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► **To cite this version:**

| Roman Novikov. A holographic uniqueness theorem. 2023. hal-04219516

HAL Id: hal-04219516

<https://hal.science/hal-04219516>

Preprint submitted on 27 Sep 2023

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A holographic uniqueness theorem

R.G. Novikov

Abstract. We consider a plane wave, a radiation solution, and the sum of these solutions (total solution) for the Helmholtz equation in an exterior region in \mathbb{R}^3 . We consider a ray in this region, such that its direction is different from the propagation direction of the plane wave. We show that the restriction of the radiation solution to this ray is uniquely determined by the intensity of the total solution on an interval of this ray. As a corollary, we also obtain that the restriction of the radiation solution to any plane in the exterior region is uniquely determined by the intensity of the total solution on an open domain in this plane. In particular, these results solve one of old mathematical questions of holography.

Keywords: Helmholtz equation, phase recovering, holography

AMS subject classification: 35J05, 35P25, 35R30

1. Introduction

We consider the Helmholtz equation

$$-\Delta\psi(x) = \kappa^2\psi(x), \quad x \in \mathcal{U}, \quad \kappa > 0, \quad (1.1)$$

where Δ is the Laplacian in x , and \mathcal{U} is a region (open connected set) in \mathbb{R}^3 consisting of all points outside a closed bounded regular surface S (as in [A], [W]).

This equation arises, in particular, in electrodynamics, acoustics, and quantum mechanics.

For equation (1.1) we consider solutions ψ_0 and ψ_1 such that:

$$\psi_0 = e^{ikx}, \quad k \in \mathbb{R}^3, \quad |k| = \kappa; \quad (1.2)$$

ψ_1 is of class C^2 and satisfies the Sommerfeld's radiation condition

$$|x|\left(\frac{\partial}{\partial|x|} - i\kappa\right)\psi_1(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty, \quad (1.3)$$

uniformly in $x/|x|$. We say that ψ_0 is a plane wave solution and ψ_1 is a radiation solution.

Let $L = L_{x_0, \theta}$ be the ray in \mathbb{R}^3 that starts at x_0 and has direction θ :

$$L = L_{x_0, \theta} = \{x \in \mathbb{R}^3 : x = x(s) = x_0 + s\theta, \quad 0 < s < +\infty\}, \quad x_0 \in \mathbb{R}^3, \quad \theta \in \mathbb{S}^2, \quad (1.4)$$

where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3 .

In the present work we show that, for a fixed plane wave solution ψ_0 , any complex-valued radiation solution ψ_1 on L is uniquely determined by the intensity $|\psi|^2$ of the total solution $\psi = \psi_0 + \psi_1$ on an arbitrary interval of L , under the assumptions that $L = L_{x_0, \theta} \subset \mathcal{U}$ and $\theta \neq k/|k|$. Here, \mathcal{U} is the region in (1.1) and k is the vector in (1.2). This result is given as Theorem 2.1 in Section 2.

As a corollary, we also obtain that, for any plane $X \subset \mathcal{U}$, any complex-valued ψ_1 on X is uniquely determined by the intensity $|\psi_0 + \psi_1|^2$ on an arbitrary open domain of X , for fixed ψ_0 ; see Theorem 2.2 in Section 2.

Our studies are motivated by holography and phaseless inverse scattering; see, e.g., [G], [Wo1], [Wo2], [JL], [Nu], [K11], [M], [N1]-[N3], [KR], [MH], [K12], [R], [HN], [NS2] and references therein. As far as we know, in the literature the aforementioned determination of ψ_1 on X from $|\psi_0 + \psi_1|^2$ on X was considered in the framework of different approximations only. Therefore, Theorems 2.1 and 2.2 solve one of old mathematical questions arising in holography. In addition, various new results on phaseless inverse scattering follow from Theorems 2.1 and 2.2 and known results on inverse scattering with phase information.

The main results of this work are presented in more detail and proved in Sections 2, 4, and 5. In our proofs we proceed from results recalled in Section 3.

2. Main results

Our key result is as follows.

Theorem 2.1. *Let ψ_0 and ψ_1 be solutions of equation (1.1) as in formulas (1.2) and (1.3). Let L be a ray as in (1.4) such that $L = L_{x_0, \theta} \subset \mathcal{U}$, $\theta \neq k/|k|$, where \mathcal{U} is the region in (1.1), k is the wave vector in (1.2). Then ψ_1 on L is uniquely determined by the intensity $|\psi_0 + \psi_1|^2$ on Λ , for fixed k , where Λ is an arbitrary non-empty open interval of L .*

As a corollary, we also get the following result.

Theorem 2.2. *Let ψ_0 and ψ_1 be solutions of equation (1.1) as in formulas (1.2) and (1.3). Let X be a two-dimensional plane in \mathbb{R}^3 such that $X \subset \mathcal{U}$. Then ψ_1 on X is uniquely determined by the intensity $|\psi_0 + \psi_1|^2$ on \mathcal{D} , for fixed k , where \mathcal{D} is an arbitrary non-empty open domain of X .*

Theorem 2.1 is proved in Sections 4 and 5 using techniques developed in [A], [W], [N1], [N2], [N4], [NS2], [NSh].

Note that ψ_0, ψ_1 are real-analytic on \mathcal{U} , and, therefore, on L in Theorem 2.1 and on X in Theorem 2.2. Therefore, the function $|\psi|^2 = |\psi_0 + \psi_1|^2 = (\psi_0 + \psi_1)(\overline{\psi_0 + \psi_1})$ is real-analytic on L in Theorem 2.1 and on X in Theorem 2.2. In view of this analyticity, Theorem 2.1 reduces to the case when $\Lambda = L$ and Theorem 2.2 reduces to the case when $\mathcal{D} = X$.

Theorem 2.2 is proved as follows (for example). We assume that $\mathcal{D} = X$. Then to determine ψ_1 at an arbitrary $x \in X$, we consider a ray $L = L_{x_0, \theta} \subset X$ such that $x \in L$, $\theta \neq k/|k|$, and use Theorem 2.1 for this L .

Remark 2.1. In the one dimensional case, an analog of Theorem 2.1 follows from results of [N3]. The two dimensional case will be considered elsewhere using results of [K] in place of the three dimensional Atkinson-Wilcox expansion (3.2) used in the present work.

To our knowledge, in the literature, rigorous mathematical results on the determination of the radiation solution ψ_1 on X from the intensity $|\psi|^2$ of the total solution $\psi = \psi_0 + \psi_1$ on X (including non-uniqueness and uniqueness), are given only in the framework of cons-

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iderable approximations (e.g., paraxial approximation for the case when k is orthogonal to X); see [Nu], [M], [MH]. Therefore, Theorem 2.2 solves an old mathematical question of holography.

In fact, an asymptotic determination of ψ_1 on $L_{x_0, \theta}$ from the intensity $|\psi_0 + \psi_1|^2$ on $L_{x_0, \theta}$ was developed in [N1], [N2], [N4], [NS1], [NS2], [NSh], for $x_0 = 0$. In Section 3 we recall formulas (3.5)-(3.8) in this connection. In the present work we proceed from these asymptotic results and classical results on the radiation solution ψ_1 , including the Atkinson-Wilcox expansion (3.2).

Many new uniqueness results on phaseless inverse (coefficient and obstacle) scattering problems follow from Theorems 2.1 and 2.2 and known results on inverse scattering with phase information in three dimensions. These new uniqueness results will be listed elsewhere.

Note that the determination in Theorems 2.1 and 2.2 includes an analytic continuation. Studies on more stable reconstructions appropriate for numerical implementation will be continued elsewhere.

3. Preliminaries

Let

$$B_r = \{x \in \mathbb{R}^3 : |x| < r\}, \quad r > 0. \quad (3.1)$$

Suppose that ψ_1 is a radiation solution of (1.1), and $\mathbb{R}^3 \setminus B_r \subset \mathcal{U}$. Then, due to results of [A], [W], we have that

$$\psi_1(x) = \frac{e^{i\kappa|x|}}{|x|} \sum_{j=1}^{\infty} \frac{f_j(\theta)}{|x|^{j-1}} \quad \text{for } x \in \mathbb{R}^3 \setminus B_r, \quad \theta = \frac{x}{|x|}, \quad (3.2)$$

where the series converges absolutely and uniformly.

Let

$$a(x, k) = |x|(|\psi(x)|^2 - 1), \quad x \in \mathcal{U}, \quad (3.3)$$

where $\psi = \psi_0 + \psi_1$, ψ_0 and ψ_1 are solutions of (1.1) as in (1.2) and (1.3), k is the wave vector in (1.2). Then (see [N1], [N4], [NSh]):

$$a(x, k) = e^{i(\kappa|x| - kx)} f_1(\theta) + e^{-i(\kappa|x| - kx)} \overline{f_1(\theta)} + O(|x|^{-1}) \quad \text{as } |x| \rightarrow +\infty, \quad (3.4)$$

uniformly in $\theta = x/|x|$;

$$f_1(\theta) = \frac{1}{D} (e^{i(ky - \kappa|y|)} a(x, k) - e^{i(kx - \kappa|x|)} a(y, k) + O(|x|^{-1})) \quad \text{as } |x| \rightarrow +\infty, \quad (3.5)$$

$$D = 2i \sin(\tau(k\theta - \kappa)), \quad \theta \in \mathbb{S}^2, \quad \tau > 0,$$

where

$$x, y \in L_{x_0, \theta}, \quad x_0 = 0, \quad y = x + \tau\theta, \quad (3.6)$$

and $D \neq 0$ for fixed θ and τ .

Remark 3.1. If an arbitrary function a on $L_{0,\theta}$ satisfies (3.4), for fixed $\theta \in \mathbb{S}^2$, $k \in \mathbb{R}^3$, $\kappa = |k| > 0$, then formula (3.5) holds.

Moreover, in fact, our work [N4] gives (see formulas (3.4) in [N4]):

$$\begin{aligned} & \text{formulas for finding } f_j(\theta) \text{ up to } O(s^{-(n-j+1)}) \text{ as } s \rightarrow +\infty, \quad j = 1, \dots, n, \quad (3.7) \\ & \text{from } |\psi(x)|^2 \text{ given at } 2n \text{ points } x = x_1(s), \dots, x_{2n}(s) \text{ on } L_{0,\theta}, \end{aligned}$$

where

$$\begin{aligned} x_i(s) &= r_i(s)(\theta), \quad i = 1, \dots, 2n, \\ r_{2j-1}(s) &= \lambda_j s, \quad r_{2j}(s) = \lambda_j s + \tau, \quad j = 1, \dots, n, \\ \lambda_1 &= 1, \quad \lambda_{j_1} < \lambda_{j_2} \quad \text{for } j_1 < j_2, \quad \tau > 0, \end{aligned} \quad (3.8)$$

and $D \neq 0$ for fixed θ and τ , where ψ is as in (3.3), D is as in (3.5).

For $n = 1$, formulas (3.7), (3.8) reduce to (3.5). In general, formulas (3.7), (3.8) are given recurrently in [N4]. Under the assumption that $\theta \neq k/|k|$, the parameter τ can be always fixed in such a way that $D \neq 0$.

4. Proof of Theorem 2.1

4.1. Case $L \subseteq L_{0,\theta}$. First, we give the proof for the case when $L \subseteq L_{0,\theta}$. In this case it is sufficient that $|\psi|^2$ on L uniquely determines $f_j(\theta)$ in (3.2) for all j via formulas (3.7), (3.8), where $\psi = \psi_0 + \psi_1$. The rest follows from the convergence of the series in (3.2) and analyticity of ψ_1 and $|\psi|^2$ on L .

Note that the recurrent formulas (3.7), (3.8) are not very simple in [N4]. Therefore, for completeness of presentation in Appendix, we give a very simple proof that $|\psi|^2$ on L uniquely determines $f_j(\theta)$ in (3.2) for all j .

4.2. General case. In fact, the general case reduces to the case of Subsection 4.1 by the change of variables

$$x' = x - q, \quad \text{such that } L \subseteq L_{q,\theta}, \quad (4.1)$$

for some fixed $q \in \mathbb{R}^3$.

In the new variables x' , we have that:

$$\psi_0 = e^{ikq} e^{ikx'}, \quad (4.2)$$

$$\begin{aligned} & \psi_1 \text{ satisfies (3) and admits presentation (3.2) with } x' \text{ in place of } x, \\ & \text{with some new } f'_j \text{ and } r' \text{ in (3.2),} \end{aligned} \quad (4.3)$$

$$|\psi|^2 = |e^{ikx'} + e^{-ikq} \psi_1(x')|^2, \quad (4.4)$$

where $x' \in \mathcal{U}' = \mathcal{U} - q$, and r' is such that $\mathbb{R}^3 \setminus B_{r'} \subset \mathcal{U}'$;

$$L \subseteq L_{q,\theta} = L_{0,\theta}. \quad (4.5)$$

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In addition,

$$e^{-ikq}\psi_1(x') = \frac{e^{i\kappa|x'|}}{|x'|} \sum_{j=1}^{\infty} \frac{f_j''(\theta)}{|x'|^{j-1}}, \quad x' \in \mathbb{R}^3 \setminus B_{r'}, \quad f_j'' = e^{-ikq} f_j', \quad \theta = \frac{x'}{|x'|}, \quad (4.6)$$

where the series converges absolutely and uniformly, the coefficients f_j' are mentioned in (4.3).

In view of (4.4)-(4.6), similarly to the case of Subsection 4.1, it is sufficient to prove that $|\psi|^2$ on L uniquely determines $f_j''(\theta)$ in (4.6) for all j . This determination of f_j'' is completely similar to the determination of f_j in Subsection 4.1.

This completes the proof of Theorem 2.1.

Appendix

In this Appendix, we give a very simple proof that $|\psi|^2$ on L uniquely determines $f_j(\theta)$ in (3.2) for all j , under the assumption that $\theta \neq k/|k|$, where $\psi = \psi_0 + \psi_1$, ψ_0 and ψ_1 are solutions of (1.1) as in (1.2) and (1.3).

The determination of f_1 follows from (3.5).

Suppose that f_1, \dots, f_n are determined, then the determination of f_{n+1} is as follows.

Let

$$\psi_{1,n}(x) = \frac{e^{i\kappa|x|}}{|x|} \sum_{j=1}^n \frac{f_j(\theta)}{|x|^{j-1}} \quad \text{for } \theta = \frac{x}{|x|}, \quad (A.1)$$

$$a_n(x, k) = |x|(|e^{ikx} + \psi_{1,n}(x)|^2 - 1), \quad (A.2)$$

$$b_n(x, k) = |x|^n(a(x, k) - a_n(x, k)), \quad (A.3)$$

where x is as in (3.2).

We have that

$$\begin{aligned} a(x, k) &= |x| \left((e^{ikx} + \psi_{1,n}(x) + \frac{e^{i\kappa|x|}}{|x|} \frac{f_{n+1}(\theta)}{|x|^n} + O(|x|^{-n-2})) \right. \\ &\times (e^{-ikx} + \overline{\psi_{1,n}(x)} + \frac{e^{-i\kappa|x|}}{|x|} \frac{\overline{f_{n+1}(\theta)}}{|x|^n} + O(|x|^{-n-2})) - 1 \Big) \end{aligned} \quad (A.4)$$

$$= a_n(x, k) + \frac{e^{i(\kappa|x|-kx)}}{|x|^n} f_{n+1}(\theta) + \frac{e^{-i(\kappa|x|-kx)}}{|x|^n} \overline{f_{n+1}(\theta)} + O(|x|^{-n-1}),$$

$$b_n(x, k) = e^{i(\kappa|x|-kx)} f_{n+1}(\theta) + e^{-i(\kappa|x|-kx)} \overline{f_{n+1}(\theta)} + O(|x|^{-1}), \quad (A.5)$$

as $|x| \rightarrow +\infty$, uniformly in $\theta = x/|x|$, where a is defined by (3.3).

Due to (A.5) and Remark 3.1, we get

$$f_{n+1}(\theta) = \frac{1}{D} (e^{i(ky-\kappa|y|)} b_n(x, k) - e^{i(kx-\kappa|x|)} b_n(y, k) + O(|x|^{-1})) \quad \text{as } |x| \rightarrow +\infty, \quad (A.6)$$

$$D = 2i \sin(\tau(k\theta - \kappa)), \quad x, y \in L, \quad y = x + \tau\theta, \quad \theta \in \mathbb{S}^2, \quad \tau > 0,$$

assuming that $D \neq 0$ for fixed θ and τ (where the parameter τ can be always fixed in such a way that $D \neq 0$, under our assumption that $\theta \neq k/|k|$).

Formulas (A.1)-(A.3) and (A.6) determine f_{n+1} , give the step of induction for finding all f_j , and complete the proof.

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