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Gelfand Type Elliptic Boundary Problems: a Synoptic Novel Codification of Solutions & Non-Solutions

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Edoardo Niccolai. Gelfand Type Elliptic Boundary Problems: a Synoptic Novel Codification of Solutions & Non-Solutions. 2024. hal-04219781

HAL Id: hal-04219781

<https://hal.science/hal-04219781>

Preprint submitted on 11 Mar 2024

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**GELFAND TYPE ELLIPTIC BOUNDARY PROBLEMS:
A SYNOPTIC NOVEL CODIFICATION
OF SOLUTIONS & NON-SOLUTIONS**

EDOARDO NICCOLAI

ABSTRACT. The Gelfand problem has its charm in its openness to multiple solutions and non-solutions. We focus our efforts on: 1-, 2-, 3- or higher-space dimensions; videlicet: $D = 1 \leq n \leq 9$, $D = 3 \leq n \leq 9$, $D = n \geq 10$, $D = n \geq 11$. Accounts are also given when the problem is dealing with a geodesic ball (spherical cap), and with no deformation of the \mathbb{R} -metric and no decrement of the Ricci scalar. Sifting through even just a part of the pertaining bibliography, curiosities, inaccuracies, and inedited observations emerge. For those in need of completeness there is an appendix on Lebesgue’s and Picard–Lindelöf’s theorems. This article presents five proofs together with a proof sketch and a hint of proof (because they are already present elsewhere). Not only. There is, for minds suited to a broad vision, a critical annotation (*in cauda venenum*).

KEYWORDS: Gelfand problem, elliptic boundary problems, classification of solutions and non-solutions from 1- to 3- or higher-space dimensions, Emden–Fowler equation and transformation, infinity Laplacian (Δ_∞), (hemi)spherical spaces, Dirichlet boundary, perturbations, Hardy’s inequality (or Hardy–Littlewood–Pólya inequality), Picard–Lindelöf theorem, mini-version of Lebesgue’s dominated convergence theorem.

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1. Intro: Euler–Lagrange Versatile Ancestry

An ancestor of the Gelfand problem (GP) can be reconstructed directly from the Euler–Lagrange formulæ, as a semi-linear partial differential equations,

$$\begin{cases} -\Delta_{\text{LB}} v = \varphi(v) \text{ in } \Omega \in \mathbb{R}^n, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \quad (1a)$$

where

$\Delta_{\text{LB}} \stackrel{\text{eqv}}{=} \Delta$ is the Laplace–Beltrami operator, which we can identify with a diffusion operator, $v \in S_\Omega \stackrel{\text{eqv}}{=} \mathcal{C}_0^2(\hat{\Omega})$, assuming S_Ω is a function (sub)space, $\Omega \in \mathbb{R}^n$ is a bounded region of the Euclidean n -space, and $\partial\Omega$ is a smooth boundary.

Eq. (1a) has a weak yet stable solution, whilst Eq. (1), in its very doubleness, is in use for the energy functional

$$\begin{aligned} \mathcal{E}_{[\Omega]}(v) &: S_\Omega \rightarrow \mathbb{R}^n, \\ \mathcal{E}_{[\Omega]}(v) &= \left(\frac{1}{2} \int_{\Omega \in \mathbb{R}^n} |\nabla^2 v| dx \right) - \left(\int_{\Omega \in \mathbb{R}^n} \varpi_{[\mathcal{E}^2]}(v) dx \right), \quad \zeta_{\lambda_k}(t) \in S_\Omega \setminus \{0\}, \end{aligned} \quad (2)$$

where

$$\varpi: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is a } \mathcal{C}^2\text{-function.}$$

We recall that certain variations of \mathcal{E} -energy, according to the flow of time $t \in \mathbb{R}^n$, have the advantage of determining when a *stable critical point* of $\mathcal{E}_{[\Omega]}$ is zero, for a given function $\zeta: \mathbb{R}^n \rightarrow \mathbb{R}^n$, viz.

$$\zeta(t) = \mathcal{E}_{[\Omega]}(v + \zeta_{\lambda_k}(t)), \quad (3)$$

with the intervention of the k -th eigenvalue ζ_{λ_k} of the Dirichlet Laplacian in Ω . In fact, it is enough to fix

$$\frac{\zeta_{\lambda_k}(t)}{t} - \frac{\zeta_{\lambda_k}(0)}{t} = \int_{\Omega \in \mathbb{R}^n} \nabla v \nabla \zeta_{\lambda_k} dx + \frac{\left(\int_{\Omega \in \mathbb{R}^n} |\nabla \zeta_{\lambda_k}|^2 dx \right) t}{2} - \int_{\Omega \in \mathbb{R}^n} \frac{\varpi(v) - \varpi(t\zeta_{\lambda_k} + v)}{t} dx. \quad (4)$$

NB. $\mathcal{E}_{[\Omega]}$ is a functional whose derivative occurs at each point in its domain, that is, its differentiability is related to $v \in S_\Omega$, while its derivative is $D\mathcal{E}_{[\Omega]}(v) = 0$, and the differentiation is a Fréchet-like process [35] on a normed spaces.

For the Lagrange theorem, one gets

$$\left| -\frac{\varpi(v) - \varpi(t\zeta_{\lambda_k} + v)}{t} \right| \leq \|\varphi\|_{L^\infty[-\alpha, \alpha]}, \quad \varphi = \dot{\varpi}, \quad (5)$$

putting

$$\begin{aligned} \alpha &= \|v\|_{L^\infty(\Omega)} + \|\zeta_{\lambda_k}\|_{L^\infty(\Omega)},^a \text{ and} \\ |t| &\leq 1. \end{aligned}$$

By Lebesgue’s dominated convergence theorem [52] (see Appendix, Theorem 6.2), one has

$$\int_{\Omega \in \mathbb{R}^n} \nabla v \nabla \zeta_{\lambda_k} dx = \int_{\Omega \in \mathbb{R}^n} \varphi(v) \zeta_{\lambda_k} dx \quad (6)$$

by establishing $t \rightarrow 0$, and $\dot{\zeta}_{\lambda_k}(0) = 0$. From which

$$\int_{\Omega \in \mathbb{R}^n} -\zeta_{\lambda_k} (\varphi(v) + \Delta v) dx = 0. \quad (7)$$

^a L^∞ is a generalization of the vector L^p space, whose norm is ensured by the essential supremum: $|\varphi_L|_\infty = \text{ess sup } |\varphi_L|$. To be precise, L^∞ is a *Banach space* of bounded sequence.

2. Gelfand Problem from Semi-linear Elliptic Formulæ

From the previous section we can move to define the constitutive Gelfand problem of stable solutions of semi-linear elliptic equations:

$$\begin{cases} -\Delta_{\mathbb{B}^n} v = \lambda e^v & \text{in } \mathbb{B}_{\rho=1}^n, \\ v = 0 & \text{on } \partial\mathbb{B}_{\rho=1}^n, \end{cases} \quad \begin{matrix} (8a) \\ (8b) \end{matrix}$$

where

$\Delta_{\mathbb{B}^n}$ is, again, the differential operator but in the ball model, and it is in the guise of diffusion operator,

$\lambda > 0$ is a (positive) parameter,

$\mathbb{B}_{\rho=1}^n \subset \mathbb{R}^{n+1}$, $n \geq 1$, is the unit ball of $\mathbb{R}^{n \geq 1}$.

The propagation is regulated by the diffusion of temperature, i.e. of heat conduction, from hot areas towards the cold boundary, according to the non-linearity of a law established by S. Arrhenius [4], under a chemical reaction with

$$\varphi_\varepsilon(v) = e^\vartheta, \quad \vartheta = \frac{v}{1 + \varepsilon v}, \quad (9)$$

being $\varepsilon > 0$ a value sufficiently small. In logarithmic appearance, Eq. (9) takes the form

$$\log \varphi_\varepsilon(v) = \frac{v \log e}{1 + \varepsilon v}. \quad (10)$$

Two perspectives open up.

(1) The first is a no steady-state, in which the pair of Eqq. (8) is not valid, because there is no stability (it may happen that the fuel undergoes a rarefaction, due to the small quantity, and the reaction is extinguished, or there may be too much fuel, which leads to a thermal explosion).

(2) The second one is a steady combustion process, as there is a stability, thanks to a very fast balance between the produced and the diffused heat, whose quantification is provided by $\lambda > 0$ (which can be considered, historically, as the Frank-Kamenetskii constant), and the pair of Eqq. (8) works.

In summary, if the value of λ is large, the pair (8) ceases to be valid; if the value of λ is small, the pair (8) is valid.

As it is well known, for $n = 1, 2, 3$, the Gelfand problem is applied to self-heating and self-ignition studies, e.g. when it is dealing with a container with a cylindrical shape, which is in a steady-state regime, viz. in an intermediate asymptotic range, see e.g. I.M. Gelfand [36], A. Farina [29].

The above can be summarized as follows.

Starting Theorem 2.1. *We say that $\check{\lambda}$ denotes the maximal value, or the extremal (parameter), of λ . Letting $\check{\lambda} = \check{\lambda}(n) > 0$, $n \geq 1$,*

- (1) *if $\lambda > \check{\lambda}$, there is no solution of (8),*
- (2) *if $0 < \lambda < \check{\lambda}$, there is a minimal solution*

$$v_\lambda^{\min} \in \mathcal{C}^2(\overline{\mathbb{B}})$$

of (8), which is a stable.

Let us go into details, by placing before the positive cases (those with solutions).

2.1. Gelfand Problem for Various Dimensions

2.1.1. Gelfand Problem in 1-Space Dimension

Theorem 2.1 *(The $n = 1$ case). When $n = 1$, the pair (8) takes this form*

$$n = 1 \begin{cases} -\ddot{v} = \lambda e^v, \\ v(-1) = v(+1) = 0. \end{cases} \quad \begin{matrix} (11a) \\ (11b) \end{matrix}$$

Let $\check{\lambda} > 0$.

(1) If $0 < \lambda < \check{\lambda}$, one has two positive solutions, which are strictly decreasing in the interval $(0, 1]$, whose determination is precisely in $x = 0$,

(i) the so-called minimal solution,

$$v_{\check{\lambda}}^{\min} \in \mathcal{C}^2[-1, +1], \quad (12)$$

which provides stability to Gelfand equation(s) in the form (11), and

(ii) a solution with Morse index [62] [2] $i_M = 1$, which can be written as

$$v_{\lambda}(i_M).$$

The curved line

$$(0, \check{\lambda}) \longrightarrow \mathcal{C}^2[-1, +1] \times \mathcal{C}^2[-1, +1], \quad \lambda \mapsto (v_{\lambda}, v_{\lambda}(i_M)),$$

turns out to be smooth, for any $x \in (-1, +1)$, so that

$$\left\{ \begin{array}{l} \lim_{\lambda \rightarrow 0^+} \{v_{\lambda}(x), v_{\lambda}(i_M)(x)\} = (0, +\infty), \\ \lim_{\lambda \rightarrow \check{\lambda}} \{v_{\lambda}(x), v_{\lambda}(i_M)(x)\} = (\check{v}(x), \check{v}(x)), \end{array} \right. \quad (13a)$$

$$\left\{ \begin{array}{l} \lim_{\lambda \rightarrow 0^+} \{v_{\lambda}(x), v_{\lambda}(i_M)(x)\} = (0, +\infty), \\ \lim_{\lambda \rightarrow \check{\lambda}} \{v_{\lambda}(x), v_{\lambda}(i_M)(x)\} = (\check{v}(x), \check{v}(x)), \end{array} \right. \quad (13b)$$

once it is specified that

$$\check{v}(x) \in \mathcal{C}^2[-1, +1]$$

is the unique solution to the pair of Eqq. (11).

(2) If $\lambda > \check{\lambda}$, there is no solution of the type (12).

Proof. For simplicity, the demonstration can be divided into four parts.

(α) We just said that v is the solution to (11), in two versions: v_{λ} and $v_{\lambda}(i_M)$. But

$$v > 0 \text{ and } -\check{v} > 0 \text{ have no point of minimum in } (-1, +1).$$

To circumvent this issue, write

$$\left\{ -\frac{1}{2}\dot{v}^2 = \lambda(e^v - e^{v(0)}) \right\} \stackrel{\text{eqv}}{=} \left\{ \frac{-\dot{v}}{\sqrt{2}\sqrt{\lambda(-e^v + e^{v(0)})}} = 1 \right\}, \quad (14)$$

so as to reach a partial solution to the pair of Eqq. (11), which is

$$\int_0^{v(0)} \frac{dt}{\llbracket \sqrt{2}\lambda(-e^t + e^{v(0)}) \rrbracket} = 1. \quad (15)$$

Let us integrate between 0 and $\varrho \in (0, 1)$. Knowing that we are dealing with v as a *even function*, and that $\dot{v}[\varrho \in (0, 1)] < 0$, the solution to the pair of Eqq. (11) will be

$$\int_{v(\varrho)}^{v(0)} \frac{dt}{\llbracket \dots \rrbracket} = \varrho \in (0, 1), \quad (16)$$

where the symbols $\llbracket \dots \rrbracket$ signify that the expression within them shall be repeated (compare with the beginning and ending repeat signs in music notation).

(β) Let us keep within the value $\check{\lambda} > 0$. One has three results:

- (i) a first solution for $\lambda = \check{\lambda}$,
- (ii) two solutions for $\lambda \in (0, \check{\lambda})$,
- (iii) no solution for $\lambda > \check{\lambda}$.

Let

$$\mathbb{R}_* = \{0\} \cup \mathbb{R}_+ \xrightarrow{n(v(0))} \mathbb{R}, \quad v(0) > 0$$

be a function in the interval $(0, \max]$, where

$$n(v(0)) = \int_0^{v(0)} \left(\frac{1}{\sqrt{e^{v(0)} - e^t}} \right) dt. \quad (17)$$

The maximum (max) lies in $\check{v}(0)$, going to meet

$$\lim_{v(0) \rightarrow 0^+} \mathfrak{h}(v(0)) = \lim_{v(0) \rightarrow +\infty} \mathfrak{h}(v(0)) = 0. \quad (18)$$

And with that we have proved the three points above requested.

(γ) Let $\zeta_\lambda \in \mathcal{C}_c^1(\Omega)$.^a We specify $Q_f(v)$ to be the quadratic form (map):

$$Q_f(v)[\zeta_\lambda] = \int_\Omega |\nabla \zeta_\lambda|^2 dx - \int_\Omega \dot{\varphi}(v) \zeta_\lambda^2 dx \geq 0. \quad (19)$$

If $\hat{E} \in \hat{\Omega} \stackrel{\text{equiv}}{=} E \in \Omega$ designates a compact subset of Ω , then all solutions v are unstable inside $E = \{0\}$ and stable outside of it. And this is because $Q_f(v)[\zeta_\lambda] \geq 0$, $\forall \zeta_\lambda \in \mathcal{C}_c^1(0, 1)$. The differentiation of the pair (11) is so equatable:

$$-\ddot{v} = \lambda e^v \lambda \dot{v}, \quad (20)$$

from which $\int_0^1 \ddot{v} (\dot{v}\varpi)^2_D d\varrho = \lambda \int_0^1 e^v (\dot{v}\varpi)^2 d\varrho$, where $\varpi \in \mathcal{C}_c^1(0, 1)$. Actually

$$Q_f(v)[\zeta_\lambda] = \int_0^1 \left\{ (d(\dot{v}\varpi) / d\varrho)^2 - \lambda e^v (\dot{v}\varpi)^2 \right\} d\varrho = \int_0^1 \left(\dot{v} (\dot{v}\varpi^2)_D + (\dot{v}\varpi)^2 - \lambda e^v (\dot{v}\varpi)^2 \right) d\varrho, \quad (21)$$

so we infer

$$Q_f(v)[\zeta_\lambda] = \int_0^1 (\dot{v}\varpi)^2 d\varrho \geq 0, \quad \zeta_\lambda = \dot{v}\varpi. \quad (22)$$

(δ) Let us look at the solution with Morse index $i_M = 1$, i.e. $v_\lambda(i_M)$ is equal to 1 or not greater than 1. Let ϖ assume the concept of *direction*, and surmise that

($\delta.1$) $\varpi \in \mathcal{C}_c^1(-1, +1)$, in such a way as to have

$$Q_f(v_\lambda(i_M))[\varpi] < 0,$$

($\delta.2$) $Q_f(\zeta_\lambda) \geq 0$, in a manner that $\zeta_\lambda(0) = 0$, $\forall \zeta_\lambda \in \mathcal{C}_c^1(-1, +1)$.

Resultantly $\tilde{\zeta}_\lambda = \zeta_\lambda - \zeta_\lambda(0)\varpi$ vanishes when the values is 0, and therefore

$$Q_f(v_\lambda(i_M))[\tilde{\zeta}_\lambda] \geq 0. \quad \square$$

Marginalia 2.1 (Bifurcation in 1-space). The bifurcation for the GP in 1-space dimension looks like this: $\|v_\lambda\|_{L^\infty(\mathbb{B})}$ on the $y =$ ordinate, with a norm $\|v\|_\infty = v(0)$, for all $v \in \{v_\lambda, v_\lambda(i_M)\}$, and $\check{\lambda} \rightarrow \lambda$ on the $x =$ abscissa. As long as $\lambda \rightarrow 0$, one sees that, due to Eq. (13b), the solution $v_\lambda(i_M)$, which is unstable, *blows-up at every x-point*, in a state where $x \in (-1, +1)$.

The shape is that of a hump-like along the y -axis, starting at the point $(0, 0)$, the origin, which extends (or rises) its $\check{\lambda}$ -peak along the x -axis, whilst the portion of the curve that decreases (or descends) from the $\check{\lambda}$ -peak approaches closer and closer to the y -axis. *

2.1.2. Gelfand–Liouville Problem in 2-Space Dimension^b

Theorem 2.2 (The $n = 2$ case). Let $\check{\lambda} = 2$, such that, w.r.t. to the pair of Eqq. (8), there is

(1) a first result v_λ , which is minimal, with a stable solution,

$$v_\lambda^{\min}(\varrho) = \ln \frac{-\frac{8(\lambda+2\sqrt{2}\sqrt{2-\lambda}-4)}{\lambda^2}}{\left\{1 + \lambda \left(-\frac{\lambda+2\sqrt{2}\sqrt{2-\lambda}-4}{\lambda^2}\right) \varrho^2\right\}^2}, \quad v_\lambda \in \mathcal{C}^2(\bar{\mathbb{B}}) \quad (23)$$

and a second result $v_\lambda(i_M)$, with an unstable solution,

$$v_\lambda(i_M)(\varrho) = \ln \frac{\frac{8(-\lambda+2\sqrt{2}\sqrt{2-\lambda}+4)}{\lambda^2}}{\left\{1 + \lambda \left(\frac{-\lambda+2\sqrt{2}\sqrt{2-\lambda}+4}{\lambda^2}\right) \varrho^2\right\}^2}, \quad v_\lambda(i_M) \in \mathcal{C}^2(\bar{\mathbb{B}}), \quad (24)$$

^a \mathcal{C}_c stands for the (linear) space of all continuous functions of compact support: the subscript c means $\mathcal{C}_{\text{compact}}$.

^b The double name comes from Liouville's survey [55]. On symmetric groups in a differential scene for the Gelfand–Liouville problem, see Yu. Bozhkov, A.C. Gilli Martins [14].

both for $0 < \lambda < \check{\lambda}$, and $\varrho \in [0, 1]$,

(2) one and only one solution under

$$v(\lambda) = \ln \frac{4}{\varrho^4 + 2\varrho^2 + 1}, \quad (25)$$

for $\lambda = \check{\lambda}$, and $\varrho \in [0, 1]$,

(3) no solution of type $v_\lambda \in \mathcal{C}^2(\bar{\mathbb{B}})$, for $\lambda > \check{\lambda}$.

Proof. The solutions we are looking for are positive with the help of the *maximum principle*, in which the usual bounded domain $\Omega \in \mathbb{R}^{n \geq 1}$ applies; inasmuch as the function $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ fulfils

$$\begin{cases} -\Delta v \geq 0 \text{ in } \Omega, \\ v \geq 0 \text{ on } \partial\Omega, \end{cases} \quad (26a)$$

$$(26b)$$

it is evident that in Ω there exists $v = 0$, or $v > 0$. Any solution requires that the boundary condition $v(1) = 0$, ergo

$$-\frac{8(\lambda + 2\sqrt{2}\sqrt{2-\lambda}-4)}{\lambda^2} = \left\{ 1 + \lambda \left(-\frac{\lambda + 2\sqrt{2}\sqrt{2-\lambda}-4}{\lambda^2} \right)^2 \right\}. \quad (27)$$

From here we can deduce that the results of our interest are

$\lambda < \check{\lambda}$ having a solution equal to 2, cf. Eqq. (23) (24),

$\lambda = \check{\lambda}$ having a solution equal to 1, cf. Eq. (25),

$\lambda > \check{\lambda}$ having a solution equal to 0, look at point (3).

Not only. The symmetrical radiality of each solution is a corollary of what was proved by Gidas–Ni–Nirenberg [37], directly including symmetry of positive solutions of second order elliptic equations. \square

NB. A study on the blow-up mechanism for solutions in 2-space of

$$\begin{cases} -\Delta v = \eta_{L^p}(x)e^v \text{ in } \Omega \subset \mathbb{R}^2, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \quad (28a)$$

$$(28b)$$

where

$v \in L^1(\Omega)$,

η_{L^p} is a certain function in $L^p(\Omega)$, for $1 < p \leq \infty$, and

$e^v \in L^{p^{ce}}(\Omega)$, with p^{ce} representing the conjugate exponent of p ,^a

is explained in H. Brezis et F. Merle [16].

Marginalia 2.2 (Bifurcation in 2-space). The *bifurcation* for the GP in 2-space dimension is $\|v_\lambda\|_{L^\infty(\mathbb{B})}$ on the $y =$ ordinate, with a norm $\|v\|_\infty = v(0)$, for all $v \in \{v_\lambda, v_\lambda(i_M)\}$, and $\check{\lambda} \rightarrow \lambda$ on the $x =$ abscissa. One sees that, due to Eq. (24), the solution $v_\lambda(i_M)$, which is unstable, *blows-up at the (space) origin*, since

$$v_\lambda(i_M)(\varrho) \rightarrow 4(-1) \ln \varrho \in (0, 1).$$

The shape is identical to that in 1-space dimension. $*$

2.1.3. Gelfand Problem in a Special Liouville-type Equation

Let

$$\begin{cases} \ddot{v}(x) + \left(\frac{n}{x} - \frac{1}{x} \right) \dot{v}(x) = -\frac{\lambda e^{-v(x)}}{A_{\mathbb{B}_{\rho=1}}^n \int_0^1 (e^{-v(x)} x^{n-1}) dx}, \\ v = 0, \dot{v}(0) = 0, \end{cases} \quad (29a)$$

$$(29b)$$

be a Liouville-type equation (cfr. see previous Section), where

$\lambda = s A_{\mathbb{B}_{\rho=1}}^n \int_0^1 e^{-v(x)} x^{n-1} dx$, in which s is a value (parameter), $A_{\mathbb{B}_{\rho=1}}^n$ is the surface area of $\mathbb{B}_{\rho=1}$, and R is the distance-radius.

^a Usually the conjugate exponent e.g. of the p -norm is symbolized with p' , but I do not like this formalism, since it is multi-semantic, and thereupon ambiguous.

Narrowing the interval to $[0, 1]$, and admitting that $v_\lambda(0) = 0$, $\dot{v}_\lambda(0) = 0$, our equation becomes

$$-\ddot{v}_\lambda(\theta) = \lambda \left(e^{v_\lambda(\theta)} \left/ \int_0^1 e^{-v_\lambda(\theta)} d\theta \right. \right), \quad (30)$$

so

$$\dot{v}_\lambda^2(\theta) = \frac{2\lambda}{\int_0^1 e^{-v_\lambda(\theta)} d\theta} e^{-v_\lambda(\theta)} - 1, \quad (31)$$

and

$$v_\lambda(\theta) = \log \cos^2 \sqrt{\frac{\lambda}{2 \int_0^1 e^{-v_\lambda(\theta)} d\theta}} \theta. \quad (32)$$

Eq. (32) is a regular solution for any eigenvalues λ_n in $0 \geq \theta < 1$. The final integral will be this:

$$\int_0^1 e^{-v_\lambda(\theta)} d\theta = \int_0^1 \left(\frac{d\theta}{\cos^2 \sqrt{\frac{\lambda}{2 \int_0^1 e^{-v_\lambda(\theta)} d\theta}} \theta} \right) = \frac{\sqrt{2}}{\sqrt{\frac{\lambda}{\int_0^1 e^{-v_\lambda(\theta)} d\theta}}} \tan \sqrt{\frac{\lambda}{2 \int_0^1 e^{-v_\lambda(\theta)} d\theta}}, \quad (33)$$

which performs as a solution to $\int_0^1 e^{-v_\lambda(\theta)} d\theta = \frac{2}{\lambda} \tan^2 \sqrt{\frac{\lambda}{2 \int_0^1 e^{-v_\lambda(\theta)} d\theta}}$.

NB. We are talking about a system in 2-space dimension.

2.1.4. Gelfand–Dirichlet Problem in 3- or Higher-Space Dimension^a

Theorem 2.3 (The $n \geq 3$ case). Put $3 \leq n \leq 9$. Then $\check{\lambda} + 4 > 2n$ such that, w.r.t. to the pair of Eqq. (8),

- (1) there are poli- v -solutions (a finite number) $v \in \mathcal{C}^2(\bar{\mathbb{B}})$, for $0 < \lambda < \check{\lambda}$, $2n \neq \lambda + 2$,
- (2) there are r -solutions, $r \in \mathbb{N}$, once it has been imposed a value $\varepsilon > 0$, so that $\varepsilon > |-2n + \lambda + 4|$,
- (3) there are poli- v -solutions (an infinite number), for $\lambda = 2n - 4$,
- (4) there is one and only one solution, for $\lambda = \check{\lambda}$,
- (5) there is no solution, for $\lambda > \check{\lambda}$,
- (5 bis) there is no $v \in \mathcal{C}^2(\mathbb{R}^n)$ -solution stable of the semi-linear partial differential equation $-\Delta = \lambda e^v$ on the Euclidean $\mathbb{R}^{n \geq 2}$ -space outside a compact set of \mathbb{R}^n , for (let us remind it once more) $3 \leq n \leq 9$, and the consequential absence of existence of solutions with finite Morse index $v_\lambda(i_M)$; thence the sequence $(v_r)_{r \in \mathbb{N}}$ appears to be bounded in $L^\infty(\Omega)$ iff (v_r) of $v_\lambda(i_M)$ is bounded in \mathbb{R} -space.^b

Proof. When there is a solution, it is positive owing to the maximum principle via *Picard–Lindelöf theorem* [67] [54] (see Appendix, Theorem 6.1)—an alternative version of which is known as *Peano theorem* [66]—,^c a demonstrative guarantee for the local existence and uniqueness of solutions with initial values and derivatives, which are obviously prescribed.

(α) For this reason we start from an equation (an initial value problem) with maximal solution,

$$\begin{cases} v(0) = \alpha, \dot{v}(0) = 0, & \alpha > 0, \end{cases} \quad (34a)$$

$$\begin{cases} \varrho^{-(n-1)} (\varrho^{n-1} \dot{v})_D + \lambda e^v = 0, \end{cases} \quad (34b)$$

generated through the fixed-point theorem. From this pair it is easy to build the integral equation (35a) and the *Emden–Fowler transformation* (35b) [28] [32] [33] [34] (see Marginalia 2.4),

$$\begin{cases} v(\varrho) = \alpha - \lambda \int_0^\varrho \int_0^\tau \frac{\tau (\frac{t}{\tau})^n}{t} e^{v(t)} dt d\tau, & \varrho \geq 0, \end{cases} \quad (35a)$$

$$\begin{cases} \alpha + \psi(t) = v(\varrho) + 2t, & \varrho = \sqrt{2} e^t \sqrt{\frac{e^{-\alpha(n-2)}}{\lambda}}. \end{cases} \quad (35b)$$

^a The double name comes from Dirichlet’s survey, explored by D.D. Joseph & T.S. Lundgren [43].

^b See E.N. Dancer and A. Farina [23].

^c The Peano existence theorem, unlike Picard–Lindelöf theorem, does without the Lipschitz continuity.

This transforms the equational pair (34) into the following expressions:

$$\left. \begin{aligned} \ddot{\psi} + n\dot{\psi} - 2\dot{\psi} + 2(n-2)e^\psi - 2n + 4 \\ \lim_{t \rightarrow -\infty} \psi(t) - 2t = \lim_{t \rightarrow -\infty} e^{-t} (\dot{\psi}(t) - 2) \end{aligned} \right\} = 0, \quad (36)$$

for which, by setting $\chi(t) + 2t = \psi(t)$, there is a unique solution in the interval $(-\infty, T)$, determined by this integral,

$$\chi(t) = 4 - 2n \left(\int_{-\infty}^t e^{(4-2n)\tau} d\tau \int_{-\infty}^\tau e^{ns+\chi(s)} ds. \right) \quad (37)$$

NB. Evidence for the aforementioned interval $(-\infty, T)$ emerges from the Lyapunov exponent [57],^a

$$\lambda_L(\psi) = 2(n-2)e^\psi - \frac{1}{2}\psi(4n - \psi - 8), \quad (38)$$

hence

$$\frac{d\lambda_L(\psi)}{dt} = 2(n-2)e^\psi \dot{\psi} - \dot{\psi}(-\ddot{\psi} + 2n - 4) = (2-n)\dot{\psi}^2 \leq 0. \quad (39)$$

If we consequently write the pair of Eqq. of \mathfrak{N}^\natural (36) as

$$\frac{d}{dt} \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \dot{\psi} \\ -(n-2)(\dot{\psi}-2) - 2(n-2)e^\psi \end{pmatrix}, \quad (40)$$

we will find that $(0, 0)$ is the *unique stationary point*, and it can be depicted as a *spiral attractor* with these eigenvalues

$$\frac{1}{2} \left(-n - i\sqrt{-(n-10)(n-2)} + 2 \right), \quad (41a)$$

$$\frac{1}{2} \left(-n + i\sqrt{-(n-10)(n-2)} + 2 \right). \quad (41b)$$

(β) The ψ -function is presumed to be the solution to the pair of Eqq. of \mathfrak{N}^\natural (36); we accordingly introduce the orbit

$$\Gamma \setminus \mathbb{U}_{\mathbb{R}}^n = \left\{ (\psi, \dot{\psi}) = (\psi(t), \dot{\psi}(t)) \right\}_{t \in \mathbb{R}}, \quad (42)$$

with an asymptotic behavior to $\dot{\psi} = 2$, during the time $t = -\infty$, in the phase plane $(\psi, \dot{\psi})$. Then the orbit $\Gamma \setminus \mathbb{U}_{\mathbb{R}}^n$ lies in the half-space

$$\mathbb{U}_{\mathbb{R}}^n = \left\{ (\psi, \dot{\psi}) \mid \dot{\psi} < 2 \right\}, \quad (43)$$

and the trajectory spirals towards the stationary $(0, 0)$ -point. And indeed

$$\lim_{t \rightarrow -\infty} \varrho^{-2} \left\{ \dot{\psi}(t) + 2 \left(e^{\psi(t)} - 1 \right) \right\} = \lambda e^\alpha \left\{ \frac{2}{(n-2)n} \right\} > 0. \quad (44)$$

It is thereby possible to subsequently satisfy, with some passages, the requirements (1) (2) (3) (4) (5) of the Theorem. Also in these dimensionalities, the symmetrical radiality of each solution is proved by Gidas–Ni–Nirenberg [37].

□

Marginalia 2.3 (Bifurcation in 3- or higher space). The bifurcation for the GP in 3- or higher-space dimension has a dual nature.

(1) In $D = 3 \leq n \leq 9$ the values approached asymptotically are fixed by $\|v_\lambda\|_{L^\infty(\mathbb{B})}$ on the $y =$ ordinate, and $2(n-2) \stackrel{\text{eqv}}{=} 2n - 4$ plus $\check{\lambda} \rightarrow \lambda$ on the $x =$ abscissa.

About the shape: the peak is in $\check{\lambda}$, but the bifurcation scheme has an approximately sinusoidal trend, which gets smaller and smaller in the multiple succession between a peak and a trough; the curve remains distant from the y -axis, as it has a winding line along the value $2(n-2) \stackrel{\text{eqv}}{=} 2n - 4$.

(2) In $D \geq 10$ the value $\|v_\lambda\|_{L^\infty(\mathbb{B})}$ is on the $y =$ ordinate, and $\check{\lambda} \rightarrow \lambda$ on the $x =$ abscissa.

About the shape: the line starts from the origin $(0, 0)$, and creates a curve that approaches indefinitely near to the vertical axis in $\check{\lambda}$, on the $x =$ abscissa. *

^a Cf. [64, margo 13.2.2].

Marginalia 2.4 (Emden–Fowler tools for non-local Gelfand system). Let

$$\begin{cases} \Delta v + \lambda e^v = 0 & \text{in } \Omega \subset \mathbb{R}^n, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (45a)$$

$$(45b)$$

indicate a Emden–Fowler equation, where Ω is circular in the \mathbb{R} -field. The pair (45) has an applicative extension to the Gelfand problem in a non-local elliptic equation of this type,

$$\begin{cases} -\Delta v = \lambda \frac{e^v}{\left(\int_{\Omega} e^v dx\right)^p}, \\ v = 0, \end{cases} \quad (46a)$$

$$(46b)$$

$x \in (\Omega, \partial\Omega)$, where the Dirichlet boundary condition is in force. If Ω coincides with a unit ball $\mathbb{B}_{\rho}^n = \{x \in \mathbb{R}^n \mid \|x\| = \rho = 1\}$, with $D = 3 \leq n \leq 9$, and $\frac{2}{n} \leq p \leq 1$, one witnesses the formation of infinitely many different bending lines in the $\lambda - v$ plane. Ascertained that Ω is a ring-shaped object, for $p \geq 1$, there is a solution for $\lambda > 0$. See K. Nagasaki & T. Suzuki [63] and T. Miyasita [61]. *

2.1.5. A Tiny Note on the (Bifurcation) Diagram in the Gelfand Problem

Several examples of diagrams related to the Gelfand problem are in [53, chapp. 9.3.2, 14.2.2, 14.3] S. Liao, D. Bartolucci & A. Jevnikar [8], and in S.-Y. Huang [42].

I always find it *humanly* funny that (*les*) *mathématiques* belonging to different branches can produce similar if not the same results. The same diagrams, *mutatis mutandis*, in different dimensions, are drawn in [47, p. 27, and p. 80 (possible non-local response diagrams)]. This is the clear symptom that our mathematics, is still laughably *too rough*.

2.1.6. Δ_{∞} , or the Infinity Laplacian

The status in which the Laplacian

$$\begin{cases} -\Delta_p v = \lambda e^v & \text{in } \Omega \subset \mathbb{R}^n, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (47a)$$

$$(47b)$$

is infinite, videlicet $p \rightarrow \infty$ of GP, is analyzed by F. Charro, B. Son, and P. Wang [19]; also compare P. Juutinen, P. Lindqvist & J.J. Manfredi [44].

The optimal way that allows to minimize the ratio is

$$\frac{\|\nabla v\|_{\infty, \Omega_{\mathbb{R}}^n}}{\|v\|_{\infty, \Omega_{\mathbb{R}}^n}} \cong \lim_{p \rightarrow \infty} \frac{\|\nabla v\|_{p, \Omega_{\mathbb{R}}^n}}{\|v\|_{p, \Omega_{\mathbb{R}}^n}}. \quad (48)$$

By the way, it is feasible [60] to find some solutions to pairs of Eqq. (47) without the rescalings

$$\begin{cases} \lambda_p^{\left(\frac{1}{p}\right)} \xrightarrow{p \rightarrow \infty} \lambda_{\max}, \\ p \\ \frac{v_{(\lambda_p)p}}{p} \xrightarrow{p \rightarrow \infty} v, \end{cases} \quad (49a)$$

$$(49b)$$

in v and λ . So, inherently to a *viscosity solution* to $\Delta_{\infty} v = 0$ of the suitable limit problem,

$$\begin{cases} \min\left(|\nabla v| - \lambda_{\max} e^v, -\Delta_{\infty} v\right) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (50a)$$

$$(50b)$$

one attains the solutions v_p , which converges uniformly in Ω as $p \rightarrow \infty$,

$$\begin{cases} \min\left(|\nabla v| - 1, -\Delta_{\infty} v\right) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (51a)$$

$$(51b)$$

concerning the distance function from the boundary of the domain at hand.

NB. It should be noted that

$$\Delta_\infty v = \langle Dv, D^2vDv \rangle = v_{x_j}v_{x_k}v_{x_jx_k} = |\nabla v|^{-2} \sum_{jk} v_{x_j}v_{x_k}v_{x_jx_k}, \quad (52)$$

emerges as a kind of Euler–Lagrange formula controlling the absolute minimizer for the L^∞ -variational problem, with the covariant and the second covariant derivatives, D and D^2 .

3. Gelfand problem in $D = 1 \leq n \leq 9$, with Dirichlet $\mathcal{C}^{2,\delta}$ -boundary

Let us move on to another version of the GP, which can be deduced from the studies of Crandall–Rabinowitz–Mignot–Puel [22] [59].

Theorem 3.1. *Let*

$$\begin{cases} -\Delta v = \lambda e^v \text{ in } \Omega \subset \mathbb{R}^{n \geq 1} \\ v = 0 \text{ on } \partial\Omega, \end{cases} \quad (53a)$$

$$(53b)$$

connote a Gelfand problem in $D = 1 \leq n \leq 9$, with Dirichlet $\mathcal{C}^{2,\delta}$ -boundary, for $\delta \in (0, 2)$. Then, there exist

(1) an extremal solution, and

(2) a stable solution v_λ inserted in the norm of this inequality, $\|v_\lambda\|_{L^\infty(\Omega)}^a \leq \varepsilon$, by selecting a constant

$$\varepsilon = \varepsilon(n, \Omega \subset \mathbb{R}^{n \geq 1}) > 0,$$

to the pair of Eqq. (53). In brief, this theorem aims to prove a regularity for elliptic problems.

Proof. We recover the quadratic form (19) for the pair (53):

$$\begin{aligned} \int_\Omega (\nabla v \nabla \varpi) dx &= \lambda \int_\Omega (e^v \varpi) dx \\ &= 2\delta \int_\Omega |\nabla v|^2 e^{2\delta v} dx = \lambda \int_\Omega e^v (e^{\delta v} - 1) (e^{\delta v} + 1) dx \\ &= \frac{2 \int_\Omega |\nabla (e^{\delta v} - 1)|^2 dx}{\delta} = \lambda \int_\Omega e^v (e^{2\delta v} - 1) dx. \end{aligned} \quad (54)$$

From this last equation we continue with (cf. Secc. 1 and 2.1.1)

$$\int_\Omega \left| \nabla \left(\underbrace{e^{\delta v} - 1}_{\zeta_\lambda} \right) \right|^2 dx \geq \lambda \int_\Omega e^v \left(\underbrace{e^{\delta v} - 1}_{\zeta_\lambda} \right)^2 dx = \lambda \int_\Omega e^v \left(-2e^{(\delta+1)v} + e^{2\delta v+v} + e^v \right) dx. \quad (55)$$

Eq. (54) plus Eq. (54) create

$$\int_\Omega e^v (e^{2\delta v} - 1) dx \geq \frac{2 \int_\Omega e^v (-2e^{(\delta+1)v} + e^{2\delta v+v} + e^v) dx}{\delta}. \quad (56)$$

Leveraging the *Hölder inequality* [41], one draws a non-equal comparison between these two expressions:

$$|\Omega|^{\frac{1}{2} - \frac{1}{2(2\delta+1)}} \left(\int_\Omega e^{(2\delta+1)v} dx \right)^{\frac{1}{2(2\delta+1)} + \frac{1}{2}} \geq \frac{1}{2} - \frac{\delta}{4} \int_\Omega (e^{(2\delta+1)v}) dx. \quad (57)$$

It presuppose that e^v is bounded in vector-valued Banach function $L^p(\Omega)$ -space, for all $p = 2\delta + 1$. In view of the fact that $D = n \leq 9$, the presence of δ suggests that $p = (2\delta + 1) > \frac{n}{2}$. Finally, the pair of Eqq. (53) together with the inequality

$$\|v_\lambda\|_{L^\infty(\Omega)} \leq \varepsilon \quad (58)$$

are in consequence both solved by v , which is equivalent to the desired regularity. It should be stressed that the *extremal solution* is

$$\check{v}(x) = -\ln|x|^2,$$

^a See footnote a, p. 3.

upon the occurrence of a coincidence between a bounded Ω -region (of the Euclidean n -space) and the unit ball \mathbb{B}_ρ^n , when the dimension is $D = n \geq 10$. What does it mean? It is simple: one can hatch *singular stable solutions* to the pair (53) in 10- or higher-space dimension. \square

4. Non-linearities Issues in (Hemi)spherical Spaces

4.1. Gelfand Problem applied to a Geodesic Ball, or Spherical Cap

The Gelfand problem, in its non-linearity, can be addressed [56] [7] [45] [46] by writing alternately

$$-\Delta_{\mathbb{S}^n} v = \lambda \varphi(v) \text{ in } \Omega \subset \mathbb{S}^n, \quad v = 0 \text{ on } \partial\Omega, \quad (59)$$

where

Δ (the Laplace–Beltrami operator) acts on the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{(n \geq 1)+1}$,
 $\varphi(0) > 0 \in \mathcal{C}^2(\mathbb{R})$ is a non-linear convex monotonically increasing function, under which the postulate

$$\lim_{q \rightarrow \infty} \frac{\varphi(q)}{q} = +\infty \begin{cases} \varphi(q) \stackrel{\text{eqv}}{=} (1+q)^{p>1}, & (60a) \\ \varphi(q) \stackrel{\text{eqv}}{=} e^q, & (60b) \end{cases}$$

is assigned, and

$\partial\Omega \neq \emptyset$.

Let us see why.

(1) Since $\hat{\mathbb{C}} \cong \mathbb{C}\mathbb{P}^1$, where $\hat{\mathbb{C}}$ is the extended complex plane, can be distinguished in topological terms, one says that it is equivalent to the 2-sphere in the 3-dimensional real space,

$$\mathbb{S}^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}, \quad (61)$$

via stereographic projection. The complex \mathbb{C} -plane is describable with the plane $x^3 = 0$ in \mathbb{R}^3 , and the number $z = (x + iy)$ is identifiable with $(x, y, 0)$, for $z \in \mathbb{C} \cong \mathbb{R}^2$, and $x, y \in \mathbb{R}$. Wherefore the system of the projective line extended by a point at infinity ∞ is also geometrically equivalent to the Riemann sphere,

$$\hat{\mathbb{C}} \stackrel{\text{eqv}}{=} \mathbb{C} \cup \{\infty\} \cong \mathbb{C}\mathbb{P}^1 \cong \mathbb{S}^2. \quad (62)$$

The Möbius group $\mathfrak{M}\ddot{\text{o}}\text{b}(\hat{\mathbb{C}}) \cong PSL_2(\mathbb{C}) \cong \frac{SL_2(\mathbb{C})}{\{\pm I\}}$ is thereby the set of all transformations

$$z \mapsto \varphi_{\alpha\beta\gamma\delta}[\hat{\mathbb{C}}](z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \text{ for } \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma \neq 0, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{C}) \quad (63)$$

of the Riemann sphere $\hat{\mathbb{C}} \cong \mathbb{C}\mathbb{P}^1 \cong \mathbb{S}^2$, so

$$\begin{aligned} \mathfrak{M}\ddot{\text{o}}\text{b}(\hat{\mathbb{C}}) &= \left\{ \varphi: \hat{\mathbb{C}} \stackrel{\text{eqv}}{=} \mathbb{C} \cup \{\infty\} \rightarrow \hat{\mathbb{C}} \stackrel{\text{eqv}}{=} \mathbb{C} \cup \{\infty\} \mid \varphi_{\alpha\beta\gamma\delta}[\hat{\mathbb{C}}](z) = \frac{\alpha z + \beta}{\gamma z + \delta} \right\} \\ &= \text{aut}(\hat{\mathbb{C}}) \cong \text{aut}(\mathbb{C}\mathbb{P}^1) \cong PSL_2(\mathbb{C}), \end{aligned} \quad (64)$$

where the bijective meromorphic mapping

$$\hat{\mathbb{C}} \stackrel{\text{eqv}}{=} \mathbb{C} \cup \{\infty\} \xrightarrow{\varphi} \hat{\mathbb{C}} \stackrel{\text{eqv}}{=} \mathbb{C} \cup \{\infty\}$$

is an automorphism of the Riemann sphere, which is purely a Möbius transformation, cf. [64, chap. 2.5.2]

(2) The hyperbolic \mathbb{H} -planes in the Beltrami–Poincaré half-space model [9] [10] [11] [68] [69],

$$\mathbb{U}_\rho^n = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\},^a \quad (65)$$

are hemispheres orthogonal to the \mathbb{C} -plane, and the geodesics are semicircles orthogonal to \mathbb{C} -plane.

The \mathbb{H} -planes are spherical caps orthogonal to the unit sphere. The *geodesic ball* $\Omega \subset \mathbb{S}^n$ goes to coincide with the *spherical cap* because of that, centered at the south pole $(0, \dots, 0, -1)$.

^a The metric on \mathbb{U}_ρ^n is given by $g_\rho^3 = \rho^2(x^n)^{-2}(dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n)$. The boundary of \mathbb{U}^n is provided by the boundary at infinity $\partial_\infty \mathbb{U}^n = (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$.

(3) The spherical cap

$$\Omega_{\varepsilon \in (0,1)} = \left\{ (x^1, \dots, x^{n+1})_{\mathbb{R}^{n+1}} \left| \sum_{j=1}^{n+1} x_j^2 = 1, -\cos\{\pi\varepsilon\} < x_{n+1} \leq 1 \right. \right\} \subset \mathbb{S}^n \quad (66)$$

is mapped onto the ball

$$\mathbb{B}_\rho^n = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid \|x\| = \rho\}, \quad (67)$$

with $\rho = \tan\{\frac{1}{2}\pi(1-\varepsilon)\}$, $\varepsilon \rightarrow 0$.

What is being sought is

(i) the minimal solution to the Gelfand problem on a spherical cap for Dirichlet's boundary value problem,

(ii) an asymptotic comportment of the solution, so as to extend to it the whole sphere by means of a sharp estimate of the *torsion function* τ_Ω of Ω_ε . For instance, the lower bound

$$\|\tau_\Omega\|_\infty > \frac{1}{\lambda_1(\Omega)},$$

is asymptotically sharp as $\varepsilon \rightarrow 0$, where $\lambda_1(\Omega)$ is the positive principal Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^n}$ in Ω , and

$$\lambda_1 = \inf_{v \neq 0} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}$$

the eigenvalue *minimizing* the quadratic functional (Dirichlet energy) on the Sobolev space $\mathfrak{S}^1(\Omega)$; it must be specified that

$$\mathfrak{S}^1(\Omega) = \{v \in L^2(\Omega), \partial_j v \in L^2(\Omega), 1 \leq j \leq n\} \quad (68)$$

is a pre-Hilbert space (isto é, a separable Hilbert space).

4.2. Gelfand Problem with no Deformation of the \mathbb{R} -metric and no Decrement of the Scalar Curvature

It is plausible to address the Gelfand problem grabbing on the fly some suggestions that come from the theorem of Hang–Wang [38], cf. M. Lai and W. Wei [49], which basically says this: taking a function, say superharmonic,

$$v \text{ on } \mathbb{R}^{n \geq 3} \stackrel{\text{eqv}}{\triangle} \leq 0,^a$$

it follows that by the maximum principle, it is *not allowed to deform the Euclidean metric in a conformal way* in a bounded region *without decreasing the (non-negative) scalar curvature*

$$R_s \geq n(n-1) \stackrel{\text{eqv}}{\triangle} n + R_s \geq n^2$$

somewhere. This echoes the *positive energy mass theorem* (in general relativity, and not only) by Schoen–Yau [73] and E. Witten [75].

Here is the graft. The above theorem states that, with small changes of perspective, for a metric g conformal to the standard metric g_0 on \mathbb{S}_+^n , its boundary is consistent with

$$g_0|_{\partial\mathbb{S}_+^n}, \text{ thus } g = g_0,$$

and (\mathbb{S}_+^n, g) is isometric to a standard hemisphere. All of this connects to GP, since it can easily be reduced to an expression of this type,

$$-\Delta v = \lambda g(v), \quad v = 0 \text{ on } \partial\Omega, \quad (69)$$

when g is non-linear in $\Omega \subset \mathbb{R}^n$, under the Dirichlet boundary condition. And we are well aware that there is an extremal $\check{\lambda}$ due to which the Eq. (69) has no solutions, for $\lambda > \check{\lambda}$. In parallel, the

^a The v is 1 near infinity, and it is identically 1 on $\mathbb{R}^{n \geq 3}$.

theorem of Hang–Wang fixes these useful values:

$$\text{(NB. Is under discussion a } g\text{-metric on } \mathbb{S}_+^n \left\{ \begin{array}{l} \check{\lambda} = 1, \text{ for } n = 2, g(v) = e^{2v}, \\ \check{\lambda} = \frac{1}{4}(n-2)n, \text{ for } n \geq 3, g(v) = (v+1)^{\frac{n}{n-2} + \frac{2}{n-2}}. \end{array} \right. \quad (70a)$$

$$(70b)$$

4.3. Gelfand Problem in a Ball-Space under Perturbation

Let us go back to the Theorem 3.1 we ascribed to Crandall–Rabinowitz–Mignot–Puel. This can give the inspiration for writing an affiliated theorem, which allows us to assert that there are singular stable solutions, if some Ω -domain preserves a \mathcal{C}^2 -diffeomorphism on a ball, in the event of perturbations. The proof of this theorem is due to J. Dávila and L. Dupaigne [24].

Starting Theorem 4.1. *Let, again,*

$$\left\{ \begin{array}{l} -\Delta v = \lambda e^v \text{ in } \Omega \subset \mathbb{R}^n, \\ v = 0 \text{ on } \partial\Omega, \end{array} \right. \quad (71a)$$

$$(71b)$$

be a Gelfand problem lying in the topology of a unit ball $\mathbb{B}_{\rho=1}$, and subject to the Dirichlet condition $v|_{\partial\mathbb{B}} = 0$, where Ω is an open set, which is bounded, with a \mathcal{C} -smooth boundary, and $\lambda \geq 0$ is the usual parameter. Then,

- (1) there is a singular solution $v + \log|x|^2 = 0$, if $\lambda = (2n-1)$
- (2) there exist a singular solution v , if $D = n \geq 4$, making small deformations to $\mathbb{B}_{\rho=1}$,
- (3) there is an extremal solution, if $D = n \geq 11$ (case in which the parameter λ is the largest possible one),
- (4) there are many cases where there is a bounded extremal solution, if $D = n \geq 10$, in the presence of non-small perturbations.

Theorem 4.1. *Let*

$$\varpi: \bar{\mathbb{B}}_{\rho=1} \rightarrow \mathbb{R}^n$$

be a \mathcal{C}^2 -map, and $\Omega_t = \{x + t\varpi(x) \mid x \in \mathbb{B}_{\rho=1}\}$, $t \in \mathbb{R}_+$, a smooth bounded region diffeomorphic to the unit ball, so the pair (71) becomes

$$\left\{ \begin{array}{l} -\Delta v = \lambda e^v \text{ in } \Omega_t \subset \mathbb{R}^n, \\ v = 0 \text{ on } \partial\Omega_t. \end{array} \right. \quad (72a)$$

$$(72b)$$

If $D = n \geq 11$, and t is small enough, then $v = \check{v}(t)$ proves to be an extremal solution in Ω_t and to (72). Plus, if $t_0 = t_0(n, \varpi) > 0$ and $t < t_0$, the solution $\check{v}(t)$ is singular in any $D = n \geq 4$; in addition, there is a value $b(t) \in \mathbb{B}_{\rho=1}$ so that

$$\left\| \check{v}(x, t) - \log \frac{1}{|x - b(t)|^2} \right\|_{L^\infty(\Omega_t)} + |\lambda(t) - 2(n-2)| \rightarrow 0, \text{ as per } t \rightarrow 0. \quad (73)$$

At the point of origin, the single solution behaves in this way: $\check{v}(x, t) = \ln \check{\lambda}(0) / \check{\lambda}(t) + \ln \frac{1}{|x - b(t)|^2} + \varepsilon|x - b(t)|$, in which $\lim_{\tau \rightarrow 0} \varepsilon(\tau) = 0$.

Hints for a Proof (via Hardy-type's inequality). Needless to repeat what others have already done. Better to make a later demonstration that paradigmatically sums up what is required. The starting axiom is *Hardy's inequality*, or *Hardy–Littlewood–Pólya inequality* [39] [40], giving it this form:

$$\varkappa_H = \frac{1}{4}(n-2)^2 \int_{\mathbb{R}^n} \frac{\zeta^2}{|x|^2} dx \leq \int_{\mathbb{R}^n} |\nabla \zeta|^2 dx, \quad (74)$$

where

$$\forall \zeta \in \left\{ \begin{array}{l} \mathcal{C}_c^1 \\ \mathcal{C}_0^\infty \end{array} \right\} (\mathbb{R}^n \setminus \{0\}), \quad (75)$$

and ε_H is a constant, for $n \geq 3$.

Let us adopt the idea of H. Brezis and J.L. Vázquez [17], who found that, referring to a semi-linear elliptic equation

$$-\Delta = \lambda \xi(v) \text{ in } \Omega, \quad (76)$$

where ξ is a continuous, positive, increasing and convex function, whose non-linearity is outlined for $v \geq 0$, $\xi(0) > 0$ such that

$$\lim_{\tau \rightarrow \infty} \frac{\xi(\tau)}{\tau} = +\infty \text{ on } [0, \infty),$$

with Dirichlet boundary $v|_{\partial\Omega} = 0$, there is a correlation between any singular extremal solution and Hardy inequality (so the former must satisfy the latter), the significance of which is that the principal eigenvalue of the linearized operator is non-negative. The correlation can be easily formulated:

$$\lambda \int_{\Omega} \dot{\xi}(v) \zeta^2 dx \leq \int_{\Omega} |\nabla \zeta|^2 dx, \quad (77)$$

with the same universal quantification explicated in (75), postulating that $\Omega \subset \mathbb{R}^n$ is a bounded smooth region, and $v \in \mathfrak{S}_0^1(\Omega)$ is a singular weak solution to (76), for $\lambda > 0$. If $D = n \geq 11$, the amount is $2(n-2) < \frac{1}{4}(n-2)^2$. As a consequence, by virtue of (73),

$$\lambda(t) e^{\heartsuit} < \frac{1}{4}(n-2)^2, \text{ where } \heartsuit = \left\| v - \log \frac{1}{|x-b(t)|^2} \right\|_{L^\infty(\Omega_t)}, \quad (78)$$

implying

$$\lambda(t) \int_{\Omega_t} e^u \zeta^2 \leq \frac{1}{4}(n-2)^2 \int_{\mathbb{R}^n} \frac{\zeta^2}{|x-b(t)|^2} \leq \int_{\mathbb{R}^n} |\nabla \zeta|^2, \quad (79)$$

And in so doing, it is stated that $v(t)$ is the extremal solution to the pair of Eqq. (72),^a and $\lambda(t)$ the extremal parameter. On this trail, see also J.L. Vázquez and E. Zuazua [74].

But the fun does not stop here. The pair (72) can be rewritten in agreement with a ball-type space. Draft

$$v(y) = \left(\chi + t \tilde{\omega}(t, y) \right), \quad (80)$$

$$\Delta_y v = \Delta_x \chi + III_t(\chi), \quad (81)$$

where

$$III_t(\chi) = t \sum_{j,l} \chi_{x_l} \left(\frac{\partial^2 \tilde{\omega}_l}{\partial y_j^2} \right) + t^2 \sum_{j,k,l} \chi_{x_k x_l} \left(\frac{\partial \tilde{\omega}_k}{\partial y_j} \right) \left(\frac{\partial \tilde{\omega}_l}{\partial y_j} \right) + 2t \sum_{j,l} \chi_{x_j x_l} \left(\frac{\partial \tilde{\omega}_l}{\partial y_j} \right) \quad (82)$$

is a second-order differential operator. The solution to the pair (72) must be

$$\chi(x) = \log \frac{1}{|x-b|^2} + \delta, \quad \delta \in \mathfrak{S}^1(\Omega), \quad \lambda = 2n - 4 + s \quad (83)$$

where

$b \in \mathbb{B}$,

δ is a small bounded perturbation, and

$s \in \mathbb{R}$ is a free parameter.

Conclusively, the Eqq. (72) are but

$$\begin{cases} -\Delta \delta - \frac{2n-4}{|x-b|^2} \delta = III_t(\delta) + \frac{2n-4}{|x-b|^2} (e^{\delta-1} - \delta) + \frac{s}{|x-b|^2} e^\delta + III_t \left(\log \frac{1}{|x-b|^2} \right) \text{ in } \mathbb{B}, \\ \delta = -\log \frac{1}{|x-b|^2} \text{ on } \partial\mathbb{B}. \end{cases} \quad (84a)$$

$$(84b)$$

^a In the light of the foregoing: $v = v(t)$.

The resolution of Eq. (84) requires a system of non-linear elliptic partial differential by way of a Laplacian with the inverse-square potential,

$$\begin{cases} -\Delta - \frac{\kappa}{|x-b|^2} \bar{\sigma} = \xi \text{ in } \mathbb{B} \stackrel{\text{eqv}}{=} \mathbb{B}_{\rho=1}(0), \\ \bar{\sigma} = u_f \text{ on } \partial\mathbb{B}, \end{cases} \quad (85a)$$

$$\quad (85b)$$

where $b, \kappa \in \mathbb{R}$, and $u_f \in \mathcal{C}^{2,\delta}(\partial\mathbb{B})$. In this respect, see L. Dupaigne [27].

Take a *little peek*. Let \mathbb{B}_ρ be an open ball of radius ρ centered at the origin, and $\varepsilon_1 > 0$ and ε_2 some constants, with $0 < \varepsilon_2 \leq \varepsilon_{2_0} \stackrel{\text{eqv}}{=} \frac{1}{4}(n-2)^2$, and χ the characteristic function.

Let $\zeta_1^{\mathbb{B}}, \dots, \zeta_n^{\mathbb{B}} \in \mathcal{C}^\infty(\Omega)$ be the solutions of

$$\zeta_k^{\mathbb{B}}(x) \geq \varepsilon_1 \zeta_0^{\mathbb{B}}(x) \begin{cases} -\Delta \zeta_k^{\mathbb{B}} - \frac{\varepsilon_2}{|x|^2} \zeta_k^{\mathbb{B}} = \chi_{\mathbb{B}_\rho}(x_k) \text{ in } \Omega \subset \mathbb{R}^n, \\ \zeta_k^{\mathbb{B}} = 0 \text{ on } \partial\Omega, \end{cases} \quad (86a)$$

$$\quad (86b)$$

for $1 \leq k \leq n$. If $\mathbb{B}_\rho(x_k) \subset \mathbb{B}_{2\rho}(x) \subset \Omega$, and $-\Delta v \geq 0$ in Ω , en conséquence,

$$v(x) \geq \int_{\mathbb{B}_{2\rho}(x)} v = \varepsilon_1 \int_{\mathbb{B}_{2\rho}(x)} v \geq \varepsilon_1 \int_{\mathbb{B}_\rho(x_k)} v = \varepsilon_1 \int_{\Omega} v \left(-\Delta \zeta_k^{\mathbb{B}} - \frac{\varepsilon_2}{|x|^2} \zeta_k^{\mathbb{B}} \right) = \varepsilon_1 \int_{\Omega} \varphi \zeta_k^{\mathbb{B}} \geq \varepsilon_1 \int_{\Omega} \varphi \zeta_0^{\mathbb{B}}. \quad (87)$$

□

5. Synopsis: Classification of Solutions and Non-Solutions in Gelfand Problem

Below is a picture of the situation, which serves as a synoptic guide.

(1) When one has to do with a *stable branch*, the solution is minimal and stable, for $\lambda \in (0, \check{\lambda})$, with L^∞ -Banach space of essentially bounded measurable functions endowed with the essential supremum norm.

(2) In $D = n \geq 2$, the GP, with $\lambda = \lambda_\tau = 2n - 4$, has a *singular solutions* $v_\tau(x) = -2 \ln |x|$.

(3) In $D = n \geq 3$, the function v_τ is a *solution among all density distributions*.

(4) In $D = 1 \leq n \leq 9$,

(i) the singular solution v_τ *does not intersect finite Morse index*,

(ii) all solutions of the curve inherent to the bifurcation map is given by $\lambda = \check{\lambda}$ with *turning points*, which means that any solution crossing the stable branch becomes *unstable*, and the Morse index is (at least) = 1,

(iii) any stable solution is ultimately *uniformly bounded*.

(5) In $D = 3 \leq n \leq 9$,^a

(i) all solutions $v_\tau(x) = -2 \ln |x|$ of the curve inherent to the bifurcation map have *infinitely many turning points* in the direction of $\lambda_\tau = 2n - 4$, and the Morse index of the trajectory must be increasing by one unit, each time the solution passes through a turning point,

(ii) there exists a *unique solution*, if λ has an (extremely) small value,

(iii) there exists a *unique solution*, for $\lambda = \check{\lambda}$,

(iv) there are *two kind of solutions*, for λ close to $\check{\lambda}$,

(v) the GP has *infinitely many solutions*, for $\lambda = 2n - 4$, and *finitely many solutions*, for $\lambda \neq 2n - 4$,

(vi) one has a *series of solutions*, for λ close enough to $2n - 4$.

(6) In $D = n \geq 10$,

(i) the solution v_τ is *stable*,

(ii) the GP has a unique solution, if $\lambda \in (0, \check{\lambda})$,

^a The paper of W. Chen and J. Dávila [20] offers an investigation into the pair of Eqq. (8), with an infinite multiplicity of regular solutions, for $3 \leq n \leq 9$, and $\lambda > 0$. Among the propositions, it is highlighted that the above-mentioned pair of equations, in which $\check{\lambda} = \check{\lambda}(n) > 0$, for $n \geq 1$,

(1) presents a minimal solution u_λ , if $0 < \lambda < \check{\lambda}$,

(2) has a unique solution, if $\lambda = \check{\lambda}$,

(3) does not present a solution, if $\lambda > \check{\lambda}$.

(iii) there exist a singular stable solution, as an *extremal solution* (with Dirichlet $\mathcal{C}^{2,\delta}$ -boundary),^a remaining within the Euclidean topology of a ball,

(iv) there exist singular stable solutions, if some Ω -domain preserves a \mathcal{C}^2 -diffeomorphism on a solid sphere (ball), even in the event of perturbations.

(7) In $D = n \geq 11$, when the parameter λ is the largest possible one, there is an *extremal solution*, within a ball-shaped space with perturbations.

(8) Now, let us turn the table here. Pay attention of the GP, adding a vector field \vec{V} , which is smooth of course:

$$-\Delta v + \vec{V}(x)\nabla v = e^v \text{ in } \mathbb{R}^n. \quad (88)$$

An apparently sophisticated triple scenario opens up.

(i) There is a stable solution, for $n \geq 4$, if the *Lyapunov-Schmidt construction* [58] [72] is called into question, which has the goodness of reducing infinite-dimensional non-linear equations in Banach spaces to finite-dimensional formulæ.

(ii) There is stable solution, for $n \geq 11$, if the vector field satisfies

$$\begin{cases} \operatorname{div} \vec{V}(x) = 0, \\ (|x| + 1) |\vec{V}(x)| \leq \epsilon, \end{cases} \quad (89a)$$

$$\quad (89b)$$

with a value ϵ sufficiently small ($0 < \epsilon$), as long as

$$\inf_{\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)} \left\{ \frac{\int_{\mathbb{R}^n} (|\nabla \varphi|^2 - e^v \varphi^2) dx}{\|\varphi\|_{L^2(\mathbb{R}^n)}} \right\} \geq 0, \quad (90)$$

$$\left\{ -\Delta \phi_+ + \vec{V} \nabla \phi_+ - e^v \phi_+ \right\}_{\mathbb{R}^n}$$

viz. the eigenvalue of the linear operator $-\Delta + \vec{V} \nabla - e^v$ is non-negative in $\mathcal{C}_0^\infty(\mathbb{R}^n)$, where ϕ_+ is a some positive function.

(iii) There is no stable \mathcal{C}^2 -solution, for $n \leq 9$.

For present purposes it is not necessary (and not interesting) to examine the demonstrative particulars, which can be found in B. Lai and L. Zhang [48].

(g) Gelfand problem for stable $W_{\text{loc}}^{1,p}$ -solutions to quasi-linear formulæ in a suitable weighted Sobolev space, was investigated by P. Le et al. [51]. Their start it is not really general, but it is this:

$$-\operatorname{div}(\eta(x)|\nabla v|^{p-2}\nabla v) = \varphi(x)e^v,$$

where $\eta, \varphi \in L_{\text{loc}}^1(\mathbb{R}^n)$ are non-negative functions, and $p \geq 2$.

(10) Viscosity solutions to the Gelfand problem for the 1-homogeneous p -Laplacian, es decir, for a pair that has a form similar to this,

$$\left(\frac{1}{1-p}|\nabla v|^{2-p}\right)\operatorname{div}(|\nabla v|^{p-2}\nabla v) = \lambda e^v \begin{cases} -\Delta_p^n v = \lambda e^v \text{ in } \Omega \subset \mathbb{R}^n, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \quad (91a)$$

$$(91b)$$

are in J. Carmona Tapia et al. [18]. For $p \in [2, \infty]$, one sets up that

(i) there exists a minimal positive solution, if $\lambda < \check{\lambda}$,

(ii) there exists no solution, if $\lambda > \check{\lambda}$.

(11) Gelfand-type quasi-linear elliptic problems with quadratic gradient terms are discussed instead by D. Arcoya et al. [3]. The couple that acts as a reasoning incipit is

$$\begin{cases} -\Delta v + c(x)|\nabla v|^2 + v = \lambda(1+v)^p \text{ in } \Omega \subset \mathbb{R}^n, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \quad (92a)$$

$$(92b)$$

once these parameters $0 < c_1 \leq c(x) \leq c_2$ are set. The results that are reached are:

(i) in the interval $(0, \check{\lambda}]$, $\check{\lambda} > 0$, there is at least one positive solution,

(ii) there are minimal regular positive solutions,

(α) for each $\lambda \in (0, \check{\lambda})$, and

^a For a general look at extremal solutions with regularity, see [21].

(β) for $\lambda = \check{\lambda}$ (but take it for granted that the spatial dimensionality, together with the parameters p , c_1 , and c_2 , are placed under sure conditions).

6. Some References on an Affine Problem: Lane–Emden Question

The Lane–Emden Problem [50] [28] is very close to the tentacular Gelfand system; so it is wise to quote some studies that may be of benefit to someone: [15] [31] [1] [65]. The second of them is written by M. Fazly and J. Wei, and it is very well done.

Appendix

Picard–Lindelöf Theorem

Theorem 6.1 (Picard–Lindelöf). *Let $\varphi_L \in \mathcal{C}_B(\Omega, \mathbb{R}^n)$ be a Lipschitz continuous function, where Ω is an open subset of \mathbb{R}^{n+1} , with $(t_0, x_0) \in \Omega$.^a We can imagine this set as a space, and more specifically as a parallelepiped, determined with these coordinates*

$$\{(t, x) \mid t_0 \leq t \leq t_0 + \alpha, |x - x_0| \leq \beta\},$$

or, without alterations of the theorem in question, as a cylinder, in which (t_0, x_0) are its internal parameter values.

If $\varphi_L: \mathcal{C}_B(\Omega, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is uniformly et locally Lipschitz continuous in x and in t , then there is a quantity $\varepsilon > 0$ such that one has an unique local solution $x(t) \in \mathcal{C}_B^1[t_0, t_0 + \varepsilon]$ of the initial value problem, for an infinitesimal interval $[t_0, t_0 + \varepsilon]$ around t_0 , where

$$\left\{x(t) = \lim_{n \rightarrow \infty} x_n(t)\right\} \stackrel{\text{eqv}}{=} \left\{x(t) = x_0 + \int_{t_0}^t \varphi_L(\tau, x(\tau)) d\tau\right\}.$$

In other words, if

$$\varphi_L: \underbrace{[t_0, T]}_{\text{interval}} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is a vector field Lipschitz function on the parallelepipedal-space, or on the cylindrical-space, w.r.t. its second variable, and once it is specified that H is the upper bound (maximum) on φ_L , ou seja $H = \max\{|\varphi_L(t, x)|\}$ on $[t_0, t_0 + \varepsilon]$, with

$$\varepsilon = \min\left\{\alpha, \frac{\beta}{H}\right\}, \quad (93)$$

then there exists a unique solution $x(t)$ of the initial value problem, for $t \in [t_0, t_0 + \varepsilon]$, or for $t \in [t_0, t_0 + T_0]$, where $T_0 = \min\left\{T, \frac{\beta}{H}\right\}$.

Proof (Sketch). I choose here to give only a demonstrative hint, because this theorem is not the core of this paper. The proof of Picard–Lindelöf theorem is built on *Banach fixed-point theorem* [5], aka *Banach–Caccioppoli theorem*, whose fixity is confirmed by a certain parameter. The proof is completed when it is shown that

$$\begin{cases} T_0 \leq T, & (94a) \\ T_0 \leq \frac{\beta}{H}, & (94b) \end{cases}$$

$$T_0 < \left(N_L = \sup_{(t,x) \neq (t,y)} \frac{|\varphi_L(t, x) - \varphi_L(t, y)|}{|x - y|}\right)^{-1}, \quad (94c)$$

where N_L is a number related to the Lipschitz continuity. □

^a The set of continuous functions \mathcal{C}_B can be thought of as a (closed) subset of a Banach space [6] $(\mathfrak{X}, \|\cdot\|)$, i.e. as a complete normed vector including a vector space \mathfrak{X} over, say, a scalar \mathbb{R} -field, with a norm $\|\cdot\|: \mathfrak{X} \rightarrow \mathbb{R}$.

Mini-version of Lebesgue's Dominated Convergence Theorem

Theorem 6.2 (Lebesgue). To put it briefly, let us consider the measure space $(\hat{\Omega}, \mathcal{B}_\sigma, \mu)$, where $\hat{\Omega}$ is a non-empty set equipped with a so-called σ -algebra, \mathcal{B}_σ is a σ -algebra of subsets of $\hat{\Omega}$, and $\mu: \mathcal{B}_\sigma \rightarrow [0, \infty]$ is a finite measure on $\hat{\Omega}$. Let $\{\varphi_{\mu_n}\}_{n=1}^\infty: \hat{\Omega} \rightarrow \mathbb{F}$ be a sequence of \mathbb{F} -valued \mathcal{B}_σ -measurable functions on $\hat{\Omega}$ pointwise converging μ -almost everywhere to φ_μ , namely

$$\varphi_{\mu_n} \rightarrow \varphi_\mu, \quad n \rightarrow \infty.$$

Finally we suppose that

$$|\varphi_{\mu_n}| \leq \mathfrak{u},$$

being $\mathfrak{u} \geq 0$ n -integrable. The function φ_μ is integrable, so that

$$\int_{\hat{\Omega}} \varphi_\mu d\mu = \lim_{n \rightarrow \infty} \int_{\hat{\Omega}} \varphi_{\mu_n} d\mu. \quad (95)$$

Proof.

$$\int_{\hat{\Omega}} 2\mathfrak{u} d\mu = \int_{\hat{\Omega}} \liminf_{n \rightarrow \infty} (2\mathfrak{u} - |\varphi_\mu - \varphi_{\mu_n}|) d\mu \leq \liminf_{n \rightarrow \infty} \int_{\hat{\Omega}} 2\mathfrak{u} - |\varphi_\mu - \varphi_{\mu_n}| d\mu, \quad (96a)$$

$$\limsup_{n \rightarrow \infty} \int_{\hat{\Omega}} |\varphi_\mu - \varphi_{\mu_n}| d\mu \leq 0, \quad (96b)$$

via inequality theorem by P. Fatou [30]. □

Those who are greedy for minutiae in a sequence of \mathbb{C} -valued measurable functions, and a demonstration, see [64, sec. 13.1.2.1].

Outro

Three Bird's Eye Remarks

(1) Today's mathematics has lost its way home, the way of a universalized science. It got lost in a regime of partially-sighted people where *ad ricercatorum* solutions prevail, and this is not nice.

Anyone who really loves the original and truthful $\mu\acute{\alpha}\theta\eta\mu\alpha$ hates the sectorization; he abhors the division of the Mathematics Palace into closed and incommunicating rooms (what I have elsewhere called *cocoon syndrome*); he disdains the knowledge that abandons the impetus of being boldly among things in a transversal manner, that knowledge that does not take—at least *in pectore*—big and daring routes, to face dangers, raising the head towards the dome from which universal questions hang, and contemplate their glare.

The reshuffling of a problem, devoid of a sense of grandeur, which is a sub-problem of a sub-problem, which is the *inàne* variant of a problem referring to the same sub-problem, brings joy exclusively to those who suffer from intellectual myopia, or to those with a siloed-brain. This is unfortunately the feudal regression in which the Gelfand problem is currently interpreted. And it cannot be otherwise: we live in an era in which mathematics has its (nefarious) flourishing within a *curtense* system.

Mathematics is no longer sailing on the *grande rio do Pensamento* nowadays, which was once the moving source of its beauty; it is like a paper boat that glides through a thousand rivulets of water. Then there are the scholars who enjoy soaking their shoes in puddles a few inches deep: scholars are like that (to echo da Ponte's words), or the school for trendies. And with this the transition from beauty to squalor is accomplished in a flash. Which is followed by the passage from intelligence to dullness, from usefulness to uselessness, and from *maculate* conception to sterility.

Surely, a (platitudinous) accusation could be brought against me: this is a simple criticism. I realize that. And so it is; it is a *pars destruens*. To silhouette a *pars construens* we must start again from the foundations, which cannot be done in an article.

(2) I ask these simple questions already conceived in 1994 by a great master of the recent past, E. De Giorgi [25, pp. 720-723, 725, e.m.], who received a very heartfelt admiration from J.F. Nash. The two men met in Povo di Trento on 6 March 1996.

What happened to the «wisdom value» (*valore sapienziale*) of mathematics? Or to the «ancient links of mathematics with all other branches of wisdom»?

«The idea of mathematics [is] not a closed learning reserved to few specialists but an open one that, while preserving its own autonomy, feeds on dialogue with the other forms of knowledge».

What happened to the «love of mathematics, as an essential component of human wisdom», which is «something more than the simple ability to calculate, [or just] manipulate numbers»?

«Mathematics serves above all to have a *qualitative idea* of what might happen, to widen the *imaginative ability* of the experimental scholars or the planner, capable of understanding that a certain mathematical model can be the right model to interpret certain phenomena».

What happened to «the task to which all of us, scholars of mathematical, physical and natural sciences and scholars of human sciences, are called»? Det vill säga, what happened to «the rediscovery of the wisdom value of our disciplines, a necessary premise for the harmonious development of all branches of knowledge»?

(3) I launch a *provocation*. Is it possible to incorporate the molecular components of a match into the theory of groups, without ending up trapped in thousands of sub-categories, or sub-fields of sub-fields? Is it possible to understand what fire is by looking at the head of a match under a microscope? What can we understand from phosphorus sulfide and potassium chlorate? They do not give us much information about the nature of fire.

Contemporary pure mathematics and (mathematical) physics have become this, an accumulation of academic rubbish-knowledge towards nothingness, which has left behind some wise teachings of the past. They have become an *exercices in style* [70] à la Queneau,^a only good for raising the insane *h*-index (the productivity and citation impact) of journal publications, without providing real stimuli to the brain.

^a R. Queneau, among other things, had a non-marginal level of mathematical competencies [71].

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