

The reformulation of Intra balanced contributions for highest contributors in Remark 2 makes it possible to understand that it requires the same principle as the Equal loss property invoked in Klijn et al. (1999), but only for agents with the maximum waste level/risk in the network. Even if this axiom focuses on a unique given agent, it reflects Gore factors 2 and 4.

The next axiom relies on the classical principle of anonymity stating that the payoffs distributed to the agents do not depend on their names. Given any hazardous waste transportation problem $(g, w, C) \in \mathfrak{P}$ and any permutation $\pi : N \to N$, construct the corresponding hazardous waste transportation problem $(g^\pi, w^\pi, C^\pi) \in \mathfrak{P}$ where the labels of the nodes are swapped according to π, *ceteris paribus*. Formally,

- $(i, i') \in E \iff (\pi(i), \pi(i')) \in E^\pi$;
- for each $i \in N$, $w^\pi_{\pi(i)} = w_i$;
- $C^\pi_{\pi(i)} = C_i$.

Axiom 5 (Anonymity). For each $(g, w, C) \in \mathfrak{P}$ and any permutation $\pi : N \to N$, it holds that

$$\forall i \in N, \forall j \in \{1, \dots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(\pi(i),j)}(g^\pi, w^\pi, C^\pi).$$

The axiom of Anonymity clearly implements an equality-is-equity principle falling into the first method of Pidot and Ratliff (2018) as mentioned in the introduction. It implies that two agents which are equal in all respects (same waste levels, identical geographical positions in the network, same costs characteristics) must pay the same share of the network maintenance costs. This axiom can be considered as a specific reading of Gore factors 1, 2 and 4: if two agents have the same degree of involvement in two problems, then they should not be able to demonstrate that their contribution can be distinguished or, equivalently, that they are equally responsible for the risks to the network, and hence should be treated equally in these two problems.

The following two axioms compare two hazardous waste transportation problems that are risk/cost-equivalent. They involve transportation problems with a null cost portion. Precisely, assume there are two agents i and i' such that $(i, i') \in E$ and $C_i = C^0$. The first axiom states that the payoffs of these agents are unchanged if they exchange their location. The second axiom states that the payoffs of each predecessor of i are unchanged if the portion between that predecessor and i is removed and replaced by a portion between that predecessor and i'. These two axioms fall under the Gore factor 2 since they rely on the quantity of waste that each agent must transport. They also fall under Gore factor 4 and Torres factor 1 since they evaluate the consequence in terms

of liability of changes in the position of some agents in the network. To state formally these axioms, some definitions are in order.

Given any hazardous waste transportation problem $(g, w, C) \in \mathfrak{P}$, first construct the problem $(g^{i\leftrightarrow i'}, w^{i\leftrightarrow i'}, C^{i\leftrightarrow i'}) \in \mathfrak{P}$ in which i and i' exchange their location without affecting their maximal amount of waste and the (null) cost of the portion linking them. Formally, let $\pi^{i\leftrightarrow i'}$ be the permutation on N that represents the operation where i exchanges its location with i': $\pi^{i\leftrightarrow i'}(i) = i'$, $\pi^{i\leftrightarrow i'}(i') = i$ and $\pi^{i\leftrightarrow i'}(\ell) = \ell$ for each $\ell \in N \setminus \{i, i'\}$. From this operation, $(g^{i\leftrightarrow i'}, w^{i\leftrightarrow i'}, C^{i\leftrightarrow i'}) \in \mathfrak{P}$ is as follows:

- $\forall \{\ell, \ell'\} \subseteq N$, $(\ell, \ell') \in E \iff (\pi^{i\leftrightarrow i'}(\ell), \pi^{i\leftrightarrow i'}(\ell')) \in E^{i\leftrightarrow i'}$;
- $w^{i\leftrightarrow i'} = w$;
- $\forall \ell \in N$, $C_\ell^{i\leftrightarrow i'} = C_{\pi^{i\leftrightarrow i'}(\ell)}$.

Next, given any $(g, w, C) \in \mathfrak{P}$ and any predecessor $p \in P_g(i)$ of i, construct the associated hazardous waste transportation problem $(g^{(p,i')}, w^{(p,i')}, C^{(p,i')}) \in \mathfrak{P}$ such that

- $E^{(p,i')} = \big(E \setminus \{(p, i)\}\big) \cup \{(p, i')\}$;
- $w^{(p,i')} = w$;
- $\forall \ell \in N$, $C_\ell^{(p,i')} = C_\ell$.

Axiom 6 (Invariance to a relocation on a null cost portion). For each $(g, w, C) \in \mathfrak{P}$ where there are two agents i and i' such that $(i, i') \in E$ and $C_i = C^0$, it holds that

$$\forall \ell \in \{i, i'\}, \forall j \in \{1, \ldots, w_\ell\}, \quad f_{(\ell,j)}(g, w, C) = f_{(\ell,j)}(g^{i\leftrightarrow i'}, w^{i\leftrightarrow i'}, C^{i\leftrightarrow i'}).$$

This axiom can also be interpreted in the light of the polluter-pays principle. If transport on a portion of the network is particularly safe and indeed costless, then an agent should not pay more if it takes that portion (no extra risk, no extra cost). Our axiom translates this principle into the requirement that swapping the positions of the endpoints of this riskless/costless portion should make no difference regarding the cost share accruing to any of their waste levels. In other words, since the two compared transportation problems are risk/cost-equivalent, the swapped agents should be invariant to which of the two networks they are part of.

Axiom 7 (Invariance to a cut-and-connect operation on a null cost portion). For each $(g, w, C) \in \mathfrak{P}$ where there are two agents i and i' such that $(i, i') \in E$ and $C_i = C^0$, it holds that

$$\forall p \in P_g(i), \forall j \in \{1, \ldots, w_p\}, \quad f_{(p,j)}(g, w, C) = f_{(p,j)}(g^{(p,i')}, w^{(p,i')}, C^{(p,i')}).$$

This axiom implements another cost-preserving change to the network. Roughly speaking, it requires an agent to be indifferent if it must use an alternative route with identical risk characteristics. In other words replacing a portion of the network by an identical (from the risk point of view) portion plus a riskless portion should not change this agent's risk assessment, and in turn its share of the network maintenance costs.

Below, we propose an allocation rule for the domain $\mathfrak{P}$ of hazardous waste transportation problems. Consider any $(g, w, C) \in \mathfrak{P}$ and any portion $i' \in N$ of g. Decompose the total cost $C_{i'}(\sum_{\ell \in U_g[i']} w_\ell)$ of maintaining and operating the portion i' into a sum of cost increases for each additional unit of waste passing through i':

$$\sum_{k=1}^{\max(w)} C_{i'}\Big(\sum_{\ell \in U_g[i']} w_\ell \wedge k\Big) - C_{i'}\Big(\sum_{\ell \in U_g[i']} w_\ell \wedge (k-1)\Big).$$

The cost increase at level k,

$$C_{i'}\Big(\sum_{\ell \in U_g[i']} w_\ell \wedge k\Big) - C_{i'}\Big(\sum_{\ell \in U_g[i']} w_\ell \wedge (k-1)\Big),$$

represents the increase of maintaining and operating cost of the portion i' when each agent ℓ located upstream of i', including i', decides to ship an additional unit of waste from $k-1$ to k, provided this additional unit is feasible for agent ℓ, that is, $\ell \in Q^w(k)$; Otherwise, agent ℓ ships its maximal amount w_ℓ.

This cost increase at level k is shared according to the following two-step procedure:

- in the first step, the cost increase is equally shared among the agents that use this portion i' and are able to ship the waste amount k. More formally, this cost increase at level k is equally shared among the agents in $U_g[i'] \cap Q^w(k)$;
- in the second step, each agent in $U_g[i'] \cap Q^w(k)$ charges an equal share of this cost to each waste amount k, $k-1, \ldots, 1$.

From this two-step procedure, it results that each agent able to ship k units of waste but currently shipping its j^{th} unit of waste, where $j \leq k$, contributes

$$\frac{1}{k} \cdot \frac{C_{i'}\big(\sum_{\ell \in U_g[i']} k \wedge w_\ell\big) - C_{i'}\big(\sum_{\ell \in U_g[i']} (k-1) \wedge w_\ell\big)}{|U_g[i'] \cap Q^w(k)|}$$

to the cost increase at each level k.

This two-step procedure is repeated for each portion i' that an agent uses, that is, for each $i' \in D_g[i]$. The overall cost charged to an agent i shipping its j^{th} unit of waste is the sum of the above contributions. Formally, the **Liability rule** is the allocation rule f^L on $\mathfrak{P}$ defined as:

$$f^L_{(i,j)}(g, w, C) = \sum_{i' \in D_g[i]} \sum_{\substack{k \geq j: \\ i \in Q^w(k)}} \frac{1}{k} \cdot \frac{C_{i'}\big(\sum_{\ell \in U_g[i']} k \wedge w_\ell\big) - C_{i'}\big(\sum_{\ell \in U_g[i']} (k-1) \wedge w_\ell\big)}{|U_g[i'] \cap Q^w(k)|}. \tag{4}$$

The rationale for this cost allocation procedure is as follows. First, agents are not held liable for the maintenance costs of the portions they do not use. Second, an agent is not held liable for the cost increases incurred by additional waste amounts higher than its maximal amount w_i as it is clearly imposed by Axiom 2. These first two principles describe the extent to which an agent's liability is limited and perfectly reflect the Gore factor 4 and the Torres factor 1. Third, the maintenance of the network takes into account the fact that it is waste profile w that will be transported. The

central regulator dimensions the maintenance costs for this profile. In order to ensure that the agents will be able to assume these maintenance costs, the central regulator progressively collects payments from agents. More precisely, the regulator requires from each agent a partial payment for each of its units of waste, say unit j, that takes into account the forthcoming maintenance costs resulting from the shipment of units $k > j$. This insurance-based system therefore leads the agents to pay multiple installments to the regulator, which is a way of improving the CERCLA regulation according to Kehne (1986) as emphasized in the introduction. Since this system can also be seen as an attempt to guarantee the ability of agents to pay they share of the maintenance costs, the Liability rule reflects the Torres factor 4. Lastly, the main and obvious objective of an allocation rule such as the Liability rule is to calibrate the contribution of each agent to the problem of maintaining the hazardous waste transportation network. As such, the Liability rule provides a relevant answer to the Gore factor 1.

The main result of this section states that that the above set of seven axioms characterize the Liability rule.

Theorem 1. *The Liability rule f^L defined as in (4) is the only allocation rule on $\mathfrak{P}$ that satisfies Efficiency, Independence of other higher waste amounts, Path consistency, Intra balanced contributions for highest contributors, Anonymity, Invariance to a relocation on a null cost portion, and Invariance to a cut-and-connect operation on a null cost portion.*

In the very special case where each agent ships one unit of waste and the sink tree is a directed path, the Liability rule f^L coincides with the *Baker-Thompson rule*, also called the *Sequential equal contributions rule*. This rule is usually applied to cost sharing problems that arise from situations in which some service is provided to a variety of different customers which differ in the amount or type of service they need. This rule arises from a report by the Baker consultancy in 1965 and Thompson's PhD thesis (Baker, 1965; Thompson, 1971) to provide a way to allocate the costs associated with an airport's runway among all types of aircraft that make (differential) use of its common infrastructure (see Thompson, 2020). Subsequently, Littlechild and Thompson (1977) show that the Baker-Thompson rule coincides with the Shapley value of a cost TU-game. Several axiomatic characterizations of the Baker-Thompson rule have been provided (see, e.g., Fragnelli and Marina, 2010; Chun et al., 2012); and the airport problem has been studied from different perspectives (see, e.g., Vázquez Brage et al., 1997; Alparslan Gök et al., 2009; Alparslan Gök, 2012; Dong et al., 2012; Hou et al., 2018). According to Thompson (2020), over 60 papers and articles deploy the airport cost game framework in some form or another to analyze various theoretical issues.

Thus, assume that, for each $i \in N$, $m_i = 1$ and the sink tree g is a directed path. Without loss of generality, let g be defined as $\{(1,2),(2,3),\ldots,(n-1,n),(n,d)\}$. Let $\mathfrak{P}^* \subseteq \mathfrak{P}$ be this subclass of hazardous transportation network problems. Then, the Liability rule is rewritten as:

$$\forall i \in N, \quad f^L_{(i,1)}(g,w,C) = \frac{C_n(n)}{n} + \frac{C_{n-1}(n-1)}{n-1} + \ldots + \frac{C(i)}{i},$$

which is the Baker-Thompson rule.[3] In view of Theorem 1, one obtains a new characterization of the Baker-Thompson rule on $\mathfrak{P}^*$.

[3] In fact, any $(g,w,C) \in \mathfrak{P}^*$ corresponds to the airport problem with costs $c_i = \sum_{k=i}^{n} C_k(k)$, for each agent $i \in N$. A formulation of the Baker-Thompson rule when $c_1 > \ldots > c_n$ can be found in Fragnelli and Marina (2010).

Theorem 2. *The Liability rule f^L (or the Baker-Thompson rule) is the only allocation rule on $\mathfrak{P}^*$ that satisfies Efficiency, Path consistency, Anonymity, and Invariance to a relocation on a null cost portion.*

If one considers the larger subclass $\mathfrak{P}^{**} \supseteq \mathfrak{P}^*$ of hazardous waste transportation network problems where for each $i \in N, m_i = 1$, and g is a sink tree, then the Liability rule is characterized by the above four axioms plus Invariance to a cut-and-connect operation on a null cost portion. This comparison shows that the added axiom is used to deal with situations where the network is tree-like, and the axioms of Independence of other higher waste amounts and Intra balanced contributions for highest contributors deal with situations where agents ship more than one unit of waste.

4.2. Comparison with the Responsibility rule

We now present the Responsibility rule f^R (Techer, 2023). This rule allocates the cost increase induced by the jth waste unit equally among the agents located upstream of this portion and shipping $w_i \geq j$ waste units. Formally,

$$\forall i \in N, \forall j \leq w_i, \quad f^R_{(i,j)}(g,w,C) = \sum_{i' \in D_g[i]} \frac{C_{i'}\big(\sum_{\ell \in U_g[i']} j \wedge w_\ell\big) - C_{i'}\big(\sum_{\ell \in U_g[i']} (j-1) \wedge w_\ell\big)}{|U_g[i'] \cap Q^w(j)|}. \quad (5)$$

The main difference with the Liability rule is that when an agent carries out its j^{th} unit of waste, it is not held liable for the extra cost incurred by the each additional unit to come. This means that cost charged for the j^{th} unit shipped does not takes into account the extra costs incurred by the future units of waste $k > j$, $Q^m(k) \ni i$. This has three consequences:

- the allocation rule f^R violates Intra balanced contributions for highest contributors. Nevertheless, it is easy to check that it satisfies all the other axioms of Theorem 1;
- The total cost charged for the jth unit of waste is equal to the total cost increase over the network at j:

$$\sum_{i \in Q^w(j)} f^R_{(i,j)}(g,w,C) = \sum_{i' \in N} \Big(C_{i'}\big(\sum_{\ell \in U_g[i']} j \wedge w_\ell \big) - C_{i'}\big(\sum_{\ell \in U_g[i']} (j-1) \wedge w_\ell \big) \Big). \quad (6)$$

  This principle is violated by the Liability rule f^L;
- the allocation rule f^R satisfies a principle of independence stronger than Independence of other higher waste amounts. This principe requires that an agent's payoff for a given waste amount j does not depend on any amount higher than j, including its own units from $j+1$ to w_i. In particular, no agent shipping j units of waste is held liable for the subsequent amounts it will carry out through the network. This principle can be expressed in the following axiom.

Axiom 8 (Independence of higher waste amounts). For each $(g,w,C) \in \mathfrak{P}$, it holds that

$$\forall i \in N, \forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g,w,C) = f_{(i,j)}(g, w \wedge \overrightarrow{j}, C).$$

It can be shown that the combination of Independence of higher waste amounts and Efficiency implies (6) (see Techer, 2023). Removing Intra balanced contributions for highest contributors and Independence of other higher waste amounts from Theorem 1 and adding Independence of higher waste amounts, one gets an axiomatic characterization of the Responsibility rule. This result is stated without proof.

Theorem 3. *The Responsibility rule f^R defined as in (5) is the only allocation rule on $\mathfrak{P}$ that satisfies Efficiency, Independence of higher waste amounts, Path consistency, Anonymity, Invariance to a relocation on a null cost portion, and Invariance to a cut-and-connect operation on a null cost portion.*

To characterize the Responsibility rule, Techer (2023) follows another route. He introduces a principle of solidarity among the agents located upstream of a portion of the network. This principle relies on the fact that each such agent is held liable for the cost incurred on this portion of the network. However, the extent to which an agent should be held liable remains to be determined. Upstream solidarity for a cost increase then requires that agents located upstream of a portion be equally impacted by a cost increase on this portion. This principle is not satisfied by the Liability rule unless agents located upstream of this portion ships the same amount of waste. This results in a strong version of Upstream solidarity for a cost increase.

Axiom 9 (**Strong upstream solidarity for a cost increase).** For each $(g, w, C) \in \mathfrak{P}$, each $i \in N$, and each pair of cost function C_i, C_i' such that

$$\forall s \in \prod_{i' \in N} \{0, \dots, w_i\}, \quad C_i(\sum_{i' \in N} s_{i'} \vee 0) - C_i(\sum_{i' \in N} (s_{i'} - 1) \vee 0) \geq C_i'(\sum_{i' \in N} s_{i'} \vee 0) - C_i'(\sum_{i' \in N} s_{i'} \vee 0),$$

it holds that, for each pair of agents $\{\ell, \ell'\} \subseteq U_g[i]$ such that $w_\ell = w_{\ell'}$,

$$\forall j \in \{1, \dots, w_\ell\}, \quad f_{(\ell,j)}(g, w, C) - f_{(\ell,j)}(g, w, C') = f_{(\ell',j)}(g, w, C) - f_{(\ell',j)}(g, w, C'),$$

where C' denotes the cost profile which only differs from C only by the fact that C_i' replaces C_i.

Techer (2023) shows that the combination of Efficiency, Independence of higher waste amounts, Path consistency and Upstream solidarity for a cost increase yields the Responsibility rule f^R. To obtain the Liability rule f^L from these principles, just weaken Independence of higher waste amounts, add Intra balanced contributions for highest contributors and strengthen Upstream solidarity for a cost increase. This result is stated without proof.

Theorem 4. *The Liability rule f^L defined as in (1) is the only allocation rule on $\mathfrak{P}$ that satisfies Efficiency, Independence of other higher waste amounts, Path consistency, Intra balanced contributions for highest contributors, and Strong upstream solidarity for a cost increase.*

The above discussion compared f^R and f^L and provided comparable axiomatics of these two allocation rules.

5. The Priority Shapley value and the Liability rule

This section introduces cooperative multi-choice games and a new Shapley-like value for this class of games, called the Priority Shapley value. Then, it is shown that the Liability rule corresponds to the Priority Shapley value of a multi-choice game constructed from a hazardous waste

transportation problem. This multi-choice game is a cost game that associates to each coalition the total cost of the network induced by this coalition. Finally, we provide two axiomatic characterizations of the Priority Shapley value. We also compare the Priority Shapley value with two other Shapley-like values for multi-choice games: the Derks-Peters value (Derks and Peters, 1993) and the Shapley value recently introduced by Lowing and Techer (2022).

5.1. Multi-choice cooperative games

Let $N = \{1, \dots, n\}$ be the fixed and finite **agent set**. Each agent $i \in N$ has a finite set of pairwise distinct activity levels $M_i := \{0, \dots, m_i\}$, $m_i \in \mathbb{N}$. These levels are linearly ordered from the lowest activity level 0 (i does not participate or cooperate) to the maximal activity level m_i. A **(multi-choice) coalition** is an element $s = (s_1, \dots, s_n) \in \mathcal{M}$, where $\mathcal{M}$ stands for the cartesian product $\prod_{i \in N} M_i$. In coalition $s = (s_1, \dots, s_n)$, each agent $i \in N$ implements its activity level $s_i \in M_i$. The coalition $m = (m_1, \dots, m_n) \in \mathcal{M}$ corresponds to the **grand coalition** in which each agent cooperates at its maximal activity level. The **null coalition** is the coalition $\overrightarrow{0}$ where no agent participates. The set $\mathcal{M}$ endowed with the usual binary relation $\leq$ on $\mathbb{R}^n$ induces a (complete) lattice with greatest element m and least element $\overrightarrow{0}$. For any two coalitions s and t of $\mathcal{M}$, $s \vee t$ and $s \wedge t$ denote their least upper bound and their greatest lower bound over $\mathcal{M}$, respectively. For $i \in N$, let $M_i^+ = M_i \setminus 0$. Further, let M^+ be the set of **agent-activity level pairs** (i, j) where $i \in N$ and $j \in M_i^+$. For any coalition $s \in \mathcal{M}$, the sets

$$\max(s) = \max\big\{s_i : i \in N\big\} \quad \text{and} \quad T(s) = \big\{i \in N : s_i = \max(s)\big\}$$

stand for the highest activity level agent in s and the set of agents which implement the highest activity levels in s, respectively. Each agent in $T(m)$ is called a **top agent**. The set of agents playing at least the activity level j in s is denoted $Q^s(j)$ and formally defined as

$$Q^s(j) = \big\{i \in N : s_i \geq j\big\}.$$

Thus, if $i \in Q^s(j)$, one necessarily have $m_i \geq j$. The cardinality of the set $Q^s(j)$ is denoted by q_j^s.

Remark 3. For any coalition $s \in \mathcal{M}$, $Q^s(\max(s)) = T(s)$; the set $Q^m(j)$ represents the set of agents such that their maximal activity level m_i is greater than or equal to j; and $T(m)$ is the subset of agents with the highest activity level in the game.

In a situation where agents have a linearly ordered set of activity levels at their disposal, it is natural to consider that if agent i deploys level s_i in a coalition s, then that level has been implemented gradually from level 1 to level s_i. With this interpretation in mind, the set of relevant agent-activity level pairs in coalition s is

$$B(s) = \big\{(i, j) \in M^+ : j \leq s_i\big\}.$$

For each coalition $s \in \mathcal{M}$ and each $i \in N$ such that $s_i \geq 1$, $s - e^i$ denotes the coalition $s' \in \mathcal{M}$ such that $s'_i = s_i - 1$ and $s'_k = s_k$ for $k \in N \setminus i$.

A **multi-choice game** on N is a pair (m, v) formed by the grand coalition m and a coalition function $v : \mathcal{M} \longrightarrow \mathbb{R}$ satisfying $v(\overrightarrow{0}) = 0$, with the interpretation that $v(s)$ represents the worth that the set of agents deploying a non-zero activity level in coalition s can generate by cooperating. Given the grand coalition m, the **null game** is the game $(m, \mathbf{0})$ such that $\mathbf{0}(s) = 0$ for each $s \in \mathcal{M}$.

Denote by $\mathfrak{G}$ the class of multi-choice games on N. Notice that TU-games can be viewed as a subclass of multi-choice games satisfying $m = \overrightarrow{1}$.

Let $t \in \mathcal{M}$, $t \neq \overrightarrow{0}$ be a multi-choice coalition. An analogue of the unanimity games in the multi-choice setting are the **minimal activity level games** $(m, u_t) \in \mathfrak{G}$ defined as:

$$\forall s \in \mathcal{M}, \quad u_t(s) = \begin{cases} 1 & \text{if } s \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

For each multi-choice game $(m, v) \in \mathfrak{G}$, it is known that the characteristic function v admits a unique linear decomposition in terms of minimal activity level games as follows:

$$v = \sum_{t \leq m} \Delta_v(t) u_t, \quad \text{where} \quad \Delta_v(t) = v(t) - \sum_{s \leq t, s \neq t} \Delta_v(s). \tag{7}$$

The quantity $\Delta_v(t)$ is the Harsanyi dividend of coalition t. Given a multi-choice game $(m, v) \in \mathfrak{G}$, a **payoff vector** $x \in \mathbb{R}^{M^+}$ assigns a payoff $x_{(i,j)} \in \mathbb{R}$ to each agent-activity level pair $(i, j) \in M^+$. A **value** on $\mathfrak{G}$ is a map that assigns a unique payoff vector to each $(m, v) \in \mathfrak{G}$.

5.2. The Priority Shapley value

To construct the Priority Shapley value, one considers a coalition formation process where the activity levels of the agents are deployed gradually and consistently with the linear order on the activity levels. This idea can be formalized as follows. Let $H^m = \{1, 2, \ldots, |M^+|\}$. An **ordering on the set of agent-activity level pairs** is a bijection $\sigma : M^+ \to H^m$ such that the rank of any pair of agent-activity level $(i, j) \in M^+$ is lower than the rank of each pair $(\ell, k) \in M^+$ such that $k > j$. In words, no agent is allowed to implement an activity level $k > j$ before all agents have already implemented either j or their maximal activity level if j is not accessible to them, that is, they do not belong to $Q^m(j)$. Formally, such an ordering σ is such that

$$\forall (i, j), (\ell, k) \in M^+, \quad [j < k] \implies [\sigma(i, j) < \sigma(\ell, k)].$$

Denote by O^m the set of such orderings. Notice that

$$|O^m| = \prod_{r=1}^{\max(m)} q_r^m!$$

Let σ be an ordering and $(i, j) \in M$. When $\sigma(i, j) \geq 2$, the predecessor $p^\sigma(i, j)$ of (i, j) is the pair (ℓ, k) such that $\sigma(\ell, k) = \sigma(i, j) - 1$. In case $\sigma(i, j) = 1$, set $p^\sigma(i, j) = \emptyset$. Let us denote by $s^{\sigma,(i,j)}$ the coalition formed when it is the turn of the pair (i, j) to be activated, that is, the coalition formed at step $\sigma(i, j)$. Formally, the coalition $s^{\sigma,(i,j)}$ is defined as:

$$\forall \ell \in N, \quad s_\ell^{\sigma,(i,j)} = \max\big\{k \in M_\ell^+ : \sigma(\ell, k) \leq \sigma(i, j); 0\big\}.$$

Note that

$$B(s^{\sigma,(i,j)}) = \big\{(\ell, k) \in M^+ : \sigma(\ell, k) \leq \sigma(i, j)\big\}.$$

Example 3. Let $N = \{1, 2\}$ and $m = (2, 2)$. If the ordering σ is such that

$$\sigma(1,1) = 1, \sigma(2,1) = 2, \sigma(1,2) = 3 \text{ and } \sigma(2,2) = 4,$$

then one has $s^{\sigma,(1,2)} = (2, 1)$ and

$$B(s^{\sigma,(1,2)}) = \big\{(1,1), (1,2), (2,1)\big\}.$$

Let $\mathcal{C} \subseteq \mathcal{M}$ be the subset of coalitions s such that

$$\forall i \in N, \quad [s_i \leq \max(s) - 2] \implies [s_i = m_i].$$

From the definition of O^m, note that $\mathcal{C}$ contains all coalitions $s^{\sigma,(i,j)}$, for $(i, j) \in M_+$ and $\sigma \in O^m$. This coalition formation process has been used in Grabisch and Xie (2007) and Lowing and Techer (2022) to extend the Weber set and the Shapley value from standard cooperative games to multi-choice cooperative games.

Given a multi-choice game $(m, v) \in \mathfrak{G}$ and an ordering $\sigma \in O^m$, define the associated **marginal vector** $\eta^\sigma(m, v) \in \mathbb{R}^{M^+}$ as

$$\begin{array}{rcl} \forall (i,j) \in M^+, \quad \eta^\sigma_{(i,j)}(m,v) & = & v(s^{\sigma,(i,j)}) - v(s^{\sigma,p^\sigma(i,j)}) \\ & = & v(s^{\sigma,(i,j)}) - v(s^{\sigma,(i,j)} - e^i). \end{array}$$

Note that if $\sigma(i, j) = 1$, then $j = 1$, $p^\sigma(i, 1) = \emptyset$, and $s^{\sigma,(i,1)} - e_i = \overrightarrow{0}$. The quantity $\eta^\sigma_{(i,j)}(m, v) \in \mathbb{R}$ is the marginal contribution of i when it increases its activity level from $j - 1$ to j with respect to the ordering σ.

We have the material to define the Priority Shapley value. It is defined through the following procedure:

• each pair $(i, j) \in M^+$ shows up according to some ordering $\sigma \in O^m$;

• at step $\sigma(i, j)$, the pair $(i, j) \in M^+$ generates the marginal contribution $\eta^\sigma_{(i,j)}(m, v)$;

• this marginal contribution is shared equally among all the activity levels $k \leq j$ of agent i.

Therefore, under $\sigma \in O^m$, the procedure generates the following payoff vector $\lambda^\sigma(m, v)$:

$$\forall (i,j) \in M^+, \quad \lambda^\sigma_{(i,j)}(m,v) = \sum_{k=j}^{m_i} \frac{\eta^\sigma_{(i,k)}(m,v)}{k}. \tag{8}$$

The **Priority Shapley value** ϕ on $\mathfrak{G}$ is the value defined as the average payoff vector λ^σ over the set of orderings O^m:

$$\begin{array}{rcl} \forall (i,j) \in M^+, \quad \phi_{(i,j)}(m,v) & = & \dfrac{1}{|O^m|} \displaystyle\sum_{\sigma \in O^m} \lambda^\sigma_{(i,j)}(m,v) \\ & = & \dfrac{1}{\prod_{r=1}^{\max(m)} q_r^m!} \displaystyle\sum_{\sigma \in O^m} \sum_{k=j}^{m_i} \frac{v(s^{\sigma,(i,k)}) - v(s^{\sigma,(i,k)} - e^i)}{k}. \end{array} \tag{9}$$

The distinctive feature of the Priority Shapley value is that it does not require that each level of activity of an agent retains its marginal contribution to a coalition. Instead, this marginal contribution is redistributed to successive lower levels of activity of that agent, which were necessary to reach that level of activity in that coalition.

Below, two alternative expressions of ϕ are provided. The first one expresses the Priority Shapley value in terms of the marginal contribution of agent i to coalitions $s \in \mathcal{C}$ where it plays at least level j. The second one expresses the Priority Shapley value in terms of the distribution of the Harsanyi dividends. The dividend of a coalition s is shared according to a two-step procedure: in the first step, the agents playing the highest activity level in s, that is the agents in $T(s)$, receive an equal share of the dividend; in the second step, this share is equally redistributed to the subset of their activity levels lower than or equal to $\max(s)$.

Proposition 1. *The Priority Shapley value ϕ on $\mathfrak{G}$ defined as in (9) admits the following equivalent expressions:*

1.

$$\forall (i,j) \in M^+, \quad \phi_{(i,j)}(m,v) = \sum_{\substack{s \in \mathcal{C} \setminus \vec{0}: \\ s_i \geq j}} \frac{(q^m_{\max(s)} - q^s_{\max(s)})!(q^s_{\max(s)} - 1)!}{q^m_{\max(s)}!} \left[\frac{v(s) - v(s - e^i)}{\max(s)} \right].$$

2.

$$\forall (i,j) \in M^+, \quad \phi_{(i,j)}(m,v), \qquad \sum_{\substack{s \in \mathcal{M} \setminus \vec{0}: \\ T(s) \ni i, \max(s) \geq j}} \frac{\Delta_s(v)}{|T(s)| \max(s)}.$$

Remark 4. In case $m_i = 1$ for each $i \in N$, the Priority Shapley value coincides with the Shapley value for TU-games. Indeed $\max(s) = 1$ for all $s \neq \vec{0}$ and, in that context, $|T(s)|$ represents the number of agents that cooperate in s, i.e., the agents choosing the activity level 1. Thus, the Priority Shapley value shares equally the dividend of a coalition among its members, what the Shapley value prescribes.

We now approach the class of hazardous waste transportation problems from a cooperative point of view. For each hazardous waste transportation problem $(g, w, C) \in \mathfrak{P}$, define the multi-choice game $(w, v_{g,C}) \in \mathfrak{G}$ where the worth $v_{g,C}(s)$ describes the total maintenance cost of the network g whenever the profile s is shipped to the delivery node d. That is:

$$\forall s \in \mathcal{M}, \quad v_{g,C}(s) = \sum_{i \in N} C_i \left(\sum_{i' \in U_g[i]} s_{i'} \right).$$

Proposition 2. *For each hazardous waste transportation problem $(g, w, C) \in \mathfrak{P}$, and its associated multi-choice game $(w, v_{g,C}) \in \mathfrak{G}$, it holds that*

$$f^L(g, w, C) = \phi(w, v_{g,C}).$$

6. Axiomatic characterizations

The first three axioms are adaptations of the principles of Efficiency, Independence of other higher waste amounts, and Intra balanced contributions for highest contributors to the context of multi-choice games.

Axiom 10 (**Efficiency).** For each $(m, v) \in \mathfrak{G}$, it holds that

$$\sum_{(i,j)\in M^+} f_{(i,j)}(m, v) = v(m).$$

Axiom 11 (**Independence of more active agents).** For each $(m, v) \in \mathfrak{G}$, and each pair of agents $\{i, i'\} \subseteq N$ such that $m_i < m_{i'}$, it holds that

$$\forall j \in M_i^+, \quad f_{(i,j)}(m, v) = f_{(i,j)}(m - e^{i'}, v).$$

Axiom 12 (**Intra balanced contributions for top agents).** For each $(m, v) \in \mathfrak{G}$, each top agent $i \in T(m)$ and each pair of activity levels $\{j, k\} \subseteq \{1, \ldots, \max(m)\}$ for i, it holds that

$$f_{(i,j)}(m, v) - f_{(i,j)}(m - e^i, v) = f_{(i,k)}(m, v) - f_{(i,k)}(m - e^i, v),$$

with the convention that, for $i \in N$, $f_{(i,m_i)}(m - e^i, v) = 0$.

The fourth axiom is new although it belongs to the family of Balanced contribution axioms, a principle introduced by Myerson (1980) to characterize the Shapley value of TU-games.

Axiom 13 (**Inter balanced contributions for top agents).** For each $(m, v) \in \mathfrak{G}$, and each pair of top agents $\{i, i'\} \subseteq T(m)$, it holds that

$$f_{(i,\max(m))}(m, v) - f_{(i,\max(m))}(m - e^{i'}, v) = f_{(i',\max(m))}(m, v) - f_{(i',\max(m))}(m - e^i, v).$$

This axiom says that, for any pair i and i' of top agents, the change in payoff for the highest level of agent i when i' decreases its activity level is equal to the change in payoff for the highest level of agent i' when i decreases its activity level available.

The next axiom is the classical principle of Additivity used to characterize numerous allocation rules in cooperative games.

Axiom 14 (**Additivity).** For each $(m, v), (m, v') \in \mathfrak{G}$, it holds that

$$f(m, v) + f(m, v') = f(m, v + v'),$$

where $v + v'$ denotes the coalitional function on m such that, for each $s \in \mathcal{M}$, $(v + v')(s) = v'(s) + v'(s)$.

The last two axioms are related to specific agents and activity levels in the game.

An agent-activity level pair $(i', j) \in M^+$ is a **maximal pair** if $j = m_{i'}$. A maximal pair $(i', m_{i'})$ is **null** in $(m, v) \in \mathfrak{G}$ if, for each $s \in \mathcal{M} \setminus \overrightarrow{0}$ such that $s_{i'} = m_{i'}$, $v(s) = v(s - e^{i'})$. That is, this pair is null if the highest activity level of i' does not generate any additional worth. The next axiom relies on an invariance principle, which states that the payoff of any other agent-activity level pair should not be affected whether i' cooperates at level $m_{i'}$ or not.

Axiom 15 (Null max activity level out). For each $(m, v) \in \mathfrak{G}$, and each null maximal pair $(i', m_{i'}) \in M^+$, it holds that

$$\forall (i,j) \in M^+ \setminus \{(i', m_{i'})\}, \quad f_{(i,j)}(m, v) = f_{(i,j)}(m - e^{i'}, v).$$

A top agent $i \in T(m)$ is **top-veto** in $(m, v) \in \mathfrak{G}$ if for each coalition $s \in \mathcal{M}$,

$$s_i < m_i \implies v(s) = 0.$$

In words, coalitions are worthless if any top-veto agent implements an activity level lower than the maximal level. Since all top-veto agents have the same veto power at level $\max(m)$, the next axiom states that any two pairs of top-veto agents with maximal activity level should receive the same payoff.

Axiom 16 (Equal treatment for top-veto agents). For each $(m, v) \in \mathfrak{G}$, and each pair $\{i, i'\} \subseteq N$ of top-veto agents in $(m, v) \in \mathfrak{G}$, it holds that

$$f_{(i,\max(m))}(m, v) = f_{(i',\max(m))}(m, v).$$

The next results provide distinct axiomatic characterizations of the Priority Shapley value.

Theorem 5. *The Priority Shapley value ϕ defined as in (9) is the only value on $\mathfrak{G}$ that satisfies Efficiency, Independence of more active agents, Intra balanced contributions for top agents, and Inter balanced contributions for top agents.*

Theorem 6. *The Priority Shapley value ϕ defined as in (9) is the only value on $\mathfrak{G}$ that satisfies Efficiency, Independence of more active agents, Intra balanced contributions for top agents, Additivity, Null max activity level out, and Equal treatment for top-veto agents.*

6.1. Comparison with two other Shapley-like values for multi-choice games

There are several extensions of Shapley value from TU-games to multi-choice cooperative games. The two closest to The Priority Shapley value are the Derks-Peters value and the Shapley value introduced in Lowing and Techer (2022). From point 2 of Proposition 1, the Priority Shapley value shares the dividend of a coalition s according to the following two-step procedure: in the first step, the agents playing the highest activity level in s, that is the agents in $T(s)$, receive an equal share of the dividend; in the second step, this share is equally redistributed to the subset of their activity levels lower than or equal to $\max(s)$. Formally,

$$\forall (i,j) \in M^+, \quad \phi_{(i,j)}(m, v) = \sum_{\substack{s \in \mathcal{M} \setminus \vec{0}: \\ T(s) \ni i, \max(s) \geq j}} \frac{\Delta_s(v)}{|T(s)| \max(s)}.$$

The Shapley-like value introduced in Lowing and Techer (2022) equally shares the dividend of a coalition s among the agents in $T(s)$; and each agent i in $T(s)$ allocates this amount to the level $s_i = \max(s)$. If ϕ^{LT} denotes this value, then

$$\forall (i,j) \in M^+, \quad \phi^{LT}_{(i,j)}(m, v) = \sum_{\substack{s \in \mathcal{M} \setminus \vec{0}: \\ T(s) \ni i, \max(s) = j}} \frac{\Delta_s(v)}{|T(s)|}.$$

In a similar way as Proposition 2, Techer (2023) shows that the Responsibility rule f^R coincides with the Shapley-like value ϕ^{LT} of the game $(w, v_{g,C})$, that is,

$$\phi^{LT}(w, v_{g,C}) = f^R(g, w, C).$$

The Derks-Peters value equally shares the dividend of a coalition s among the the activity-level pairs (i, j) such that $j \leq s_i$. That is, it shares the dividend of a coalition s among its set of relevant agent-activity level pairs $B(s)$. If ϕ^{DP} denotes this value, then

$$\forall (i,j) \in M^+, \quad \phi^{DP}_{(i,j)}(m, v) = \sum_{\substack{s \in \mathcal{M} \setminus \vec{0}: \\ s_i \geq j}} \frac{\Delta_s(v)}{|B(s)|} = \sum_{\substack{s \in \mathcal{M} \setminus \vec{0}: \\ s_i \geq j}} \frac{\Delta_s(v)}{\sum_{i \in N} s_i}.$$

With regard to the distribution procedure of the dividend $\Delta_s(v)$, the Priority Shapley value ϕ and the Lowing-Techer value ϕ^{LT} consider only the agents playing the highest activity level in s. But unlike ϕ^{LT}, the Priority Shapley value ϕ shares the dividend among all activity level $j \leq s_i = \max(s)$. This last principle is shared by the Priority value and the Derks-Peters value ϕ^{DP}. The main difference between ϕ^{DP} and ϕ is that the former takes into account all agents in coalition s whereas ϕ takes into account only the agents in $T(s)$, that is those playing the activity level $\max(s)$. With these remarks in mind, it is possible to compare these three values axiomatically.

Let us consider the characterization of ϕ given in Theorem 5. The Lowing-Techer value satisfies all axioms except Intra balanced contributions for top agents. It satisfies a principle of independence stronger than the Independence of more active agents. To be precise, it satisfies the following principle.

Axiom 17 (Independence of higher activity levels). For each $(m, v) \in \mathfrak{P}$, it holds that

$$\forall i \in N, \forall j \in \{1, \ldots, m_i\}, \quad f_{(i,j)}(m, v) = f_{ij}(m \wedge \overrightarrow{j}, v),$$

where $(m \wedge \overrightarrow{j}, v)$ is read as the restriction of v to the profiles induced by $m \wedge \overrightarrow{j}$.

Clearly this axiom is a direct adaptation of Independence of higher waste amounts from $\mathfrak{P}$ to $\mathfrak{G}$. The Priority Shapley value ϕ fails to satisfy this axiom.

The Derks-Peters value ϕ^{DP} satisfies all axioms of Theorem 5 except Independence of more active agents. Furthermore, it satisfies a stronger principle than Inter balanced contributions for top agents, called Upper balanced contributions.

Axiom 18 (Upper balanced contributions). For each $(m, v) \in \mathfrak{G}$, and each distinct maximal pairs $(i, m_i), (i', m_{i'}) \in M_+$, it holds that

$$f_{(i,m_i)}(m, v) - f_{(i,m_i)}(m - e^{i'}, v) = f_{(i,m_{i'})}(m, v) - f_{(i,m_{i'})}(m - e^{i}, v).$$

The Priority Shapley value ϕ fails to satisfy this axiom unless i and i' are in $T(m)$. Note also that both the Priority Shapley value ϕ and the Derks-Peters value ϕ^{DP} satisfy a stronger version of Intra balanced contributions for top agents. In fact, the principle embodied in this axiom can be applied to any agent, not just top agents. This version of the axiom is called Intra balanced contributions (also known as Equal loss axiom).

Axiom 19 (Intra balanced contributions). For each $(m, v) \in \mathfrak{G}$, each $i \in N$ and each pair of activity levels $\{j, k\} \subseteq \{1, \ldots, m_i\}$, it holds that

$$f_{(i,j)}(m, v) - f_{(i,j)}(m - e^i, v) = f_{(i,k)}(m, v) - f_{(i,k)}(m - e^i, v).$$

Theorem 7. *(Klijn et al., 1999) The Derks-Peters value ϕ^{DP} is the only value on $\mathfrak{G}$ that satisfies Efficiency, Intra balanced contributions, and Upper balanced contributions.*

Although Lowing and Techer (2022) do not axiomatize ϕ^{LT} in terms of the Balanced contributions principle, the following result holds. Its proof follows similar arguments to the proof of Theorem 5, and therefore, it is omitted.

Theorem 8. *The Lowing-Techer value ϕ^{LT} is the only value on $\mathfrak{G}$ that satisfies Efficiency, Inter balanced contributions for top agents, and Independence of higher activity levels.*

Table 1 summarizes the axiomatic comparison of the above three Shapley-like values. The symbol ✓ means that the axiom is used in the characterization result; the symbol ∘ means that the axiom is satisfied by the value though not used in the characterization result; and a white cell means that the axiom is not satisfied by the value.

Axioms	ϕ^{LT}	ϕ^{DP}	ϕ
Efficiency	✓	✓	✓
Upper balanced contributions		✓	
Inter balanced contributions for top agents	✓	∘	✓
Intra balanced contributions for top agents		∘	✓
Intra balanced contributions (Equal loss axiom)		✓	∘
Independence of more active agents	∘		✓
Independence of higher activity levels	✓		

Table 1: Axiomatic comparison of the Shapley-like values.

7. Conclusion

There are several plausible directions in which our work can be extended. In this section, we briefly allude to one of them. It would make sense to endow each portion of the network with a (transport) capacity as in a flow network. This can only be done by relaxing the assumption that the sink tree is given, so as to allow waste that cannot travel on a portion with saturated transport capacity to travel on alternative portions. This is left for future work.

Compliance with Ethical Standards

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8. Appendix

8.1. Proof of Theorem 1

This proof is based on several intermediate results stated in the following lemmas.

Lemma 1. *Let f be an allocation rule on $\mathfrak{P}$ satisfying Independence of other higher waste amounts, and Intra balanced contributions for highest contributors. Then, for each $(g, w, C) \in \mathfrak{P}$, each agent $i \in N$, the following holds:*

$$\forall j \in \{1, \ldots, w_i - 1\}, \quad f_{(i,j)}(g, w, C) = f_{(i,w_i)}(g, w, C) + f_{(i,j)}(g, w \wedge \overrightarrow{(w_i - 1)}, C).$$

Proof. Pick any $(g, w, C) \in \mathfrak{P}$ and any agent $i \in N$. Thanks to Remark 1,

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(i,j)}(g, w \wedge \overrightarrow{w_i}, C).$$

Starting from $(g, w \wedge \overrightarrow{w_i}, C) \in \mathfrak{P}$, Remark 2 leads to:

$$\forall j \in \{1, \ldots, w_i - 1\}, \quad f_{(i,j)}(g, w \wedge \overrightarrow{w_i}, C) = f_{(i,w_i)}(g, w \wedge \overrightarrow{w_i}, C) + f_{(i,j)}(g, (w \wedge \overrightarrow{w_i}) - e^i, C),$$

so that

$$\forall j \in \{1, \ldots, w_i - 1\}, \quad f_{(i,j)}(g, w, C) = f_{(i,w_i)}(g, w \wedge \overrightarrow{w_i}, C) + f_{(i,j)}(g, (w \wedge \overrightarrow{w_i}) - e^i, C). \tag{10}$$

Finally, consider the right-hand side of (10). Using Remark 1 again, one obtains that

$$f_{(i,w_i)}(g, w, C) = f_{(i,j)}(g, w \wedge \overrightarrow{w_i}, C) \text{ and } f_{(i,j)}(g, (w \wedge \overrightarrow{w_i}) - e^i, C) = f_{(i,j)}(g, (w \wedge \overrightarrow{w_i}) \wedge \overrightarrow{(w_i - 1)}, C).$$

By noting that

$$f_{(i,j)}(g, (w \wedge \overrightarrow{w_i}) \wedge \overrightarrow{(w_i - 1)}, C) = f_{(i,j)}(g, w \wedge \overrightarrow{(w_i - 1)}, C),$$

the result follows. ■

Lemma 2. *Consider any $(g, w, C) \in \mathfrak{P}$, and any two leaves ℓ and ℓ' of g with the same successor i in g such that $C_\ell = C_{\ell'}$ and $w_\ell = w_{\ell'}$. If an allocation rule f satisfies Anonymity, then*

$$\forall j \in \{1, \ldots, w_\ell\}, \quad f_{(\ell,j)}(g, w, C) = f_{(\ell',j)}(g, w, C).$$

Proof. Pick any $(g, w, C) \in \mathfrak{P}$, and two leaves ℓ and ℓ' in g as hypothesized. Let π be the permutation on N such that $\pi(\ell) = \ell'$, $\pi(\ell') = \ell$, and $\pi(\ell'') = \ell''$ for each other $\ell'' \in N \setminus \{\ell, \ell'\}$. From $(g, w, C) \in \mathfrak{P}$, construct the corresponding hazardous transportation waste problem $(g^\pi, w^\pi, C^\pi) \in \mathfrak{P}$ as defined before the statement of Anonymity. From the structure of g and the fact that $C_\ell = C_{\ell'}$ and $w_\ell = w_{\ell'}$, (g^π, w^π, C^π) coincides with (g, w, C). By Anonymity, one obtains that

$$\forall j \in \{1, \ldots, w_\ell\}, \quad f_{(\ell,j)}(g, w, C) = f_{(\pi(\ell),j)}((g^\pi, w^\pi, C^\pi) = f_{(\ell',j)}(g, w, C).$$

■

Lemma 3. *Let f be an allocation rule on $\mathfrak{P}$ satisfying Anonymity, and Invariance to a relocation on a null cost portion. For each $(g, w, C) \in \mathfrak{P}$, each pair $\{i, i'\} \subseteq N$ such that $(i, i') \in E$, $C_i = C^0$, and $w_i = w_{i'}$, it holds that*

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(i',j)}(g, w, C).$$

Proof. Consider $(g, w, C) \in \mathfrak{P}$ and i and i' as hypothesized. Let π be the permutation on N such that $\pi(i) = i'$, $\pi(i') = i$ and $\pi(\ell) = \ell$ for each other $\ell \in N \setminus \{i, i'\}$. Because $w_i = w_{i'}$, the following equality holds:

$$(g^{i \leftrightarrow i'}, w^{i \leftrightarrow i'}, C^{i \leftrightarrow i'}) = (g^{\pi}, w^{\pi}, C^{\pi}).$$

By Invariance to a relocation on a null cost portion,

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(i,j)}(g^{i \leftrightarrow i'}, w^{i \leftrightarrow i'}, C^{i \leftrightarrow i'}).$$

By Anonymity,

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(i',j)}(g^{\pi}, w^{\pi}, C^{\pi}) = f_{(i',j)}(g^{i \leftrightarrow i'}, w^{i \leftrightarrow i'}, C^{i \leftrightarrow i'}).$$

Thus,

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g^{i \leftrightarrow i'}, w^{i \leftrightarrow i'}, C^{i \leftrightarrow i'}) = f_{(i',j)}(g^{i \leftrightarrow i'}, w^{i \leftrightarrow i'}, C^{i \leftrightarrow i'}).$$

In particular, by setting $(g', w', C') := (g^{i \leftrightarrow i'}, w^{i \leftrightarrow i'}, C^{i \leftrightarrow i'})$, one obtains that

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i',j)}((g')^{i' \leftrightarrow i}, (w')^{i' \leftrightarrow i}, (C')^{i' \leftrightarrow i}) = f_{(i,j)}((g')^{i' \leftrightarrow i}, (w')^{i' \leftrightarrow i}, (C')^{i' \leftrightarrow i}).$$

Since $((g')^{i' \leftrightarrow i}, (w')^{i' \leftrightarrow i}, (C')^{i' \leftrightarrow i}) = (g, w, C)$, then

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i',j)}(g, w, C) = f_{(i,j)}(g, w, C),$$

■

Proof of Theorem 1. **Uniqueness part**: The combination of the seven axioms induces at most one allocation rule on $\mathfrak{P}$. So, let f be an allocation rule on $\mathfrak{P}$ satisfying these seven axioms. To show that f is uniquely determined. Pick any hazardous waste transportation problem $(g, w, C) \in \mathfrak{P}$. Set $\mathfrak{K}(g, w, C)$ as the set of portions in g with a non-null cost:

$$\mathfrak{K}(g, w, C) = \big\{i \in N : C_i \neq C^0\big\}.$$

The proof proceeds by induction on the number of elements in $\mathfrak{K}(g, w, C)$.

Induction basis: $\mathfrak{K}(g, w, C)$ is empty. By Efficiency,

$$\sum_{i \in N} \sum_{j=1}^{w_i} f_{(i,j)}(g, w, C) = 0.$$

Consider any directed path of length two of the form (i', i, d). Cut the directed link $(i', i) \in E$ and connect i' to the treatment facility d by adding the directed link (i', d), *ceteris paribus*. Repeat this operation until there is no directed path of length two leading to d. The resulting sink tree $g^{\star}$ is such that each $i \in N$ is directly connected to the treatment facility d. Apply repeatedly

Invariance to a cut-and-connect operation on a null cost portion to obtain (with a slight abuse of notation),

$$\forall i \in N, \forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g, w, C) = f_{(i,j)}(g^\star, w, C).$$

To complete the induction basis, it suffices to prove that $f_{(i,j)}(g^\star, w, C) = 0$, for each agent $i \in N$, and each $j \in \{1, \ldots, w_i\}$. To that end, one again proceeds by induction on the value of $\max(w)$.

Induction basis: If $\max(w) = 1$, then $Q^w(1) = N$. The Efficiency condition applied to $(g^\star, w, C)$ becomes

$$\sum_{i \in N} f_{(i,1)}(g^\star, w, C) = 0.$$

From Lemma 2, all agents in N obtain the same payoff. Together with the Efficiency condition, this forces that $f_{(i,1)}(g^\star, w, C) = 0$ for each $i \in N$.

Induction hypothesis: Assume that the assertion is true whenever $\max(w) \leq k$, for $k \geq 1$.

Induction step: Consider the situation where $\max(w) = k + 1$. If $i \in N \setminus Q^w(k+1)$, then Remark 1 and the induction hypothesis yield the following equalities:

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g^\star, w, C) = f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C) = 0.$$

It thus remains to deal with the agents in $Q^w(k+1)$. By Lemma 2,

$$\forall \{i, i'\} \subseteq Q^w(k+1), \quad f_{(i,k+1)}(g^\star, w, C) = f_{(i',k+1)}(g^\star, w, C).$$

By the induction hypothesis, for each agent $i \in Q^w(k+1)$, one has

$$\forall j \in \{1, \ldots, k\}, \quad f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C) = 0.$$

Using the above equality and Lemma 1, one obtains

$$\begin{aligned} \forall i \in Q^w(k+1), \forall j \in \{1, \ldots, k\}, \quad f_{(i,j)}(g^\star, w, C) &= f_{(i,k+1)}(g^\star, w, C) + f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C) \\ &= f_{(i,k+1)}(g^\star, w, C). \end{aligned} \tag{11}$$

It amounts to saying that, for any $i' \in Q^w(k+1)$,

$$\sum_{i \in Q^w(k+1)} \sum_{j=1}^{k+1} f_{(i,j)}(g^\star, w, C) = (k+1)|Q^w(k+1)| f_{(i',k+1)}(g^\star, w, C).$$

On the other hand, by Efficiency,

$$\sum_{i \in Q^w(k+1)} \sum_{j=1}^{k+1} f_{(i,j)}(g^\star, w, C) = \sum_{i \in N} \sum_{j=1}^{w_i} f_{(i,j)}(g^\star, w, C) = 0.$$

Therefore, one concludes that

$$\forall i' \in Q^w(k+1), \quad f_{(i',k+1)}(g^\star, w, C) = 0,$$

and so, thanks to (11), for each $j \in \{1, \ldots, k+1\}$, $f_{(i',j)}(g^\star, w, C) = 0$, as desired.

INDUCTION HYPOTHESIS: Suppose that $f(g,w,C)$ is uniquely determined when $\mathfrak{K}(g,w,C)$ contains t elements, for $t \geq 0$.

INDUCTION STEP: Suppose that $\mathfrak{K}(g,w,C)$ contains $t+1$ elements. Let

$$U_g = \bigcap_{i \in \mathfrak{K}(g,w,C)} U_g[i].$$

Case 1. If $U_g = \emptyset$, then, for each $i \in N$, there exists $i' \in \mathfrak{K}(g,w,C)$ such that $i \notin U_g[i']$, and so $D_g[i] \cap U_g[i'] = \emptyset$. Now define the cost profile $C - \lambda^{C,i'}$ (see (2) for the definition of $\lambda^{C,i'}$). Because $i' \notin D_g[i]$, by Path consistency, one obtains

$$\forall j \in \{1,\ldots,w_i\}, \quad f_{(i,j)}(g,w,C) = f_{(i,j)}(g,w,C - \lambda^{C,i'}).$$

By the induction hypothesis,

$$\forall j \in \{1,\ldots,w_i\}, \quad f_{(i,j)}(g,w,C - \lambda^{C,i'}) \text{ is uniquely determined}$$

and so is $f_{(i,j)}(g,w,C)$, as desired.

Case 2. If $U_g \neq \emptyset$, then, since g is a sink tree, there exists $i' \in \mathfrak{K}(g,w,C)$ such that $U_g = U_g[i']$. If $i \notin U_g$, then proceeding as in **Case 1**, one concludes that, for each $j \in \{1,\ldots,w_i\}$, $f_{(i,j)}(g,w,C)$ is uniquely determined. It remains to deal with the payoffs of agents belonging to U_g. By definition of U_g, for each $i \in U_g \setminus i'$, $C_i = C^0$. Consider any directed path of length two of the form (ℓ,ℓ',i'). Cut the directed link $(\ell,\ell') \in E$ and connect ℓ to i by adding the directed link (ℓ,i'), *ceteris paribus*. Repeat this operation until there is no directed path of length two leading to i'. The resulting sink tree $g^\star$ is such that each $i \in U_g \setminus i'$ is now directly connected to i'. Apply repeatedly Invariance to a cut-and-connect operation on a null cost portion to obtain

$$\forall i \in U_g \setminus i', \forall j \in \{1,\ldots,w_i\}, \quad f_{(i,j)}(g,w,C) = f_{(i,j)}(g^\star,w,C).$$

To complete the proof, one again proceeds by induction on the value of $\max(w_{U_g})$ where w_{U_g} stands for the restriction of w to the components of U_g.

Induction basis: If $\max(w_{U_g}) = 1$, then, the Efficiency condition applied to (g,w,C) and the above equalities give

$$\sum_{i \in U_g} f_{(i,1)}(g^\star,w,C) = \sum_{i \in N} C_i\Big(\sum_{\ell \in U_g[i]} w_\ell\Big) - \sum_{i \in N \setminus U_g} \sum_{j=1}^{w_i} f_{(i,j)}(g,w,C).$$

Note that the right-hand side of the above equality is uniquely determined. On the other hand, by Lemma 2, for each pair $\{\ell,\ell'\} \subseteq U_g \setminus i'$, $f_{(\ell,1)}(g^\star,w,C) = f_{(\ell',1)}(g^\star,w,C)$. Furthermore, by Lemma 3, $f_{(\ell,1)}(g^\star,w,C) = f_{(i',1)}(g^\star,w,C)$, meaning that each agent in U_g obtains the same payoff. Together with the Efficiency condition, this forces that

$$\forall \ell \in U_g, \quad f_{(\ell,1)}(g^\star,w,C) = \frac{1}{|U_g|}\Big[\sum_{i \in N} C_i\Big(\sum_{\ell' \in U_g[i]} w_{\ell'}\Big) - \sum_{i \in N \setminus U_g} \sum_{j=1}^{w_i} f_{(i,j)}(g,w,C)\Big].$$

Therefore, for each $\ell \in U$, $f_{(\ell,1)}(g^\star,w,C)$ is uniquely determined, as desired.

Induction hypothesis: Assume that the assertion is true whenever $\max(w_{U_g}) \leq k$, for $k \geq 1$.

Induction step: Consider the situation where $\max(w_{U_g}) = k+1$. If $i \in U_g \setminus Q^w(k+1)$, then by Remark 1,

$$\forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g^\star, w, C) = f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C),$$

which is uniquely determined by the induction hypothesis. So, it remains to deal with the agents in $U_g \cap Q^w(k+1)$. By Lemma 1,

$$\forall i \in U_g \cap Q^w(k+1), \forall j \in \{1, \ldots, k\}, \quad f_{(i,j)}(g^\star, w, C) = f_{(i,k+1)}(g^\star, w, C) + f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C). \tag{12}$$

By the induction hypothesis, for each $j \in \{1, \ldots, k\}$, $f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C)$ is uniquely determined. It remains to prove that $f_{(i,k+1)}(g^\star, w, C)$ is uniquely determined. By Lemma 2,

$$\forall \{\ell, \ell'\} \subseteq (U_g \setminus i) \cap Q^w(k+1), \quad f_{(\ell,k+1)}(g^\star, w, C) = f_{(\ell',k+1)}(g^\star, w, C).$$

Furthermore, if $i' \in Q^w(k+1)$, then, by Lemma 3, one obtains

$$f_{(\ell,k+1)}(g^\star, w, C) = f_{(i',k+1)}(g^\star, w, C).$$

All in all, all agents of $U_g \cap Q^w(k+1)$ obtain the same payoff at level $k+1$. Let α_{k+1} denote this common value. Using the Efficiency condition and the fact that, for each pair (i,j) such that $i \in N \setminus (U_g \cap Q^w(k+1))$ and $1 \leq j \leq w_i$, the payoff $f_{(i,j)}(g^\star, w, C)$ is uniquely determined (see above), one concludes that the quantity

$$\sum_{i \in U_g \cap Q^w(k+1)} \sum_{j=1}^{k+1} f_{(i,j)}(g^\star, w, C) = \sum_{i \in N} \sum_{j=1}^{w_i} f_{(i,j)}(g^\star, w, C) - \sum_{i \in N \setminus (U_g \cap Q^w(k+1))} \sum_{j=1}^{w_i} f_{(i,j)}(g^\star, w, C)$$

is uniquely determined. Furthermore, by (12), one obtains

$$\sum_{i \in U_g \cap Q^w(k+1)} \sum_{j=1}^{k+1} f_{(i,j)}(g^\star, w, C) = \sum_{i \in U_g \cap Q^w(k+1)} \sum_{j=1}^{k} \left(f_{(i,j)}(w \wedge \overrightarrow{k}, C) + \alpha_{k+1} \right) + \sum_{i \in U_g \cap Q^w(k+1)} \alpha_{k+1},$$

which leads to

$$\sum_{i \in U_g \cap Q^w(k+1)} \sum_{j=1}^{k+1} f_{(i,j)}(g^\star, w, C) = \sum_{i \in U_g \cap Q^w(k+1)} \sum_{j=1}^{k} f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C) + (k+1) |U_g \cap Q^w(k+1)| \alpha_{k+1}.$$

By the induction hypothesis,

$$\sum_{i \in U_g \cap Q^w(k+1)} \sum_{j=1}^{k} f_{(i,j)}(g^\star, w \wedge \overrightarrow{k}, C)$$

is uniquely determined, which allows to conclude that α_{k+1} is uniquely determined. The proof is complete.

Existence part: One verifies that f^L satisfies the seven axioms. From the definition given in (4) of f^L, the cost increase at level k,

$$C_{i'}\Big(\sum_{\ell\in U_g[i']} w_\ell \wedge k\Big) - C_{i'}\Big(\sum_{\ell\in U_g[i']} w_\ell \wedge (k-1)\Big),$$

is paid only by the agents i located upstream of i' provided that $w_i \geq k$, that is, this cost increase is paid only by the agents belonging to $U_g[i']\cap Q^w(k)$. Therefore, f^L trivially satisfies Path consistency and Independence of other higher waste amounts. Each $i \in U_g[i'] \cap Q^w(k)$ pays an equal share of this cost increase by charging it equally to each waste amount $k, k-1, \ldots, 1$, that is, the Liability rule charges an amount

$$\frac{1}{k}\cdot\frac{C_{i'}(\sum_{\ell\in U_g[i']} w_\ell \wedge k) - C_{i'}(\sum_{\ell\in U_g[i']} w_\ell \wedge (k-1))}{|U_g[i'] \cap Q^w(k)|},$$

to each waste amount $j \in \{1, \ldots, k\}$. It follows that if agent i ships $w_i - 1$ units of waste instead of w_i, each unit $w_i - 1, \ldots, 1$, will reduce the amount to be paid by the same amount,

$$\frac{1}{w_i}\cdot\frac{C_{i'}(\sum_{\ell\in U_g[i']} w_\ell \wedge w_i) - C_{i'}(\sum_{\ell\in U_g[i']} w_\ell \wedge (w_i-1))}{|U_g[i'] \cap Q^w(w_i)|}.$$

This shows that f^L satisfies Intra balanced contributions for highest contributors. It must also be clear that the cost charged to each waste amount j of i does not depend on the label i. Thus, f^L satisfies Anonymity as well. From the above discussion, it follows that if the cost function $C_{i'} = C^0$ is the null cost function, the portion connecting each predecessor $i \in P_g(i')$ to i' can be removed and replaced by a portion connecting each of these predecessors to the successor of i', without modifying the cost share to be paid for i under f^L. This ensures that f^L satisfies Invariance to a cut-and-connect operation on a null cost portion. Furthermore, i has trivially no cost to pay when it passes its waste on its portion, and each successor of i has nothing to pay either, since it does not use that portion. Then, under f^L, these agents will pay part of the cost increases of the portions from i's successor to d. As a result, the positions of i and its successor can be exchanged without modifying their payoffs under f^L. This ensures that f^L satisfies Invariance to a relocation on a null cost portion. It remains to prove that f^L indeed satisfies Efficiency. First the total cost paid by $i \in N$ in (g, w, C) under f^L is

$$\begin{aligned}
\sum_{j=1}^{w_i} f^L_{(i,j)}(g,w,C) &= \sum_{j=1}^{w_i}\sum_{i'\in D_g[i]}\sum_{\substack{k\geq j:\\ i\in Q^w(k)}} \frac{1}{k}\cdot\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big) - C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|U_g[i']\cap Q^w(k)|}\\
&= \sum_{i'\in D_g[i]}\sum_{k=1}^{w_i}\sum_{j=1}^{k}\frac{1}{k}\cdot\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big) - C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|U_g[i']\cap Q^w(k)|}\\
&= \sum_{i'\in D_g[i]}\sum_{k=1}^{w_i}\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big) - C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|U_g[i']\cap Q^w(k)|}.
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{i\in N}\sum_{j=1}^{w_i} f^L_{(i,j)}(g,w,C) &= \sum_{i\in N}\sum_{i'\in D_g[i]}\sum_{k=1}^{w_i}\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big)-C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|U_g[i']\cap Q^w(k)|}\\
&= \sum_{i'\in N}\sum_{i\in U_g[i']}\sum_{k=1}^{w_i}\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big)-C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|U_g[i']\cap Q^w(k)|}\\
&= \sum_{i'\in N}\sum_{k=1}^{\max(w)}\sum_{i\in U_g[i']\cap Q^w(k)}\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big)-C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\Big)}{|U_g[i']\cap Q^w(k)|}\\
&= \sum_{i'\in N}\sum_{k=1}^{\max(w)} C_{i'}\Big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\Big)-C_{i'}\Big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\Big)\\
&= \sum_{i'\in N} C_{i'}\Big(\sum_{\ell\in U_g[i']} w_\ell\Big),
\end{aligned}$$

which proves that f^L satisfies Efficiency. This completes the proof of Theorem 1. ■

Logical independence of the axioms. The axioms invoked in Theorem 1 are logically independent, as shown by the following alternative allocation rules.

- The allocation rule f on $\mathfrak{P}$ defined as:

  $$\forall (i,j)\in M^+,\quad f_{(i,j)}(g,w,C)=0$$

  satisfies all the axioms except Efficiency.

- The allocation rule f on $\mathfrak{P}$ defined as:

  $$\forall (i,j)\in M^+,\quad f_{(i,j)}(g,w,C)=\sum_{i'\in D_g[i]}\sum_{k=j}^{\max(w)}\frac{1}{w_i\wedge k}\cdot\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big)-C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|U_g[i']|}$$

  satisfies all the axioms except Independence of other higher waste amounts.

- The allocation rule f on $\mathfrak{P}$ defined as:

  $$\forall (i,j)\in M^+,\quad f_{(i,j)}(g,w,C)=\sum_{i'\in N}\sum_{\substack{k\geq j:\\ i\in Q^w(k)}}\frac{1}{k}\cdot\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big)-C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|Q^w(k)|}$$

  satisfies all the axioms except Path consistency.

- The Responsibility rule f^R on $\mathfrak{P}$ defined as:

  $$\forall (i,j)\in M^+,\quad f^R_{(i,j)}(g,w,C)=\sum_{i'\in D_g[i]}\sum_{\substack{k\geq j:\\ i\in Q^w(k)}}\frac{C_{i'}\big(\sum_{\ell\in U_g[i']} k\wedge w_\ell\big)-C_{i'}\big(\sum_{\ell\in U_g[i']}(k-1)\wedge w_\ell\big)}{|U_g[i']\cap Q^w(k)|}$$

  satisfies all the axioms except Intra balanced contributions for highest contributors.

- The allocation rule f on $\mathfrak{P}$ defined as:

$$\forall (i,j) \in M^+, \quad f_{(i,j)}(g,w,C) = \sum_{i' \in D_g[i]} \sum_{\substack{k \geq j: \\ i \in Q^w(k)}} i \cdot \frac{C_{i'}\big(\sum_{\ell \in U_g[i']} k \wedge w_\ell\big) - C_{i'}\big(\sum_{\ell \in U_g[i']} (k-1) \wedge w_\ell\big)}{\sum_{\ell \in U_g[i'] \cap Q^w(k)} \ell}$$

satisfies all the axioms except Anonymity.

- To define the next allocation rule, recall that by (2), C can be decomposed as:

$$C = \sum_{i' \in N} \lambda^{C,i'}.$$

And by (3), one has the following decomposition of C:

$$\lambda^{C,i'} = \sum_{k=1}^{\max(w)} \lambda^{C,i',k} \text{ and so } C = \sum_{i' \in N} \sum_{k=1}^{\max(w)} \lambda^{C,i',k},$$

where,

$$\forall s \in \prod_{i \in N} \{0, \ldots, w_i\}, \quad \lambda^{C,i',k}(s) = \lambda^{C,i'}\Big(\sum_{i \in U_g[i']} s_i \wedge k\Big) - \lambda^{C,i'}\Big(\sum_{i \in U_g[i']} s_i \wedge (k-1)\Big).$$

From this, define the additive allocation rule f as follows: if i' is a successor of a leaf of g and $i' \in Q^w(k)$, then

$$\forall j \in \{1, \ldots, k\}, \quad f_{(i',j)}(g,w,\lambda^{C,i',k}) = \frac{1}{k}\Big(C_{i'}\big(\sum_{\ell \in U_{[i']}} k \wedge w_\ell\big) - C_{i'}\big(\sum_{\ell \in U_{[i']}} (k-1) \wedge w_\ell\big)\Big),$$

and $f_{(i,j)}(g,w,\lambda^{C,i',k}) = 0$ if either $i \in N \setminus i'$ or $i = i'$ and $j \in \{k+1, \ldots, w_i\}$.
In any other case, that is, if i' is neither a successor of a leaf of g nor $i' \in Q^w(k)$,

$$\forall i \in N, \forall j \in \{1, \ldots, w_i\}, \quad f_{(i,j)}(g,w,\lambda^{C,i',k}) = f^L_{(i,j)}(g,w,\lambda^{C,i',k}).$$

By additivity of f,

$$f(g,w,C) = \sum_{i' \in N} \sum_{k=1}^{\max(w)} f(g,w,\lambda^{C,i',k}).$$

This allocation rule f on $\mathfrak{P}$ satisfies all the axioms except Invariance to a relocation on a null cost portion.

- Define the allocation rule f as follows. Pick any $(g,w,C) \in \mathfrak{P}$. Consider the set $P_g(d)$ of predecessors of the sink d and define $S_g = \{i_1, i_2, \ldots, i_p\} \subseteq P_g(d)$ as the possibly empty subset of predecessors of the sink d that are also leaves of g and such that $w_{i_t} = 1$ for $t \in \{1, \ldots, p\}$; and the complementary subset $P_g(d) \setminus S_g = \{i'_1, i'_2, \ldots, i'_q\}$. One distinguishes two cases.

Case 1. If $P_g(d) \setminus S_g \neq \emptyset$ and $S_g \neq \emptyset$, define the cost function C' as:

$$C'_i = \begin{cases} C_i + \frac{1}{q} & \text{if } i \in P_g(d) \setminus S_g, \\ C_i & \text{otherwise.} \end{cases}$$

In this case,

$$f_{(i,j)}(g,w,C) = \begin{cases} f^L_{(i,1)}(g,w,C) - \frac{1}{p} & \text{if } i \in S_g, \\ \\ f^L_{(i,j)}(g,w,C') & \text{if } i \in N \setminus S_g \text{ and } j \in \{1,\ldots,w_i\}. \end{cases}$$

Case 2. Either $P_g(d) \setminus S_g = \emptyset$ or $S_g = \emptyset$. Then, set $f(g,w,C) = f^L(g,w,c)$.

This allocation rule f on $\mathfrak{P}$ satisfies all the axioms except Invariance to a cut-and-connect operation on a null cost portion.

8.2. Proof of Theorem 2

Proof of Theorem 2. Only the uniqueness part has to be shown. So, let f be an allocation rule on $\mathfrak{P}^*$ satisfying the axioms of the statement of Theorem 2. Pick any hazardous waste transportation problem $(g,w,C) \in \mathfrak{P}^*$. As in the proof of Theorem 1, set $\mathfrak{K}(g,w,C)$ as the set of portions in g with a non-null cost:

$$\mathfrak{K}(g,w,C) = \left\{ i \in N : C_i \neq C^0 \right\}.$$

The proof proceeds by induction on the number of elements in $\mathfrak{K}(g,w,C)$.

INDUCTION BASIS: $\mathfrak{K}(g,w,C)$ is empty. By Efficiency,

$$\sum_{i \in N} f_{(i,1)}(g,w,C) = 0.$$

By Lemma 3, all agents in N obtain the same payoff. By Efficiency, it follows that $f(g,w,C) = \overrightarrow{0}$.

INDUCTION HYPOTHESIS: Suppose that $f(g,w,C)$ is uniquely determined when $\mathfrak{K}(g,w,C)$ contains t elements, for $t \geq 0$.

INDUCTION STEP: Suppose that $\mathfrak{K}(g,w,C)$ contains $t+1$ elements. Let ℓ be the most distant agent from d such that $C_k = C^0$. This means that each $i \in U_g[\ell]$ is such that $C_i = C^0$. By Lemma 3, all agents in $U_g[\ell]$ obtain the same payoff, say $c^f \in \mathbb{R}$. Next, consider any $i \in N \setminus U_g[k]$. By Path consistency,

$$f_{(i,1)}(g,w,C) = f_{(i,1)}(g,w,C - \lambda^{\ell,C}).$$

By induction hypothesis, $f_{(i,1)}(g,w,C - \lambda^{\lambda,C})$ is uniquely determined, so is $f_{(i,1)}(g,w,C)$, for each $i \in N \setminus U_g[k]$. By Efficiency,

$$c^f = \frac{\sum_{i \in N} C_i(i) - \sum_{i \in N \setminus U_g[k]} f_{(i,1)}(g,w,C)}{|U_g[k]|}$$

is uniquely determined. This complete the proof of the induction step. ■

8.3. Proof of Proposition 1

Before starting with the proof of Proposition 1, a definition is required. A coalition $s \in \mathcal{C}$ is **induced by an ordering** $\sigma \in O^m$ if there is a pair $(i,j) \in M^+$ such that $s^{\sigma,(i,j)} = s$. Let I^σ be the subset of coalitions in $\mathcal{C}$ induced by σ. Obviously, the grand coalition m and the null coalition $\overrightarrow{0}$ belong to I^σ whatever $\sigma \in O^m$.

Proof of Proposition 1. Point 1. From the definition of the Priority Shapley value given in (9), one observes that $s^{\sigma,(i,j)}$ and $s^{\sigma,(i,j)} - e^i$ are in $\mathcal{C}$, and coalition $s^{\sigma,(i,j)}$ is such that $\max(s^{\sigma,(i,j)}) = j$. From this observation, one deduces that

$$\forall (i,j) \in M^+, \quad \phi_{(i,j)}(m,v) = \frac{1}{\prod_{r=1}^{\max(m)} q_r^m!} \sum_{\substack{s \in \mathcal{C}: \\ s_i \geq j}} \sum_{\substack{\sigma \in O^m: \\ I^\sigma \ni s}} \frac{v(s) - v(s-e^i)}{\max(s)}.$$

Using Remark 3, for any coalition $s \in \mathcal{C}$, the number of orderings $\sigma \in O^m$ such that $I^\sigma \ni s$ is given by:

$$\Big(\prod_{r=1}^{\max(s)-1} q_r^m!\Big)\big(q^s_{\max(s)} - 1\big)!\big(q^m_{\max(s)} - q^s_{\max(s)}\big)!\Big(\prod_{r=\max(s)+1}^{\max(m)} q_r^m!\Big).$$

From this, one directly gets the desired result:

$$\forall (i,j) \in M^+, \quad \phi_{(i,j)}(m,v) = \sum_{\substack{s \in \mathcal{C}: \\ s_i \geq j}} \frac{(q^m_{\max(s)} - q^s_{\max(s)})!(q^s_{\max(s)} - 1)!}{q^m_{\max(s)}!} \left[\frac{v(s) - v(s-e^i)}{\max(s)}\right].$$

Point 2. The Priority Shapley value ϕ is obviously a linear function in v. By (7), it follows that

$$\forall (i,j) \in M^+, \quad \phi_{(i,j)}(m,v) = \sum_{s \leq m} \Delta_v(s) \phi_{(i,j)}(m, u_s). \tag{13}$$

Note that, for any ordering $\sigma \in O^m$, $\eta^\sigma_{(i,j)}(m,u_s) = 1$ if and only if the following conditions hold:

- $i \in T(s)$;
- $j = \max(s)$;
- for any other agent $\ell \in T(s)$, $\sigma(i, \max(s)) > \sigma(\ell, \max(s))$.

In all other cases, $\eta^\sigma_{(i,j)}(m,u_s) = 0$. By the above fact and definition (8) of $\lambda^\sigma_{(i,j)}(m,u_s)$, it follows that

$$\forall (i,j) \in M^+, \quad \lambda^\sigma_{(i,j)}(m,u_s) = \begin{cases} \dfrac{1}{\max(s)} & \text{if } i \in T(s), j \leq \max(s), \forall \ell \in T(s) \setminus i, \sigma(i,\max(s)) > \sigma(\ell, \max(s)), \\ 0 & \text{otherwise.} \end{cases}$$

By definition (9) of $\phi_{(i,j)}(m,u_s)$, it holds that

$$\begin{aligned} \forall (i,j) \in M^+ : i \in T(s), j \leq \max(s), \quad \phi_{(i,j)}(m,u_s) &= \frac{1}{|O^m|} \sum_{\sigma \in O^m} \lambda^\sigma_{(i,j)}(m,u_s) \\ &= \frac{1}{|O^m|} \sum_{\substack{\sigma \in O^m: \\ \forall \ell \in T(s), \sigma(i,\max(s)) > \sigma(\ell,\max(s))}} \frac{1}{\max(s)}. \end{aligned}$$

And, for each $i \in N \setminus T(s)$ and $j \in M_i$, $\phi_{(i,j)}(m, u_t) = 0$.

To complete the proof, note that, for $i, \ell \in T(s)$, the number of restricted orders σ such that $\sigma(i, \max(s)) > \sigma(\ell, \max(s))$ is equal to the number of restricted orders σ such that $\sigma(\ell, \max(s)) > \sigma(i, \max(s))$, and it is given by

$$\frac{|O^m|}{|T(s)|}.$$

It follows that

$$\begin{aligned}
\forall (i,j) \in M^+ : i \in T(s), j \leq \max(s), \quad \phi_{(i,j)}(m, u_s) &= \frac{1}{|O^m|} \sum_{\substack{\sigma \in O^m: \\ \forall \ell \in T(s), \sigma(i,\max(s)) > \sigma(\ell,\max(s))}} \frac{1}{\max(s)} \\
&= \frac{1}{|O^m|} \frac{|O^m|}{|T(s)|} \frac{1}{\max(s)} \\
&= \frac{1}{|T(s)| \max(s)},
\end{aligned}$$

and so

$$\forall (i,j) \in M^+, \quad \phi_{(i,j)}(m, u_s) = \begin{cases} \frac{1}{|T(s)| \max(s)} & \text{if } i \in T(s), j \leq \max(s), \\ 0 & \text{otherwise.} \end{cases}$$

The desired result follows by (13).

■

8.4. Proof of Proposition 2

Proof of Proposition 2. Consider any hazardous waste transportation problem $(g, w, C) \in \mathfrak{P}$ and its associated multi-choice game $(w, v_{g,C}) \in \mathfrak{G}$. By (2), C can be decomposed as:

$$C = \sum_{i' \in N} \lambda^{C,i'}.$$

By (3), one has the following decomposition of C:

$$\lambda^{C,i'} = \sum_{k=1}^{\max(w)} \lambda^{C,i',k}, \text{ and so } C = \sum_{i' \in N} \sum_{k=1}^{\max(w)} \lambda^{C,i',k},$$

where

$$\forall s \in \prod_{i \in N} \{0, \ldots, w_i\}, \quad \lambda^{C,i',k}(s) = \lambda^{C,i'}\left(\sum_{i \in U_g[i']} s_i \wedge k\right) - \lambda^{C,i'}\left(\sum_{i \in U_g[i']} s_i \wedge (k-1)\right).$$

Pick another hazardous waste transportation problem $(g, w, C') \in \mathfrak{P}$ with the same sink tree and waste profile as (g, w, C) and define the hazardous waste transportation problem $(g, w, C+C') \in \mathfrak{P}$. By definition of the coalition functions $v_{g,C}$ and $v_{g,C'}$, it holds that $v_{g,C+C'} = v_{g,C} + v_{g,C'}$. Observe also that the Priority Shapley value ϕ given in (9) is additive in v and the Liability rule f^L given in (4) is additive in C. Therefore, using (3), it suffices to prove that

$$\forall k \in \{1, \ldots, \max(w)\}, \quad \phi(w, v_{g,\lambda^{C,i',k}}) = f^L(g, w, \lambda^{C,i',k}).$$

From the definition of the Liability rule f^L, one has that

$$f^L_{(i,j)}(g,w,\lambda^{C,i',k}) = \begin{cases} \dfrac{C_{i'}\big(\sum\limits_{\ell\in U_g[i']} w_\ell \wedge k\big) - C_{i'}\big(\sum\limits_{\ell\in U_g[i']} w_\ell \wedge (k-1)\big)}{|U_g[i']\cap Q^w(k)|} & \text{if } i\in U_g[i']\cap Q^w(k), j\in\{1,\dots,k\}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} \forall s\in\mathcal{M},\quad v_{g,\lambda^{C,i',k}}(s) &= \lambda^{C,i',k}(s) \\ &= C_{i'}\Big(\sum_{\ell\in U_g[i']} s_\ell\wedge k\Big) - C_{i'}\Big(\sum_{\ell\in U_g[i']} s_\ell\wedge(k-1)\Big) \\ &= C_{i'}\Big(\sum_{\ell\in U_g[i']} s_\ell\wedge(k-1) + |U_g[i']\cap Q^s(k)|\Big) - C_{i'}\Big(\sum_{\ell\in U_g[i']} s_\ell\wedge(k-1)\Big). \end{aligned}$$

Let σ be any ordering in O^m. Using the definition of $v_{g,\lambda^{C,i',k}}$, one has that

- for each $i\in N\setminus U_g[i']$ and each $j\in\{1,\dots,w_i\}$, $\quad \eta^\sigma_{(i,j)}(w,v_{g,\lambda^{C,i',k}})=0$;
- for each $i\in N$ and each $j\in\{1,\dots,w_i\}\setminus k$, $\quad \eta^\sigma_{(i,j)}(w,v_{g,\lambda^{C,i',k}})=0$.

By definition of the Priority Shapley value ϕ given in (9), one obtains the following payoffs:

$$\phi_{(i,j)}(w,v_{g,\lambda^{C,i',k}}) = \begin{cases} \dfrac{\sum\limits_{\sigma\in O^w}\eta^\sigma_{(i,k)}(w,v_{g,\lambda^{C,i',k}})}{|O^w|} & \text{if } i\in U_g[i']\cap Q^w(k), j\in\{1,\dots,k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Furthermore, by the definition of a marginal vector,

$$\begin{aligned} \forall\sigma\in O^w,\ \sum_{(i,j)\in M^+}\eta^\sigma_{(i,j)}(w,v_{g,\lambda^{C,i',k}}) &= \sum_{i\in U_g[i']\cap Q^w(k)}\eta^\sigma_{(i,k)}(w,v_{g,\lambda^{C,i',k}}) \\ &= v_{g,\lambda^{C,i',k}}(w) \\ &= C_{i'}\Big(\sum_{\ell\in U_g[i']} w_\ell\wedge k\Big) - C_{i'}\Big(\sum_{\ell\in U_g[i']} w_\ell\wedge(k-1)\Big), \end{aligned}$$

and so,

$$\begin{aligned} \sum_{\sigma\in O^w}\sum_{i\in U_g[i']\cap Q^w(k)}\eta^\sigma_{(i,k)}(w,v_{g,\lambda^{C,i',k}}) &= \sum_{i\in U_g[i']\cap Q^w(k)}\sum_{\sigma\in O^w}\eta^\sigma_{(i,k)}(w,v_{g,\lambda^{C,i',k}}) \\ &= |O^w|\cdot\Big(C_{i'}\big(\sum_{\ell\in U_g[i']} w_\ell\wedge k\big) - C_{i'}\big(\sum_{\ell\in U_g[i']} w_\ell\wedge(k-1)\big)\Big) \end{aligned} \tag{15}$$

Next, let ℓ and ℓ' be two agents in $U_g[i']\cap Q^w(k)$ and let $inv^{\ell,\ell'}(\sigma)\in O^w$ be defined as:

- $inv^{\ell,\ell'}(\sigma)(\ell,k) = \sigma(\ell',k)$;
- $inv^{\ell,\ell'}(\sigma)(\ell',k) = \sigma(\ell,k)$;
- $inv^{\ell,\ell'}(\sigma)(i,j) = \sigma(i,j), \quad \forall (i,j) \in M^+ \setminus \{(\ell,k),(\ell',k)\}$.

Because $C_{i'}$ only depends on the sum of the amount of waste shipped by the agents of $U_g[i']$, it holds that

$$\eta^{\sigma}_{(i,k)}(w, v_{g,\lambda^{C,i',k}}) = \eta^{inv^{\ell,\ell'}(\sigma)}_{(i,k)}(w, v_{g,\lambda^{C,i',k}}).$$

From this observation, one concludes that

$$\begin{aligned} \sum_{\sigma \in O^w} \eta^{\sigma}_{(i,k)}(w, v_{g,\lambda^{C,i',k}}) &= \sum_{\sigma \in O^w} \eta^{inv^{\ell,\ell'}(\sigma)}_{(i,k)}(w, v_{g,\lambda^{C,i',k}}) \\ &= \sum_{\sigma' \in O^w} \eta^{\sigma'}_{(i,k)}(w, v_{g,\lambda^{C,i',k}}). \end{aligned} \tag{16}$$

By (15) and (16), it holds that

$$\sum_{\sigma \in O^w} \eta^{\sigma}_{(i,k)}(w, v_{g,\lambda^{C,i',k}}) = \frac{|O^w| \cdot \Big(C_{i'}\big(\sum_{\ell \in U_g[i']} w_\ell \wedge k\big) - C_{i'}\big(\sum_{\ell \in U_g[i']} w_\ell \wedge (k-1)\big)\Big)}{|U_g[i'] \cap Q^w(k)|}.$$

Substituting the above equality in (14), one obtains that $\phi(w, v_{g,\lambda^{C,i',k}}) = f^L(g, w, \lambda^{C,i',k})$, as desired. ■

8.5. Proof of Theorem 5

Proof of Theorem 5. **Uniqueness part**: We prove that Efficiency, Independence of more active agents, Inter balanced contributions for top agents, and Intra balanced contributions for top agents determine ϕ in a unique way. Assume that a value f on $\mathfrak{G}$ satisfies the above four axioms. The proof is by induction on $\sum_{i \in N} m_i$.

Induction basis: If $\sum_{i \in N} m_i = 1$, then there is exactly one active agent in the game, say i', with $m_{i'} = 1$. Therefore, $M^+ = \{(i',1)\}$ and, by Efficiency,

$$f_{(i',1)}(m, v) = v(0, \ldots, 1, \ldots, 0) = v(m),$$

meaning that $f = \phi$.

Induction hypothesis: Assume that $f = \phi$ for all multi-choice games in $\mathfrak{G}$ such that $\sum_{i \in N} m_i \leq p$, for some $p \geq 1$.

Induction step: Consider any multi-choice game $(m, v) \in \mathfrak{G}$ such that $\sum_{i \in N} m_i = p + 1$. Two separate cases are distinguished.

Case 1. Consider any $i \in N \setminus T(m)$. Consider any activity level $j \in M_i^+$ and any agent $i' \in T(m)$. By Independence of more active agents and the induction hypothesis, one obtains

$$f_{(i,j)}(m, v) = f_{(i,j)}(m - e^{i'}, v) = \phi_{(i,j)}(m - e^{i'}, v) = \phi_{(i,j)}(m, v),$$

where the first equality follows from Independence of more active agents applied to f, the second equality is a consequence of the induction hypothesis, and the third equality follows from Independence of more active agents applied to ϕ.

Case 2. Consider any $i \in T(m)$. First, Intra balanced contributions for top agents applied to f implies that

$$\forall i \in T(m), \forall j \in M_i^+, \quad f_{(i,j)}(m,v) - f_{(i,j)}(m-e^i,v) = f_{(i,\max(m))}(m,v). \tag{17}$$

In a similar way, Intra balanced contributions for top agents applied to ϕ leads to

$$\forall i \in T(m), \forall j \in M_i^+, \quad \phi_{(i,j)}(m,v) - \phi_{(i,j)}(m-e^i,v) = \phi_{(i,\max(m))}(m,v). \tag{18}$$

By induction hypothesis, $f_{(i,j)}(m-e^i,v) = \phi_{(i,j)}(m-e^i,v)$. Thus, subtracting (18) from (17), one obtains that

$$\forall i \in T(m), \forall j \in M_i^+, \quad f_{(i,j)}(m,v) - \phi_{(i,j)}(m,v) = f_{(i,\max(m))}(m,v) - \phi_{(i,\max(m))}(m,v), \tag{19}$$

which shows that the difference $f_{(i,j)}(m,v) - \phi_{(i,j)}(m,v)$ does not depend on the activity level j. On the other hand, for any pair $\{i,i'\} \subseteq T(m)$, Inter balanced contributions for top agents applied to f implies that

$$f_{(i,\max(m))}(m,v) - f_{(i,\max(m))}(m-e^{i'},v) = f_{(i',\max(m))}(m,v) - f_{(i',\max(m))}(m-e^i,v). \tag{20}$$

In a similar way, Inter balanced contributions for top agents applied to ϕ leads to

$$\phi_{(i,\max(m))}(m,v) - \phi_{(i,\max(m))}(m-e^{i'},v) = \phi_{(i',\max(m))}(m,v) - \phi_{(i',\max(m))}(m-e^i,v). \tag{21}$$

Subtracting (21) from (20) and using the induction hypothesis as above, one obtains that

$$f_{(i,\max(m)))}(m,v) - \phi_{(i,\max(m))}(m,v) = f_{(i',\max(m))}(m,v) - \phi_{(i',\max(m))}(m,v),$$

which proves that $f_{(i,\max(m))}(m,v) - \phi_{(i,\max(m))}(m,v)$ does not depend on the agent $i \in T(m)$. Taking into account (19), one finally obtains that

$$\exists c^{f,\phi} \in \mathbb{R} : \forall i \in T(m), \forall j \in \{1,\ldots,\max(m)\}, \quad f_{(i,j)}(m,v) - \phi_{(i,j)}(m,v) = c^{f,\phi}. \tag{22}$$

By Efficiency,

$$\begin{aligned}
v(m) &= \sum_{(i,j)\in M^+} f_{(i,j)}(m,v) \\
&= \sum_{\substack{(i,j)\in M^+: \\ i\in T(m)}} f_{(i,j)}(m,v) + \sum_{\substack{(i,j)\in M^+: \\ i\notin T(m)}} f_{(i,j)}(m,v) \\
&\overset{(22),\text{Case 1}}{=} \sum_{\substack{(i,j)\in M^+: \\ i\in T(m)}} (\phi_{(i,j)}(m,v) + c^{f,\phi}) + \sum_{\substack{(i,j)\in M^+: \\ i\notin T(m)}} \phi_{(i,j)}(m,v) \\
&= v(m) + \max(m)|T(m)|c^{f,\phi},
\end{aligned} \tag{23}$$

which implies that $c^{f,\phi} = 0$. Therefore,

$$\forall i \in T(m), \forall j \in M_i^+, \quad f_{(i,j)}(m,v) = \phi_{(i,j)}(m,v),$$

which concludes the proof of the induction step.

Existence part: We prove that ϕ satisfies the four axioms. Consider any $(m,v) \in \mathfrak{G}$. For each ordering $\sigma \in O^m$,

$$\begin{aligned}
\sum_{(i,j)\in M^+} \lambda^\sigma_{(i,j)}(m,v) &= \sum_{i\in N}\sum_{j=1}^{m_i}\sum_{k=j}^{m_i} \frac{\eta^\sigma_{(i,k)}(m,v)}{k} \\
&= \sum_{i\in N}\sum_{k=1}^{m_i}\sum_{j=1}^{k} \frac{\eta^\sigma_{(i,k)}(m,v)}{k} \\
&= \sum_{i\in N}\sum_{k=1}^{m_i} \eta^\sigma_{(i,k)}(m,v) \\
&= \sum_{(i,j)\in M^+} \eta^\sigma_{(i,k)}(m,v).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{(i,j)\in M^+} \lambda^\sigma_{(i,j)}(m,v) &= \sum_{(i,k)\in M^+} \big(v(s^{\sigma,(i,k)} - v(s^{\sigma,p^\sigma(i,k)}\big) \\
&= \sum_{(i,k)\in M^+} v(s^{\sigma,(i,k)}) - \sum_{(i,k)\in M^+,\sigma(i,k)\neq 1} v(s^{\sigma,p^\sigma(i,k)}), \\
&= \sum_{(i,k)\in M^+} v(s^{\sigma,(i,k)}) - \sum_{(i,k)\in M^+,\sigma(i,k)\neq |M^+|} v(s^{\sigma,(i,k)}) \\
&= v(s^{\sigma,\sigma^{-1}(|M^+|)}) \\
&= v(m). \qquad (24)
\end{aligned}$$

By (24) and the definition of ϕ, it follows that

$$\begin{aligned}
\sum_{(i,j)\in M^+} \phi_{(i,j)}(m,v) &= \frac{1}{|O^m|}\sum_{\sigma\in O^m}\sum_{(i,j)\in M^+} \lambda^\sigma_{(i,j)}(m,v) \\
&= \frac{1}{|O^m|}\sum_{\sigma\in O^m} v(m) \\
&= v(m),
\end{aligned}$$

which proves that ϕ satisfies Efficiency.

Pick any pair of agents $\{i,i'\} \subseteq N$ such that $m_i < m_{i'}$, and any $(i,j) \in M^+$. By definition of O^m, for each $\sigma \in O^m$, $\sigma(i,j) < \sigma(i',m_{i'})$. Therefore, for each $\sigma \in O^m$, $\eta^\sigma_{ij}(m,v) = \eta^\sigma_{ij}(m-e^{i'},v)$, and so, $\lambda^\sigma_{(i,j)}(m,v) = \lambda^\sigma_{(i,j)}(m-e^{i'},v)$. Because the latter equality holds for each $\sigma \in O^m$, one gets $\phi_{(i,j)}(m,v) = \phi_{(i,j)}(m-e^{i'},v)$. This show that ϕ satisfies Independence of more active agents.

Now, pick any pair of top agents $\{i, i'\} \subseteq T(m)$. By Point 2. of Proposition 1,

$$\phi_{(i,\max(m))}(m,v) - \phi_{(i,\max(m))}(m-e^{i'},v) = \sum_{\substack{t \leq m : T(t) \ni i, \\ \max(t) = \max(m)}} \frac{\Delta_v(t)}{\max(t)|T(t)|} - \sum_{\substack{t \leq m - e^{i'}, T(t) \ni i, \\ \max(t) = \max(m - e^{i'})}} \frac{\Delta_v(t)}{\max(t)|T(t)|}.$$

A coalition t appears in the first sum and not in the second one if and only if $t_{i'} = \max(m)$. Thus,

$$\begin{aligned} \phi_{(i,\max(m))}(m,v) - \phi_{(i,\max(m))}(m-e^{i'},v) &= \sum_{\substack{t \leq m, T(t) \supseteq \{i,i'\}, \\ \max(t) = \max(m)}} \frac{\Delta_v(t)}{\max(t)|T(t)|} \\ &= \phi_{(i',\max(m))}(m,v) - \phi_{(i',\max(m))}(m-e^{i},v), \end{aligned}$$

where the second equality follows from a symmetric argument. This ensures that ϕ satisfies Inter balanced contributions for top agents.

Finally, similarly as above, by choosing a top agent $i \in T(m)$ and any pair of distinct activity levels $\{j, j'\} \subseteq \{1, \ldots, \max(m)\}$, one obtains that

$$\begin{aligned} \phi_{(i,j)}(m,v) - \phi_{(i,j)}(m-e^{i},v) &= \sum_{t \leq m, T(t) \ni i, \max(t) = \max(m)} \frac{\Delta_v(t)}{\max(t)|T(t)|} \\ &= \phi_{(i,j')}(m,v) - \phi_{(i,j')}(m-e^{i},v), \end{aligned}$$

which entails that ϕ satisfies Intra balanced contributions for top agents.

■

Logical independence of the axioms. The axioms invoked in Theorem 5 are logically independent, as shown by the following alternative values. The second and third examples are taken from Table 1.

- The null value on $\mathfrak{G}$ that distributes a zero payoff to each pair $(i,j) \in M_+$ satisfies all the axioms except Efficiency.
- The Shapley-like value ϕ^{DP} on $\mathfrak{G}$ defined as:

$$\forall (i,j) \in M^+, \quad \phi^{DP}_{(i,j)}(m,v) = \sum_{\substack{s \in \mathcal{M} \setminus \vec{0} : \\ s_i \geq j}} \frac{\Delta_s(v)}{|B(s)|}$$

  satisfies all the axioms except Independence of more active agents.

- The Shapley-like value ϕ^{LT} on $\mathfrak{G}$ defined as:

$$\forall (i,j) \in M_+, \quad \phi^{LT}_{(i,j)}(m,v) = \sum_{\substack{s \in \mathcal{M} \setminus \vec{0} : \\ T(s) \ni i, \max(s) = j}} \frac{\Delta_s(v)}{|T(s)|},$$

  satisfies all the axioms except Intra balanced contributions for top agents.

- Associate with each $i \in N$ a positive real number $\beta_i \in \mathbb{R}_{++}$. The value f^β on $\mathfrak{G}$ defined as:

$$\forall (i,j) \in M_+, \quad f^\beta_{(i,j)}(m,v) = \sum_{\substack{s \in \mathcal{M} \setminus \vec{0}: \\ T(s) \ni i, \max(s) = j}} \frac{\beta_i}{\sum_{i' \in T(s)} \beta_{i'}} \cdot \frac{\Delta_s(v)}{|\max(s)|},$$

satisfies all the axioms except Inter balanced contributions for top agents.

8.6. Proof of Theorem 6

Proof of Theorem 6. **Uniqueness part**: Consider any value f on $\mathfrak{G}$ satisfying Efficiency, Independence of more active agents, Intra balanced contributions for top agents, Additivity, Null max activity level out, and Equal treatment for top-veto agents. Pick any $(m,v) \in \mathfrak{G}$. By Additivity,

$$f(m,v) = \sum_{t \leq m} f(m, \Delta_v(t)u_t).$$

It must be shown that $\forall t \leq m$, $f(m, \Delta_v(t)u_t) = \phi(m, \Delta_v(t)u_t)$. To that end, consider any minimum activity level game $(m, \Delta_v(t)u_t)$. Each pair (i, m_i) where $m_i > t_i$, if it exists, is a null pair in $(m, \Delta_v(t)u_t)$. Applying successively Null max activity level out, one obtains that

$$\forall (i,j) \in M_+ : j \leq t_i, \quad f_{(i,j)}(m, \Delta_v(t)u_t) = f_{(i,j)}(t, \Delta_v(t)u_t).$$

Null max activity level out and Efficiency leads to

$$\begin{aligned} \forall (i,j) \in M_+ : j > t_i, \quad f_{(i,j)}(m, \Delta_v(t)u_t) &= 0 \\ &= \phi_{(i,j)}(m, \Delta_v(t)u_t), \end{aligned}$$

as desired. Pick any $i' \in T(t)$. By Independence of more active agents,

$$\forall i \notin T(t), \forall j \in \{1, \ldots, t_i\}, \quad f_{(i,j)}(t, \Delta_v(t)u_t) = f_{(i,j)}(t - e^{i'}, \Delta_v(t)u_t).$$

Note that $(t - e^{i'}, \Delta_v(t)u_t)$ is the null game $(t - e^{i'}, \mathbf{0})$. Additivity applied to a null game implies zero payoffs for each pair $(i,j) \in M_+$. It follows that

$$\begin{aligned} \forall i \in N \setminus T(t), \forall j \in \{1, \ldots, t_i\}, \quad f_{(i,j)}(t, \Delta_v(t)u_t) &= 0 \\ &= \phi_{(i,j)}(t, \Delta_v(t)u_t). \end{aligned}$$

It remains to deal with the agents in $T(t)$. Note that each $i \in T(t)$ is a top-veto agent in $(t, \Delta_v(t)u_t)$. By Equal treatment of top-veto agents applied to f and ϕ,

$$\forall i, i' \in T(t), \quad f_{(i,\max(t))}(t, \Delta_v(t)u_t) = f_{(i',\max(t))}(t, \Delta_v(t)u_t),$$

$$\forall i, i' \in T(t), \quad \phi_{(i,\max(t))}(t, \Delta_v(t)u_t) = \phi_{(i',\max(t))}(t, \Delta_v(t)u_t),$$

and so

$$f_{(i,\max(t))}(t, \Delta_v(t)u_t) - \phi_{(i,\max(t))}(t, \Delta_v(t)u_t) = f_{(i',\max(t))}(t, \Delta_v(t)u_t) - \phi_{(i',\max(t))}(t, \Delta_v(t)u_t). \quad (25)$$

Using Intra balanced contributions for top agents, Independence of more active agents, and proceeding as in **Case 2** in the proof of Theorem 5, one obtains that

$$\forall i \in T(t), \exists c_i^{f,\phi} \in \mathbb{R}, \forall j \in \{1, \ldots, \max(m)\}, \quad f_{(i,j)}(t, \Delta_v(t)u_t) - \phi_{(i,j)}(t, \Delta_v(t)u_t) = c_i^{f,\phi}, \quad (26)$$

which shows that the above difference does not depend of the activity level j. Combining (25) and (26), one deduces that

$$\exists c^{f,\phi} \in \mathbb{R} : \forall i \in T(m), \forall j \in \{1, \ldots, \max(m)\}, \quad f_{(i,j)}(t, \Delta_v(t)u_t) - \phi_{(i,j)}(t, \Delta_v(t)u_t) = c^{f,\phi},$$

which ensures that the above difference does not depend upon either j or i. Using Efficiency as in the proof of Theorem 5, one easily obtains $c^{f,\phi} = 0$, and so $f_{(i,j)}(t, \Delta_v(t)u_t) = \phi_{(i,j)}(t, \Delta_v(t)u_t)$ for each $i \in T(t)$, as desired.

Existence part: One verifies that the Priority Shapley value ϕ satisfies the six axioms of the statement of Theorem 6. It was seen earlier that ϕ satisfies Efficiency, Independence of more active agents, Intra balanced contributions for top agents, and Additivity.

Let any $(m, v) \in \mathfrak{G}$, and let $(i', m_{i'})$ be a null maximal pair. For each $\sigma \in O^m$, let $p^{i'}(\sigma)$ be the ordering of $O^{m-e^{i'}}$ induced by σ. Obviously, by definition of a null maximal pair, $\eta^{\sigma}_{(i',m_{i'})}(m, v) = 0$, and it holds that

$$\forall (i,j) \in M^+ \setminus \{(i', m^{i'})\}, \quad \eta^{\sigma}_{(i,j)}(m, v) = \eta^{p^{i'}(\sigma)}_{(i,j)}(m - e^{i'}, v),$$

and so

$$\forall (i,j) \in M^+ \setminus \{(i', m^{i'})\}, \quad \lambda^{\sigma}_{(i,j)}(m, v) = \lambda^{p^{i'}(\sigma)}_{(i,j)}(m - e^{i'}, v). \tag{27}$$

Moreover, for each $\sigma' \in O^{m-e^{i'}}$, there are $q^m_{m_{i'}}$ orderings $\sigma \in O^m$ such that $p^{i'}(\sigma) = \sigma'$. Thus,

$$\begin{aligned}
\forall (i,j) \in M^+ \setminus \{(i', m^{i'})\}, \quad \phi_{(i,j)}(m, v) &= \frac{1}{|O^m|} \sum_{\sigma \in O^m} \lambda^{\sigma}_{(i,j)}(m, v) \\
&= \frac{1}{|O^m|} \sum_{\sigma' \in O^{m-e^{i'}}} \sum_{\substack{\sigma \in O^m : \\ p^{i'}(\sigma) = \sigma'}} \lambda^{\sigma}_{(i,j)}(m, v) \\
&\overset{(27)}{=} \frac{1}{q^m_{m_{i'}} |O^{m-e^{i'}}|} \sum_{\sigma' \in O^{m-e^{i'}}} q^m_{m_{i'}} \lambda^{\sigma'}_{(i,j)}(m - e^{i'}, v) \\
&= \phi_{(i,j)}(m - e^{i'} v),
\end{aligned}$$

which ensures that ϕ satisfies Null max activity level out.

Now, consider any $(m, v) \in \mathfrak{G}$ that contains two distinct top-veto agents $i, i' \in T(m)$. For $\sigma \in O^m$, let $inv^{i,i'\max(m)}(\sigma) \in O^m$ defined as:

- $inv^{i,i',\max(m)}(\sigma)(i, \max(m)) = \sigma(i', \max(m))$;
- $inv^{i,i',\max(m)}(\sigma)(i', \max(m)) = \sigma(i, \max(m))$;
- $inv^{i,i',\max(m)}(\ell, j) = \sigma(\ell, j), \quad \forall (\ell, j) \in M^+ \setminus \{(i, \max(m)), (i', \max(m))\}$.

Note that

$$s^{\sigma,(i',\max(m))} = s^{inv^{i,i',\max(m)}(\sigma),(i,\max(m))}.$$

From the above equality, one deduces that

$$\eta^{\sigma}_{(i,\max(m))}(m, v) = \eta^{inv^{i,i'\max(m)}(\sigma)}_{(i',\max(m))}(m, v).$$

Therefore, one obtains that

$$\begin{aligned}\forall \sigma \in O^m, \quad \lambda^{\sigma}_{(i,\max(m))}(m,v) &= \frac{\eta^{\sigma}_{(i,\max(m))}(m,v)}{\max(m)} \\ &= \frac{\eta^{inv^{i,i'\max(m)}(\sigma)}_{(i',\max(m))}(m,v)}{\max(m)} \\ &= \lambda^{inv^{i,i'\max(m)}(\sigma)}_{(i',\max(m))}(m,v), \end{aligned} \tag{28}$$

and so

$$\begin{aligned}\phi_{(i,\max(m))}(m,v) &= \frac{1}{|O^m|}\sum_{\sigma\in O^m}\lambda^{\sigma}_{(i,\max(m))}(m,v) \\ &= \frac{1}{|O^m|}\sum_{inv^{i,i'\max(m)}(\sigma)\in O^m}\lambda^{inv^{i,i',\max(m)}(\sigma)}_{(i',\max(m))}(m,v) \\ &= \phi_{(i',\max(m))}(m,v),\end{aligned}$$

which proves that ϕ satisfies Equal treatment for top-veto agents.

■

Logical independence of the axioms. The axioms invoked in Theorem 6 are logically independent, as shown by the following alternative values. The first four are those used to show the logical independence of the axioms invoked in Theorem 5.

- The null value on $\mathfrak{G}$ that distributes a zero payoff to each pair $(i,j) \in M_+$ satisfies all the axioms except Efficiency.
- The Shapley-like value ϕ^{DP} on $\mathfrak{G}$ satisfies all the axioms except Independence of more active agents.
- The Shapley-like value ϕ^{LT} on $\mathfrak{G}$ satisfies all the axioms except Intra balanced contributions for top agents.
- The value f^{β} on $\mathfrak{G}$ satisfies all the axioms except Equal treatment of top-veto agents.
- The value f on $\mathfrak{G}$ defined as:

$$\forall (i,j)\in M_+, \quad f_{(i,j)}(m,v) = \sum_{\substack{s\in\mathcal{M}\setminus\vec{0}:\\ T(s)\ni i,\max(s)\geq j}} \frac{(v((m\wedge\overrightarrow{(\max(s)-1)})+e^i))^2+1}{\sum_{i'\in T(s)}(v((m\wedge\overrightarrow{(\max(s)-1)})+e^{i'}))^2+1}\cdot\frac{\Delta_s(v)}{|\max(s)|},$$

  satisfies all the axioms except Additivity.

- The value f on $\mathfrak{G}$ defined as:

$$\forall (i,j)\in M^+, \quad f_{(i,j)}(m,v) = \sum_{\substack{k\geq j:\\ Q^m(k)\ni i}} \frac{1}{k}\cdot\frac{v(m\wedge\overrightarrow{k})-v(m\wedge\overrightarrow{(k-1)})}{|Q^w(k)|}$$

  satisfies all the axioms except Null max activity level out.