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# Revisiting energy estimates of the CPR scheme for the Shallow Water equations

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The present notes concern some improvements concerning the CPR (Centred Potential Regularization) schemes proposed in [2] and [1] in the particular case of the Shallow Water equations. We propose here a totally explicit variant of the scheme introduced in [1], and clarify the time step condition required to get energy dissipation (in other terms, entropy stability). This work also results in a slight modification of the viscosity terms, that may help improving numerical results without restricting to low-Froude regimes. This document is organized as follows: in Section 1 we introduce the main notations as well as elementary technical lemmas concerning the discrete operators. The numerical scheme is given in Section 2, together with preliminary results regarding the discrete energy. The main stability results are discussed in Section 3, and extended in 2D in Section 4.

## 1 Introduction and general settings

We start here from the 1d Shallow Water system with topography:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{gh^2}{2}\right) = -gh\partial_x z, \end{cases} \quad (1)$$

where  $h = h(x, t)$  is the water height,  $u = u(x, t)$  is the horizontal average velocity and  $z = z(x)$  accounts for the topography variations. This system admits the following energy conservation law:

$$\partial_t(he) + \partial_x\left(\left(e + \frac{1}{2}gh\right)hu\right) = 0, \quad (2)$$

where

$$e = \frac{1}{2}u^2 + \frac{1}{2}gh + gz.$$

In view of the design of the numerical scheme, we now introduce the discrete operators used in the next sections, as well as elementary estimates and duality formulas. In the one dimensional case, we first consider a uniform grid with space step  $\Delta x$ , and denote  $K \in \mathbb{Z}$  the index of a reference element. First, for any sequence of scalar interface quantity  $(b_{K+1/2})$ , we define the operator:

$$\partial_K b = \frac{1}{\Delta x} (b_{K+1/2} - b_{K-1/2}), \quad (3)$$

with the specific notation  $\partial_K^c b$  when the interface quantities correspond to the half sum of the values sharing the interface, that is:

$$\partial_K^c b = \frac{1}{\Delta x} (\bar{b}_{K+1/2} - \bar{b}_{K-1/2}) , \quad (4)$$

where  $\bar{b}_{K+1/2} = \frac{1}{2}(b_{K+1} + b_K)$ . We also introduce the interface jump  $[b]_{K+1/2} = \frac{1}{2}(b_{K+1} - b_K)$ , so that  $b_K = \bar{b}_{K+1/2} - [b]_{K+1/2} = \bar{b}_{K-1/2} + [b]_{K-1/2}$  for all  $K \in \mathbb{Z}$ . When no confusion is possible, these two definitions will extend to multiple variables by considering the difference of term to term interface products, leading to, in the case of two variables:

$$\partial_K(a, b) = \frac{1}{\Delta x} (a_{K+1/2} b_{K+1/2} - a_{K-1/2} b_{K-1/2}) . \quad (5)$$

We also define the discrete upwind derivative, for any collocated sequence of scalars  $(a_K)$

$$\partial_K^{up}(a, b) = \frac{1}{\Delta x} \left( \mathcal{F}_{K+1/2}^{up}(a, b) - \mathcal{F}_{K-1/2}^{up}(a, b) \right) , \quad (6)$$

where  $\mathcal{F}_{K+1/2}^{up}(a, b) = a_K b_{K+1/2}^+ + a_{K+1} b_{K+1/2}^-$ , and  $w^+ = \frac{1}{2}(w + |w|)$ ,  $w^- = \frac{1}{2}(w - |w|)$  for any scalar quantity  $w \in \mathbb{R}$ . In what follows we will regularly use the notation:

$$\mathcal{S}_K(a, b) = a_{K+1/2} b_{K+1/2} + a_{K-1/2} b_{K-1/2} , \quad (7)$$

When no confusion is possible, we will use the convention  $\mathcal{S}_K(a^2) = \mathcal{S}_K(a, a)$ . Remark that  $a_K \mathcal{S}_K(b) = \bar{a}_{K+1/2} b_{K+1/2} + \bar{a}_{K-1/2} b_{K-1/2} - [a]_{K+1/2} b_{K+1/2} + [a]_{K-1/2} b_{K-1/2}$ , which gives a first duality formula:

$$a_K \mathcal{S}_K(b) = \mathcal{S}_K(\bar{a}, b) - \Delta x \partial_K([a], b) . \quad (8)$$

**Lemma 1.1** (*Centred fluxes duality*). *We have the following duality formula:*

$$a_K \partial_K^c b + b_K \partial_K^c a = \widetilde{\partial}_K^c(a, b) ,$$

where

$$\widetilde{\partial}_K^c(a, b) = \partial_K^c(a, b) - \partial_K([a], [b]) . \quad (9)$$

*Proof.* We have, using  $a_K = \bar{a}_{K+1/2} - [a]_{K+1/2} = \bar{a}_{K-1/2} + [a]_{K-1/2}$ :

$$\begin{aligned} a_K \partial_K^c b + b_K \partial_K^c a &= \frac{1}{\Delta x} [a_K (\bar{b}_{K+1/2} - \bar{b}_{K-1/2}) + b_K (\bar{a}_{K+1/2} - \bar{a}_{K-1/2})] \\ &= \frac{1}{\Delta x} [a_K (\bar{b}_{K+1/2} - \bar{b}_{K-1/2}) + b_K ([a]_{K+1/2} + [a]_{K-1/2})] \\ &= \frac{1}{\Delta x} \left[ \bar{a}_{K+1/2} \bar{b}_{K+1/2} - \bar{a}_{K-1/2} \bar{b}_{K-1/2} \right. \\ &\quad \left. + [a]_{K+1/2} (b_K - \bar{b}_{K+1/2}) + [a]_{K-1/2} (b_K - \bar{b}_{K-1/2}) \right] \\ &= \partial_K^c(a, b) - \partial_K([a], [b]) . \end{aligned}$$

□

**Lemma 1.2** (*Upwind fluxes duality*). *We have the following duality formula:*

$$\partial_K^{up}(a, b) = a_K \partial_K b + (b \partial a)_K^{up},$$

where

$$(b \partial a)_K^{up} = b_{K+1/2}^- \left( \frac{a_{K+1} - a_K}{\Delta x} \right) + b_{K-1/2}^+ \left( \frac{a_K - a_{K-1}}{\Delta x} \right). \quad (10)$$

*Proof.* We have:

$$\begin{aligned} \Delta x [\partial_K^{up}(a, b) - a_K \partial_K b] &= \left[ \left( a_K b_{K+1/2}^+ + a_{K+1} b_{K+1/2}^- \right) - \left( a_{K-1} b_{K-1/2}^+ + a_K b_{K-1/2}^- \right) \right] \\ &\quad - \left[ a_K \left( b_{K+1/2}^+ + b_{K+1/2}^- \right) - a_K \left( b_{K-1/2}^+ + b_{K-1/2}^- \right) \right] \\ &= b_{K+1/2}^- (a_{K+1} - a_K) + b_{K-1/2}^+ (a_K - a_{K-1}). \end{aligned}$$

□

**Lemma 1.3** (*Upwind fluxes duality - Squares*). *We have the following equality:*

$$2a_K (b \partial a)_K^{up} + (a_K)^2 \partial_K b = \partial_K^{up}(a^2, b) + \frac{4}{\Delta x} \mathcal{R}_K^+([a]^2, b),$$

where

$$\mathcal{R}_K^+([a]^2, b) := (-b_{K+1/2})^+ [a]_{K+1/2}^2 + (b_{K-1/2})^+ [a]_{K-1/2}^2. \quad (11)$$

*Proof.* Using the classical relation

$$(y - x)x = \frac{1}{2}y^2 - \frac{1}{2}x^2 - \frac{1}{2}(x - y)^2, \quad (12)$$

we have:

$$\begin{aligned} \Delta x (2a_K (b \partial a)_K^{up} + (a_K)^2 \partial_K b) &= 2a_K \left( b_{K+1/2}^- (a_{K+1} - a_K) + b_{K-1/2}^+ (a_K - a_{K-1}) \right) + (a_K)^2 (b_{K+1/2} - b_{K-1/2}) \\ &= b_{K+1/2}^- \left( (a_{K+1})^2 - (a_K)^2 - (a_{K+1} - a_K)^2 \right) - b_{K-1/2}^+ \left( (a_{K-1})^2 - (a_K)^2 - (a_{K-1} - a_K)^2 \right) \\ &\quad + (a_K)^2 (b_{K+1/2}^+ + b_{K+1/2}^-) - (a_K)^2 (b_{K-1/2}^+ + b_{K-1/2}^-) \\ &= \left( b_{K+1/2}^- (a_{K+1})^2 + b_{K+1/2}^+ (a_K)^2 \right) - \left( b_{K-1/2}^- (a_K)^2 + b_{K-1/2}^+ (a_{K-1})^2 \right) \\ &\quad + \left( -b_{K+1/2}^- (a_{K+1} - a_K)^2 + b_{K-1/2}^+ (a_K - a_{K-1})^2 \right). \end{aligned}$$

Going back to the definition (6) and using  $-(w^-) = -\frac{1}{2}(w - |w|) = (-w)^+$ , we get the announced result. □

**Lemma 1.4** (*Square estimate for upwind fluxes*). *With the notations of the previous Lemma, we have the following inequality:*

$$[(b \partial a)_K^{up}]^2 \leq \frac{4}{\Delta x^2} \left( (-b_{K+1/2})^+ + b_{K-1/2}^+ \right) \mathcal{R}_K^+([a]^2, b).$$

*Proof.* We write:

$$[(b\partial a)_K^{up}]^2 = \frac{4}{\Delta x^2} \left[ (-b_{K+1/2})^+ \times -[a]_{K+1/2} + b_{K-1/2}^+ [a]_{K-1/2} \right]^2,$$

and use Cauchy-Schwarz inequality with the vectors:

$$\mathbf{u} = \begin{pmatrix} \sqrt{(-b_{K+1/2})^+} \\ \sqrt{b_{K-1/2}^+} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -\sqrt{(-b_{K+1/2})^+} [a]_{K+1/2} \\ \sqrt{b_{K-1/2}^+} [a]_{K-1/2} \end{pmatrix}$$

to obtain

$$|\mathbf{u} \cdot \mathbf{v}|^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 = \left( (-b_{K+1/2})^+ + b_{K-1/2}^+ \right) \cdot \left( (-b_{K+1/2})^+ [a]_{K+1/2}^2 + b_{K-1/2}^+ [a]_{K-1/2}^2 \right).$$

□

## 2 Numerical scheme and preliminary energy estimates

Introducing the scalar potential  $\phi = g(h+z)$ , we consider the following numerical scheme for the model (1):

$$\begin{cases} h_K^{n+1} = h_K^n - \Delta t \partial_K (hu^*), \\ (hu)_K^{n+1} = (hu)_K^n - \Delta t \partial_K^{up}(u, hu^*) - \Delta t h_K^n \partial_K \phi^*. \end{cases} \quad (13)$$

Referring to (3) and (6), we only need to define  $(hu)_{K+1/2}^*$  and  $\phi_{K+1/2}^*$  to characterize the discrete operators. Following [1], we set:

$$(hu)_{K+1/2}^* = \overline{hu}_{K+1/2} - \Pi_{K+1/2}, \quad (14)$$

and

$$\phi_{K+1/2}^* = \bar{\phi}_{K+1/2} - \Lambda_{K+1/2}, \quad (15)$$

where we recall that the superscript “ $-$ ” refers to the mean interface value taken at time  $n$  (the scheme being totally explicit, the time index will be omitted when possible in the following). The quantities  $\Pi_{K+1/2}$  and  $\Lambda_{K+1/2}$  govern the numerical viscosity of the scheme, and will be defined in Proposition 3.1 (formulas (39) and (40)) in order to guarantee energy dissipation. Using the mass equation in (13), we easily get:

$$(hu)_K^{n+1} - (hu)_K^n = h_K^{n+1} (u_K^{n+1} - u_K^n) - \Delta t u_K^n \partial_K (hu^*).$$

Invoking the momentum equation:

$$h_K^{n+1} (u_K^{n+1} - u_K^n) = -\Delta t (\partial_K^{up}(u, hu^*) - u_K^n \partial_K (hu^*)) - \Delta t h_K^n \partial_K \phi^*,$$

and using Lemma 1.2, we obtain the velocity equation:

$$u_K^{n+1} = u_K^n - \Delta t \frac{1}{h_K^{n+1}} (hu^* \partial u)_K^{up} - \Delta t \frac{h_K^n}{h_K^{n+1}} \partial_K \phi^*, \quad (16)$$

the operator  $(hu^* \partial u)_K^{up}$  being based on (10). In what follows, we will consider that holds a condition of the form (we refer to Section §3.2 for a rigorous clarification on the time step condition).

$$\frac{\Delta t}{\Delta x} (|hu_{K+1/2}^*| + |hu_{K-1/2}^*|) \leq \frac{\rho}{1+\rho} h_K^n, \quad (17)$$

where  $\rho$  is a strictly positive constant. Let's remark that the time constraint (17) implies:

$$h_K^n - h_K^{n+1} = \frac{\Delta t}{\Delta x} (hu_{K+1/2}^* - hu_{K-1/2}^*) \leq \frac{\Delta t}{\Delta x} (|hu_{K+1/2}^*| + |hu_{K-1/2}^*|) \leq \frac{\rho}{1+\rho} h_K^n,$$

which gives

$$h_K^n \leq (1+\rho)h_K^{n+1}. \quad (18)$$

In particular, we have, under (17):

$$\frac{\Delta t}{\Delta x} \left( (-hu_{K+1/2}^*)^+ + (hu_{K-1/2}^*)^+ \right) \leq \rho h_K^{n+1}. \quad (19)$$

The next proposition gives local estimates for the potential energy:

**Proposition 2.1.** *Let  $\mathcal{E}_K^n = \frac{1}{2}g(h_K^n)^2 + gh_K^n z_K$  the potential energy at time  $n$  at the level of the cell  $K$ . Setting  $\lambda = \frac{\Delta t}{\Delta x}$ , we have the following estimate:*

$$\begin{aligned} \mathcal{E}_K^{n+1} - \mathcal{E}_K^n + \Delta t \widetilde{\partial}_K(\phi, hu^*) - \Delta t (hu)_K^n \partial_K^c \phi \\ \leq 2g\lambda^2 \mathcal{S}_K([hu]^2) + 2g\lambda^2 \mathcal{S}_K(\Pi^2) - \lambda \mathcal{S}_K(\Pi, [\phi]), \end{aligned} \quad (20)$$

where the operator  $\mathcal{S}_K$  is defined according to (7). The flux term  $\widetilde{\partial}_K(\phi, hu^*)$  is defined by

$$\widetilde{\partial}_K(\phi, hu^*) = \partial_K(\phi, hu^*) - \partial_K([\phi], [hu]),$$

where  $\partial_K(\phi, hu^*)$  and  $\partial_K([\phi], [hu])$  are defined by (5). The term  $\partial_K([\phi], [hu])$  is a lower order contribution with respect to the mesh size, seen as a bias on the leading flux term  $\partial_K(\phi, hu^*)$ .

*Proof.* A basic computation give:

$$\mathcal{E}_K^{n+1} - \mathcal{E}_K^n = (h_K^{n+1} - h_K^n) \phi_K^n + \frac{1}{2}g(h_K^{n+1} - h_K^n)^2. \quad (21)$$

On a first hand we have:

$$\begin{aligned} (h_K^{n+1} - h_K^n) \phi_K^n &= -\frac{\Delta t}{\Delta x} (hu_{K+1/2}^* - hu_{K-1/2}^*) \phi_K^n \\ &= -\frac{\Delta t}{\Delta x} (hu_{K+1/2}^* \bar{\phi}_{K+1/2} - hu_{K-1/2}^* \bar{\phi}_{K-1/2}) \\ &\quad + \frac{\Delta t}{\Delta x} (hu_{K+1/2}^* [\phi]_{K+1/2} + hu_{K-1/2}^* [\phi]_{K-1/2}) \\ &= -\Delta t \partial_K(\phi, hu^*) + \frac{\Delta t}{\Delta x} (\bar{hu}_{K+1/2} [\phi]_{K+1/2} + \bar{hu}_{K-1/2} [\phi]_{K-1/2}) - \frac{\Delta t}{\Delta x} \mathcal{S}_K(\Pi, [\phi]) \\ &= -\Delta t \partial_K(\phi, hu^*) + \frac{\Delta t}{\Delta x} (hu)_K^n ([\phi]_{K+1/2} + [\phi]_{K-1/2}) \\ &\quad + \Delta t \partial_K([\phi], [hu]) - \frac{\Delta t}{\Delta x} \mathcal{S}_K(\Pi, [\phi]) \\ &= -\Delta t \widetilde{\partial}_K(\phi, hu^*) + \Delta t (hu)_K^n \partial_K^c \phi - \lambda \mathcal{S}_K(\Pi, [\phi]). \end{aligned}$$

Using standard convexity inequalities, the remaining term of the right hand side in (21) can be estimated as follows:

$$\begin{aligned} \frac{g}{2} (h_K^{n+1} - h_K^n)^2 &= \frac{g}{2} \lambda^2 (\overline{hu}_{K+1/2} - \overline{hu}_{K-1/2} - (\Pi_{K+1/2} - \Pi_{K-1/2}))^2 \\ &= \frac{g}{2} \lambda^2 ([hu]_{K+1/2} + [hu]_{K-1/2} - (\Pi_{K+1/2} - \Pi_{K-1/2}))^2 \\ &\leq 2g\lambda^2 \mathcal{S}_K ([hu]^2) + 2g\lambda^2 \mathcal{S}_K (\Pi^2) . \end{aligned} \quad (22)$$

Gathering the two previous contributions, we get the announced result.  $\square$

We now turn to kinetic energy estimates:

**Proposition 2.2.** *Define  $\mathcal{K}_K^n = \frac{1}{2} h_K^n (u_K^n)^2$  the local kinetic energy at time  $n$ . Let  $\xi_K \in ]0, 1[$  and assume that the local CFL condition (17) is ensured with  $\rho = \xi_K$  at the level of the cell  $K$ . We have the following estimation:*

$$\begin{aligned} \mathcal{K}_K^{n+1} - \mathcal{K}_K^n + \Delta t \widetilde{\partial}_K^{up} \left( \frac{1}{2} u^2, hu^* \right) + \Delta t (hu)_K^n \partial_K^c \phi \\ \leq -\lambda \mathcal{S}_K ([hu], \Lambda) + 2\lambda^2 \mathcal{S}_K ([\phi]^2, h\nu) + 2\lambda^2 \mathcal{S}_K (\Lambda^2, h\nu) . \end{aligned} \quad (23)$$

Having in mind definition (6), the upwind flux is of the form

$$\widetilde{\partial}_K^{up} \left( \frac{1}{2} u^2, hu^* \right) = \partial_K^{up} \left( \frac{1}{2} u^2, hu^* \right) + \mathcal{A}_K ,$$

where term  $\mathcal{A}_K$  gathers  $\mathcal{O}(\Delta t, \Delta x)$  terms, corresponding to a consistent perturbation of the leading flux. The term  $\nu_K$  implied in the right hand side of the estimate (23) is defined by:

$$\nu_K = \frac{1 + \xi_K}{1 - \xi_K} . \quad (24)$$

*Proof.* We start writing:

$$\begin{aligned} h_K^{n+1} (u_K^{n+1} - u_K^n) u_K^n &= h_K^{n+1} \left( \frac{1}{2} (u_K^{n+1})^2 - \frac{1}{2} (u_K^n)^2 - \frac{1}{2} (u_K^{n+1} - u_K^n)^2 \right) \\ &= \mathcal{K}_K^{n+1} - \mathcal{K}_K^n - \frac{1}{2} (u_K^n)^2 (h_K^{n+1} - h_K^n) - \frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2 \\ &= \mathcal{K}_K^{n+1} - \mathcal{K}_K^n + \frac{1}{2} \Delta t (u_K^n)^2 \partial_K (hu^*) - \frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2 . \end{aligned}$$

Hence, with the velocity equation (16):

$$\begin{aligned} \mathcal{K}_K^{n+1} - \mathcal{K}_K^n + \frac{1}{2} \Delta t (u_K^n)^2 \partial_K (hu^*) - \frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2 \\ = u_K^n (-\Delta t (hu^* \partial u)_K^{up} - \Delta t h_K^n \partial_K \phi^*) , \end{aligned}$$

or equivalently:

$$\begin{aligned} \mathcal{K}_K^{n+1} - \mathcal{K}_K^n + \Delta t \left( \frac{1}{2} (u_K^n)^2 \partial_K (hu^*) + u_K^n (hu^* \partial u)_K^{up} \right) + \Delta t (hu)_K^n \partial_K \phi^* \\ = \frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2 . \end{aligned} \quad (25)$$

We focus on the terms of the left hand side. With the help of Lemma 1.3, we write:

$$\begin{aligned} \frac{1}{2}(u_K^n)^2 \partial_K(hu^*) + u_K^n (hu^* \partial u)_K^{up} &= \frac{1}{2} \left( (u_K^n)^2 \partial_K(hu^*) + 2u_K^n (hu^* \partial u)_K^{up} \right) \\ &= \partial_K^{up} \left( \frac{1}{2} u^2, hu^* \right) + \frac{2}{\Delta x} \mathcal{R}_K^+([u]^2, hu^*) . \end{aligned} \quad (26)$$

The last term of the left hand side in (25) can be subject to the following reformulation:

$$\begin{aligned} (hu)_K^n \partial_K \phi^* &= (hu)_K^n \partial_K^c \phi - \frac{1}{\Delta x} (hu)_K^n (\Lambda_{K+1/2} - \Lambda_{K-1/2}) \\ &= (hu)_K^n \partial_K^c \phi - \frac{1}{\Delta x} (\bar{h}u_{K+1/2} \Lambda_{K+1/2} - \bar{h}u_{K-1/2} \Lambda_{K-1/2}) \\ &\quad + \frac{1}{\Delta x} ([hu]_{K+1/2} \Lambda_{K+1/2} + [hu]_{K-1/2} \Lambda_{K-1/2}) \\ &= (hu)_K^n \partial_K^c \phi - \partial_K(\Lambda, hu) + \frac{1}{\Delta x} \mathcal{S}_K([hu], \Lambda) . \end{aligned} \quad (27)$$

Injecting (26) and (27) in (25), we get:

$$\mathcal{K}_K^{n+1} - \mathcal{K}_K^n + \Delta t \widetilde{\partial}_K^{up} \left( \frac{1}{2} u^2, hu^* \right) + \Delta t (hu)_K^n \partial_K^c \phi = -\lambda \mathcal{S}_K([hu], \Lambda) + \mathcal{U}_K , \quad (28)$$

where

$$\mathcal{U}_K = -2 \frac{\Delta t}{\Delta x} \mathcal{R}_K^+([u]^2, hu^*) + \frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2 , \quad (29)$$

and  $\widetilde{\partial}_K^{up} \left( \frac{1}{2} u^2, hu^* \right) = \partial_K^{up} \left( \frac{1}{2} u^2, hu^* \right) - \partial_K(\Lambda, hu)$ . Remark that, as will be established rigorously later on, the viscous term  $\Lambda$  is scaled as  $\mathcal{O}(\Delta t)$ , so that the bias on the flux  $\partial_K(\Lambda, hu)$  is of the expected form, according to the statement of the Proposition.

The following step relies on the estimation of the term  $\frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2$ . Using again the velocity equation (16), we have:

$$\frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2 = \frac{1}{2} h_K^{n+1} \Delta t^2 \left[ \frac{1}{h_K^{n+1}} (hu^* \partial u)_K^{up} + \frac{h_K^n}{h_K^{n+1}} \partial_K \phi^* \right]^2 .$$

Using the equality

$$\partial_K \phi^* = \frac{1}{\Delta x} (\phi_{K+1/2}^* - \phi_{K-1/2}^*) = \frac{1}{\Delta x} ([\phi]_{K+1/2} + [\phi]_{K-1/2}) - \frac{1}{\Delta x} (\Lambda_{K+1/2} - \Lambda_{K-1/2}) ,$$

we write

$$\frac{1}{2} h_K^{n+1} (u_K^{n+1} - u_K^n)^2 = \frac{1}{2} \Delta t^2 h_K^{n+1} (C_1 + C_2 + C_3)^2 ,$$

with

$$\begin{aligned} C_1 &= \frac{1}{h_K^{n+1}} (hu^* \partial u)_K^{up} , \\ C_2 &= \frac{1}{\Delta x} \frac{h_K^n}{h_K^{n+1}} ([\phi]_{K+1/2} + [\phi]_{K-1/2}) , \\ C_3 &= -\frac{1}{\Delta x} \frac{h_K^n}{h_K^{n+1}} (\Lambda_{K+1/2} - \Lambda_{K-1/2}) . \end{aligned}$$



We now use the convexity of the square function with the weights  $\left(\xi_K, \frac{1-\xi_K}{2}, \frac{1-\xi_K}{2}\right)$ , so that:

$$\frac{1}{2}h_K^{n+1}(u_K^{n+1} - u_K^n)^2 \leq \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3, \quad (30)$$

where

$$\begin{aligned} \mathcal{C}_1 &= \frac{1}{2}\Delta t^2 \frac{1}{\xi_K h_K^{n+1}} [(hu^* \partial u)_K^{up}]^2, \\ \mathcal{C}_2 &= \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{h_K^n}{1-\xi_K} \left(\frac{h_K^n}{h_K^{n+1}}\right) ([\phi]_{K+1/2} + [\phi]_{K-1/2})^2, \\ \mathcal{C}_3 &= \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{h_K^n}{1-\xi_K} \left(\frac{h_K^n}{h_K^{n+1}}\right) (\Lambda_{K+1/2} - \Lambda_{K-1/2})^2. \end{aligned}$$

Regarding the first term  $\mathcal{C}_1$ , a direct use of Lemma 1.4 give:

$$\mathcal{C}_1 \leq 2 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{1}{\xi_K h_K^{n+1}} ((-hu_{K+1/2}^*)^+ + (hu_{K-1/2}^*)^+) \mathcal{R}_K^+([u]^2, hu^*), \quad (31)$$

so that, considering the first term of (29):

$$\begin{aligned} &\mathcal{C}_1 - 2 \frac{\Delta t}{\Delta x} \mathcal{R}_K^+([u]^2, hu^*) \\ &\leq 2 \frac{\Delta t}{\Delta x} \mathcal{R}_K^+([u]^2, hu^*) \left(\frac{\Delta t}{\Delta x} ((-hu_{K+1/2}^*)^+ + (hu_{K-1/2}^*)^+) / (\xi_K h_K^{n+1}) - 1\right). \end{aligned} \quad (32)$$

The right hand side is negative under the condition (17) with  $\rho = \xi_K$  in virtue of (19). Now, considering that (18) holds, the ratio  $h_K^n/h_K^{n+1}$  can be bounded by  $1 + \xi_K$  in  $\mathcal{C}_2$  and  $\mathcal{C}_3$ , leading to:

$$\mathcal{C}_2 \leq 2\lambda^2 h_K^n \frac{1 + \xi_K}{1 - \xi_K} \mathcal{S}_K([\phi]^2).$$

Using the notation  $\nu_K = \frac{1 + \xi_K}{1 - \xi_K}$  and (8), this estimation can be reformulated as:

$$\mathcal{C}_2 \leq 2\lambda^2 \mathcal{S}_K([\phi]^2, h\nu) - 2\lambda^2 \Delta x \partial_K([\phi]^2, [h\nu]). \quad (33)$$

In a similar way, we obtain:

$$\mathcal{C}_3 \leq 2\lambda^2 \mathcal{S}_K(\Lambda^2, h\nu) - 2\lambda^2 \Delta x \partial_K(\Lambda^2, [h\nu]). \quad (34)$$

Gathering estimates (32), (33) and (34) we get, going back to (29):

$$\begin{aligned} \mathcal{U}_K &\leq 2\lambda^2 \mathcal{S}_K([\phi]^2, h\nu) + 2\lambda^2 \mathcal{S}_K(\Lambda^2, h\nu) \\ &\quad - 2\lambda^2 \Delta x \partial_K(\Lambda^2, [h\nu]) - 2\lambda^2 \Delta x \partial_K([\phi]^2, [h\nu]). \end{aligned} \quad (35)$$

Plugging this estimate in (28), we recognize the right hand side of (23) up to antisymmetric contributions scaled as  $\mathcal{O}(\Delta t^2)$ , that are incorporated in the fluxes  $\widetilde{\partial}_K^{up}\left(\frac{1}{2}u^2, hu^*\right)$ .  $\square$

### 3 Global stability result

#### 3.1 Choice of the stabilization terms

Denoting  $E_K^n = \mathcal{E}_K^n + \mathcal{K}_K^n$  the local mechanical energy, the sum of energy estimates (20) and (23), can be written under the compact form:

$$E_K^{n+1} - E_K^n + \Delta t \mathfrak{d}_K \left( e + \frac{1}{2}gh, hu^* \right) \leq \lambda (\mathcal{G}_K + \mathcal{L}_K) , \quad (36)$$

where the energy flux is given by (we remark that  $e + \frac{1}{2}gh = \frac{1}{2}u^2 + \phi$ ):

$$\mathfrak{d}_K \left( e + \frac{1}{2}gh, hu^* \right) = \widetilde{\partial}_K^{up} \left( \frac{1}{2}u^2, hu^* \right) + \widetilde{\partial}_K(\phi, hu^*) ,$$

and

$$\mathcal{G}_K = 2g\lambda \mathcal{S}_K(\Pi^2) - \mathcal{S}_K(\Pi, [\phi]) + 2\lambda \mathcal{S}_K([\phi]^2, h\nu) , \quad (37)$$

$$\mathcal{L}_K = 2g\lambda \mathcal{S}_K([hu]^2) - \mathcal{S}_K([hu], \Lambda) + 2\lambda \mathcal{S}_K(\Lambda^2, h\nu) . \quad (38)$$

We thus recognize at the left hand side a discrete equivalent of the continuous energy equation (2). We now discuss the conditions ensuring the negativity of the terms appearing in the right hand side.

**Proposition 3.1** (*Control of the terms  $\mathcal{G}_K$  and  $\mathcal{L}_K$* ). *Suppose that the viscous terms involved in formulas (14), (15) have the following form:*

$$\Pi_{K+1/2} = 2\gamma \left( \frac{\Delta t}{\Delta x} \right) (\overline{h\nu})_{K+1/2} [\phi]_{K+1/2} , \quad (39)$$

and

$$\Lambda_{K+1/2} = 2\alpha g \left( \frac{\Delta t}{\Delta x} \right) [hu]_{K+1/2} , \quad (40)$$

where  $\gamma, \alpha$  are positive constants. Assume that the following time constraint holds:

$$\frac{\Delta t}{\Delta x} \left( \sqrt{g(\overline{h\nu})_{K+1/2}} \right) \leq \frac{1}{4} , \quad (41)$$

where we recall that  $\nu_K = \frac{1 + \xi_K}{1 - \xi_K}$  (see (24), Proposition 2.2). Then, if  $\alpha, \gamma$  satisfy the following bounds:

$$\gamma, \alpha \in [r^-, r^+] \quad , \quad \text{with } r^\pm(\vartheta) = \frac{1 \pm \sqrt{1 - \vartheta}}{\vartheta/2} , \quad (42)$$

where  $\vartheta = 16\lambda^2 g(\overline{h\nu})_{K+1/2}$ , the terms  $\mathcal{G}_K$  and  $\mathcal{L}_K$  in (36) are both negative.

*Proof.* Considering the definition of  $\mathcal{S}_K$  (7), the term  $\mathcal{G}_K$  (37) may be written as the sum of two interface contributions. More explicitly, we have  $\mathcal{G}_K = \mathfrak{g}_{k+1/2} + \mathfrak{g}_{k-1/2}$  with

$$\mathfrak{g}_{K+1/2} = 2g\lambda (\Pi_{K+1/2})^2 - \Pi_{K+1/2} [\phi]_{K+1/2} + 2\lambda (\overline{h\nu})_{K+1/2} ([\phi]_{K+1/2})^2 ,$$

wich leads to identify conditions ensuring  $\mathbf{g}_{K+1/2} \leq 0$  at the level of each cell interface. In what follows, we will focus at a generic interface  $K + 1/2$  and remove the corresponding subscripts for the sake of clarity. We thus rewrite  $\Pi = 2\gamma\lambda(\overline{h\nu})[\phi]$ , to get:

$$\mathbf{g} = 2\lambda(\overline{h\nu})[\phi]^2 [1 - \gamma + (4\lambda^2 g(\overline{h\nu})) \gamma^2] .$$

We easily verify that the condition to have real roots leads to the CFL condition (41) and the negativity is ensured with the bounds (42) for the constant  $\gamma$ . In a similar way, we can write the term (38) as  $\mathcal{L}_K = \mathfrak{l}_{K+1/2} + \mathfrak{l}_{K-1/2}$ , where:

$$\mathfrak{l}_{K+1/2} = 2g\lambda ([hu]_{K+1/2})^2 - [hu]_{K+1/2}\Lambda_{K+1/2} + 2\lambda(\overline{h\nu})_{K+1/2} (\Lambda_{K+1/2})^2 .$$

This motivates a definition of the form (40), from which we extract exactly the same conditions, namely (41) and  $\alpha \in [r^-, r^+]$ .  $\square$

**Remark 3.2.** We have  $\lim_{\vartheta \rightarrow 0} r^-(\vartheta) = 1$ ,  $r^\pm(1) = 2$  and  $\lim_{\vartheta \rightarrow 0} r^+(\vartheta) = +\infty$ . The function  $\vartheta \mapsto r^-(\vartheta)$  being increasing and  $\vartheta \mapsto r^+(\vartheta)$  decreasing on the interval  $[0, 1]$ , we get that the value  $\gamma = \alpha = 2$  is always admissible, independently from  $K$  (see Fig. 1. Note also the possibility of taking  $\gamma, \alpha$  close to 1 by diminishing  $\vartheta$  (that is the time step).

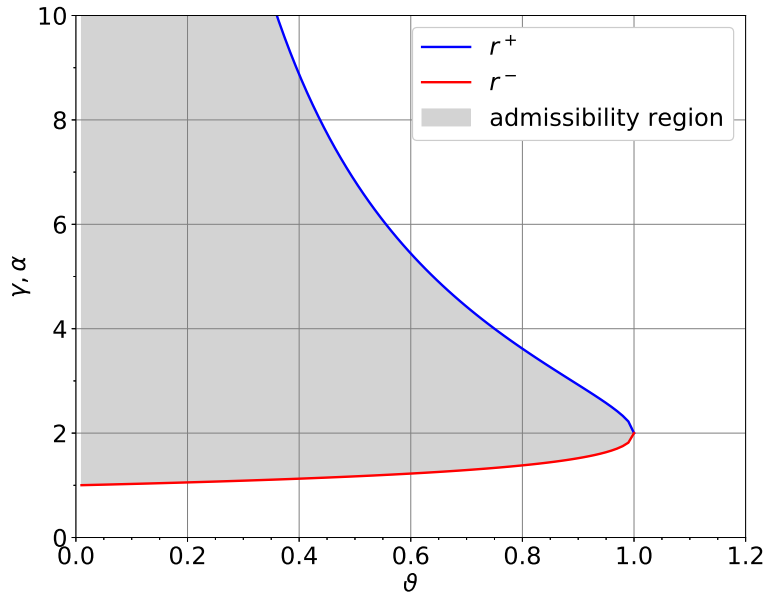


Figure 1: Admissibility region for the viscosity constants  $\gamma, \alpha$  as function of  $\vartheta = 16\lambda^2 g(\overline{h\nu})_{K+1/2}$ , issuing from (42).

### 3.2 Time step conditions

We start by recalling that the negativity of the right hand side of the discrete energy budget (36) is ensured under the advective time constraint (17) with  $\rho = \xi_K \in ]0, 1[$

(required to obtain the bounds on the kinetic energy, see Proposition 2.2), supplemented by the condition (41), implying the local quantity  $\nu_K = \frac{1 + \xi_K}{1 - \xi_K}$ . A first difficulty when considering (17) is that the time step is also involved in the interface velocities through the stabilization terms (39). The following result clarifies the conditions to impose on  $\lambda = \Delta t / \Delta x$ :

**Lemma 3.3.** *Let  $c_K = \sqrt{gh_K}$  and assume that*

$$\nu_K \leq \frac{1}{4\lambda c_K}. \quad (43)$$

*Then, a sufficient condition to have (17) is given by the following interface inequality*

$$2\lambda u_e^* \leq \frac{\xi_K}{1 + \xi_K}, \quad (44)$$

where  $u_e^* = \left( |(\overline{hu})_e| + \frac{1}{2g} \gamma \bar{c}_e |[\phi]_e| \right) / \check{h}_e$ , where  $e = K \pm 1/2$  and  $\check{h}_e = \min(h_K^n, h_L^n)$ , with the convention  $L = K \pm 1$ .

*Proof.* First, it is clear that (17) is ensured under the local condition  $\lambda |(hu)_e^*| \leq \frac{1}{2} \frac{\xi_K}{1 + \xi_K} h_K^n$  for  $e = K \pm 1/2$ , which, according to (14) and (39), is itself satisfied under:

$$2\lambda \left( |(\overline{hu})_e| + 2\gamma \lambda (\overline{h\nu})_e |[\phi]_e| \right) / \check{h}_e \leq \frac{\xi_K}{1 + \xi_K}. \quad (45)$$

Taking now (43) into account:

$$\lambda (\overline{h\nu})_e \leq \frac{1}{8} (h_K/c_K + h_L/c_L) = \frac{1}{4g} \bar{c}_e,$$

which gives the announced result.  $\square$

From this, a first basic approach to calibrate the time step consists in choosing  $\xi_K = \xi$  as constant over the mesh and consider separately the associated time constraints (44), (41), as done in [1]. This strategy appears to be relevant in low Froude regimes, where the time step is mainly governed by (41), which suggests a low value for  $\xi$  (so that  $\nu$  is taken close to 1 in practice) to minimize diffusive losses. More generally, it is possible to exploit the local character of  $\xi_K$  to express a balance between (44) and (41) through a unique CFL condition. This is the purpose of the following result:

**Proposition 3.4.** *Assume that the following CFL condition holds:*

$$\lambda \max_{e=K \pm 1/2} (u_e^* + \hat{c}_e) \leq \frac{1}{4}, \quad (46)$$

*with  $u_e^*$  given by (44) and  $\hat{c}_e = \max(c_K, c_L)$ . Then, conditions (41) and (44) are both satisfied.*

*Proof.* We start by introducing the notation  $\hat{u}_K^* = \max_{e=K\pm 1/2} u_e^*$ . We first remark that if  $\hat{u}_K^* = 0$ , then  $(hu)_e^* = 0$  for  $e = K \pm 1/2$ . In that case, the term  $\mathcal{C}_1$  appearing in the kinetic energy estimate (30) is zero and thus no longer needs to be controlled. This allows to take the weights  $(1/2, 1/2)$  to bound the other terms  $\mathcal{C}_2$  and  $\mathcal{C}_3$ , which amounts to take  $\nu_K = 1$  (notice that we have also  $[\phi]_e = 0$  in that case, so that  $\mathcal{C}_2$  is also equal to zero). As a result, (44) is automatically satisfied and (46) directly implies (41).

We now focus on the case  $\hat{u}_K^* > 0$ . First, under (46), we have  $2\lambda\hat{u}_K^* < 1/2$ , which allows to define

$$\xi_K := \frac{2\lambda\hat{u}_K^*}{1 - 2\lambda\hat{u}_K^*} \in ]0, 1[. \quad (47)$$

Then, the definition  $\nu_K = (1 + \xi_K)/(1 - \xi_K)$  leads to:

$$1/\nu_K = 1 - 4\lambda\hat{u}_K^*.$$

On the other side, noticing that  $\hat{c}_e = \max(c_K, c_L) \geq c_K$ , we remark that (46) also implies:

$$4\lambda \max_{e=K\pm 1/2} u_e^* + 4\lambda c_K \leq 1,$$

so that  $1/\nu_K \geq 4\lambda c_K$ , which, according to (43), ensures that we are within the frame of Lemma 3.3. Then, from the definition of  $\xi_K$  (47) we have  $2\lambda\hat{u}_K^* = \xi_K/(1 + \xi_K)$ , so that condition (44) is fulfilled.

The remaining condition (41) is almost immediate. Using  $\nu_K \leq 1/(4\lambda c_K)$ , a basic computation leads to the following estimation:

$$\lambda\sqrt{g(\overline{h\nu})_e} \leq \frac{1}{2}\sqrt{\lambda\bar{c}_e}.$$

As a result, to guarantee (41) it is sufficient to have  $\lambda\bar{c}_e \leq 1/4$ , which is also covered by (46).  $\square$

**Remark 3.5.** *Of course, these results stand as soon as we do not consider dry fronts. A simple strategy to avoid the problem due to the appearance of dry cells is to consider locally a standard upwind scheme whenever  $\check{h}_e$  falls below a certain tolerance value.*

To summarize, equipped with definitions (39), (40) and the notations

$$u_e^* = \left( |\overline{hu}_e| + \frac{1}{2g}\gamma\bar{c}_e|[\phi]_e| \right) / \check{h}_e, \quad c_K = \sqrt{gh_K^n},$$

the scheme (13) is stable under the CFL condition (46) in the sense of existence of local energy estimates of the form:

$$E_K^{n+1} - E_K^n + \Delta t \mathfrak{d}_K \left( e + \frac{1}{2}gh, hu^* \right) \leq 0. \quad (48)$$

The constants  $\gamma, \alpha$  involved in (39), (40) have to satisfy the bounds (42), and the quantity  $\nu_K$  is defined by  $\nu_K = \frac{1}{1 - 4\lambda\hat{u}_K^*}$ , where  $\hat{u}_K^* = \max_{e=K\pm 1/2} u_e^*$ .

**Remark 3.6.** We finally remark that condition (46) entails:

$$4\lambda\hat{u}_K^* \leq \frac{\hat{u}_K^*}{\hat{u}_K^* + c_K}$$

leading to  $1/\nu_K = 1 - 4\lambda\hat{u}_K^* \geq 1 - \frac{\hat{u}_K^*}{\hat{u}_K^* + c_K} = \frac{c_K}{\hat{u}_K^* + c_K}$ . This implies:

$$\nu_K \leq 1 + (Fr)_K,$$

where  $(Fr)_K = \frac{\hat{u}_K^*}{c_K}$  stands for a local Froude number. This alternative definition of the viscous terms (with respect to [1]) makes it possible to adapt the viscosity magnitude according to the flow regimes, as a function of the Froude number.

### 3.3 Addition of passive transport

To complete the picture, we show how the method naturally extends to passive transport considering the following model:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{gh^2}{2}\right) = -gh\partial_x z, \\ \partial_t(h\omega) + \partial_x(hu\omega) = 0, \end{cases} \quad (49)$$

$\omega = \omega(x, t)$  referring to any scalar variable. The augmented energy relies on the quantity:

$$\underline{e} = \frac{1}{2}u^2 + \frac{1}{2}gh + gb + \kappa\omega^2,$$

where  $\kappa > 0$ , and the energy equation remains:

$$\partial_t(h\underline{e}) + \partial_x\left(\left(\underline{e} + \frac{1}{2}gh\right)hu\right) = 0. \quad (50)$$

The last equation in (49) can be integrated using the upwind fluxes defined in (6):

$$\begin{cases} h_K^{n+1} = h_K^n - \Delta t \partial_K(hu^*), \\ (hu)_K^{n+1} = (hu)_K^n - \Delta t \partial_K^{up}(u, hu^*) - \Delta t h_K^n \partial_K \phi^*, \\ (h\omega)_K^{n+1} = (h\omega)_K^n - \Delta t \partial_K^{up}(\omega, hu^*). \end{cases} \quad (51)$$

More precisely, we have the following result:

**Proposition 3.7.** Let  $\mathcal{W}_K^n = \kappa h_K^n (\omega_K^n)^2$  the local energy at time  $n$  associated with the variable  $\omega$ . Under the CFL condition (17) with  $\rho = 1$ , we have:

$$\mathcal{W}_K^{n+1} - \mathcal{W}_K^n + \Delta t \partial_K^{up}(\kappa\omega^2, hu^*) \leq 0. \quad (52)$$

*Proof.* The proof follows the first steps of the one of Proposition 2.2 for the kinetic energy. First, by similar arguments as those employed to get (16), we easily get:

$$\omega_K^{n+1} = \omega_K^n - \Delta t \frac{1}{h_K^{n+1}} (hu^* \partial \omega)_K^{up}. \quad (53)$$

Using the mass equation and the relation (12), we get:

$$2\kappa h_K^{n+1} (\omega_K^{n+1} - \omega_K^n) \omega_K^n = \mathcal{W}_K^{n+1} - \mathcal{W}_K^n + \kappa \Delta t (\omega_K^n)^2 \partial_K(hu^*) - \kappa h_K^{n+1} (\omega_K^{n+1} - \omega_K^n)^2,$$

and using the scheme (53), together with Lemma 1.3:

$$\begin{aligned} \mathcal{W}_K^{n+1} - \mathcal{W}_K^n &= -\kappa \Delta t (2\omega_K^n (hu^* \partial \omega)_K^{up} + (\omega_K^n)^2 \partial_K hu^*) + \kappa h_K^{n+1} (\omega_K^{n+1} - \omega_K^n)^2 \\ &= -\kappa \Delta t \partial_K^{up}(\omega^2, hu^*) - 4\kappa \frac{\Delta t}{\Delta x} \mathcal{R}_K^+([\omega]^2, hu^*) + \kappa h_K^{n+1} (\omega_K^{n+1} - \omega_K^n)^2. \end{aligned}$$

This previous equality can be reformulated as

$$\mathcal{W}_K^{n+1} - \mathcal{W}_K^n + \Delta t \partial_K^{up}(\kappa \omega^2, hu^*) = \kappa \mathcal{T}_K, \quad (54)$$

with  $\mathcal{T}_K = h_K^{n+1} (\omega_K^{n+1} - \omega_K^n)^2 - 4 \frac{\Delta t}{\Delta x} \mathcal{R}_K^+([\omega]^2, hu^*)$ . Thus it remains to establish that  $\mathcal{T}_K \leq 0$  under the CFL condition (17). We have, using (53) and Lemma 1.4 :

$$\begin{aligned} h_K^{n+1} (\omega_K^{n+1} - \omega_K^n)^2 &= \frac{(\Delta t)^2}{h_K^{n+1}} [(hu^* \partial \omega)_K^{up}]^2 \\ &\leq 4 \left( \frac{\Delta t}{\Delta x} \right)^2 \frac{1}{h_K^{n+1}} ((-hu_{K+1/2}^*)^+ + (hu_{K-1/2}^*)^+) \mathcal{R}_K^{up}([\omega]^2, hu^*) \end{aligned}$$

Hence:

$$\mathcal{T}_K \leq 4 \frac{\Delta t}{\Delta x} (\mathcal{R}_K^+([\omega]^2, hu^*)) \left( \frac{\Delta t}{\Delta x} ((-hu_{K+1/2}^*)^+ + (hu_{K-1/2}^*)^+) / h_K^{n+1} - 1 \right),$$

which is negative under the CFL (17) in the case  $\rho = 1$ , according to (19).  $\square$

It directly follows that the scheme stability is ensured under the same conditions (discussed in Section §3.2), with this time a discrete energy estimate of the form:

$$\underline{E}_K^{n+1} - \underline{E}_K^n + \Delta t \mathfrak{d}_K \left( \underline{e} + \frac{1}{2} gh, hu^* \right) \leq 0, \quad (55)$$

where  $\underline{E}_K^n = \mathcal{E}_K^n + \mathcal{K}_K^n + \mathcal{W}_K^n$ , and the energy flux is given by:

$$\mathfrak{d}_K \left( \underline{e} + \frac{1}{2} gh, hu^* \right) = \widetilde{\partial}_K^{up} \left( \frac{1}{2} u^2, hu^* \right) + \widetilde{\partial}_K(\phi, hu^*) + \partial_K^{up}(\kappa \omega^2, hu^*).$$

## 4 Two dimensional extension

To extend the present approach in the 2D case, we consider a 2D domain meshed with polygonal cells indexed by  $K \in \mathbf{Z}$ . In what follows,  $m_K$  and  $m_{\partial K}$  will stand for the area and perimeter of a reference cell  $K$ . For every edge  $e \in \partial K$ , we will employ the notations  $m_e$  and  $\mathbf{n}_{e,K}$  respectively for the length and the outward normal of the corresponding boundary interface (see Fig 2). Denoting  $\mathbf{u} = {}^t(u, v)$  the velocity field, the set of equations we now consider is the following:

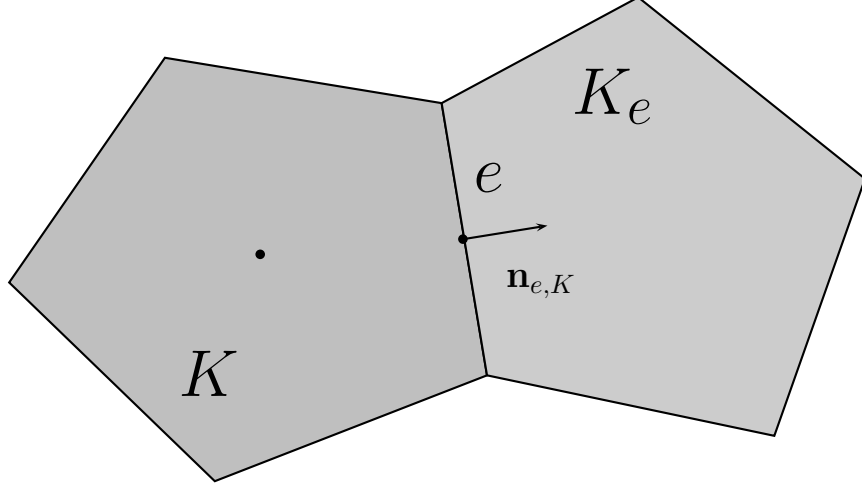


Figure 2: Geometric settings - focus on the interface  $e$ .

$$\begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t (h\mathbf{u}) + \nabla \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla \left( \frac{gh^2}{2} \right) = -gh\nabla z, \end{cases} \quad (56)$$

which can be written in the compact form:

$$\partial_t W + \nabla \cdot \mathbf{F}(W) + hS(W) = 0, \quad (57)$$

where  $W = \begin{pmatrix} h \\ h\mathbf{u} \end{pmatrix}$ ,  $\mathbf{F}(W) = \begin{pmatrix} h\mathbf{u} \\ h\mathbf{u} \otimes \mathbf{u} \end{pmatrix}$  and  $S(W) = \begin{pmatrix} 0 \\ \nabla \phi \end{pmatrix}$ , where we recall that  $\phi = g(h + z)$ .

#### 4.1 Derivation of the scheme

Denoting  $W_K^n$  the approximate solution on the cell  $K$  at time  $t^n$ , we seek for a two-dimensional formulation in terms of convex combination of one dimensional schemes, that is:

$$W_K^{n+1} = \sum_{e \in \partial K} \alpha_{e,K} W_{e,K}^{n+1}, \quad (58)$$

where  $W_{e,K}^{n+1}$  results from the application of the one dimensional approach in the normal direction  $\mathbf{n}_{e,K}$ , and  $\alpha_{e,K}$  are positive coefficients such that  $\sum_{e \in \partial K} \alpha_{e,K} = 1$ . Before going

further, we first remark that the one dimensional scheme with passive transport (51) can be rewritten in the following form:

$$\begin{aligned} W_K^{n+1} = W_K^n - \frac{\Delta t}{\Delta x} & (\mathcal{F}(W_K^n, W_{K+1}^n) - \mathcal{F}(W_{K-1}^n, W_K^n)) \\ & - \frac{\Delta t}{\Delta x} h_K^n (\mathcal{S}(W_K^n, W_{K+1}^n) - \mathcal{S}(W_{K-1}^n, W_K^n)), \end{aligned} \quad (59)$$



where, using notation (6):

$$\mathcal{F}(W_K^n, W_{K+1}^n) = \begin{pmatrix} (hu)_{K+1/2}^* \\ \mathcal{F}_{K+1/2}^{up}(u, (hu)^*) \\ \mathcal{F}_{K+1/2}^{up}(\omega, (hu)^*) \end{pmatrix}, \quad \mathcal{S}(W_K^n, W_{K+1}^n) = \begin{pmatrix} 0 \\ \phi_{K+1/2}^* \\ 0 \end{pmatrix}. \quad (60)$$

Going back to the 2D formalism, let's now select a reference interface  $e$  at the level of a cell  $K$  and apply this scheme to the states  $W_K^n, W_K^n, W_{K_e}^n$  in the reference associated with the outward normal  $\mathbf{n}_{e,K}$ . The velocity vector is subject to a change of coordinates which can be operated through the following mapping (we note  $\mathbf{n}_{e,K} = {}^t(n_x, n_y)$ ):

$$\tau : W = \begin{pmatrix} h \\ hu \end{pmatrix} \mapsto \widehat{W} = \begin{pmatrix} h \\ hu^\tau \end{pmatrix}, \quad (61)$$

where  $\mathbf{u}^\tau = {}^t(u^\tau, v^\tau)$  with:

$$\begin{aligned} u^\tau &= un_x + vn_y, \\ v^\tau &= -un_y + vn_x. \end{aligned}$$

It can easily be checked that the return to original  $(x, y)$  coordinates through  $\tau^{-1}$  is characterized by:

$$\begin{aligned} u &= u^\tau n_x - v^\tau n_y, \\ v &= u^\tau n_y + v^\tau n_x. \end{aligned}$$

Applying the basis scheme (59) in this reference to the states  $\widehat{W}_K^n, \widehat{W}_K^n, \widehat{W}_{K_e}^n$ , with a space step  $\Delta_K = \frac{m_K}{m_{\partial K}}$  and considering a passive transport for  $v^\tau$ , we directly get the following auxiliary update:

$$\begin{aligned} \widehat{W}_{e,K}^{n+1} &= \widehat{W}_K^n - \frac{\Delta t}{\Delta_K} \left( \mathcal{F}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) - \mathcal{F}(\widehat{W}_K^n, \widehat{W}_K^n) \right) \\ &\quad - \frac{\Delta t}{\Delta_K} h_K^n \left( \mathcal{S}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) - \mathcal{S}(\widehat{W}_K^n, \widehat{W}_K^n) \right). \end{aligned} \quad (62)$$

By consistency of the numerical fluxes  $\mathcal{F}$  (60) with the 1D physical advective fluxes  $F(h, hu, hu) = {}^t(hu, hu^2, huw)$ , we have  $\mathcal{F}(\widehat{W}_K^n, \widehat{W}_K^n) = F(\widehat{W}_K^n)$ , and it can easily be checked that  $\tau^{-1}F(\widehat{W}_K^n) = \mathbf{F}(W_K^n) \cdot \mathbf{n}_{e,K}$ . In a second hand, according to (60) and the definition (14), the computation of  $\mathcal{F}(\widehat{W}_K^n, \widehat{W}_{K_e}^n)$  implies the following interface value:

$$(hu^\tau)_e^* = (\overline{hu^\tau})_e - \Pi_e,$$

where  $(\overline{hu^\tau})_e = \frac{1}{2} (hu_{K_e}^\tau + hu_K^\tau)$ . The discussion on the rigorous calibration of the viscosity term  $\Pi_e$  and associated time constraints is postponed to the next section. Remarking that  $u^\tau = \mathbf{u} \cdot \mathbf{n}_{e,K}$ , we get  $(hu^\tau)_e^* = (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K}$ , where

$$(h\mathbf{u})_e^* = (\overline{h\mathbf{u}})_e - \Pi_e, \quad \Pi_e = \Pi_e \mathbf{n}_{e,K}, \quad (63)$$

As a result, we can write:

$$\mathcal{F}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) = \begin{pmatrix} (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} \\ \mathcal{F}_e^{up}(u^\tau, (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K}) \\ \mathcal{F}_e^{up}(v^\tau, (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K}) \end{pmatrix}, \quad (64)$$

so that going back to the  $(x, y)$  coordinates we can define the 2D numerical fluxes as:

$$\mathcal{F}(W_K^n, W_{K_e}^n, \mathbf{n}_{e,K}) := \tau^{-1} \mathcal{F}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) = \begin{pmatrix} (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} \\ \mathcal{F}_e^{up}(\mathbf{u}, (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K}) \end{pmatrix}, \quad (65)$$

where

$$\mathcal{F}_e^{up}(\mathbf{u}, (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K}) = \mathbf{u}_K \left( (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} \right)^+ + \mathbf{u}_{K_e} \left( (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} \right)^-.$$

Remains the second component of  $\mathcal{S}(\widehat{W}_K^n, \widehat{W}_{K_e}^n)$ , which is, still according to the basis scheme (59) and (15):

$$\phi_e^* = \bar{\phi}_e - \Lambda_e, \quad (66)$$

with  $\bar{\phi}_e = \frac{1}{2}(\phi_{K_e} + \phi_K)$  and a stabilization term  $\Lambda_e$ , which explicit form will be exhibited later on. Note finally that we have  $\tau^{-1} \mathcal{S}(\widehat{W}_K^n, \widehat{W}_K^n) = \begin{pmatrix} 0 \\ \phi_K \mathbf{n}_{e,K} \end{pmatrix}$  and  $\tau^{-1} \mathcal{S}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) = \begin{pmatrix} 0 \\ \phi_e^* \mathbf{n}_{e,K} \end{pmatrix}$ . Putting this all together, we are ready to use formula (58) with  $\alpha_{e,K} = \frac{m_e}{m_{\partial K}}$ , which gives, with the support of Green formula:

$$\begin{aligned} W_K^{n+1} &= \sum_{e \in \partial K} \frac{m_e}{m_{\partial K}} \tau^{-1} \left( \widehat{W}_{e,K}^{n+1} \right) \\ &= W_K^n - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}(W_K^n, W_{K_e}^n, \mathbf{n}_{e,K}) m_e - \frac{\Delta t}{m_K} h_K^n \sum_{e \in \partial K} \begin{pmatrix} 0 \\ \phi_e^* \mathbf{n}_{e,K} \end{pmatrix} m_e. \end{aligned}$$

This leads to the scheme proposed in [1]:

$$\left\{ \begin{array}{l} h_K^{n+1} = h_K^n - \frac{\Delta t}{m_K} \sum_{e \in \partial K} (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} m_e \\ (h\mathbf{u})_K^{n+1} = (h\mathbf{u})_K^n - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \left( \mathbf{u}_K \left( (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} \right)^+ + \mathbf{u}_{K_e} \left( (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} \right)^- \right) m_e \\ \quad - \frac{\Delta t}{m_K} h_K^n \sum_{e \in \partial K} \phi_e^* \mathbf{n}_{e,K} m_e, \end{array} \right. \quad (67)$$

where

$$(h\mathbf{u})_e^* = (\overline{h\mathbf{u}})_e - \mathbf{\Pi}_e, \quad \phi_e^* = \bar{\phi}_e - \Lambda_e, \quad (68)$$

We now address the issues relating to the stabilisation terms involved in (68) and associated time step conditions to ensure fully discrete energy estimates.

## 4.2 Stability of the 2D scheme

Denoting  $\eta(W) = h\underline{e}(W) = \frac{1}{2}gh^2 + ghb + \frac{1}{2}hu^2 + \frac{1}{2}hv^2$  the energy functional, the classical idea is to exploit (58) and the convexity of  $\eta$  to inherit the stability properties of the 1D approach. In particular, this prompts us to re-examine the stability conditions for each auxiliary update (62) by integrating its specific features, i.e. the formulation on non-regular grids and the presence of two identical states. We will rely on the following proposition:

**Proposition 4.1.** *With the notation  $\lambda_K = \Delta t / \Delta_K$ , and introducing the variable  $h\omega$  as passive transport, we consider the scheme (51) in the case of non regular meshes. We denote the vector state  $W_K^n = {}^t(h_K^n, (hu)_K^n, (h\omega)_K^n)$  and suppose that  $W_{K-1}^n = W_K^n$ . Assume that the viscous terms (14), (15) are given by:*

$$\Pi_{K+1/2} = \gamma(\overline{\lambda h\nu})_{K+1/2}[\phi]_{K+1/2} \quad , \quad \Lambda_{K+1/2} = \alpha g \bar{\lambda}_{K+1/2}[hu]_{K+1/2} \quad , \quad (69)$$

and that the time step is chosen such that:

$$\lambda_K \left( -(hu)_{K+1/2}^* \right)^+ \leq \xi_K h_K^{n+1} \quad , \quad (70)$$

and

$$\sqrt{g \bar{\lambda}_{K+1/2} (\overline{\lambda h\nu})_{K+1/2}} \leq 1/2 \quad , \quad (71)$$

Then, if the constants  $\gamma, \alpha$  satisfying the bounds:

$$\gamma, \alpha \in [r^-, r^+] \quad , \quad \text{with } r^\pm(\vartheta) = \frac{1 \pm \sqrt{1 - \vartheta}}{\vartheta/2} \quad , \quad (72)$$

where  $\vartheta = 4g \bar{\lambda}_{K+1/2} (\overline{\lambda h\nu})_{K+1/2}$ , we have the following estimation:

$$\underline{E}_K^n \leq \underline{E}_K^n - \lambda_K \left( \tilde{\mathcal{G}}(W_K^n, W_{K+1}^n) - G(W_K^n) \right) \quad , \quad (73)$$

Above, we recall the notation

$$\underline{E}_K^n = \mathcal{E}_K^n + \mathcal{K}_K^n + \mathcal{W}_K^n = \frac{1}{2}g(h_K^n)^2 + gh_K^n z_K + \frac{1}{2}h_K^n (u_K^n)^2 + \kappa h_K^n (\omega_K^n)^2 \quad ,$$

the function  $G$  corresponds to the physical entropy flux, namely  $G(W) = \left( \frac{1}{2}u^2 + \kappa\omega^2 + \phi \right) hu$  and  $\tilde{\mathcal{G}}$  refers to a numerical entropy flux function consistent with  $G$ , defined as:

$$\tilde{\mathcal{G}}(W_K^n, W_{K+1}^n) = \mathcal{F}_{K+1/2}^{up} \left( \frac{1}{2}u^2 + \kappa\omega^2, hu^* \right) + \bar{\phi}_{K+1/2} (hu)_{K+1/2}^* + \tilde{\mathcal{B}}(W_K^n, W_{K+1}^n) \quad . \quad (74)$$

The upwind flux in this last equality is given by (6), while the contribution  $\tilde{\mathcal{B}}$  refers to lower order corrective fluxes terms.

*Proof.* First note that, provided we replace  $\Delta x$  by  $\Delta_K$ , all the results stated in Section §1 are still valid in the case of non regular meshes. Without going again through all the technical steps, some comments are nonetheless necessary regarding potential and kinetic energy estimates in the particular case where the space step  $\Delta_K$  is not constant and the vector state at the level of the cell  $K - 1$  is equal to the one on  $K$ . As a first step, regarding Proposition 2.1 dedicated to potential energy, we have to mention that the jumps located at the interface  $K - 1/2$  (corresponding to the “double” state  $W_K^n, W_K^n$  in the present context) vanish in (22). This allows to save a factor of two in the inequalities and leads to:

$$\begin{aligned} \mathcal{E}_K^{n+1} - \mathcal{E}_K^n + \lambda_K \left( \tilde{\mathcal{G}}^\mathcal{E}(W_K^n, W_{K+1}^n) - G^\mathcal{E}(W_K^n) \right) - \Delta t (hu)_K^n \partial_K^c \phi \\ \leq g \lambda_K^2 [hu]_{K+1/2}^2 + g \lambda_K^2 \Pi_{K+1/2}^2 - \lambda_K \Pi_{K+1/2} [\phi]_{K+1/2} \quad . \end{aligned}$$

where  $G^\mathcal{E}(W_K^n) = \phi_K^n(hu)_K^n$  and

$$\tilde{\mathcal{G}}^\mathcal{E}(W_K^n, W_{K+1}^n) = \bar{\phi}_{K+1/2}(hu)_{K+1/2}^* - [\phi]_{K+1/2}[hu]_{K+1/2}.$$

The two first terms of the right hand side may now be rewritten as follows:

$$\begin{aligned} g\lambda_K^2[hu]_{K+1/2}^2 &= g\lambda_K[hu]_{K+1/2}^2\bar{\lambda}_{K+1/2} - g\lambda_K[hu]_{K+1/2}^2[\lambda]_{K+1/2}, \\ g\lambda_K^2\Pi_{K+1/2}^2 &= g\lambda_K\Pi_{K+1/2}^2\bar{\lambda}_{K+1/2} - g\lambda_K\Pi_{K+1/2}^2[\lambda]_{K+1/2}. \end{aligned}$$

Seeing the last terms of the two previous equalities as a bias on the leading flux term, we incorporate them in  $\tilde{\mathcal{G}}^\mathcal{E}(W_K^n, W_{K+1}^n)$ , so that we can write:

$$\begin{aligned} \mathcal{E}_K^{n+1} - \mathcal{E}_K^n + \lambda_K \left( \tilde{\mathcal{G}}^\mathcal{E}(W_K^n, W_{K+1}^n) - G^\mathcal{E}(W_K^n) \right) - \Delta t(hu)_K^n \partial_K^c \phi \\ \leq \lambda_K \left[ g\bar{\lambda}_{K+1/2}([hu]_{K+1/2})^2 + g\bar{\lambda}_{K+1/2}(\Pi_{K+1/2})^2 - \Pi_{K+1/2}[\phi]_{K+1/2} \right]. \end{aligned} \quad (75)$$

In a similar way, using the same argument to bound  $\mathcal{C}_2$  and  $\mathcal{C}_3$  in (30), the kinetic energy estimate (23) is now:

$$\begin{aligned} \mathcal{K}_K^{n+1} - \mathcal{K}_K^n + \lambda_K \left( \tilde{\mathcal{G}}^\mathcal{K}(W_K^n, W_{K+1}^n) - G^\mathcal{K}(W_K^n) \right) + \Delta t(hu)_K^n \partial_K^c \phi \\ \leq \lambda_K \left[ (\bar{\lambda}h\nu)_{K+1/2}([\phi]_{K+1/2})^2 + (\bar{\lambda}h\nu)_{K+1/2}(\Lambda_{K+1/2})^2 - [hu]_{K+1/2}\Lambda_{K+1/2} \right], \end{aligned} \quad (76)$$

with  $G^\mathcal{K}(W_K^n) = \frac{1}{2}(u_K^n)^2(hu)_K^n$  and

$$\tilde{\mathcal{G}}^\mathcal{K}(W_K^n, W_{K+1}^n) = \mathcal{F}_{K+1/2}^{up} \left( \frac{1}{2}u^2, hu^* \right) + \tilde{\mathcal{B}}(W_K^n, W_{K+1}^n).$$

Above we make use of the generic notation  $\tilde{\mathcal{B}}(W_K^n, W_{K+1}^n)$  to gather all the lower order correction terms. Also, it may be important to emphasize that in this case, the quantity  $\mathcal{R}_K^+([u]^2, hu^*)$  defined in (11) is equal to  $\mathcal{R}_K^+([u]^2, hu^*) = (-hu_{K+1/2}^*)^+ [u]_{K+1/2}^2$  and that (1.4) give  $[(hu^* \partial u)_K^{up}]^2 = \frac{4}{(\Delta_K)^2} (-hu_{K+1/2}^*)^+ \mathcal{R}_K^+([u]^2, hu^*)$ . The time step condition issuing from (32) is thus (70). Gathering (75) and (76), and considering the variable  $\omega$  subject to passive transport (which has no impact on the energy estimates, according to Proposition 3.7), the energy budget is of the form:

$$\underline{E}_K^{n+1} - \underline{E}_K^n + \lambda_K \left( \tilde{\mathcal{G}}(W_K^n, W_{K+1}^n) - G(W_K^n) \right) \leq \lambda_K \left[ \mathcal{G}_{K+1/2} + \mathcal{L}_{K+1/2} \right], \quad (77)$$

with the numerical energy flux  $\tilde{\mathcal{G}}$  given by (74), consistent with the physical flux  $G$ . To get (73), it thus remains to examine the conditions ensuring the negativity of the right hand side in the previous inequality:

$$\mathcal{G}_{K+1/2} = (\bar{\lambda}h\nu)_{K+1/2}([\phi]_{K+1/2})^2 + g\bar{\lambda}_{K+1/2}(\Pi_{K+1/2})^2 - \Pi_{K+1/2}[\phi]_{K+1/2} \leq 0, \quad (78)$$

$$\mathcal{L}_{K+1/2} = g\bar{\lambda}_{K+1/2}([hu]_{K+1/2})^2 + (\bar{\lambda}h\nu)_{K+1/2}(\Lambda_{K+1/2})^2 - [hu]_{K+1/2}\Lambda_{K+1/2} \leq 0. \quad (79)$$

Reproducing the analysis carried out in Proposition 3.1, we directly fall on the definition  $\Pi_{K+1/2} = \gamma(\bar{\lambda}h\nu)_{K+1/2}[\phi]_{K+1/2}$  and  $\Lambda_{K+1/2} = \alpha g\bar{\lambda}_{K+1/2}[hu]_{K+1/2}$ , associated with the bounds (72) and the CFL condition (71).  $\square$

It should therefore be noted that in this particular case, the gain is a factor of two in the calibration of the viscosity terms (see (39), (40)) and in the time step based on gravity waves (41).

Based on this, returning to (62), we now apply the result of the previous proposition to the states  $\widehat{W}_K^n, \widehat{W}_{K_e}^n, \widehat{W}_{K_e}^n$ , with  $v^\tau$  playing the role of passive transport. By substituting the reference interface  $K + 1/2$  by  $e \in \partial K$  We thus define:

$$\Pi_e = \gamma(\overline{\lambda h \nu})_e [\phi]_e \quad , \quad \Lambda_e = \alpha g \bar{\lambda}_e [hu^\tau]_e, \quad (80)$$

and assume that the time step conditions

$$\lambda_K (-(hu^\tau)_e^*)^+ \leq \xi_K h_{e,K}^{n+1} \quad , \quad \sqrt{g \bar{\lambda}_e (\overline{\lambda h \nu})_e} \leq 1/2, \quad (81)$$

are satisfied at the interface  $e \in \partial K$ . Provided the bounds (72) are verified with  $\vartheta = 4g \bar{\lambda}_e (\overline{\lambda h \nu})_e$ , energy estimate (73) give:

$$\eta(\widehat{W}_{e,K}^{n+1}) \leq \eta(\widehat{W}_K^n) - \frac{\Delta t}{\Delta_K} \left( \tilde{\mathcal{G}}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) - G(\widehat{W}_K^n) \right) \quad (82)$$

where

$$\tilde{\mathcal{G}}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) = \mathcal{F}_e^{up} \left( \frac{1}{2} \|\mathbf{u}\|^2, (hu^\tau)^* \right) + \bar{\phi}_e (hu^\tau)_e^* + \tilde{\mathcal{B}}(\widehat{W}_K^n, \widehat{W}_{K_e}^n), \quad (83)$$

where  $\tilde{\mathcal{B}}$  contains all the low order jump corrections. Now, recalling that  $(h\mathbf{u}) \cdot \mathbf{n}_{e,K} = hu^\tau$ , we introduce the notation:

$$\mathcal{G}(W_K^n, W_{K_e}^n, \mathbf{n}_{e,K}) := \tilde{\mathcal{G}}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) = \mathcal{F}_e^{up} \left( \frac{1}{2} \|\mathbf{u}\|^2, (h\mathbf{u})^* \cdot \mathbf{n}_{e,K} \right) + \phi_e (h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} + \tilde{\mathcal{B}}(\widehat{W}_K^n, \widehat{W}_{K_e}^n), \quad (84)$$

and remark that  $G(\widehat{W}_K^n) = \left( \frac{1}{2} \|\mathbf{u}_K\|^2 + \phi_K \right) (h\mathbf{u})_K \cdot \mathbf{n}_{e,K}$ , which can be seen as the projection of the 2D entropy flux  $\mathbf{G}$  on the outward normal vector associated with the interface  $e$ , namely  $\mathbf{G}(W_K^n) \cdot \mathbf{n}_{e,K}$ . Having this in mind, formula (58) and the convexity of  $\eta$  leads to  $\eta(W_K^{n+1}) \leq \sum_{e \in \partial K} \alpha_{e,K} \eta(W_{e,K}^{n+1})$ . Since  $\eta$  is invariant by change of coordinates, the use of (82) and Green formula allows to write:

$$\begin{aligned} \eta(W_K^{n+1}) &\leq \sum_{e \in \partial K} \alpha_{e,K} \left[ \eta(W_K^n) - \frac{\Delta t}{\Delta_K} \left( \tilde{\mathcal{G}}(\widehat{W}_K^n, \widehat{W}_{K_e}^n) - G(\widehat{W}_K^n) \right) \right] \\ &\leq \eta(W_K^n) - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{G}(W_K^n, W_{K_e}^n, \mathbf{n}_{e,K}) m_e. \end{aligned} \quad (85)$$

It is worth stressing at this point that the flux term (84) appearing in (85) is based on centred and upwind fluxes (6). All the corrective terms formally integrated through the notation  $\tilde{\mathcal{B}}(W_K^n, W_{K_e}^n)$  are products involving interface jump quantities, and enjoy the same symmetry properties as the leading flux terms. The numerical entropy fluxes  $\mathcal{G}$  defined in (84) thus inherit from the consistency of the upwind and centred fluxes:

$$\mathcal{G}(W_K^n, W_{K_e}^n, \mathbf{n}_{e,K}) = \left( \frac{1}{2} \|\mathbf{u}_K\|^2 + \phi_K \right) hu_K^\tau = \mathbf{G}(W_K^n) \cdot \mathbf{n}_{e,K}, \quad (86)$$

and satisfy the conservativity property:

$$\mathcal{G}(W_K^n, W_{K_e}^n, \mathbf{n}_{e,K}) = -\mathcal{G}(W_{K_e}^n, W_K^n, -\mathbf{n}_{e,K}). \quad (87)$$

As a result, the stability of the scheme (67) is ensured through the existence of fully discrete local energy estimates (85) provided the time step satisfies conditions (81), written as:

$$\lambda_K (-(h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K})^+ \leq \xi_K h_{e,K}^{n+1}, \quad (88)$$

$$\sqrt{g\bar{\lambda}_e(\bar{\lambda}h\nu)_e} \leq 1/2. \quad (89)$$

Based on (80), the viscosity terms in (68) can be written as follows:

$$\mathbf{\Pi}_e = \gamma(\bar{\lambda}h\nu)_e \boldsymbol{\delta}_e \phi \quad , \quad \Lambda_e = \alpha g \bar{\lambda}_e \delta_e(h\mathbf{u}), \quad (90)$$

with the jump operators  $\boldsymbol{\delta}_e \phi = \frac{1}{2} (\phi_{K_e}^n - \phi_K^n) \mathbf{n}_{e,K}$  and  $\delta_e(h\mathbf{u}) = \frac{1}{2} (h\mathbf{u}_{K_e}^n - h\mathbf{u}_K^n) \cdot \mathbf{n}_{e,K}$ . The constants  $\gamma, \alpha$  are subject to the bounds (72) with  $\vartheta = 4g\bar{\lambda}_e(\bar{\lambda}h\nu)_e$ . We finally discuss explicit conditions ensuring (88), (89) in this last proposition:

**Proposition 4.2.** *Conditions (88), (89) are satisfied under:*

$$\Delta t \max \left( \frac{m_{\partial K}}{m_K}, \frac{m_{\partial K_e}}{m_{K_e}} \right) (u_e^* + \hat{c}_e) \leq 1/2, \quad (91)$$

where  $\hat{c}_e = \max(c_K, c_{K_e})$ ,  $c_K = \sqrt{gh_K^n}$ . Advection terms are carried by the quantity:

$$u_e^* = \left( \frac{1}{2} (|(h\mathbf{u})_{K_e}^n \cdot \mathbf{n}_{e,K}| + |(h\mathbf{u})_K^n \cdot \mathbf{n}_{e,K}|) + \frac{\gamma}{2g} \|\boldsymbol{\delta}_e \phi\|_{\bar{c}_e} \right) / \check{h}_e, \quad (92)$$

with  $\check{h}_e = \min(h_K^n, h_{K_e}^n)$ , and the term  $\nu_K$  involved in (90) is defined as:

$$\nu_K = \frac{1}{1 - 2M_K} \quad \text{with} \quad M_K = \max_{e \in \partial K} \left( \hat{\lambda}_e u_e^* \right). \quad (93)$$

*Proof.* The proof of this results follows the same lines as those of Proposition 3.4. Recalling that  $\lambda_K = \Delta t / \Delta_K$  with  $\Delta_K = m_K / m_{\partial K}$ , we start rewriting (91) as follows:

$$\max_{e \in \partial K} \hat{\lambda}_e (u_e^* + \hat{c}_e) \leq 1/2. \quad (94)$$

with  $\hat{\lambda}_e = \max(\lambda_K, \lambda_{K_e})$ . The case  $M_K = 0$  can be handled rapidly remarking that it corresponds to the steady state  $(h\mathbf{u})_K^n = (h\mathbf{u})_{K_e}^n = 0$ ,  $\phi_K^n = \phi_{K_e}^n$  for all  $e \in \partial K$ , which is exactly preserved by the scheme. Focusing now on the case  $M_K > 0$ , we first remark that (94) implies  $M_K < 1/2$ , which allows to define:

$$\xi_K = \frac{M_K}{1 - M_K} \in ]0, 1[, \quad (95)$$

and gives  $\nu_K = \frac{1 + \xi_K}{1 - \xi_K} = \frac{1}{1 - 2M_K}$ . On a second hand, it is clear that (94) also implies  $2M_K + 2\lambda_K c_K \leq 1$ , so that:

$$\nu_K \leq \frac{1}{2\lambda_K c_K}. \quad (96)$$

Keeping in mind this estimation, we look for a sufficient condition to guarantee (88). We note that:

$$-(h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K} \leq |(\mathbf{h}\mathbf{u})_e^* \cdot \mathbf{n}_{e,K}| \leq \frac{1}{2} (|(\mathbf{h}\mathbf{u})_K^* \cdot \mathbf{n}_{e,K}| + |(\mathbf{h}\mathbf{u})_{K_e}^* \cdot \mathbf{n}_{e,K}|) + |\mathbf{\Pi}_e \cdot \mathbf{n}_{e,K}|,$$

and using (96) in the definition (90):  $|\mathbf{\Pi}_e \cdot \mathbf{n}_{e,K}| \leq \frac{\gamma}{2g} \bar{c}_e \|\boldsymbol{\delta}_e \phi\|$ . This eventually leads to

$(-(h\mathbf{u})_e^* \cdot \mathbf{n}_{e,K})^+ \leq \check{h}_e u_e^*$ . On the other hand, the 3-point mass scheme (62) we considered when getting (81) can be expressed through the following equality:

$$h_K^n - h_{e,K}^{n+1} = \lambda_K \left( \frac{1}{2} ((\mathbf{h}\mathbf{u})_{K_e}^* \cdot \mathbf{n}_{e,K} - (\mathbf{h}\mathbf{u})_K^n \cdot \mathbf{n}_{e,K}) - \mathbf{\Pi}_e \cdot \mathbf{n}_{e,K} \right). \quad (97)$$

By similar arguments, one easily obtains  $h_{e,K}^{n+1} \geq h_K^n - \lambda_K \check{h}_e u_e^*$ . Then we can deduce the following sufficient condition for (88):

$$\lambda_K \check{h}_e u_e^* \leq \xi_K \left( h_K^n - \lambda_K \check{h}_e u_e^* \right), \quad (98)$$

which can be simply reformulated as  $\lambda_K \check{h}_e u_e^* \leq M_K h_K^n$ , according to the relation  $M_K = \xi_K / (1 + \xi_K)$  issuing from (95). This latter condition is obvious in view of the definition of  $M_K$  (93).

As regards now the time step restriction (89), using (96) we have  $(\overline{\lambda h \nu})_e \leq \bar{c}_e / 2g$ , leading to the sufficient condition  $\bar{\lambda}_e \bar{c}_e \leq 1/2$ , which is also provided by (94).  $\square$

**Remark 4.3.** *The main difference with [1] lies in the choice of the coefficients implied in the Jensen inequalities of the kinetic energy estimates (30). The choices made here lead to a fully explicit version of the scheme, give the same bounds governing the choice of  $\gamma$  and  $\alpha$ , and energy stability is ensured under a unique standard global CFL condition (94). Also, as stated previously in the 1d case (Remark 3.6), the estimation of the form  $\nu_K \leq 1 + (Fr)_K$  where  $(Fr)_K = \hat{\mathbf{u}}_K^* / c_K$ , still valid in this case (setting  $\hat{\mathbf{u}}_K^* = M_K / \lambda_K$ ), allows the viscosity to be adjusted naturally according to the flow regime under consideration.*

## References

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