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Uniqueness of time-dependent inclusions in anisotropic heat conductive bodies

O. Poisson*

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Abstract

We consider an inverse boundary value problem for the heat equation with a non-smooth coefficient of conductivity which models the displacement of a moving body inside a nonhomogeneous background. We prove the uniqueness of the moving inclusion from the knowledge of the Dirichlet-to-Neumann operator by using a dynamical probe method.

Keywords: Inverse problem, Heat equation, Dynamical probe method.

AMS : 35R30, 35K05.

1 Introduction

1.1 Inverse heat conductivity problem

Let $T > 0$ and let Ω be a bounded domain in \mathbb{R}^3 , with a lipschitzian boundary $\Gamma = \partial\Omega$. Let us consider the anisotropic heat equation

$$\partial_t v - \operatorname{div}(\mathbf{a}\nabla v) = 0 \quad \text{in } \Omega_{0,T} \equiv \Omega \times (0, T), \quad (1)$$

where the operators div , the divergence, and ∇ , the gradient, are relative to the spatial variable x . In our model, the conductivity $\mathbf{a} = (a_{ij})_{1 \leq i, j \leq 3}$ is a 3×3 real symmetric matrix with positive bounded measurable coefficients of x . It satisfies the uniform elliptic condition:

there exists $\gamma_\infty > 0$ such that

$$\gamma_\infty^{-1}|\xi|^2 \leq \mathbf{a}\xi \cdot \xi \leq \gamma_\infty|\xi|^2, \quad \xi \in \mathbb{R}^3. \quad (2)$$

It is well-known that, for all $f \in L^2(0, T; H^{1/2}(\Gamma))$ and $v_0 \in L^2(\Omega)$, there exists only one solution $v = v(\mathbf{a}, v_0; f) \in H^1((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$ of (1) with the following initial boundary value problem:

$$\begin{cases} v = f & \text{on } \Gamma_{0,T} \equiv \Gamma \times (0, T), \\ v|_{t=0} = v_0 & \text{on } \Omega. \end{cases} \quad (3)$$

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See for example the book of Wloka[21]. Then, we can define the Dirichlet-to-Neumann map (D-N map) as

$$\Lambda_{\mathbf{a};v_0} : L^2((0, T); H^{1/2}(\Gamma)) \ni f \mapsto \mathbf{a} \nabla v(\mathbf{a}, v_0; f) \cdot \nu \in L^2((0, T); H^{-1/2}(\Gamma)),$$

where ν denotes the outer unit normal to Γ . In physical terms, $f = f(t, x)$ is the temperature distribution on the boundary and $\Lambda_{\mathbf{a};v_0}(f)$ is the resulting heat flux through the boundary.

In this article we are concerned with the Calderón inverse problem for (1) which is to determine \mathbf{a} from the knowledge of the D-N map $\Lambda_{v_0, \mathbf{a}}$. The conductivity \mathbf{a} consists in a non necessarily smooth background and an unknown inclusion $t \mapsto D_t \subset \Omega$ which moves continuously inside the body Ω . Thus, in our inverse problem, the function $\mathbf{a}|_{\Omega \setminus D_t}$ coincides with a measurable real matrix-function $\mathbf{b} \in L^\infty(\Omega)$ which satisfies (2) and represents the conductivity of a background medium, and so, is known. The inverse problem we address is to determine the moving inclusion $D = \cup_{0 \leq t \leq T} (D_t \times \{t\}) \subset \Omega_{0,T}$ from the knowledge of $\Lambda_{\mathbf{a};v_0}$.

Remark 1. *In our problem the value of the conductivity inside the inclusion, $\mathbf{a}|_{D_t}$, and the initial value of v, v_0 , are unknown but the article does not deal with their determination.*

1.2 Main assumptions

The two following assumptions were already considered by several authors in the isotropic situation [3],[15],[19].

(H0): there exists a positive constant δ_1 such that

$$(H0a) : \quad \mathbf{b}^{-1} - \mathbf{a}^{-1} \leq -\delta_1 < 0, \quad \mathbf{b} - \mathbf{a} \geq \delta_1 > 0 \quad \text{in } D,$$

or

$$(H0b) : \quad \mathbf{b}^{-1} - \mathbf{a}^{-1} \geq \delta_1 > 0, \quad \mathbf{b} - \mathbf{a} \leq -\delta_1 < 0 \quad \text{in } D.$$

(H1): for all $t \in [0, T]$, the set $\mathbb{R}^3 \setminus \overline{D_t}$ is connected.

Because of technical limitations of our method when \mathbf{b} is not sufficiently smooth, we need some additional geometrical assumptions on D . For a point $x \in \mathbb{R}^3$ and a non-empty set $E \subset \mathbb{R}^3$ we denote by $d(x, E)$ the quantity $\inf_{z \in E} |x - z|$ and by $|E|$ the Lebesgue-measure of E .

(H2): $t \mapsto D_t$ is lipschitzian in the following sense:

there exists $K_D > 0$ such that for all $x \in \overline{\Omega}$ the mapping $t \mapsto d(x, \Omega \setminus D_t)$ is lipschitzian in $[0, T]$ with lipschitzian constant K_D and the mapping $t \mapsto d(x, D_t)$ is lipschitzian at all $s \in [0, T]$ such that $D_s \neq \emptyset$ with lipschitzian constant K_D .

(H3):

(H3a): for all $t \in [0, T]$, D_t satisfies the exterior cone property, i.e., there exists $\rho(t) > 0$ such that for all $z \in \partial D_t$, there exists an open cylindrical cone $C_o(z, \rho) \subset \mathbb{R}^3 \setminus \overline{D_t}$ with summit z , hightness ρ and volume ρ^3 ,

and

(H3b): there exists $L_D \in (0, 1)$ such that $|D_t \cap B(z, r)| \geq L_D \min(|D_t|, |B(z, r)|), \forall r > 0, z \in \partial D_t, t \in [0, T]$.

The Runge approximation method in the dynamical probe method is based on the uniqueness property (UC) which holds if the conductivity is constant. However V. Isakov has shown that (UC) can fail if the conductivity is not sufficiently regular [15]. Therefore we add the following assumption on \mathbf{b} :

(UC) in Ω - Let $\omega \subset \Omega$ be a sufficiently smooth domain, let $a < b$ and let $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ be such that $\partial_t u - \operatorname{div}(\mathbf{b}\nabla u) = 0$ in $\omega \times (a, b)$ and $u = \mathbf{a}\nabla u \cdot \nu = 0$ on $S \times (a, b)$, where S is a non-empty open subset of $\partial\omega$. Then, necessarily, $u = 0$ in $\omega \times (a, b)$.

Remark 2. *The above definition of (UC) is independent of the choice of the time-interval $[0, T]$ since in our work we assume that \mathbf{b} does not depend on the variable t .*

Remark 3. *Condition (UC) holds if \mathbf{b} is lipschitzian or piecewise smooth: see the results of Vessella [20, chap 5].*

1.3 Main Result

Here we state our uniqueness result for the above inverse problem. Let $v_0, v'_0 \in L^2(\Omega)$, be two conductivities \mathbf{a}, \mathbf{a}' satisfying (H0)-(H3) and (UC). Let D' be the inclusion related to \mathbf{a}' .

Theorem 1. *Assume that $\Lambda_{v_0, \mathbf{a}} = \Lambda_{v'_0, \mathbf{a}'}$. Then, $D = D'$.*

Remark 4. *Our proof of Theorem 1 is not completely constructive, although it is based on the same dynamical method developed by the author who showed a (theoretical) reconstruction of D from the knowledge of $\Lambda_{v_0, \mathbf{a}}$ [19].*

Remark 5. *We shall proof Theorem 1 with the following assumption:*

$$\overline{D(t)} \subset \Omega, \quad t \in [0, T].$$

Therefore we replace (H1) by:

(H1'): *one has $\overline{D(t)} \subset \Omega$, and the set $\Omega \setminus \overline{D_t}$ is connected, for all $t \in [0, T]$.*

The general proof of Theorem 1 where $D(t)$ may touch $\partial\Omega$ is easily get from the following modification on the case (H1'):

- We consider a large smooth bounded domain Ω' containing $\overline{\Omega}$ and we put $\mathbf{b} = I_3$ (the 3×3 identity matrix) in $\Omega' \setminus \Omega$.
- (If necessary)¹ (UC) is assumed with Ω replaced by Ω' .

Remark 6. *The proof of Theorem 1 will show that (H0) can be extended to the following situation:*

(H0') *There exist positive constants ε_0, δ_1 , such that for $(x, t) \in \overline{D}$,*

$$\mathbf{b}^{-1}(x) - \mathbf{a}^{-1}(x) \leq -\delta_1 < 0, \quad \mathbf{b}(x) - \mathbf{a}(x) \geq \delta_1 > 0 \quad \text{if } d(x, \partial D_t) \leq \varepsilon_0,$$

or

$$\mathbf{b}^{-1}(x) - \mathbf{a}^{-1}(x) \geq \delta_1 > 0, \quad \mathbf{b}(x) - \mathbf{a}|_{D_t}(x) \leq -\delta_1 < 0 \quad \text{if } d(x, \partial D_t) \leq \varepsilon_0,$$

¹the question that (UC) in Ω would imply (UC) in Ω' is out of the scope of this article

1.4 Outline

In Section 2 we recall the basis of the dynamical probe method, the Runge approximation method and we construct indicator and pre-indicator functions from special Cauchy boundary data. In Section 3 we state the lower and upper estimates on the indicator function from which the proof of our main Theorem 1 can be achieved in Section 4. In Section 5 we develop the technical results on which the proof of the estimates of Section 3 is based.

2 The dynamical probe method (DPM) with special solutions of the heat equation

2.1 Notations

Let us give some notations for this paper. For $E \subset \mathbb{R}^3$, $a < b$, and for $U \subset \mathbb{R}^3 \times \mathbb{R}$, we put $E_{a,b} = E \times (a, b)$ and $U_t \equiv \{x \in \mathbb{R}^3 \mid (x, t) \in U\}$.

For non-negative integers p, q or $p = 1/2$, $H^p(\Omega)$, $H^p(\partial\Omega)$ and $H^{p,q}(\Omega_{(a,b)})$ denote the usual Sobolev spaces where the superscripts p and q indicate the regularity with respect to x and t , respectively. For an open set $U \subset \mathbb{R}^4$ with Lipschitz boundary ∂U , $H^{p,q}(U)$ is defined likewise. More precisely, $g \in H^{p,q}(U)$ if and only if there exists $G \in H^{p,q}(\mathbb{R}^4)$ with $G = g$ in U . If it is the case, $\|g\|_{H^{p,q}(U)}$ is defined to be

$$\|g\|_{H^{p,q}} := \inf \|G\|_{H^{p,q}(\mathbb{R}^4)},$$

where the infimum is taken over all G such that $G = g$ in U . Let X be a normed space of functions. A function $f(x, t)$ is said to be in $L^2((0, T); X)$ if $f(\cdot, t) \in X$ for almost all $t \in (0, T)$ and

$$\|f\|_{L^2((0,T);X)}^2 := \int_0^T \|f(\cdot, t)\|_{L^2(X)}^2 dt < \infty.$$

(For more details, we refer to J.L. Lions and E. Magenes [17]).

We write $\mathcal{L}_{\mathbf{a}} := \partial_t - \operatorname{div}(\mathbf{a}\nabla \cdot)$, so $\mathcal{L}_I := \partial_t - \Delta$ for the homogeneous case. Similarly, we consider operator for the backward related heat equation, $\mathcal{L}_{\mathbf{a}}^* := -\partial_t - \operatorname{div}(\mathbf{a}\nabla \cdot)$. We denote by $B(r)$ any ball of radius $r > 0$ in \mathbb{R}^3 . The open ball $\{x \in \mathbb{R}^3; |y - x| < r\}$, $r > 0$, is denoted $B(y, r)$.

We denote by $\mathbf{d}(t)$ the distance between $y(t)$ and D_t if $D_t \neq \emptyset$, i.e., $\mathbf{d}(t) = d(y(t), D_t)$. If $D_t = \emptyset$ then we put $\mathbf{d}(t) = +\infty$, $1/\mathbf{d}(t) = 0$.

If $\xi \in \mathbb{R}^3$ then $|\xi|$ denotes the euclidian norm of ξ and if \mathbf{m} is a 3×3 real matrix then $|\mathbf{m}| := \sup_{\xi \in \mathbb{R}^3, |\xi|=1} |\mathbf{m}\xi \cdot \xi|$.

2.2 Brief history of the determination of an inclusion from the D-N map

The determination of a sufficiently smooth moving inclusion inside an homogeneous body was stated by A. Elayyan and V. Isakov [4]. Their proof is by contradiction. DPM for (1) is an extension of Ikehata's probe method which was developed for the elliptic equation $\operatorname{div}(\mathbf{a}\nabla v) = 0$ where \mathbf{a} may be tensorial [13]. In the parabolic situation, DPM was firstly presented by Y. Daido, H. Kang and G. Nakamura in the case where the background is homogeneous and $D_t \in C^2$ for all t [3]. But there,

although a part of DPM works for all spatial dimension n , the reconstruction of D was proved only in the case $n = 1$. DPM of Y. Daido, H. Kang and G. Nakamura made the Runge approximation of the fundamental solution of the operator \mathcal{L}_I . (Note that an error in this work was corrected by V. Isakov, K. Kim, G. Nakamura [15]). Unlike to the DPM of Y. Daido, H. Kang, G. Nakamura, extending the method of A. Elayyan and V. Isakov by a more quantitative version which requires more regularity, M. Di Cristo and S. Vessella proved the log-stability of $\Lambda_{\mathbf{a},0} \mapsto D$ in the scalar case ($\mathbf{a} = aI_3$) [2].

Returning to DPM, the author used "special solutions" for the classical heat operator which are more convenient functions than the basic fundamental solutions $\Gamma(x - y, t - s)$, because their behaviour in time and space are sufficiently separated [19]. Since the background was homogeneous, the DPM of the author can reconstruct any spatially irregular inclusion as in the elliptic situation [19].

However, in our situation we are limited to inclusions with some kind of lipshitzian regularity (see (H2), (H3)). Moreover the negative part " $-C_M \mathbf{d}(t)^2$ " in (30) makes the reconstruction process unclear so the proof of Theorem 1 is by contradiction only.

2.3 Runge approximation method

The Runge approximation method for the operator unperturbed operator \mathcal{L}_I with the homogeneous conductivity $\mathbf{a} = I_3$ was developed first by Y. Daido, H. Kang, G. Nakamura, then by V. Isakov, K. Kim, G. Nakamura [3], [15].

Let $\Sigma : [0, T] \ni t \mapsto y(t) \in \mathbb{R}^3 \setminus \overline{D}_t$ be a lipshitzian curve which does not touch D . We extend Σ to $t \in \mathbb{R}$ by putting $y(t) = y(T)$ for $t \geq T$ and $y(t) = y(0)$ for $t \leq 0$. Then, thanks to (H1'), there exists an open set $U \subset \Omega \times \mathbb{R}$ containing \overline{D} and satisfying

$$\begin{cases} \partial U \text{ is lipshitzian,} \\ \text{dist}(U, \Sigma) := \inf\{|x - y|; x \in U, y \in \Sigma\} > 0, \\ \Omega \setminus \overline{U}_t \text{ is connected, } t \in \mathbb{R}. \end{cases}$$

The Runge approximation method works thanks to (UC) notably, and gives the following result [3, 15, 19]. For $\tau > 0$ we denote $\Sigma^\tau = \cup_{t \in \mathbb{R}} B(y(t), 1/\tau) \times \{t\}$.

Proposition 1. *Assume (H1') and (UC). Let Σ and U be as above. Let $u \in H^{1,0}(\Omega_{(0,T)}) \cap H^{0,1}(\Omega_{(0,T)})$ be a solution of $\mathcal{L}_{\mathbf{b}}u = 0$ in $\Omega_{(-1,T+1)} \setminus \Sigma^\tau$. Then for $\tau > \inf\{r > 0 \mid \text{dist}(U, \Sigma^r) > 0\}$ there exists a sequence $u_j \in H^{1,0}(\Omega_{(-1,T+1)}) \cap H^{0,1}(\Omega_{(-1,T+1)})$ such that*

$$\begin{cases} \mathcal{L}_{\mathbf{b}}u_j = 0 & \text{in } \Omega_{(-1,T+1)}, \\ u_j \rightarrow u & \text{in } H^{1,0}(U) \cap H^{0,1}(U), \\ u_j(0) = u(0) & \text{in } L^2(\Omega). \end{cases}$$

2.4 Heat Kernels

In many researchs devoted to inverse problems for parabolic equations, the background is homogeneous, i.e., $\mathbf{b} = I_3$. In such a classical situation, the heat operator is $\partial_t - \Delta$ and its usual kernel $\Gamma(x, t)$ has many properties, as

1. It is explicit:

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^3.$$

2. It satisfies

$$\Gamma(x, t) \leq \frac{C}{\sqrt{t}} |\nabla \Gamma(x, t)|, \quad t > 0, \quad x \in \mathbb{R}^3,$$

for some $C > 0$. Hence, $\Gamma(x, t)$ is small compared to $|\nabla \Gamma(x, t)|$ as $t \rightarrow 0$.

3. Thanks to the Laplace transform $\int_0^\infty \cdot e^{-\tau^2 t} dt$ of $\partial_t - \Delta$, we consider similarly the elliptic operator $-\Delta + \tau^2$ with the (large) real parameter $\tau > 0$. Its kernel $E(x; \tau)$ is explicit too:

$$E(x; \tau) = \int_0^\infty \Gamma(x, t) e^{-\tau^2 t} dt = \frac{e^{-\tau|x|}}{4\pi|x|}, \quad x \in \mathbb{R}^3.$$

4. It satisfies

$$E(x; \tau) \leq \tau |\nabla E(x; \tau)|, \quad x \in \mathbb{R}^3.$$

Hence, $E(x; \tau)$ is small compared to $|\nabla E(x; \tau)|$ as $\tau \rightarrow \infty$, uniformly in all bounded set of $\mathbb{R}^3 \setminus \{0\}$. This fact was exploited by the author [19].

Let us come back to the heat equation with a general conductivity \mathbf{b} . We put $\mathbf{b}(x) = I_3$ for $x \in \mathbb{R}^3 \setminus \bar{\Omega}$.

For $y \in \mathbb{R}^3$, we denote by $G_y \in C(\mathbb{R}; L^2(\mathbb{R}^3))$ the fundamental solution of

$$\mathcal{L}_{\mathbf{b}} G_y = \delta_{(y, 0)},$$

which satisfies

$$G_y(x, t) = 0, \quad t < 0.$$

We have the estimate:

$$\frac{\kappa e^{-\frac{|x-y|^2}{4\kappa^2 t}}}{t^{3/2}} \leq G_y(x, t) \leq \frac{e^{-\frac{\kappa^2 |x-y|^2}{4t}}}{\kappa t^{3/2}}, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (4)$$

for some constant $\kappa = \kappa(\mathbf{b}) \in (0, 1)$. See the famous results of D. G. Aronson and J. Nash [1, 18].

For $\tau > 0$ we put the Laplace Transform of $G_y(x, t)$ as

$$p_\tau(x; y) := e^{-\tau^2 t} \int_{-\infty}^t e^{\tau^2 s} G_y(x, t-s) ds = \int_0^\infty e^{-\tau^2 s} G_y(x, s) ds. \quad (5)$$

Let us observe that $p_\tau(\cdot; y)$ belongs to $H_{loc}^1(\mathbb{R}^3 \setminus \{y\})$ and, thanks to (4), satisfies

$$(-\operatorname{div}(\mathbf{b}\nabla \cdot) + \tau^2) p_\tau(\cdot; y) = \delta_y(\cdot), \quad (6)$$

$$2\sqrt{\pi} \frac{\kappa^2 e^{-\frac{\tau}{\kappa}|x-y|}}{|x-y|} \leq p_\tau(x; y) \leq 2\sqrt{\pi} \frac{e^{-\kappa\tau|x-y|}}{\kappa^2|x-y|}, \quad x \in \mathbb{R}^3 \setminus \{y\}. \quad (7)$$

This is also a consequence of the works of Nash and Aronson.

2.5 Special solutions

Let us consider a lipschitzian curve $\Sigma \subset \mathbb{R}^3 \times \mathbb{R}$ as in Section 2.3, and fix $\theta \in (0, T)$. Let $\mu \geq 1$ be another positive parameter that we shall precise later.

The author considered special solutions related to the following functions (with other notations and with $\mathbf{b} \equiv I_3$):

$$\begin{aligned} U_{OP}(x, t) &:= e^{\tau^2(T+t)} \int_0^\infty e^{\tau\mu(|t-\theta-s|-|t-\theta|)} \Gamma(x - y(t-s), s) e^{-\tau^2 s} ds, \\ U_{OP}^*(x, t) &:= e^{-\tau^2(T+t)} \int_0^\infty e^{\tau\mu(|t-\theta+s|-|t-\theta|)} \Gamma(x - y(t+s), s) e^{-\tau^2 s} ds, \end{aligned}$$

[19]. In fact, U_{OP} and U_{OP}^* are respectively solutions of the following forward and backward heat equations:

$$\begin{aligned} \mathcal{L}_I U_{OP}(x, t) &= e^{\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \mathcal{L}_I^* U_{OP}^*(x, t) &= e^{-\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}. \end{aligned}$$

Moreover they satisfies

$$\begin{aligned} U_{OP}(x, t) &= \varphi(x, t) e^{\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)), \\ U_{OP}^*(x, t) &= \varphi^*(x, t) e^{-\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)), \end{aligned}$$

such that, for some $C = C(R, \mu) > 0$ and all $\tau \geq C$,

$$\frac{1}{C} \leq |\varphi(x, t)| + |\varphi^*(x, t)| \leq C \quad \text{in } B(0, R) \times \mathbb{R}, \quad (8)$$

$$|\nabla \varphi(x, t)| + |\nabla \varphi^*(x, t)| \leq C \quad \text{in } B(0, R) \times \mathbb{R}, \quad (9)$$

[19, Lemma 1]. With the general conductivity \mathbf{b} , we construct here special solutions u_τ and u_τ^* as follows. Let us put

$$m_\tau(x, t) = M_0(\tau|x - y(t)|), \quad t \in \mathbb{R}, \quad (10)$$

where M_0 is defined by $M_0(r) = |1 - r| 1_{|r| \leq 1}$. Hence m_τ is a lipschitzian function with support closed to Σ as $\tau \gg 1$. We then put, for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$\begin{aligned} u_\tau(x, t) &= \int_{s \in \mathbb{R}} \int_{y \in \mathbb{R}^3} e^{\tau^2(s+T)} e^{-\tau\mu|s-\theta|} m(y, s) G_y(x, t-s) dy ds \quad (11) \\ &= \int_{s=0}^\infty \int_{y \in \mathbb{R}^3} e^{\tau^2(T+t-s)} e^{-\tau\mu|t-\theta-s|} m(y, t-s) G_y(x, s) dy ds, \end{aligned}$$

$$\begin{aligned} u_\tau^*(x, t) &= \int_{s \in \mathbb{R}} \int_{y \in \mathbb{R}^3} e^{-\tau^2(T+s)} e^{-\tau\mu|s-\theta|} m_\tau(y, s) G_y(x, s-t) dy ds \quad (12) \\ &= \int_{s=0}^\infty \int_{y \in \mathbb{R}^3} e^{-\tau^2(T+t+s)} e^{-\tau\mu|t-\theta+s|} m_\tau(y, t+s) G_y(x, s) dy ds. \end{aligned}$$

The functions u_τ and $u_\tau^*(x, t)$ are positive and satisfy

$$\begin{aligned} \mathcal{L}_\mathbf{b} u_\tau(x, t) &= e^{\tau^2(t+T)} e^{-\tau\mu|t-\theta|} m(x, t) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \quad (13) \\ \mathcal{L}_\mathbf{b}^* u_\tau^*(x, t) &= e^{-\tau^2(T+t)} e^{-\tau\mu|t-\theta|} m_\tau(x, t) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}. \end{aligned}$$

Remark 7. If $m_\tau(x, t)$ was replaced by $\delta(x - y(t))$ then it would be difficult to make the estimation of $\dot{y}(s)\nabla_y G_{y(t-s)}(x, t)$ that would appear in the expression of $\partial_t u_\tau$.

We then expect that

$$u_\tau(x, t) \stackrel{\tau \rightarrow \infty}{\simeq} e^{\tau^2(T+t)} e^{-\tau\mu|t-\theta|} \tau^{-3} p_\tau(x, y(t)), \quad (14)$$

$$u_\tau^*(x, t) \stackrel{\tau \rightarrow \infty}{\simeq} e^{-\tau^2(T+t)} e^{-\tau\mu|t-\theta|} \tau^{-3} p_\tau(x, y(t)), \quad (15)$$

where the meaning of " \simeq " will be clarified shortly. Since the comparison requires the time-derivatives of $u_\tau(x, t)$ or $u_\tau^*(x, t)$ and remembering Remark 7, we introduce the following smooth approximation of $p_\tau(x; y(t))$:

$$P_\tau(x, t) := \int_0^\infty \int_{\mathbb{R}^3} e^{-\tau^2 s} m_\tau(y, t) G_y(x, s) dy ds. \quad (16)$$

We then put

$$q_\tau(x, t) := e^{-\tau^2(T+t)} u_\tau(x, t) - e^{-\tau\mu|t-\theta|} P_\tau(x, t), \quad (17)$$

$$q_\tau^*(x, t) := e^{\tau^2(T+t)} u_\tau^*(x, t) - e^{-\tau\mu|t-\theta|} P_\tau(x, t). \quad (18)$$

The main difficulty in the proof of Theorem 1 is to prove that the quantity

$$R_0 := \int_D (|\nabla q_\tau(x, t)|^2 + |\nabla q_\tau^*(x, t)|^2) dx dt, \quad (19)$$

is negligible compared to $\int_D \tau^{-6} e^{-2\tau\mu|t-\theta|} |\nabla p_\tau(x, t)|^2 dx dt$ or, in an equivalent way (see Lemmas 5.3 and 5.4), to $\int_D \tau^{-4} e^{-2\tau\mu|t-\theta|} |p_\tau(x, t)|^2 dx dt$. We shall prove in Appendix the following Lemma.

Lemma 2.1. (Estimate of ∇q_τ in D_t). Let $M > 0$ and assume that $|\dot{y}|_\infty \leq M$. There exist positive constants $C_M, \tau_0(\Sigma)$ such that if $t \in [0, T]$ and $\tau > \tau_0$ then

$$\int_{D_t} (|\nabla q_\tau(x, t)|^2 + |\nabla q_\tau^*(x, t)|^2) dx \leq C_M \mathbf{d}(t)^2 \tau^{-4} e^{-2\tau\mu|t-\theta|} \int_{D_t} |p_\tau(x, y(t))|^2 dx. \quad (20)$$

(Remember that $\mathbf{d}(t) = d(y(t), D_t)$.) So R_0 is effectively "negligible" when the curve Σ is sufficiently close to D at least at time θ . This constraint is new compared to consequences of (8) and (9) (for which Assumption (H3) is in addition superfluous) and makes a theoretical reconstruction of D problematic, as opposite to the possible reconstruction proposed by the author [19].

2.6 Pre-indicator sequence and indicator function

As in section 2.3, we can consider sequences $(u_j)_j$ and $(u_j^*)_j$ such that $u_j \rightarrow u_\tau$ and $u_j^* \rightarrow u_\tau^*$ in the sense of Proposition 1. Considering $v_j = v(\mathbf{a}, v_0; u_j|_{\Gamma_{(0,T)}})$ and the solution $v_\tau \in H^1((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$ of

$$\begin{cases} \mathcal{L}_\mathbf{a} v_\tau &= \mathcal{L}_\mathbf{b} u_\tau, \\ v_\tau &= u_\tau \quad \text{on } \Gamma_{0,T}, \\ v_\tau|_{t=0} &= v_0 \quad \text{on } \Omega, \end{cases} \quad (21)$$

we put

$$w_\tau = v_\tau - u_\tau \quad (22)$$

and

$$\begin{aligned} I_j(\tau) &:= \int_{\Gamma \times [0, T]} (\Lambda_{\mathbf{a}; v_0}(u_j|_{\Gamma_{[0, T]}}) - \mathbf{b} \nabla u_j \cdot \nu) u_j^*|_{\Gamma \times [0, T]} d\sigma(x) dt, \\ I_\infty(\tau) &:= \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_\tau \nabla u_\tau^* dx dt + \int_{\Omega} [w_\tau u_\tau^*]_0^T dx, \end{aligned} \quad (23)$$

where $d\sigma(x)$ is the usual measure on the boundary Γ . The knowledge of $\Lambda_{\mathbf{a}; v_0}$ involves that of $I_j(\tau)$'s. Furthermore, as for the proofs in similar situations, Proposition 1 implies that

$$I_j(\tau) \rightarrow I_\infty(\tau) \in \mathbb{R} \quad \text{as } j \rightarrow \infty, \quad (24)$$

For a proof, see the works based on DPM [3, 19]. Hence, if (UC) holds, then the knowledge of $\Lambda_{\mathbf{a}; v_0}$ involves that of $I_\infty(\tau)$'s.

3 Estimates on the indicator function

In the following results the positive constants c, C, C_1 may depend on T, Ω, μ , on the the conductivity, but not on τ . We indicate when they depend on an upper bound M of $|\dot{y}|_\infty$ or on the initial data v_0 .

Lemma 3.1. *Under assumption (H0b) we have*

$$\begin{aligned} I_\infty(\tau) &\leq C \int_D e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dx dt \\ &\quad + C \int_D (|\nabla q_\tau|^2 + |\nabla q_\tau^*|^2) dx dt + 10(\|v_0\|_{L^2(\Omega)}^2 + d_\Omega) e^{-\tau\mu \min(T-\theta, \theta)}, \end{aligned} \quad (25)$$

and

$$\begin{aligned} I_\infty(\tau) &\geq \frac{1}{C} \int_D e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dx dt \\ &\quad - C \int_D (|\nabla q_\tau|^2 + |\nabla q_\tau^*|^2) dx dt - 10(\|v_0\|_{L^2(\Omega)}^2 + d_\Omega) e^{-\tau\mu \min(T-\theta, \theta)}, \end{aligned} \quad (26)$$

for some $C \geq 1$, for all $\tau > \mu + 1$, $\mu > 0$.

Proof in Appendix.

We put also

$$d_\Omega := \sup\{|x - y|; x, y \in \Omega\}, \quad (27)$$

$$\varepsilon_\Sigma := \inf_{t \in [0, T]} \mathbf{d}(t) > 0. \quad (28)$$

Lemma 3.2. *Let $M > 0$ and assume that $|\dot{y}|_\infty \leq M$. Then, under assumption (H0b), there exist positive constants $c = c(M)$, $C_1 = C_1(v_0)$, C_M , $\tau_0 = \tau_0(\Sigma)$, such that if $\tau > \tau_0$ we then have*

$$\begin{aligned} I_\infty(\tau) &\leq c\tau^{-4} \int_D e^{-2\tau\mu|t-\theta|} |p_\tau(x, y(t))|^2 dx dt \\ &\quad + C_1 e^{-\tau\mu \min(T-\theta, \theta)}, \end{aligned} \quad (29)$$

and

$$I_\infty(\tau) \geq \frac{1}{c} \tau^{-4} \int_D (1 - C_M \mathbf{d}(t)^2) e^{-2\tau\mu|t-\theta|} |p_\tau(x, y(t))|^2 dx dt \quad (30)$$

$$- C_1 e^{-\tau\mu \min(T-\theta, \theta)}.$$

The proof of Lemma 3.2 requires the developments of Section 5. Let us extend the function \ln to $(-\infty, 0]$ by putting $\ln(I) = -\infty$ if $I \leq 0$.

Lemma 3.3. *We assume that (H0b) is true. Let $\theta \in (0, T)$. Let us fix $\mu \geq 4\kappa^{-1}d_\Omega \max((T-\theta)^{-1}, \theta^{-1})$.*

1) *Let Σ be a lipschitzian curve such that $\varepsilon_\Sigma > 0$. We then have*

$$\limsup_{\tau \rightarrow \infty} \tau^{-1} \ln(I_\infty(\tau)) \leq -2\kappa\varepsilon_\Sigma. \quad (31)$$

2) *Assume that $D_\theta \neq \emptyset$. Let $M > 0, \alpha > 0$. Let $(\Sigma = \Sigma(\varepsilon))_{0 < \varepsilon \leq \alpha^2}$ be a family of curves as in Section 2.3, such that we have*

$$\begin{cases} |\dot{y}(\cdot)|_\infty \leq M, \\ \mathbf{d}(t) \leq \frac{2}{\alpha^3} \varepsilon \quad \text{for } |t - \theta| \leq \varepsilon, \\ \mathbf{d}(t) \in \left[\frac{1}{2\alpha} |t - \theta|, \frac{2}{\alpha^3} |t - \theta| \right] \quad \text{for } \varepsilon \leq |t - \theta| \leq \alpha^2, \\ \mathbf{d}(t) \geq \alpha/2 \quad \text{for } |t - \theta| \geq \alpha^2, \end{cases} \quad (32)$$

where $\mathbf{d}(t) := d(y(t), D_t)$. Then there exists $\varepsilon_1 \in (0, \alpha^2]$ such that for $0 < \varepsilon \leq \varepsilon_1$ we have

$$\liminf_{\tau \rightarrow \infty} \tau^{-1} \ln(I_\infty(\tau)) \geq -(8\kappa^{-1}\alpha^{-3} + 4\mu)\varepsilon. \quad (33)$$

Proof in Appendix.

4 Proof of Theorem 1

We may assume that (H0b) holds, since the case where (H0a) holds is similar. Thanks to Remark 5 we have $\overline{D_t} \cup \overline{D'_t} \subset \Omega$, $t \in [0, T]$. Let us assume that $D \neq D'$. Then there exists $(z, \theta) \in \Omega \times [0, T]$ with $D_\theta \neq \emptyset$, $z \in \partial D_\theta$ and $z \notin \overline{D'_\theta}$ or with $D'_\theta \neq \emptyset$, $z \in \partial D'_\theta$ and $z \notin \overline{D_\theta}$. Thus, we consider for simplicity that $z \in \partial D_\theta$ and $z \notin \overline{D'_\theta}$. Thanks to (H2), $t \mapsto (D_t, D'_t)$ is continuous so we consider also that $0 < \theta < T$. In fact let us explain why can consider also that $z \in \partial D_\theta \setminus \overline{D'_t}$ if $|t - \theta| < \beta$ for some $\beta > 0$. If D'_t is void for $|t - \theta|$ sufficiently small then it is immediate, but if D'_t is not void for $|t - \theta|$ sufficiently small then we can't be sure that $d(z, D'_t) > 0$ when $t \simeq \theta$. However, in such a case, thanks to (H2), there exists a sequence $\theta_n \rightarrow \theta$ satisfying $D'_{\theta_n} \neq \emptyset$ and $\overline{D_{\theta_n}} \setminus \overline{D'_{\theta_n}} \neq \emptyset$. We then replace (z, θ) by another couple (z_n, θ_n) with $z_n \in \partial D_{\theta_n} \setminus \overline{D'_{\theta_n}}$. Then, since $D'_{\theta_n} \neq \emptyset$ and thanks to (H2), we have $z_n \notin \partial D'_t$ if $t \simeq \theta_n$.

So we can consider that

$$z \in \partial D_\theta \setminus \overline{D'_t} \text{ if } |t - \theta| \leq \beta \text{ for some } \beta > 0. \quad (34)$$

Let us construct a family of curves $\Sigma = \Sigma(\varepsilon)$ for $0 < \varepsilon \leq \alpha^2$, for some positive α such that (32) and

$$\varepsilon'_\Sigma := \inf_{0 \leq t \leq T} d(y(t), D'_t) \geq \alpha/2 \quad (35)$$

hold. In fact, since D_θ and D'_θ satisfy Assumptions (H1') and (H3b) then there exists a lipschitzian curve $\tilde{y} : [0, 1] \ni s \mapsto \tilde{y}(s) \in \mathbb{R}^3$ with Lipschitz constant \tilde{M} such that $\tilde{y}(0) = z$, $\tilde{y}(1) \notin \Omega$, $\tilde{y}(s) \notin \overline{D_\theta}$ for $s \neq 0$ and $\tilde{y}(s) \notin \overline{D'_t}$ for $s \in [0, 1]$ and $|t - \theta| \leq \beta$. Thanks to (H2), (H3a) and to (34) we have for all $s \in [0, 1]$

$$\begin{aligned} d(\tilde{y}(s), D'_t) &\geq \alpha - K_{D'}|t - \theta|, \\ d(\tilde{y}(s), D_t) &\in [\alpha s - K_D|t - \theta|, \frac{1}{\alpha}s + K_D|t - \theta|], \end{aligned} \quad (36)$$

where $\alpha > 0$ is sufficiently small. We may consider that

$$\alpha \leq \min(1, (2K_D)^{-1}, (2K_{D'})^{-1}, d(\partial\Omega, D_t), d(\partial\Omega, D'_t)), \quad t \in [0, T]. \quad (37)$$

Then we have

$$d(\tilde{y}(s), D'_t) \geq \alpha/2 \quad \text{for } |t - \theta| \leq \frac{\alpha}{2K_{D'}}, \quad s \in [0, 1]. \quad (38)$$

We put $y_0(t) = \tilde{y}(|t - \theta|/\alpha^2)$ for $|t - \theta| \leq \alpha^2$ and $y_0(t) = \tilde{y}(1)$ for $|t - \theta| \geq \alpha^2$. From (37), (38) or (36), and since $y_0(r) \notin \Omega$ for $|r - \theta| \geq \alpha^2$ we obtain

$$d(y_0(r), D'_t) \geq \alpha/2, \quad \text{for } t, r \in [0, T], \quad (39)$$

$$d(y_0(r), D_t) \geq \alpha/2, \quad \text{for } t, r \in [0, T], |r - \theta| \geq \alpha^2. \quad (40)$$

Then for all $\varepsilon \in (0, \alpha^2]$ we put

$$y(t) = \begin{cases} y_0(\theta + \varepsilon) & \text{for } |t - \theta| \leq \varepsilon \\ y_0(t) & \text{for } |t - \theta| \geq \varepsilon. \end{cases}$$

Thanks to (37), (36) (39), (40) we then obtain all the conditions of (32) with $M = \tilde{M}/\alpha^2$, and (35).

Let us denote by $I'_\infty(\tau)$ the indicator function for the conductivity \mathbf{a}' . Thanks to (31) of Lemma 3.3 we have

$$\limsup_{\tau \rightarrow \infty} \tau^{-1} \ln(I'_\infty(\tau)) \leq -\kappa\alpha,$$

and there exists $\varepsilon_1 \in (0, \alpha^2)$ such that for $\varepsilon \in (0, \varepsilon_1]$ we have, from (33),

$$\liminf_{\tau \rightarrow \infty} \tau^{-1} \ln(I_\infty(\tau)) \geq -(8\kappa^{-1}\alpha^{-3} + 4\mu)\varepsilon.$$

Then, $I'_\infty(\tau) \neq I_\infty(\tau)$ for all $\varepsilon < \min(\varepsilon_1, \frac{\kappa\alpha}{8\kappa^{-1}\alpha^{-3} + 4\mu})$ and τ sufficiently large. The result at §2.6 implies that $\Lambda_{v_0, \mathbf{a}} \neq \Lambda_{v'_0, \mathbf{a}'}$. \square

5 Technical Results

5.1 Basic estimates

Lemma 5.1. *Let $t \in [0, T]$ be such that $D_t \neq \emptyset$. Then there exists a non empty finite family I , and points $x_i \in D_t$, $i \in I$, such that*

$$\cup_{i \in I} B_i(1/\tau) \subset \overline{D_t} \subset \cup_{i \in I} B_i(3/\tau),$$

and $B_i(1/\tau) \cap B_j(1/\tau) = \emptyset$ if $i, j \in I$, $i \neq j$, where $B_i(R)$ denotes the open euclidian ball of radius $R > 0$ and centered at x_i .

Proof. The lemma is a straightforwardly consequence of the compactness of $\overline{D_t}$ and Vitali's lemma.

We have the following proposition:

Proposition 2. (*Parabolic Harnack's inequality*). *There exists $c > 0$ such that if $r > 0$, $t \in \mathbb{R}$, and if $y \in \mathbb{R}^3 \setminus B(2r)$ or $0 \notin (t - r^2, t + r^2)$ then we have*

$$\max_{x \in \overline{B(r)}, s \in [t - \frac{3}{4}r^2, t - \frac{1}{4}r^2]} G_y(x, s) \leq c \min_{x \in \overline{B(r)}, s \in [t + \frac{1}{4}r^2, t + r^2]} G_y(x, s). \quad (41)$$

For a proof, see for example the work of E.B. Fabes and D.W. Stroock [5].

Let us remember that p_τ is defined by (5). From Proposition 2, we prove the following Lemma.

Lemma 5.2. (*Elliptic Harnack's inequality*). *Let $\beta > 0$. There exists $c > 0$ such that for all $\tau > 0$, for all ball $B(\beta/\tau) \subset \mathbb{R}^N$, if $y \notin B(2\beta/\tau)$ we then have*

$$\max_{x \in B(\beta/\tau)} p_\tau(x; y) \leq c \min_{x \in B(\beta/\tau)} p_\tau(x; y). \quad (42)$$

Proof. Applying (41) with $s = t$, $r = \beta/\tau$, we have, for all $x, z \in \overline{B(\beta/\tau)}$,

$$\begin{aligned} p_\tau(z; y) &= \int_0^\infty e^{-\tau^2 s} G_y(z, s) ds \\ &= \int_{\frac{1}{2}\beta^2/\tau^2}^\infty e^{-\tau^2(s - \frac{1}{2}\beta^2/\tau^2)} G_y(z, s - \frac{1}{2}\beta^2/\tau^2) ds \\ &\leq \int_{\frac{1}{2}\beta^2/\tau^2}^\infty e^{-\tau^2(s - \frac{1}{2}\beta^2/\tau^2)} c G_y(x, s) ds \\ &\leq c e^{\frac{1}{2}\beta^2} \int_0^\infty e^{-\tau^2 s} G_y(x, s) ds \\ &= c e^{\frac{1}{2}\beta^2} p_\tau(x; y). \end{aligned}$$

We then obtain (42). \square

Let us remember that $y(\cdot)$ and Σ were defined in Section 2.5 and P_τ by (16).

Lemma 5.3. (*Caccioppoli's inequality for P_τ*). *Let P_τ be defined by (16). Let $\beta > 0$. Then there exists $c > 0$ such that for all $\tau > 0$, if $B(\beta/\tau) \cap B(y(t); \frac{1}{\tau}) = \emptyset$ we then have*

$$\frac{1}{c} \int_{B(\frac{\beta}{4\tau})} \tau^2 P_\tau^2(x, t) dx \leq \int_{B(\frac{\beta}{2\tau})} |\nabla P_\tau|^2(x, t) dx \leq c \int_{B(\frac{\beta}{\tau})} \tau^2 P_\tau^2(x, t) dx. \quad (43)$$

Proof in Appendix.

5.2 Comparison between u_τ , P_τ and p_τ

Lemma 5.4. (*Comparison between P_τ and p_τ*). *There exists $c > 0$ such that for all $\tau > 0$, $t \in \mathbb{R}$, if $x \notin B(y(t); \frac{2}{\kappa\beta\tau})$ we then have*

$$\frac{1}{c} \tau^3 P_\tau(x, t) \leq p_\tau(x, y(t)) \leq c \tau^3 P_\tau(x, t), \quad (44)$$

where κ is the constant of (4) or (7).

Proof in Appendix.

Lemma 5.5. (Comparison between u_τ and p_τ). Let $M > 0$ and assume that $|\dot{y}|_\infty \leq M$. Then there exist positive constants $C(M)$, $C_1(M)$, $\tau_0(M)$ such that for $\tau \geq \tau_0$, $t \in [0, T]$, $x \in \Omega \setminus B(y(t), C_1/\tau)$, we have:

$$e^{-\tau^2(T+t)} u_\tau(x, t) \leq C e^{-\tau\mu|t-\theta|} \tau^{-3} p_\tau(x, y(t)). \quad (45)$$

Proof in Appendix.

Lemma 5.6. Let $t \in [0, T]$. Let $M > 0$ and assume that $|\dot{y}|_\infty \leq M$. Then there exist positive constants $C(M)$, $C_1(M)$, $\tau_0(M)$ such that for $\tau \geq \tau_0$, $t \in [0, T]$ and $x \in \Omega \setminus B(y(t), C_1/\tau)$ we have:

$$|\partial_t(e^{-\tau^2(t+T)} u_\tau(x, t))| \leq C e^{-\tau\mu|t-\theta|} \tau^{-2} p_\tau(x, y(t)). \quad (46)$$

Proof in Appendix.

Let us remember that q_τ is defined by (17).

Lemma 5.7. (Estimate of q_τ). Let $t \in [0, T]$. Let $M > 0$, assume that $|\dot{y}|_\infty \leq M$. Then there exist positive constants $C(M)$, $C_1(M)$, $\tau_0(M)$ such that for $\tau \geq \tau_0$, $t \in [0, T]$ and $x \in \Omega \setminus B(y(t), C_1/\tau)$, we have

$$|q_\tau(x, t)| \leq C \tau^{-3} e^{-\tau\mu|t-\theta|} |x - y(t)| p_\tau(x, y(t)). \quad (47)$$

Proof in Appendix.

5.3 Estimates of special function in D_t

Lemma 5.8. (Estimates of P_τ in D_t). Let $t \in [0, T]$. Then, there exists $c \geq 1$ such that for all $\tau > \frac{12}{\kappa^5 \mathbf{d}(t)}$, we have

$$\frac{1}{c} \int_{D_t} \tau^{-4} p_\tau^2(x, y(t)) dx \leq \int_{D_t} |\nabla P_\tau(x, t)|^2 dx \leq c \int_{D_t} \tau^{-4} p_\tau^2(x, y(t)) dx. \quad (48)$$

Proof in Appendix.

Lemma 5.9. (Estimate of ∇q_τ in D_t). Let $M > 0$, assume that $|\dot{y}|_\infty \leq M$. Then there exist two positive constants C_M and $\tau_0 = \tau_0(\Sigma)$ such that if $\tau > \tau_0$, $t \in [0, T]$, then

$$\int_{D_t} |\nabla q_\tau|^2(x, t) dx \leq C_M \tau^{-4} e^{-2\tau\mu|t-\theta|} \int_{D_t} |x - y(t)|^2 |p_\tau(x, y(t))|^2 dx. \quad (49)$$

Proof in Appendix.

Lemma 5.10. There exist positive constant C , $\tau_0(\Sigma)$ such that for $\tau > \tau_0$, $t \in [0, T]$, we have

$$\int_{D_t} |x - y(t)|^2 |p_\tau(x, y(t))|^2 dx \leq C \mathbf{d}(t)^2 \int_{D_t} |p_\tau(x, y(t))|^2 dx. \quad (50)$$

Proof in Appendix.

Now we are ready to prove Lemma 3.2.

5.4 Proof of Lemma 3.2

We obtain (30) and (29) from (26), (25), (20) of Lemma 2.1, (48) of Lemma 5.8. \square

Conclusion

So we have proven the injectivity of $D \mapsto \Lambda_{v_0, \mathbf{a}}$ by extending the Dynamical Probe Method. We already know that, in the case where \mathbf{a} is scalar and the background $\mathbf{b} = 1$, the DPM is effective in reconstructing the inclusion D from the Dirichlet-to-Neumann mapping $\Lambda_{v_0, \mathbf{a}}$ even when D_t has no regularity according to the space variable. But in the more general case where \mathbf{b} is not constant the behaviour of the special functions u_τ, u_τ^* as τ tends to infinity is not so obvious anymore, which technically requires us to prove that the product $\nabla u_\tau \nabla u_\tau^*$ is positive not punctually but in a weaker sense, and with additional conditions. In our work the main new constraint that allows the uniqueness proof to work is on the geometry of D : some kind of uniform lipschitzian regularity of $D_t, t \in [0, T]$. By looking carefully at the various technical elements of the multiple Lemmas we can hope to improve this condition a little, perhaps by replacing it by a geometric constraint of the Holder type with coefficient in $(\frac{1}{2}, 1)$. The question of reconstructing D from $\Lambda_{v_0, \mathbf{a}}$ remains delicate for two reasons. First, the negative term $C_M \mathbf{d}(t)^2$ in (30) forces the curve to be partly sufficiently close enough to the inclusion to obtain a good lower bound of the indicator function $I_\infty(\tau)$, which complicates a strategy for detecting the unknown D . Then, Runge's method allows only a theoretical reconstruction. Nevertheless, the reconstruction of points of D sufficiently close to the lateral boundary of the cylinder becomes possible, and this without the use of the Runge approximation. However, such a study would burden the article. Another question is to be able to weaken the condition that $t \mapsto D_t$ is lipschitzian. It is open.

Appendix

Proof of Lemma 3.1. We put

$$X_1 := \int_{\Omega \times [0, T]} (\mathbf{b}^{-1} - \mathbf{a}^{-1}) (\mathbf{b} \nabla u_\tau)^2 dx e^{-2\tau^2(T+t)} dt,$$

$$X_2 := \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) (\nabla u_\tau)^2 dx e^{-2\tau^2(T+t)} dt,$$

$$w_\tau := v_\tau - u_\tau, \tag{51}$$

$$\begin{aligned} \Psi_\tau &:= (\mathbf{a} - \mathbf{b}) \nabla v_\tau + \mathbf{b} \nabla w_\tau = \mathbf{a} \nabla v_\tau - \mathbf{b} \nabla u_\tau \\ &= (\mathbf{a} - \mathbf{b}) \nabla u_\tau + \mathbf{a} \nabla w_\tau, \end{aligned} \tag{52}$$

$$B_1 := \int_{\Omega \times [0, T]} \mathbf{a}^{-1} (\Psi_\tau)^2 dx e^{-2\tau^2(T+t)} dt$$

$$B_2 := \int_{\Omega \times [0, T]} \mathbf{a} (\nabla w_\tau)^2 dx e^{-2\tau^2(T+t)} dt,$$

$$B_3 := \int_{\Omega \times [0, T]} \tau^2 w_\tau^2 dx e^{-2\tau^2(T+t)} dt,$$

and

$$\begin{aligned}
R_1 &:= \int_{\Omega} [w_{\tau} u_{\tau}^*]_0^T dx, \\
R_2 &:= \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_{\tau} \nabla (e^{\tau^2(T+t)} u_{\tau}^* - e^{-\tau^2(T+t)} u_{\tau}) dx e^{-\tau^2(t+T)} dt \\
&= \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_{\tau} \cdot (\nabla q_{\tau}^*(x, t) - \nabla q_{\tau}(x, t)) dx e^{-\tau^2(t+T)} dt, \\
R_3 &:= \frac{1}{2} \int_{\Omega} [w_{\tau}^2 e^{-2\tau^2(T+t)}]_0^T dx.
\end{aligned}$$

Step 1. We prove that

$$I_{\infty}(\tau) = X_1 + B_1 + B_3 + R_1 + R_2 + R_3, \quad (53)$$

$$I_{\infty}(\tau) = X_2 - B_2 - B_3 + R_1 + R_2 - R_3. \quad (54)$$

From (23) we have

$$I_{\infty}(\tau) = \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_{\tau} \nabla u_{\tau} dx e^{-2\tau^2(T+t)} dt + R_1 + R_2. \quad (55)$$

1. We put

$$\begin{aligned}
A_1 &:= \int_{\Omega \times [0, T]} \mathbf{a}^{-1} \Psi_{\tau} \cdot (\mathbf{a} - \mathbf{b}) \nabla u_{\tau} dx e^{-2\tau^2(T+t)} dt, \\
A_2 &:= \int_{\Omega \times [0, T]} \nabla w_{\tau} \Psi_{\tau} dx e^{-2\tau^2(T+t)} dt.
\end{aligned}$$

Then, since $(\mathbf{a} - \mathbf{b}) \nabla u_{\tau} = \Psi_{\tau} - \mathbf{a} \nabla w_{\tau}$, we then have $A_1 = B_1 - A_2$.

By integration by parts we have

$$\begin{aligned}
A_2 &= - \int_{\Omega \times [0, T]} w_{\tau} \operatorname{div} \Psi_{\tau} dx e^{-2\tau^2(T+t)} dt \\
&= - \int_{\Omega \times [0, T]} w_{\tau} \partial_t w_{\tau} dx e^{-2\tau^2(T+t)} dt = -B_3 - R_3.
\end{aligned} \quad (56)$$

We thus have

$$A_1 = B_1 + B_3 + R_3. \quad (57)$$

For any 3×3 real matrix \mathbf{m} we have $\mathbf{m} \nabla u_{\tau} \cdot \nabla u_{\tau} = \mathbf{m}_S \nabla u_{\tau} \cdot \nabla u_{\tau}$. Then, thanks to

$$\nabla v_{\tau} = \mathbf{a}^{-1} \Psi_{\tau} + \mathbf{a}^{-1} \mathbf{b} \nabla u_{\tau}, \quad (58)$$

we obtain (53) from (55) and (57).

2. We consider (55) again. Thanks to (51) then to (52) we have

$$\begin{aligned}
(\mathbf{a} - \mathbf{b}) \nabla v_{\tau} \nabla u_{\tau} &= (\mathbf{a} - \mathbf{b}) \nabla u_{\tau} \nabla u_{\tau} + \mathbf{a} \nabla w_{\tau} \nabla u_{\tau} - \mathbf{b} \nabla w_{\tau} \nabla u_{\tau} \\
&= (\mathbf{a} - \mathbf{b}) \nabla u_{\tau} \nabla u_{\tau} + \mathbf{a} \nabla w_{\tau} (\nabla v_{\tau} - \nabla w_{\tau}) - \mathbf{b} \nabla w_{\tau} \nabla u_{\tau} \\
&= (\mathbf{a} - \mathbf{b}) \nabla u_{\tau} \nabla u_{\tau} - \mathbf{a} \nabla w_{\tau} \nabla w_{\tau} + \nabla w_{\tau} \Psi_{\tau}.
\end{aligned}$$

Hence

$$I_\infty(\tau) = \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b})(\nabla u_\tau)^2 dx e^{-2\tau^2(T+t)} dt - B_2 + A_2 + R_1 + R_2,$$

which gives (54) with the help of (56). \square

Step 2. We put

$$X_0 := \int_D e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dx dt \quad (59)$$

and

$$\begin{aligned} R_4 &:= \frac{1}{2} \int_\Omega e^{4\tau^2 T} |u_\tau^*(T)|^2 dx + \frac{1}{2} \int_\Omega e^{2\tau^2 T} |u_\tau^*(0)|^2 dx \\ &\quad + 2 \int_\Omega e^{-2\tau^2 T} |u_\tau(0)|^2 dx + 2 \int_\Omega e^{-2\tau^2 T} |v_0|^2 dx, \\ R_5 &:= \int_D |\nabla q_\tau|^2 dx dt, \quad R_5^* := \int_D |\nabla q_\tau^*|^2 dx dt. \end{aligned}$$

Thanks to (2) and to Assumption (H0b) we have the following estimates:

$$I_\infty(\tau) \geq CX_0 + \frac{1}{2}B_1 + B_3 - 2R_4 - \frac{1}{C}(R_5 + R_5^*), \quad (60)$$

$$I_\infty(\tau) \leq \frac{1}{C}X_0 - \frac{1}{2}B_2 - B_3 + 2R_4 + \frac{1}{C}(R_5 + R_5^*), \quad (61)$$

for some $C \in (0, 1)$.

Proof. Thanks to Cauchy-Minkovski inequality and to the definition (22) we have

$$\begin{aligned} R_1 + R_3 &= \int_\Omega (w_\tau u_\tau^* + \frac{1}{2}w_\tau^2 e^{-4\tau^2 T})|_{t=T} dx \\ &\quad - \int_\Omega (w_\tau u_\tau^* + \frac{1}{2}w_\tau^2 e^{-2\tau^2 T})|_{t=0} dx \\ &\geq -R_4. \end{aligned} \quad (62)$$

Similarly we have

$$R_1 - R_3 \leq R_4. \quad (63)$$

We observe that, thanks to (53) and (54),

$$X_1 = I_\infty(\tau) - B_1 - B_3 - R_1 - R_2 - R_3, \quad (64)$$

$$X_2 = I_\infty(\tau) + B_2 + B_3 - R_1 - R_2 + R_3. \quad (65)$$

Thanks to (58) again we have

$$\begin{aligned} |R_2| &\leq \int_{\Omega \times [0, T]} e^{-\tau^2(T+t)} |\mathbf{a} - \mathbf{b}| |\mathbf{a}^{-1}| |\Psi_\tau| |\nabla q_\tau^* - \nabla q_\tau| dx dt \\ &\quad + \int_{\Omega \times [0, T]} e^{-\tau^2(T+t)} |\mathbf{a} - \mathbf{b}| |\mathbf{a}^{-1}| |\mathbf{b}| |\nabla u_\tau| |\nabla q_\tau^* - \nabla q_\tau| dx dt \\ &\leq \frac{1}{2}B_1 + \frac{1}{2}X_1 + C(R_5 + R_5^*). \end{aligned} \quad (66)$$

From (64) and (66) we get

$$|R_2| \leq \frac{1}{2}I_\infty(\tau) - \frac{1}{2}(B_3 + R_3 + R_1 + R_2) + C(R_5 + R_5^*). \quad (67)$$

Estimates (53), (62) and (67) imply

$$I_\infty(\tau) \geq \frac{1}{2}X_1 + \frac{1}{2}B_1 + B_3 - R_4 - C(R_5 + R_5^*). \quad (68)$$

By using (17), (18), (H0b), and the basic estimate $a^2 \geq \frac{1}{2}(a+b)^2 - b^2$, we have

$$\begin{aligned} X_1 &\geq \int_D \delta_1 e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dx dt \\ &\quad - \int_{\Omega \times [0, T]} |\mathbf{b}| |\mathbf{a}^{-1}| |\mathbf{a} - \mathbf{b}| |\nabla q_\tau|^2 dx dt \\ &\geq CX_0 - \frac{1}{C}R_5, \end{aligned}$$

for some $C \in (0, 1)$. Then with (68) we obtain (60).

Similarly, by using (54), (59), (65) we obtain (61).

Step 3. We prove that for $\tau > \mu + 1$ we have

$$|R_4| \leq (2\|v_0\|_{L^2(\Omega)}^2 + 5d_\Omega) e^{-\tau\mu \min(T-\theta, \theta)}. \quad (69)$$

Proof. Firstly, we have

$$\begin{aligned} 0 \leq u_\tau(x, 0) &= \int_0^\infty \int_{\mathbb{R}^3} e^{\tau^2(T-s)} e^{-\tau\mu|\theta+s|} m(y, -s) G_y(x, s) dy ds \\ &\leq e^{\tau^2 T} e^{-\tau\mu\theta} \int_0^\infty \int_{\mathbb{R}^3} e^{-\tau^2 s} G_y(x, s) dy ds = \frac{1}{\tau^2} e^{\tau^2 T} e^{-\tau\mu\theta}. \end{aligned}$$

Here we used the notorious relation

$$\int_{\mathbb{R}^3} G_y(x, s) dy = 1. \quad (70)$$

Hence

$$0 \leq e^{-\tau^2 T} u_\tau(x, 0) \leq e^{-\tau\mu\theta}, \quad \tau \geq 1. \quad (71)$$

Similarly we have

$$0 \leq e^{2\tau^2 T} u_\tau^*(x, T) \leq e^{-\tau\mu(T-\theta)}, \quad \tau \geq 1. \quad (72)$$

Secondly, since $\tau > \mu + 1 > 1$ we have

$$\begin{aligned} 0 \leq u_\tau^*(x, 0) &= \int_0^\infty \int_{\mathbb{R}^3} e^{-\tau^2(T+s)} e^{-\tau\mu|\theta-s|} m(y, s) G_y(x, s) dy ds \\ &\leq e^{-\tau^2 T} e^{-\tau\mu\theta} \int_0^\infty e^{-(\tau^2 - \tau\mu)s} \int_{\mathbb{R}^3} G_y(x, s) dy ds \\ &= \frac{1}{\tau^2 - \tau\mu} e^{-\tau^2 T} e^{-\tau\mu\theta} \leq e^{-\tau^2 T} e^{-\tau\mu\theta}. \end{aligned} \quad (73)$$

From (73), (72), (71), we obtain for $\tau > \mu + 1 > 1$:

$$R_4 \leq 2\|v_0\|_{L^2(\Omega)}^2 e^{-2\tau^2 T} + 2d_\Omega e^{-\tau\mu(T-\theta)} + 3d_\Omega e^{-\tau\mu\theta},$$

which implies (69).

Estimates (26) and (25) come immediately from (60), (61), (69) and the fact that $B_j \geq 0$ for $j = 1, 2, 3$. \square

Proof of Lemma 5.3. We observe that for all t , the function $P_\tau(\cdot; t)$ is the unique solution in $H^1(\mathbb{R}^3)$ of

$$(-\operatorname{div}(\mathbf{b}\nabla\cdot) + \tau^2)P_\tau(\cdot; t) = m_\tau(\cdot, t). \quad (74)$$

Let $\phi \in C^1(\mathbb{R}; [0, 1])$ with $\phi(r) = 1$ for $|r| \leq 1/2$ and $\phi(r) = 0$ for $|r| \geq 1$. Put $\psi(x) = \phi(\tau(x - x_0)/\beta)$ where x_0 is the center of the ball $B(\beta/\tau)$. We multiply (74) by $P_\tau(\cdot, t)\psi^2$ and integrate it over Ω . Since $\operatorname{supp}(\psi) \cap \operatorname{supp}(m_\tau(\cdot, t))$ has Lebesgue measure zero, we then have

$$\int_{\Omega} [\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + 2\mathbf{b}\nabla P_\tau(\cdot, t)\psi P_\tau(\cdot, t)\nabla\psi + \tau^2 P_\tau^2(\cdot, t)\psi^2] = 0. \quad (75)$$

Then, from Cauchy-Minkovski's inequality,

$$\begin{aligned} \int_{\Omega} [\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + \tau^2 P_\tau^2(\cdot, t)\psi^2] &\leq \int_{\Omega} |2\mathbf{b}\nabla P_\tau(\cdot, t)\psi P_\tau(\cdot, t)\nabla\psi| \\ &\leq \int_{\Omega} [\frac{1}{2}\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + 2\mathbf{b}P_\tau^2(\cdot, t)(\nabla\psi)^2]. \end{aligned}$$

Thus, for some $C' > 0$,

$$\int_{\Omega} [|\nabla P_\tau(\cdot, t)|^2 + \tau^2 P_\tau^2(\cdot, t)]\psi^2(x)dx \leq C' \int_{\Omega} P_\tau^2(\cdot, t)|\nabla\psi|^2(x)dx.$$

Since $\operatorname{supp} \psi \subset B(\beta/\tau)$ with $|\nabla\psi(x)| \leq \frac{\tau}{\beta} \max|\phi'|$, $\psi \geq 0$, and $\psi = 1$ in $B(\frac{\beta}{2\tau})$, we then have

$$\int_{B(\frac{\beta}{2\tau})} |\nabla P_\tau(\cdot, t)|^2(x)dx \leq C''\tau^2 \int_{B(\frac{\beta}{\tau})} P_\tau^2(\cdot, t)dx,$$

which proves the second inequality in (43).

From (75) and thanks to Cauchy-Minkovski's inequality we have also

$$\begin{aligned} \int_{\Omega} [\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + \tau^2 P_\tau^2(\cdot, t)\psi^2] &\leq \int_{\Omega} |2\mathbf{b}\nabla P_\tau(\cdot, t)\nabla\psi P_\tau(\cdot, t)\psi| \\ &\leq \int_{\Omega} [\frac{2}{\tau^2}\gamma_\infty^2 |\nabla P_\tau(\cdot, t)|^2 |\nabla\psi|^2 + \frac{1}{2}\tau^2 P_\tau^2(\cdot, t)\psi^2]. \end{aligned}$$

Thus,

$$\int_{\Omega} \tau^2 P_\tau^2(\cdot, t)\psi^2(x)dx \leq C\tau^{-2} \int_{\Omega} |\nabla P_\tau(\cdot, t)|^2 |\nabla\psi|^2(x)dx.$$

We then obtain

$$\int_{B(\frac{\beta}{2\tau})} \tau^2 P_\tau^2(\cdot, t)dx \leq C' \int_{B(\frac{\beta}{\tau})} |\nabla P_\tau(\cdot, t)|^2 dx$$

which proves the first inequality in (43) with β replaced by 2β . \square

Proof of Lemma 5.4. Since $G_x(y, s) = G_y(x, s)$ and thanks to (41) with $r = 1/\tau$, we have for all $x \notin B(y(t), 2/\tau)$:

$$\begin{aligned} \int_{B(y(t), 1/\tau)} G_y(x, s) dy &= \int_{B(y(t), 1/\tau)} G_x(y, s) dy \leq |B(1/\tau)| \max_{B(y(t), 1/\tau)} G_x(\cdot, s) \\ &\leq c\tau^{-3} G_x(y(t), s + \frac{1}{2\tau^2}) = c\tau^{-3} G_{y(t)}(x, s + \frac{1}{2\tau^2}). \end{aligned}$$

Then, since $\tau|x - y(t)| \geq 2/\kappa^5 \geq 2$, since $m_\tau \leq 1$ and $\text{supp } m_\tau = B(y(t), \frac{1}{\tau})$ we have

$$\begin{aligned} P_\tau(x, t) &\leq \int_0^\infty e^{-\tau^2 s} \int_{B(y(t), \frac{1}{\tau})} G_y(x, s) dy ds \quad (76) \\ &\leq c\tau^{-3} \int_0^\infty e^{-\tau^2 s} G_{y(t)}(x, s + \frac{1}{2\tau^2}) ds \\ &= c\tau^{-3} \int_{\frac{1}{2\tau^2}}^\infty e^{-\tau^2(s - \frac{1}{2\tau^2})} G_{y(t)}(x, s) ds \\ &\leq c'\tau^{-3} \int_0^\infty e^{-\tau^2 s} G_{y(t)}(x, s) ds = c'\tau^{-3} p_\tau(x; y(t)). \end{aligned}$$

We obtain the first inequality of (44). Let us prove the second one. Since $m_\tau \geq 1/2$ in $B(y(t), \frac{1}{2\tau})$ we then have

$$\begin{aligned} P_\tau(x, t) &\geq \frac{1}{2} \int_0^\infty e^{-\tau^2 s} \int_{B(y(t), \frac{1}{2\tau})} G_y(x, s) dy ds \\ &\geq c\tau^{-3} \int_0^\infty e^{-\tau^2 s} \inf_{y \in B(y(t), \frac{1}{2\tau})} G_y(x, s) ds. \end{aligned}$$

By applying (41) with $r = 1/\tau$ and observing that $G_y(x, s) = G_x(y, s)$ we then have for all $x \notin B(y(t), 2/\tau)$:

$$\begin{aligned} P_\tau(x, t) &\geq c\tau^{-3} \int_0^\infty e^{-\tau^2 s} \inf_{y \in B(y(t), \frac{1}{2\tau})} G_y(x, s) ds \\ &\geq c\tau^{-3} \int_0^\infty e^{-\tau^2 s} G_{y(t)}(x, s - \frac{1}{2\tau^2}) ds \\ &= c\tau^{-3} \int_{\frac{1}{2\tau^2}}^\infty e^{-\tau^2(s + \frac{1}{2\tau^2})} G_{y(t)}(x, s) ds \\ &= c'\tau^{-3} \left(p_\tau(x, y(t)) - \int_0^{\frac{1}{2\tau^2}} e^{-\tau^2 s} G_{y(t)}(x, s) ds \right), \quad (77) \end{aligned}$$

where $c' > 0$. We put $R := \int_0^{\frac{1}{2\tau^2}} e^{-\tau^2 s} G_{y(t)}(x, s) ds$. Thanks to (4) and (7) we have

$$\begin{aligned} R &\leq \int_0^{\frac{1}{2\tau^2}} \frac{e^{-\frac{\kappa^2|x-y(t)|^2}{4s}}}{\kappa s^{3/2}} ds \leq \sqrt{2}\kappa^{-1}\tau \int_1^\infty e^{-\kappa^2|x-y(t)|^2\tau^2 r/2} dr \\ &= 2\sqrt{2}\kappa^{-3}\tau^{-1}|x-y(t)|^{-2} e^{-\kappa^2|x-y(t)|^2\tau^2/2} \\ &\leq \frac{1}{2} p_\tau(x, y(t)) \frac{2}{\kappa^5\tau|x-y(t)|} \exp(-\kappa^{-1}\tau|x-y(t)|(\kappa^2\tau|x-y(t)|/2 - 1)). \end{aligned}$$

Since $\tau|x - y(t)| \geq 2/\kappa^5 > 2/\kappa^2$, we then have $R \leq \frac{1}{2}p_\tau(x, y(t))$. Hence

$$P_\tau(x, t) \geq \frac{1}{2}c'\tau^{-3}p_\tau(x, y(t)).$$

The conclusion follows. \square

Proof of Lemma 5.5. Let us observe that

$$e^{-\tau\mu|t-\theta-s|} \leq e^{-\tau\mu|t-\theta|}e^{\tau\mu s} \quad s > 0, t \in \mathbb{R}. \quad (78)$$

Hence

$$\begin{aligned} e^{-\tau^2(T+t)}u_\tau(x, t) &= \int_0^\infty e^{-\tau^2s}e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} m_\tau(y, t-s)G_y(x, s)dyds \\ &\leq e^{-\tau\mu|t-\theta|}H, \end{aligned} \quad (79)$$

where we put

$$H := \int_0^\infty e^{-(\tau^2-\tau\mu)s} \int_{B(y(t-s), 1/\tau)} G_y(x, s)dyds \equiv H_1 + H_2 \quad (80)$$

with

$$\begin{aligned} H_1 &:= \int_{s>\lambda/\tau} e^{-(\tau^2-\tau\mu)s} \int_{B(y(t-s), 1/\tau)} G_y(x, s)dyds, \\ H_2 &:= \int_0^{\lambda/\tau} e^{-(\tau^2-\tau\mu)s} \int_{B(y(t-s), 1/\tau)} G_y(x, s)dyds, \end{aligned}$$

and where $\lambda := 2|x - y(t)|/\kappa$. We put also $M' := M\lambda + 1$, $C_1 = \max(1, 8\kappa^{-7}(1 + M^2d_\Omega^2))$. Since $|y(t-s) - y(t)| \leq Ms$, we then have $B(y(t-s), 1/\tau) \subset B(y(t), Ms + 1/\tau)$ and so

$$\begin{aligned} H_2 &\leq e^{\mu\lambda} \int_0^{\lambda/\tau} e^{-\tau^2s} \int_{B(y(t), Ms+1/\tau)} G_y(x, s)dyds \\ &\leq e^{\mu\lambda} \int_0^{\lambda/\tau} e^{-\tau^2s} \int_{B(y(t), M'/\tau)} G_y(x, s)dyds. \end{aligned}$$

Since $\tau \geq 2\kappa^{-1}M$, $|x - y(t)| \geq 1/\tau$, we then have $|x - y(t)| \geq M'/\tau$ and so we can apply (41) where x and y are exchanged and with $r = M'/(2\tau)$. Hence

$$\begin{aligned} H_2 &\leq ce^{\mu\lambda} \int_0^\infty e^{-\tau^2s} |B(y(t), M'/\tau)| G_{y(t)}(x, s + M'^2/(2\tau^2)) ds \\ &\leq ce^{\mu\lambda} \int_0^\infty e^{-\tau^2s} (2M')^3 |B(y(t), 1/(2\tau))| G_{y(t)}(x, s + M'^2/(2\tau^2)) ds \\ &= ce^{\mu\lambda} (2M')^3 \int_{M'^2/(2\tau^2)}^\infty e^{-\tau^2(s-M'^2/(2\tau^2))} |B(y(t), 1/(2\tau))| G_{y(t)}(x, s) ds \\ &= ce^{\mu\lambda+M'^2/2} M'^3 \tau^{-3} (p_\tau(x, y(t)) - R), \end{aligned}$$

with $R := \int_0^{M'^2/(2\tau^2)} e^{-\tau^2 s} G_{y(t)}(x, s) ds$ and c is the constant (41). Thanks to (4) we have

$$\begin{aligned} R &\leq \int_0^{\frac{M'^2}{2\tau^2}} \frac{e^{-\frac{\kappa^2|x-y(t)|^2}{4s}}}{\kappa s^{3/2}} ds \leq \sqrt{2}\kappa^{-1}\tau M'^{-1} \int_1^\infty e^{-\kappa^2|x-y(t)|^2\tau^2 r/(2M'^2)} dr \\ &= 2\sqrt{2}M'\kappa^{-3}\tau^{-1}|x-y(t)|^{-2}e^{-\kappa^2|x-y(t)|^2\tau^2/(2M'^2)} \\ &\leq \frac{1}{2}p_\tau(x, y(t)) \frac{M'}{\kappa^5\tau|x-y(t)|} \exp(-\kappa^{-1}\tau|x-y(t)|(M'^{-2}\kappa^2\tau|x-y(t)|/2-1)). \end{aligned}$$

Since $\tau|x-y(t)| \geq C_1$ then $M'^{-2}\kappa^2\tau|x-y(t)| \geq 2$ and $M'^{-1}\kappa^5\tau|x-y(t)| \geq 1$. Hence $R \leq \frac{1}{2}p_\tau(x, y(t))$ and

$$\begin{aligned} H_2 &\leq CM'^3 e^{2\kappa^{-1}\mu|x-y(t)|} \tau^{-3} p_\tau(x, y(t)) \\ &\leq CM'^3 e^{2\kappa^{-1}\mu d_\Omega} \tau^{-3} p_\tau(x, y(t)), \quad (x, t) \in \Omega_{0,T} \\ &= C'(M) \tau^{-3} p_\tau(x, y(t)), \quad (x, t) \in \Omega_{0,T}. \end{aligned} \tag{81}$$

Let us estimate H_1 . Since $G_y(x, s) \leq \kappa^{-1}s^{-3/2}$ and $\tau \geq 2\mu$ we then have

$$\begin{aligned} H_1 &\leq \kappa^{-1}\tau^{-3} \int_{s>\lambda/\tau} s^{-3/2} e^{-\tau^2 s/2} ds \\ &\leq \kappa^{-1}\tau^{-3} \begin{cases} \int_{s>\lambda/\tau} (\lambda/\tau)^{-3/2} e^{-\tau^2 s/2} ds &= 2(\lambda/\tau)^{-3/2} \tau^{-2} e^{-\tau\lambda/2} \\ \int_{s>\lambda/\tau} s^{-3/2} e^{-\tau\lambda/2} ds &= 2(\lambda/\tau)^{-1/2} e^{-\tau\lambda/2} \end{cases}. \end{aligned}$$

Hence

$$H_1 \leq 2\kappa^{-1}\lambda^{-1}\tau^{-3} e^{-\tau\lambda/2}.$$

Thanks to (7) we then obtain

$$H_1 \leq \kappa^{-2}\tau^{-3} p_\tau(x, y(t)), \quad (x, t) \in \Omega_{0,T}. \tag{82}$$

Then, thanks to Lemma 5.4 and from (82), (81), (79), the conclusion follows. \square

Proof of Lemma 5.6. We have

$$\partial_t(e^{-\tau^2(t+T)} u_\tau(x, t)) = Y_1 + Y_2,$$

with

$$\begin{aligned} Y_1 &:= -\tau\mu \int_{s=0}^t e^{-\tau^2 s} \text{sign}(t-\theta-s) e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} m_\tau(y, t-s) G_y(x, s) dy ds, \\ Y_2 &:= \int_{s=0}^t e^{-\tau^2 s} e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} \partial_t m_\tau(y, t-s) G_y(x, s) dy ds. \end{aligned}$$

Let us estimate Y_1 . We have

$$\begin{aligned} |Y_1(x, t)| &\leq \tau\mu \int_{s=0}^\infty e^{-\tau^2 s} e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} m_\tau(y, t-s) G_y(x, s) dy ds \\ &= \tau\mu e^{-\tau^2(t+T)} u_\tau(x, t). \end{aligned}$$

Thanks to Lemma 5.5 we obtain

$$|Y_1(x, t)| \leq C e^{-\tau\mu|t-\theta|} \tau^{-2} p_\tau(x, y(t)). \quad (83)$$

Let us estimate Y_2 . Remember that $\text{supp } m_\tau(\cdot, t) \subset \overline{B(y(t), 1/\tau)}$ and that

$$|\partial_t m_\tau(y, t)| = \tau |\dot{y}(t) \nabla M_0(\tau(y - y(t)))| \leq M\tau.$$

Hence we have, as in the estimates of (76) we obtain

$$\begin{aligned} |Y_2| &\leq C\tau \int_{s=0}^{\infty} e^{-\tau^2 s} e^{-\tau\mu|t-\theta-s|} \int_{B(y(t-s), 1/\tau)} G_y(x, s) dy ds \\ &\leq e^{-\tau\mu|t-\theta|} H, \end{aligned}$$

where H is defined by (80). Hence

$$|Y_2| \leq C' e^{-\tau\mu|t-\theta|} \tau^{-2} p_\tau(x, y(t)). \quad (84)$$

From (83), (84) we obtain (46). \square

Proof of Lemma 5.7. We write $q_\tau(x, t) = \int_0^\infty e^{-\tau^2 s} \int_{\mathbb{R}^3} (A - B) G_y(x, s) dy ds$ with

$$\begin{aligned} A &\equiv e^{-\tau\mu|t-\theta-s|} m_\tau(y, t-s), \\ B &\equiv e^{-\tau\mu|t-\theta|} m_\tau(y, t). \end{aligned}$$

Let us observe that, since $e^{\tau\mu s} - 1 \leq \mu\tau s e^{\tau\mu s}$ and thanks to (78), then

$$\begin{aligned} |A - B| &\leq e^{-\tau\mu|t-\theta|} \left(\mu\tau s e^{\tau\mu s} 1_{B(y(t-s), 1/\tau)} \right. \\ &\quad \left. + M\tau s \max(1_{B(y(t), 1/\tau)}, 1_{B(y(t-s), 1/\tau)}) \right). \end{aligned}$$

Hence

$$|q_\tau(x, t)| \leq \tau e^{-\tau\mu|t-\theta|} (\mu R_1 + M R_2) \quad (85)$$

with

$$R_1 := \int_0^\infty e^{-\tilde{\tau}^2 s} \int_{B(y(t-s), 1/\tau)} s G_y(x, s) dy ds, \quad (86)$$

$$R_2 := \int_0^\infty e^{-\tau^2 s} \int_{\tilde{B}} s G_y(x, s) dy ds, \quad (87)$$

where $\tilde{\tau} := \sqrt{\tau^2 - \tau\mu}$ and $\tilde{B} := B(y(t-s), 1/\tau) \cup B(y(t), 1/\tau)$.

Let us put again $\lambda = 2\kappa^{-1}|x - y(t)|$. We write $R_2 = R_{21} + R_{22}$ with

$$\begin{aligned} R_{21} &:= \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{\tilde{B}} s G_y(x, s) dy ds, \\ R_{22} &:= \int_{\lambda/\tau}^\infty e^{-\tau^2 s} \int_{\tilde{B}} s G_y(x, s) dy ds. \end{aligned}$$

As for the estimate of H_1 in the proof of Lemma 5.5 we have

$$\begin{aligned} |R_{22}| &\leq 2\kappa^{-1} \tau^{-3} \int_{\lambda/\tau}^\infty s^{-1/2} e^{-\tau^2 s/2} ds \\ &\leq 2\kappa^{-1} \tau^{-3+1/2} \lambda^{-1/2} \int_{\lambda/\tau}^\infty e^{-\tau^2 s/2} ds = 4\kappa^{-1} \tau^{-5+1/2} \lambda^{-1/2} e^{-\tau\lambda/2} \\ &= 2\sqrt{2}\kappa^{-1/2} \tau^{-5+1/2} |x - y(t)|^{-1/2} e^{-\tau\kappa^{-1}|x-y(t)|}. \end{aligned}$$

Thanks to (7) and since $\tau|x - y(t)| \geq 1$ we then have

$$\begin{aligned} |R_{22}| &\leq \kappa^{-5/2}\tau^{-4}(\tau|x - y(t)|)^{-1/2}|x - y(t)|p_\tau(x, y(t)) \\ &\leq C\tau^{-4}|x - y(t)|p_\tau(x, y(t)). \end{aligned} \quad (88)$$

For $s \leq \lambda/\tau$ we have $\tilde{B} \subset B(y(t), M'/\tau)$ with $M' := \lambda M + 1$. Hence, as for the estimate of H_2 in the proof of Lemma 5.5 and since $\tau|x - y(t)| \geq 2M'^2\kappa^{-5}$ we then have

$$\begin{aligned} |R_{21}| &\leq \lambda\tau^{-1} \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{B(y(t), M'/\tau)} G_y(x, s) dy ds \\ &\leq \lambda\tau^{-1} C(M)\tau^{-3} p_\tau(x, y(t)) \\ &\leq C(M)\tau^{-4} |x - y(t)| p_\tau(x, y(t)). \end{aligned} \quad (89)$$

From (88) and (89), we obtain that for $\tau|x - y(t)| \geq 2M'^2\kappa^{-5}$ we have

$$|R_2| \leq C(M)\tau^{-4} |x - y(t)| p_\tau(x, y(t)). \quad (90)$$

Now, we estimate R_1 as R_2 by splitting the integral in (87) with $s < \lambda/\tau$ or $s > \lambda/\tau$. We observe that $\tilde{\tau} = \tau\sqrt{1 - \mu\tau^{-1}} \geq \frac{1}{\sqrt{2}}\tau$. Hence $R_1 = R_{11} + R_{12}$ with, since $\tilde{\tau}|x - y(t)| \geq \sqrt{2}\kappa^{-5}$ and $\tau \geq 2\mu$,

$$\begin{aligned} |R_{12}| &\leq \tau^{-3+1/2}\lambda^{-1/2} \int_{\lambda/\tau}^{\infty} e^{-\tilde{\tau}^2 s/2} ds \\ &= 2\kappa^{-1}\tilde{\tau}^{-2}\tau^{-3+1/2}\lambda^{-1/2} e^{-\tau\lambda+\lambda\mu} \\ &\leq C\tau^{-4} |x - y(t)| p_\tau(x, y(t)). \end{aligned} \quad (91)$$

Finally, since $\tau|x - y(t)| \geq 2\kappa^{-5}$, we have, as for the estimate of R_{21} ,

$$\begin{aligned} |R_{11}| &\leq \lambda\tau^{-1} e^{\lambda\mu} \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{B(y(t), 1/\tau)} G_y(x, s) dy ds \\ &\leq C(M)\tau^{-4} |x - y(t)| p_\tau(x, y(t)). \end{aligned} \quad (92)$$

Thanks to (91) and (92), we obtain

$$|R_1| \leq C(M)\tau^{-4} |x - y(t)| p_\tau(x, y(t)). \quad (93)$$

Thanks to (93) and (90), (85), we obtain (47) for $\tau \geq 2\mu$, $t \in [0, T]$ and $x \in \Omega \setminus B(y(t), C_1/\tau)$. \square

Proof of Lemma 5.8. We consider a family of balls $B_i(1/\tau)$, $i \in I$, as in Lemma 5.1. By using (42) (with $\beta = 6$), (43) and (44), and by observing that $B_i(6/\tau) \cap$

$B(y(t), 2\kappa^5/\tau) = \emptyset$ for $\tau > 12\kappa^{-5}\mathbf{d}(t)^{-1}$, we can write

$$\begin{aligned}
\int_{D_t} |\nabla P_\tau|^2(x, t) dx &\leq \sum_{i \in I} \int_{B_i(3/\tau)} |\nabla P_\tau|^2(x, t) dx \\
&\leq C_1 \sum_{i \in I} \int_{B_i(6/\tau)} \tau^2 P_\tau^2(x, t) dx \\
&\leq C_2 \sum_{i \in I} \int_{B_i(6/\tau)} \tau^{-4} p_\tau^2(x, y(t)) dx \\
&\leq C_3 \sum_{i \in I} |B_i(6/\tau)| \tau^{-4} p_\tau^2(x_i, y(t)) \\
&\leq C_4 \sum_{i \in I} |B_i(1/\tau)| \tau^{-4} \min_{B_i(1/\tau)} p_\tau^2(\cdot, y(t)) \\
&\leq C_5 \sum_{i \in I} \int_{B_i(1/\tau)} \tau^{-4} p_\tau^2(x, y(t)) dx \\
&\leq C_5 \int_{D_t} \tau^{-4} p_\tau^2(x, y(t)) dx. \tag{94}
\end{aligned}$$

Hence, the second inequality of (48) is proved. The proof of the first one is similar. \square

Proof of Lemma 5.9. We put $C'_1(M) = C_1 + 6$, $C_2 = \max(C'_1, 12\kappa^{-5})$ where $C_1(M)$ is the constant in Lemma 5.7. We consider again the balls $B(1/\tau)$, $B(3/\tau)$, defined in Lemma 5.1. Thus

$$J := \int_{D_t} |\nabla q_\tau(x, t)|^2 dx \leq \sum_i \int_{B_i(3/\tau)} |\nabla q_\tau(x, t)|^2 dx. \tag{95}$$

Let us fix i and denote $B(3/\tau) = B_i(3/\tau)$. We consider again the functions $\phi \in C^1(\mathbb{R}; [0, 1])$ and $\psi(x) = \phi(\tau(x - x_0)/6)$ where x_0 is the center of a ball $B(6/\tau)$, as in the proof of Lemma 5.3 (with $\beta = 6$).

Thanks to Lemma 5.6, there exists a positive constant $C(M)$ such that for $\tau \geq 2\mu$, $t \in [0, T]$, $x \in \Omega \setminus B(y(t); C_1/\tau)$, we have

$$\begin{aligned}
|(-\operatorname{div} \mathbf{b}\nabla + \tau^2)q_\tau(x, t)| &= \left| \partial_t(e^{-\tau^2(t+T)}u_\tau(x, t)) \right| \\
&\leq C\tau^{-2}e^{-\tau\mu|t-\theta|}p_\tau(x, y(t)). \tag{96}
\end{aligned}$$

We observe that

$$\begin{aligned}
x \in \operatorname{supp}(\psi) = \overline{B(6/\tau)} &\Rightarrow |x - y(t)| \geq |x_0 - y(t)| - 6/\tau \geq \\
&\mathbf{d}(t) - 6/\tau > C'_1/\tau - 6/\tau = C_1/\tau.
\end{aligned}$$

Hence we can multiply (96) by $q_\tau(x, t)\psi^2(x)$ and integrate it over Ω . This implies

$$\begin{aligned}
\int_{\Omega} (\mathbf{b}(\nabla q_\tau(\cdot, t))^2 \psi^2 + 2\mathbf{b}\nabla q_\tau(\cdot, t)\psi q_\tau(\cdot, t)\nabla\psi + \tau^2 q_\tau^2(\cdot, t)\psi^2) \\
\leq C\tau^{-2}e^{-\tau\mu|t-\theta|} \int_{\Omega} |q_\tau(\cdot, t)|p_\tau(\cdot, y(t))\psi^2.
\end{aligned}$$

Then, from Cauchy-Minkovski's inequality, and as in the proof of Lemma 5.3, we obtain

$$\begin{aligned} \int_{\Omega} (|\nabla q_{\tau}(\cdot, t)|^2 + \tau^2 q_{\tau}^2(\cdot, t)) \psi^2 &\leq C \int_{\Omega} q_{\tau}^2(\cdot, t) (\nabla \psi)^2 + C e^{-\tau \mu |t - \theta|} \tau^{-2} \cdot \\ &\quad \left(\int_{\Omega} |q_{\tau}(\cdot, t)|^2 \psi^2 \right)^{1/2} \left(\int_{\Omega} |p_{\tau}(\cdot, y(t))|^2 \psi^2 \right)^{1/2}. \end{aligned}$$

Since $\text{supp } \psi = \overline{B(6/\tau)}$ with $|\nabla \psi(x)| \leq \tau \max |\phi'|/6$, $\psi \geq 0$, and $\psi = 1$ in $B(\frac{3}{\tau})$, we then have

$$\begin{aligned} \int_{B(\frac{3}{\tau})} |\nabla q_{\tau}(\cdot, t)|^2 &\leq C \tau^2 \int_{B(\frac{6}{\tau})} q_{\tau}^2(\cdot, t) + C e^{-\tau \mu |t - \theta|} \tau^{-2} \cdot \\ &\quad \cdot \left(\int_{B(\frac{6}{\tau})} |q_{\tau}(\cdot, t)|^2 \right)^{1/2} \left(\int_{B(\frac{6}{\tau})} |p_{\tau}(\cdot, y(t))|^2 \right)^{1/2}. \end{aligned}$$

Thanks to Lemma 5.7 and by using $\tau^{-1} \leq C_1 |x - y(t)|$ for $x \in B(\frac{6}{\tau})$, we then have

$$\begin{aligned} \int_{B(\frac{3}{\tau})} |\nabla q_{\tau}(\cdot, t)|^2 &\leq C \tau^{-4} e^{-2\tau \mu |t - \theta|} \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx + \\ &\quad C \tau^{-5} e^{-2\tau \mu |t - \theta|} \cdot \left(\int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx \right)^{1/2} \cdot \\ &\quad \cdot \left(\int_{B(\frac{6}{\tau})} |p_{\tau}(x, y(t))|^2 dx \right)^{1/2} \\ &\leq C' e^{-2\tau \mu |t - \theta|} \tau^{-4} \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx. \quad (97) \end{aligned}$$

By putting (97) in (95) we obtain

$$J \leq C' e^{-2\tau \mu |t - \theta|} \tau^{-4} \sum_i \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx. \quad (98)$$

Finally, as in (94) we have

$$\sum_i \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx \leq C \int_{D_t} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx.$$

This with (98) prove (49). \square

Proof of Lemma 5.10. We can assume that $D_t \neq \emptyset$. We put $\lambda = 2\kappa^{-2} \mathbf{d}(t)$ and

$$\begin{aligned} J &:= \int_{D_t} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx = J_1 + J_2, \\ J_1 &:= \int_{D_t \cap B(y(t), \lambda)} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx, \\ J_2 &:= \int_{D_t \setminus B(y(t), \lambda)} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx, \\ \tilde{J} &:= \int_{D_t} |p_{\tau}(x, y(t))|^2 dx. \end{aligned}$$

We then have

$$J_1 \leq \lambda^2 \tilde{J} = 4\kappa^{-4} \mathbf{d}(t)^2 \tilde{J}. \quad (99)$$

On the one hand thanks to (7) we have

$$J_2 \leq |D_t| 4\pi\kappa^{-4} \exp(-2\kappa\lambda\tau),$$

and, on the other hand,

$$\begin{aligned} J_2 &\leq 4\pi\kappa^{-4} \int_{|x-y(t)|>\lambda} e^{-2\kappa\tau|x-y(t)|} dx \leq 16\pi^2\kappa^{-4} \int_{r>\lambda} e^{-2\kappa\tau r} r^2 dr \\ &\leq 20\pi^2\lambda^2\kappa^{-5}\tau^{-1} \exp(-2\kappa\tau\lambda) \leq 20\pi^2\lambda^3\kappa^{-4} \exp(-2\kappa\tau\lambda) \\ &\leq 20\pi^2\kappa^{-10} (2\mathbf{d}(t))^3 \exp(-2\kappa\tau\lambda). \end{aligned}$$

Here we used $\lambda \geq \kappa^{-1}\tau^{-1}$. Hence

$$J_2 \leq 4\pi\kappa^{-4} \min(40\pi\kappa^{-6} \mathbf{d}(t)^3, |D_t|) \exp(-2\kappa\tau\lambda). \quad (100)$$

Let us fix $x_0 \in \partial D_t$ such that $\mathbf{d}(t) = |x_0 - y(t)|$. Then $B(y(t), 2\mathbf{d}(t)) \supset B(x_0, \mathbf{d}(t))$ so, thanks to (7) and to (H3b), we have

$$\begin{aligned} \tilde{J} &\geq \int_{D_t \cap B(y(t), 2\mathbf{d}(t))} |p_\tau(x, y(t))|^2 dx \\ &\geq |D_t \cap B(y(t), 2\mathbf{d}(t))| 4\pi\kappa^4 (2\mathbf{d}(t))^{-2} \exp(-4\kappa^{-1}\tau\mathbf{d}(t)) \\ &\geq \pi\kappa^4 |D_t \cap B(x_0, \mathbf{d}(t))| \mathbf{d}(t)^{-2} \exp(-2\kappa\tau\lambda) \\ &\geq \pi\kappa^4 L_D \min(|D_t|, |B(x_0, \mathbf{d}(t))|) \mathbf{d}(t)^{-2} \exp(-2\kappa\tau\lambda) \\ &\geq \pi\kappa^4 L_D \min(|D_t|, \frac{4}{3}\pi\mathbf{d}(t)^3) \mathbf{d}(t)^{-2} \exp(-2\kappa\tau\lambda). \end{aligned}$$

Then

$$\frac{J_2}{\mathbf{d}(t)^2 \tilde{J}} \leq CL_D^{-1} \kappa^{-14}, \quad (101)$$

for some numerical parameter $C > 0$. From (101) and (99) we obtain

$$J \leq C' L_D^{-1} \mathbf{d}(t)^2 \tilde{J},$$

which is the estimate to prove. \square

Proof of Lemma 2.1. It is the direct consequence of Lemma 5.9 and Lemma 5.10.

Proof of Lemma 3.3

1) Thanks to (7) and to Lemma 3.2 we have, for all $\tau > \tau_0$,

$$\begin{aligned} I_\infty(\tau) &\leq c\tau^{-4} \int_0^T \int_{D_t} 16\kappa^{-4} \varepsilon_\Sigma^{-2} e^{-2\kappa\tau\varepsilon_\Sigma} dx dt + C_1 e^{-4\kappa^{-1}d_\Omega\tau} \\ &\leq C_2 \varepsilon_\Sigma^{-2} \tau^{-4} e^{-2\kappa\varepsilon_\Sigma\tau}. \end{aligned}$$

We then obtain (31).

2) Let us fix $T_1 \in (0, \min(\alpha|D_\theta|^{1/3}, \frac{1}{2}\theta, \frac{1}{2}(T-\theta), \alpha^2, \frac{\alpha^3}{\sqrt{8C_M}}))$ sufficiently small such that we have, thanks to (H2),

$$|D_t| \geq \frac{1}{2}|D_\theta| > 0 \quad \text{for } |t-\theta| \leq T_1. \quad (102)$$

Thanks to (30) in Lemma 3.2 we have, for $\tau > \tau_0$,

$$I_\infty(\tau) \geq I_0 - R_0$$

with

$$\begin{aligned} R_0 &:= \frac{1}{c}\tau^{-4} \int_{|t-\theta| \geq T_1} C_M \mathbf{d}(t)^2 e^{-2\tau\mu|t-\theta|} \int_{D_t} p_\tau^2(x, y(t)) dx dt \\ &\quad + C_1 e^{-\tau\mu \min(\theta, T-\theta)}, \\ I_0 &:= \frac{1}{c}\tau^{-4} \int_{|t-\theta| \leq T_1} (1 - C_M \mathbf{d}(t)^2) e^{-2\tau\mu|t-\theta|} \int_{D_t} p_\tau^2(x, y(t)) dx dt. \end{aligned}$$

Thanks to (7) we have

$$R_0 \leq C_3(\Sigma, v_0)\tau^{-4} e^{-2\tau\mu T_1}.$$

By observing that, thanks to (32), $|t-\theta| \leq T_1$ implies $\mathbf{d}(t) \leq \frac{2}{\alpha^3}T_1 \leq \frac{1}{\sqrt{2C_M}}$. Thus, putting

$$B(t) := \int_{D_t \cap B(y(t), 2\mathbf{d}(t))} e^{-4\tau\kappa^{-1}\mathbf{d}(t)} dx, \quad \varepsilon \leq |t-\theta| \leq 2\varepsilon,$$

and restricting ε to the interval $(0, T_1/2)$, we obtain

$$\begin{aligned} I_0 &\geq \frac{1}{2c'} \sup_{0 \leq r \leq T} \{d(y(r), D_r)\}^{-2\tau-4} \int_{|t-\theta| \leq T_1} e^{-2\tau\mu|t-\theta|} B(t) dt \\ &\geq c(M, \alpha, \Omega, T, \kappa)\tau^{-4} \int_{\varepsilon \leq |t-\theta| \leq 2\varepsilon} e^{-2\tau\mu|t-\theta|} B(t) dt. \end{aligned}$$

Let us give a lower bound for $B(t)$, $\varepsilon \leq |t-\theta| \leq 2\varepsilon$. We have, thanks to (32),

$$D_t \cap B(y(t), 2\mathbf{d}(t)) \supset D_t \cap B(x(t), \mathbf{d}(t)) \supset D_t \cap B(x(t), \frac{\varepsilon}{2\alpha}),$$

for some $x(t) \in \partial D_t$. Thanks to (H3b) we have

$$|D_t \cap B(x(t), \mathbf{d}(t))| \geq L_D \min(|D_t|, \frac{\pi\varepsilon^3}{6\alpha^3}).$$

Thus, thanks to (102) and since $\varepsilon \leq \frac{1}{2}T_1 \leq \frac{1}{2}\alpha|D_\theta|^{1/3}$, we have

$$|D_t| \geq \frac{1}{2}|D_\theta| \geq \frac{\pi\varepsilon^3}{6\alpha^3} \quad \text{for } \varepsilon \leq |t-\theta| \leq 2\varepsilon,$$

and so

$$|D_t \cap B(y(t), 2\mathbf{d}(t))| \geq L_D \frac{\pi\varepsilon^3}{6\alpha^3}, \quad \varepsilon \leq |t-\theta| \leq 2\varepsilon. \quad (103)$$

Then, thanks to (32),

$$B(t) \geq L_D \frac{\pi \varepsilon^3}{6\alpha^3} e^{-4\tau \kappa^{-1} \mathbf{d}(t)} \geq L_D \frac{\pi \varepsilon^3}{6\alpha^3} e^{-8\tau \kappa^{-1} \alpha^{-3} \varepsilon}, \quad \varepsilon \leq |t - \theta| \leq 2\varepsilon.$$

Finally we obtain

$$I_0 \geq C(\Sigma_\varepsilon) \tau^{-4} e^{-(8\kappa^{-1} \alpha^{-3} + 4\mu)\varepsilon \tau}, \quad (104)$$

with $C(\Sigma_\varepsilon) > 0$. Let us put $\varepsilon_1 = \min(\frac{1}{2}T_1, \frac{\mu T_1}{\kappa^{-1} \alpha^{-3} + 4\mu})$. For $\varepsilon \in (0, \varepsilon_1]$ we then have

$$R_0/I_0 \leq C'(\Sigma) e^{-\tau \mu T_1},$$

and so $R_0/I_0 \leq \frac{1}{2}$ for $\tau \geq \tau_0$ (eventually modified). Thus, for $\varepsilon \in (0, \varepsilon_1]$,

$$I_\infty(\tau) \geq \frac{1}{2} C(\Sigma_\varepsilon) \tau^{-4} e^{-(8\kappa^{-1} \alpha^{-3} + 4\mu)\varepsilon \tau}$$

which implies (33). □

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