An optimal spectral inequality for degenerate operators

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Abstract .- In this paper we establish a Lebeau-Robbiano spectral inequality for a degenerate one dimensional elliptic operator. Carleman techniques and moment method are combined. Application to null controllability on a measurable set in time for the degenerated heat equation is described.

1 Introduction and main results

The purpose of this article is to prove a spectral inequality for a family of degenerate operators acting on the interval $(0,1)$. In arbitrary dimension, for a second-order symmetric elliptic operator $\mathcal{P}$ on a bounded domain Ω with homogeneous Dirichlet or Neumann boundary conditions, the spectral inequality also called Lebeau-Robbiano estimate takes the form

$$\|u\|_{L^2(\Omega)} \leq ce^{c\sqrt{\lambda}} \|u\|_{L^2(\omega)} \ , \quad \forall u \in \operatorname{span}\{\Phi_j; \lambda_j \leq \lambda\} \tag{1.1}$$

where $\omega \subset \Omega$ is an open subset and where the functions Φ_j form a Hilbert basis of $L^2(\Omega)$ of eigenfunctions of $\mathcal{P}$ associated with the nonnegative eigenvalues λ_j , $j \in \mathbb{N}$, counted with their multiplicities. In other words, the family of spectral projectors associated to $\mathcal{P}$ enjoys an observability inequality on a set $\omega \subset \Omega$ for low frequencies $\lambda_j \leq \lambda$ with a constant cost as $ce^{c\sqrt{\lambda}}$.

The state of art to prove (1.1) is either Carleman inequalities for elliptic equations (see [LR], [LZ], [JL], [L], [LRL], [LRLeR1], [LRLeR2], [Le], [LL] and [Q], [FQZ]) or observation estimate at one point in time for parabolic equations (see [AEWZ], [BaP] and [BP]).

One of the key applications of (1.1) is either observability for parabolic systems or controllability for parabolic systems, knowing that both are equivalent properties by a duality argument (see [Zu], [FZ], [FI], [Mi] and [Mi2]).

Observability and controllability for the one-dimensional degenerate parabolic operator has been extensively studied in many ways: Backstepping approach for closed-loop control (see [GLM] and [LM]); Carleman inequalities (see [ABCF], [CMV], [CMV2] and [CTY]); Flatness approach (see [Mo] and [BLR]); Moment method (see [CMV3] and [CMV4]).

We shall consider the linear unbounded operators $\mathcal{P}$ in $L^2(0,1)$, defined by

$$\begin{cases} \mathcal{P} = -\frac{d}{dx}\left(x^\alpha \frac{d}{dx}\right) \text{ , with } \alpha \in [0,2) \ , \\ D(\mathcal{P}) = \left\{\vartheta \in H^1_{\alpha,0}(0,1)\,;\, \mathcal{P}\vartheta \in L^2(0,1) \text{ and } \mathrm{BC}_\alpha(\vartheta) = 0\right\} \ , \end{cases}$$

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where

$$H^1_{\alpha,0}(0,1) := \left\{ \vartheta \in L^2(0,1);\, \vartheta \text{ is loc. absolutely continuous in } (0,1]\,,\ \int_0^1 x^\alpha |\vartheta'|^2 < \infty,\, \vartheta(1) = 0 \right\}\ ,$$

and

$$\mathrm{BC}_\alpha(\vartheta) = \begin{cases} \vartheta_{|_{x=0}}\ , & \text{for } \alpha \in [0,1)\ , \\ (x^\alpha \vartheta')_{|_{x=0}}\ , & \text{for } \alpha \in [1,2)\ . \end{cases}$$

Such $\mathcal{P}$ is a closed self-adjoint positive densely defined operator, with compact resolvent. As a consequence, the following spectral decomposition holds: There exists a countable family of eigenfunctions Φ_j associated with eigenvalues λ_j such that

- $\{\Phi_j\}_{j\geq 1}$ forms a Hilbert basis of $L^2(0,1)$
- $\mathcal{P}\Phi_j = \lambda_j \Phi_j$
- $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty$.

An explicit expression of the eigenvalues is given in [Gu] for the weakly degenerate case $\alpha \in (0,1)$, and in [Mo] for the strongly degenerate case $\alpha \in [1,2)$, and depends on the Bessel functions of first kind (see [MM]). The eigenvalues are simples and more proprieties are emphasized by Cannarsa, Martinez and Vancostenoble: First, a uniform bound for the first eigenvalue

$$\exists c_1, c_2 > 0 \quad \forall \alpha \in [0,2) \quad c_1 \leq \lambda_1 \leq c_2 \tag{1.2}$$

(see [CMV3] (10) at page 176 and (34) at page 183 for $\alpha \in [0,1)$; see [CMV4] proposition 2.13 at page 10 and (3.8)-(3.9) at page 13 for $\alpha \in [1,2)$); Secondly, a uniform spectral gap

$$\exists \gamma > 0 \quad \forall \alpha \in [0,2) \quad \forall k \geq 1 \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma(2-\alpha) \tag{1.3}$$

(see [CMV3] (74) at page 198 for $\alpha \in [0,1)$; see [CMV4] at page 30 for $\alpha \in [1,2)$).

We are interested in the spectral inequality for the sum of eigenfunctions. Such Lebeau-Robbiano estimate is done with explicit dependence on $\alpha \in [0,2)$. Our main result is as follows.

Theorem 1.1 .- *Let ω be an open and nonempty subset of $(0,1)$. There exists a constant $C > 0$ such that*

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq C e^{C\frac{1}{(2-\alpha)^2}\sqrt{\Lambda}} \int_\omega \left| \sum_{\lambda_j \leq \Lambda} a_j \Phi_j \right|^2\ ,$$

for any $\alpha \in [0,2)$, $\{a_j\} \in \mathbb{R}$ and any $\Lambda > 0$.

This is equivalent to

$$\sum_{j=1,\cdot,N} |a_j|^2 \leq C e^{C\frac{1}{(2-\alpha)^2}\sqrt{\lambda_N}} \int_\omega \left| \sum_{j=1,\cdot,N} a_j \Phi_j \right|^2\ ,$$

for any $\alpha \in [0,2)$, $\{a_j\} \in \mathbb{R}$ and any $N > 0$.

Here, our approach is based on a combinaison of both Carleman techniques and the moment method for an elliptic equation. In one hand, it seems difficult to find the appropriate weight function in Carleman techniques or logarithmic convexity methods for getting directly the desired spectral inequality. On the other hand, the moment method is an appropriate tool to get the cost of controllability for the one-dimensional degenerate parabolic operator.

As a consequence of Theorem 1.1, we have the following observability estimate from a measurable set in time for the one-dimensional degenerate parabolic operator.

Theorem 1.2 .- *Let ω be an open and nonempty subset of $(0,1)$ and $E \subset (0,T)$ be a measurable set of positive measure. There exists a constant $C>0$ such that*

$$\left\|e^{-T\mathcal{P}}y_0\right\|_{L^2(0,1)} \leq Ce^{C\frac{1}{(2-\alpha)^4}} \int_{\omega\times E} \left|e^{-t\mathcal{P}}y_0\right| \ ,$$

for any $\alpha \in [0,2)$ and any $y_0 \in L^2(0,1)$.

This is equivalent to

$$\|y(\cdot,T)\|_{L^2(0,1)} \leq Ce^{C\frac{1}{(2-\alpha)^4}} \int_{\omega\times E} |y(x,t)|\, dxdt \ ,$$

for any $\alpha \in [0,2)$ and any $y_0 \in L^2(0,1)$ where y is the weak solution of the degenerate heat equation

$$\begin{cases} \partial_t y - \partial_x\left(x^\alpha \partial_x y\right) = 0 \ , & \text{in } (0,1)\times(0,T) \ , \\ \mathrm{BC}_\alpha(y) = 0 \ , & \text{on } \{0\}\times(0,T) \ , \\ y(1,t) = 0 \ , & t \in (0,T) \ , \\ y(x,0) = y_0 \ , & x \in (0,1) \ . \end{cases}$$

In the last years, a lot of works have been devoted to the observability on measurable sets (see e.g. [AE], [EMZ], [PW], [WZ], [LiZ]). Applications to impulse control and finite-time stabilization can be rewritten as in [BP].

2 Elliptic observation estimates (proof of Theorem 1.1)

In this Section, our aim is to prove Theorem 1.1 and we start with presenting the following three results: We first have an uniform observability estimate for a single eigenfunction given by Proposition 2.1; Proposition 2.2 establishes a quantitative Holder type of estimate for an elliptic equation far from the degeneracy; Proposition 2.3 is an uniform observability estimate for the elliptic equation; We end this Section with the proof of Theorem 1.1.

Proposition 2.1 .- *For any ω open and nonempty subset of $(0,1)$,*

$$\exists \rho > 0 \quad \forall \alpha \in [0,2) \quad \forall j \geq 1 \quad \int_\omega |\Phi_j|^2 \geq \rho(2-\alpha) \ .$$

Given $T>0$ arbitrary, we now consider the following homogeneous elliptic problem:

$$\begin{cases} \partial_t^2\varphi - \mathcal{P}\varphi = 0 \ , & \text{in } (0,1)\times(0,T) \ , \\ \mathrm{BC}_\alpha(\varphi) = 0 \ , & \text{on } \{0\}\times(0,T) \ , \\ \varphi_{|x=1} = 0 \ , & \text{on } \{1\}\times(0,T) \ , \\ \varphi(\cdot,0) = \varphi_0 \ , & \text{in } (0,1) \ , \\ \partial_t\varphi(\cdot,0) = \varphi_1 \ , & \text{in } (0,1) \ , \end{cases} \tag{2.1}$$

where φ_0 and φ_1 belong to $\mathrm{span}\{\Phi_j; 1 \leq j \leq N\}$.

Proposition 2.2 .- *Let $0<a<b<1$ and $T>0$. There exist $c>0$ and $\delta \in (0,1)$ such that for all $\alpha \in [0,2)$, the solution φ of (2.1) satisfies*

$$\|\varphi\|_{H^1\left(\left(\frac{2a+b}{3},\frac{a+2b}{3}\right)\times(0,T/4)\right)} \leq c\,\|\varphi\|^{1-\delta}_{H^1((a.b)\times(0,T))} \left(\|\varphi_0\|_{H^1(a,b)} + \|\varphi_1\|_{L^2(a,b)}\right)^\delta \ ,$$

Proposition 2.3 .- *Let ω be an open and nonempty subset of $(0,1)$. For any $N \geq 1$, $T>0$, and any $\alpha \in [0,2)$, the solution φ of (2.1) satisfies*

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)} \leq \frac{C(1+\lambda_N)}{\rho^2(2-\alpha)^2}\left(1+\frac{1}{T}\right) e^{C\sqrt{\lambda_N}\left(T+\frac{1}{T\gamma^2(2-\alpha)^2}\right)} \int_0^T\int_\omega |\varphi|^2 \ ,$$

where $C > 0$ is independent of $N, T > 0$ and $\alpha \in [0, 2)$. Here ρ is given by Proposition 2.1 and γ comes from (1.3).

Now, we are able to present the proof of Theorem 1.1.

Our strategy is as follows. We will use Proposition 2.3 and we observe the whole domain (including the region where the ellipticity degenerates) from one region where the operator $\partial_t^2 + \mathcal{P}$ is uniformly elliptic; there, we use classical global Carleman techniques to observe from the boundary $(a, b) \times \{0\}$ with Proposition 2.2. That observation region provides precisely the right hand side of Theorem 1.1.

Proof of Theorem 1.1 .- We consider the above homogeneous elliptic problem with $\varphi_0(x) = 0$ and $\varphi_1(x) = \sum_{j=1,\cdot\cdot,N} a_j \Phi_j(x)$ where $\{a_j\} \in \mathbb{R}$. Recall that φ can be explicitly written by Fourier series: For any $x \in (0, 1)$,

$$\varphi(x,t) = \sum_{j=1,\cdot\cdot,N} \frac{1}{\sqrt{\lambda_j}} \sinh\left(\sqrt{\lambda_j} t\right) a_j \Phi_j(x) \ .$$

Let $0 < a < b < 1$ and set $\omega = (a, b)$ and $\widetilde{\omega} = \left(\frac{2a+b}{3}, \frac{a+2b}{3}\right)$. We have , for some constants $C, C_1, C_2 > 0$ independent on N and α,

$$\begin{aligned}
\sum_{j=1,\cdot\cdot,N} |a_j|^2 &\leq \sum_{j=1,\cdot\cdot,N} |a_j|^2 \frac{1}{\lambda_j} \sinh^2\left(\sqrt{\lambda_j}/4\right) C e^{C\sqrt{\lambda_N}} && \text{by (1.2)} \\
&= C e^{C\sqrt{\lambda_N}} \left\|\varphi(\cdot, 1/4)\right\|^2_{L^2(0,1)} \\
&\leq C_1 e^{C_1 \frac{1}{(2-\alpha)^2}\sqrt{\lambda_N}} \int_0^{1/4} \int_{\widetilde{\omega}} |\varphi|^2 && \text{by Proposition 2.3 applied to } \widetilde{\omega} \times (0, 1/4) \\
&\leq C_2 e^{C_2 \frac{1}{(2-\alpha)^2}\sqrt{\lambda_N}} \left\|\varphi\right\|^{2(1-\delta)}_{H^1(\omega\times(0,1))} \left\|\varphi_1\right\|^{2\delta}_{L^2(\omega)} && \text{by Proposition 2.2.}
\end{aligned}$$

But,

$$\left\|\varphi_1\right\|^2_{L^2(\omega)} = \int_\omega \left| \sum_{j=1,\cdot\cdot,N} a_j \Phi_j \right|^2$$

and for some constants $c_1, c_2 > 0$ independent on N and α, it holds

$$\begin{aligned}
\left\|\varphi\right\|^2_{H^1(\omega\times(0,1))} &= \left\|\varphi\right\|^2_{L^2(\omega\times(0,1))} + \left\|\partial_x \varphi\right\|^2_{L^2(\omega\times(0,1))} + \left\|\partial_t \varphi\right\|^2_{L^2(\omega\times(0,1))} \\
&\leq \left\|\varphi\right\|^2_{L^2\left((0,1)^2\right)} + c_1 \left\|x^{\alpha/2} \partial_x \varphi\right\|^2_{L^2\left((0,1)^2\right)} + \left\|\partial_t \varphi\right\|^2_{L^2\left((0,1)^2\right)} \\
&\leq c_2 e^{c_2\sqrt{\lambda_N}} \sum_{j=1,\cdot\cdot,N} |a_j|^2 \ .
\end{aligned}$$

Combining the above estimates completes the proof of Theorem 1.1.

3 Elliptic observability by the moment method (proof of Proposition 2.3)

In this section, we shall prove Proposition 2.3. Let ω be an open and nonempty subset of $(0, 1)$. Given $T > 0$ arbitrary, we consider the following non-homogeneous elliptic problem:

$$\begin{cases}
\partial_t^2 u - \mathcal{P} u = h \ , & \text{in } (0,1) \times (0,T) \ , \\
\mathrm{BC}_\alpha(u) = 0 \ , & \text{on } \{0\} \times (0,T) \ , \\
u_{|_{x=1}} = 0 \ , & \text{on } \{1\} \times (0,T) \ , \\
u(\cdot, 0) = u_0 \ , & \text{in } (0,1) \ , \\
\partial_t u(\cdot, 0) = u_1 \ , & \text{in } (0,1) \ ,
\end{cases} \tag{3.1}$$

where

$$\begin{cases} h(x,t) = \sum\limits_{j=1,\cdots,N} \sum\limits_{k=1,\cdots,N} g_k(t) \left(\int_\omega \Phi_j \Phi_k\right) \Phi_j(x) \text{ with } g(x,t) = \sum\limits_{k=1,\cdots,N} g_k(t)\, \Phi_k(x) \ , \\ u_0(x) = \sum\limits_{j=1,\cdots,N} a_j \Phi_j(x) \ , \\ u_1(x) = \sum\limits_{j=1,\cdots,N} b_j \Phi_j(x) \ . \end{cases} \tag{3.2}$$

3.1 Well-posedness property

Definition 3.1 .- Let $N \in \mathbb{N}^*$. We denote $\Pi_N L^2 = \text{span}\{\Phi_j; 1 \le j \le N\}$. The space $\Pi_N L^2$ endowed with the $L^2(\Omega)$ norm is a finite dimensional Hilbert space.

It is well-known that when $g_j \in L^2(0,T)$, the unique solution of (3.1) verifies $u \in H^2(0,T;\Pi_N L^2)$ and is given by the Duhamel formula

$$u(\cdot,t) = \sum_{j=1,\cdots,N} \cosh(\sqrt{\lambda_j}t) a_j \Phi_j + \sum_{j=1,\cdots,N} \frac{\sinh(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}} b_j \Phi_j \\ + \sum_{j=1,\cdots,N} \sum_{k=1,\cdots,N} (\int_\omega \Phi_j \Phi_k) \int_0^t \frac{\sinh(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} g_k(s) ds \Phi_j \ .$$

3.2 Construction of the control

Definition 3.2 .- We say that system (3.1) is controllable at time T if for any $(u_0, u_1) \in (\Pi_N L^2)^2$ there is $g \in L^2(0,T;\Pi_N L^2)$ as in (3.2) such that

$$u(\cdot,T) = \partial_t u(\cdot,T) = 0 \ .$$

Lemma 3.1 .- *Equation (3.1) is controllable in time T if and only if, for any $(u_0,u_1) \in (\Pi_N L^2)^2$ there is $g \in L^2(0,T;\Pi_N L^2)$ as in (3.2) such that the following relation holds*

$$-\int_0^1 u_1 \varphi(\cdot,T) - \int_0^1 u_0 \partial_t \varphi(\cdot,T) = \int_0^T \int_\omega g(x,t)\, \varphi(x,T-t)\, dxdt \tag{3.3}$$

for any $(\varphi_0,\varphi_1) \in (\Pi_N L^2)^2$, where φ is the solution of (2.1).

Further, if the system (3.1) is controllable at time T with a control $g \in L^2(0,T;\Pi_N L^2)$ satisfying the bound

$$\|g\|^2_{L^2((0,1)\times(0,T))} := \sum_{j=1,\cdots,N} \int_0^T |g_j(t)|^2 \le K \|(u_0,u_1)\|^2_{(L^2(0,1))^2} := K \sum_{j=1,\cdots,N} \left(a_j^2 + b_j^2\right)$$

for some $K > 0$, then the solution φ of (2.1) satisfies

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)} + \|\partial_t \varphi(\cdot,T)\|^2_{L^2(0,1)} \le K \int_0^T \int_\omega |\varphi|^2 \ .$$

Proof of Lemma 3.1 .- Let $g \in L^2(0,T;\Pi_N L^2)$ be arbitrary and u be the solution of (3.1). Given φ the solution of (2.1) then, by multiplying (3.1) by $\varphi(x,T-t)$ and by integrating by parts we obtain that

$$\int_0^1 \partial_t u(\cdot,T)\, \varphi_0 + \int_0^1 u(\cdot,T)\, \varphi_1 - \int_0^1 u_1 \varphi(\cdot,T) - \int_0^1 u_0 \partial_t \varphi(\cdot,T) = \int_0^T \int_0^1 h(x,t)\, \varphi(x,T-t)\, dxdt$$

and

$$\int_0^T \int_0^1 h(x,t)\,\varphi(x,T-t)\,dxdt = \int_0^T \int_\omega g(x,t)\,\varphi(x,T-t)\,dxdt\ .$$

Now, if (3.3) is verified, it follows that

$$\int_0^1 \partial_t u(\cdot,T)\,\varphi_0 + \int_0^1 u(\cdot,T)\,\varphi_1 = 0$$

for any $(\varphi_0,\varphi_1) \in (\Pi_N L^2)^2$ which implies that $u(\cdot,T) = \partial_t u(\cdot,T) = 0$. Hence, the solution is controllable at time T and g is a control for (3.1). Reciprocally, if $g \in L^2(0,T;\Pi_N L^2)$ is a control for (3.1), we have that $u(\cdot,T) = \partial_t u(\cdot,T) = 0$. It implies that (3.3) holds. Finally, one can choose $(u_0,u_1) = (\partial_t\varphi(\cdot,T), \varphi(\cdot,T))$ and apply (3.3) to get the desired estimate thanks to Cauchy-Schwarz inequality and the proof finishes.

Proof of Proposition 2.3 .- Our aim is to construct a control g given by $g(x,t) = \sum\limits_{k=1,\cdot\cdot,N} g_k(t)\,\Phi_k(x)$ such that (3.3) holds. Let

$$\begin{cases} \varphi_0(x) = \sum\limits_{j=1,\cdot\cdot,N} c_j\Phi_j(x)\ , \\ \varphi_1(x) = \sum\limits_{j=1,\cdot\cdot,N} d_j\Phi_j(x) \end{cases}$$

be the initial data of (2.1). Then, recall that φ can be explicitly written by Fourier series: For any $x \in (0,1)$,

$$\varphi(x,t) = \sum_{j=1,\cdot\cdot,N} \left(e^{\sqrt{\lambda_j}t}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right) + e^{-\sqrt{\lambda_j}t}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)\right)\Phi_j(x)\ .$$

First, let us clarify the expression $-\int_0^1 u_1\varphi(\cdot,T) - \int_0^1 u_0\partial_t\varphi(\cdot,T)$:

$$\begin{aligned} -\int_0^1 u_1\varphi(\cdot,T) - \int_0^1 u_0\partial_t\varphi(\cdot,T) \ &= \sum_{j=1,\cdot\cdot,N} e^{\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right)\int_0^1 \left(-u_1 - \sqrt{\lambda_j}u_0\right)\Phi_j \\ &\quad + \sum_{j=1,\cdot\cdot,N} e^{-\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)\int_0^1 \left(-u_1 + \sqrt{\lambda_j}u_0\right)\Phi_j\ . \end{aligned} \tag{3.4}$$

Next, let us clarify the expression $\int_0^T\int_\omega g(x,t)\,\varphi(x,T-t)\,dxdt$, that is $\int_0^T\int_0^1 h(x,t)\,\varphi(x,T-t)\,dxdt$:

$$\begin{aligned} &\int_0^T \int_\omega g(x,t)\,\varphi(x,T-t)\,dxdt \\ = \ &\sum_{k=1,\cdot\cdot,N}\sum_{j=1,\cdot\cdot,N} e^{\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right)\int_\omega \Phi_k\Phi_j\int_0^T g_k(t)\,e^{-\sqrt{\lambda_j}t}dt \\ &+ \sum_{k=1,\cdot\cdot,N}\sum_{j=1,\cdot\cdot,N} e^{-\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)\int_\omega \Phi_k\Phi_j\int_0^T g_k(t)\,e^{\sqrt{\lambda_j}t}dt\ . \end{aligned}$$

Now, suppose that $g_k(t) = \alpha_k\sigma_k^0(t) + \beta_k\sigma_k^1(t)$ where σ_k^0, σ_k^1 belong to $L^2(0,T)$ and that the following moment formula holds:

$$\begin{cases} \int_0^T \sigma_k^0(t)\,e^{-\sqrt{\lambda_j}t}dt = 0 \text{ and } \int_0^T \sigma_k^1(t)\,e^{-\sqrt{\lambda_j}t}dt = \delta_{jk}\ ; \\ \int_0^T \sigma_k^0(t)\,e^{\sqrt{\lambda_j}t}dt = \delta_{jk} \text{ and } \int_0^T \sigma_k^1(t)\,e^{\sqrt{\lambda_j}t}dt = 0\ , \end{cases} \tag{3.5}$$

then, we obtain

$$\begin{aligned} \int_0^T \int_\omega g(x,t)\,\varphi(x,T-t)\,dxdt \ &= \sum_{j=1,\cdot\cdot,N} e^{\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j + \frac{1}{\sqrt{\lambda_j}}d_j\right)\beta_j\int_\omega |\Phi_j|^2 \\ &\quad + \sum_{j=1,\cdot\cdot,N} e^{-\sqrt{\lambda_j}T}\frac{1}{2}\left(c_j - \frac{1}{\sqrt{\lambda_j}}d_j\right)\alpha_j\int_\omega |\Phi_j|^2\ . \end{aligned} \tag{3.6}$$

By comparing the identities (3.4) and (3.6), one can deduce that if

$$\int_0^1 \left(-u_1 - \sqrt{\lambda_j} u_0\right) \Phi_j = \beta_j \int_\omega |\Phi_j|^2 \text{ and } \int_0^1 \left(-u_1 + \sqrt{\lambda_j} u_0\right) \Phi_j = \alpha_j \int_\omega |\Phi_j|^2$$

for any $j = 1, \cdot\cdot, N$, then (3.3) holds for any (φ_0, φ_1) which implies by Lemma 3.1 that (3.1) is controllable in time T.

Therefore, one can conclude that the control given by $g(x,t) := \sum_{j=1,\cdot\cdot,N} \left[\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)\right] \Phi_j(x)$ where

$$\alpha_j := \frac{\int_0^1 \left(-u_1 + \sqrt{\lambda_j} u_0\right) \Phi_j}{\int_\omega |\Phi_j|^2} = \frac{-b_j + \sqrt{\lambda_j} a_j}{\int_\omega |\Phi_j|^2} \text{ and } \beta_j := \frac{\int_0^1 \left(-u_1 - \sqrt{\lambda_j} u_0\right) \Phi_j}{\int_\omega |\Phi_j|^2} = \frac{-b_j - \sqrt{\lambda_j} a_j}{\int_\omega |\Phi_j|^2} ,$$

is an appropriate candidate. Notice that by Proposition 2.1, $\int_\omega |\Phi_j|^2 \neq 0$. It remains to construct the sequence of functions $\left(\sigma_k^0, \sigma_k^1\right)_{k\geq 1}$ in $\left(L^2(0,T)\right)^2$ such that (3.5) holds. Such property is called biorthogonality of the family $\left(\sigma_k^0, \sigma_k^1\right)_{k\geq 1}$. To do so, we apply the following result from Cannarsa, Martinez and Vancostenoble (see [CMV3] Theorem 2.4 at page 179) :

Theorem 3.1 .- (Existence of a suitable biorthogonal family and upper bounds) Assume that

$$\forall n > 0,\ \mu_n \geq 0$$

and that there is some $r > 0$ such that

$$\forall n > 0,\ \sqrt{\mu_{n+1}} - \sqrt{\mu_n} \geq r \ .$$

Then there exists a family $(\theta_m)_{m>0}$ which is biorthogonal to the family $(e^{\mu_n t})_{n>0}$ in $L^2(0,T)$:

$$\forall m, n > 0,\ \int_0^T \theta_m(t)\, e^{\mu_n t} dt = \delta_{mn} \ .$$

Moreover, it satisfies: there is some universal constant c independent of T, r and m such that, for all $m > 0$, we have

$$\|\theta_m\|^2_{L^2(0,T)} \leq c e^{-2\mu_m T} e^{c\frac{1}{r}\sqrt{\mu_m}} B(T,r)$$

with

$$B(T,r) = \begin{cases} \left(\frac{1}{T} + \frac{1}{T^2 r^2}\right) e^{c\frac{1}{Tr^2}} & \text{if } T \leq \frac{1}{r^2}, \\ cr^2 & \text{if } T \geq \frac{1}{r^2} \ . \end{cases}$$

Now, define the increasing sequence of non negative real numbers $(\mu_n)_{n\geq 1}$ as follows:

$$\mu_n = \begin{cases} \sqrt{\lambda_N} - \sqrt{\lambda_{N-(n-1)}} & \text{if } 1 \leq n \leq N, \\ \sqrt{\lambda_N} + \sqrt{\lambda_{n-N}} & \text{if } N+1 \leq n \leq 2N, \\ \left(\sqrt{\mu_{n-1}} + \gamma (\lambda_N)^{-1/4}\right)^2 & \text{if } n \geq 2N+1 \ . \end{cases}$$

We need to check that such sequence fulfills the assumption of Theorem 3.1 thanks to the fact that $\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma(2-\alpha)$ given by (1.3). Indeed, for any $1 \leq n \leq N-1$,

$$\sqrt{\mu_{n+1}} - \sqrt{\mu_n} = \frac{\sqrt{\lambda_{N-(n-1)}} - \sqrt{\lambda_{N-n}}}{\sqrt{\sqrt{\lambda_N} - \sqrt{\lambda_{N-n}}} + \sqrt{\sqrt{\lambda_N} - \sqrt{\lambda_{N-(n-1)}}}} \geq \frac{\gamma(2-\alpha)}{2(\lambda_N)^{1/4}} \ ;$$

for any $N+1 \leq n \leq 2N-1$,

$$\sqrt{\mu_{n+1}} - \sqrt{\mu_n} = \frac{\sqrt{\lambda_{n+1-N}} - \sqrt{\lambda_{n-N}}}{\sqrt{\sqrt{\lambda_N} + \sqrt{\lambda_{n+1-N}}} + \sqrt{\sqrt{\lambda_N} + \sqrt{\lambda_{n-N}}}} \geq \frac{\gamma(2-\alpha)}{2\sqrt{2}(\lambda_N)^{1/4}} \ ;$$

for any $n \geq 2N$, $\sqrt{\mu_{n+1}} - \sqrt{\mu_n} = \gamma \left(\lambda_N\right)^{-1/4}$ and

$$\sqrt{\mu_{N+1}} - \sqrt{\mu_N} = \frac{2\sqrt{\lambda_1}}{\sqrt{\sqrt{\lambda_N} + \sqrt{\lambda_1}} + \sqrt{\sqrt{\lambda_N} - \sqrt{\lambda_1}}} \geq \frac{2\sqrt{\lambda_1}}{\left(1+\sqrt{2}\right)\left(\lambda_N\right)^{1/4}} \ .$$

Consequently, it fulfills by a straightforward computation the assumptions of the above Theorem 3.1: Precisely,

$$\forall n > 0,\ \mu_n \geq 0 \text{ and } \sqrt{\mu_{n+1}} - \sqrt{\mu_n} \geq r \ ,$$

with

$$r = \frac{\varsigma}{\left(\lambda_N\right)^{1/4}} \ , \text{ and } \quad \varsigma = \min\left(\frac{\gamma(2-\alpha)}{2\sqrt{2}}, \frac{2\sqrt{\lambda_1}}{1+\sqrt{2}}\right) \ . \tag{3.7}$$

By Theorem 3.1, we have a family $\left(\theta_m\right)_{m>0}$ which is biorthogonal to the family $\left(e^{\mu_n t}\right)_{n>0}$ in $L^2(0,T)$:

$$\forall m, n > 0,\ \int_0^T \theta_m\left(t\right) e^{\mu_n t} dt = \delta_{mn} \ .$$

Therefore,

$$\text{if } 1 \leq n \leq N, \text{ then } \int_0^T \theta_m\left(t\right) e^{\sqrt{\lambda_N} t} e^{-\sqrt{\lambda_{N-(n-1)}} t} dt = \delta_{mn} \ ;$$

$$\text{if } N+1 \leq n \leq 2N, \text{ then } \int_0^T \theta_m\left(t\right) e^{\sqrt{\lambda_N} t} e^{\sqrt{\lambda_{n-N}} t} dt = \delta_{mn} \ .$$

That is, for any $j = 1, \cdot\cdot, N$,

$$\int_0^T \theta_{N-(j-1)}\left(t\right) e^{\sqrt{\lambda_N} t} e^{-\sqrt{\lambda_j} t} dt = 1 \ ; \ \int_0^T \theta_m\left(t\right) e^{\sqrt{\lambda_N} t} e^{-\sqrt{\lambda_j} t} dt = 0 \text{ when } m \neq N-(j-1) \ ; \tag{3.8}$$

$$\int_0^T \theta_{N+j}\left(t\right) e^{\sqrt{\lambda_N} t} e^{\sqrt{\lambda_j} t} dt = 1 \ ; \ \int_0^T \theta_m\left(t\right) e^{\sqrt{\lambda_N} t} e^{\sqrt{\lambda_j} t} dt = 0 \text{ when } m \neq N+j \ . \tag{3.9}$$

Finally, we set for any $k = 1, \cdot\cdot, N$,

$$\sigma_k^0\left(t\right) = \theta_{N+k}\left(t\right) e^{\sqrt{\lambda_N} t} \text{ and } \sigma_k^1\left(t\right) = \theta_{N-(k-1)}\left(t\right) e^{\sqrt{\lambda_N} t}$$

in order that by (3.8), for $k, j = 1, \ldots, N$,

$$\int_0^T \sigma_k^0\left(t\right) e^{-\sqrt{\lambda_j} t} dt = 0 \text{ and } \int_0^T \sigma_k^1\left(t\right) e^{-\sqrt{\lambda_j} t} dt = \delta_{jk}$$

and by (3.9)

$$\int_0^T \sigma_k^0\left(t\right) e^{\sqrt{\lambda_j} t} dt = \delta_{jk} \text{ and } \int_0^T \sigma_k^1\left(t\right) e^{\sqrt{\lambda_j} t} dt = 0 \ .$$

Further, it holds that for any $k = 1, \cdot\cdot, N$,

$$\left\|\sigma_k^0\right\|_{L^2(0,T)}^2 \leq e^{2\sqrt{\lambda_N} T} \left\|\theta_{N+k}\right\|_{L^2(0,T)}^2 \text{ and } \left\|\sigma_k^1\right\|_{L^2(0,T)}^2 \leq e^{2\sqrt{\lambda_N} T} \left\|\theta_{N-(k-1)}\right\|_{L^2(0,T)}^2 \ . \tag{3.10}$$

This completes the construction of our control given by $g\left(x,t\right) := \sum_{j=1,\cdot\cdot,N} \left[\alpha_j \sigma_j^0\left(t\right) + \beta_j \sigma_j^1\left(t\right)\right] \Phi_j\left(x\right)$.

3.3 Cost of the control

Theorem 3.1 with (3.7) implies that there is some universal constant c independent of T and N such that for any $m = 1, \cdot\cdot, 2N$,

$$\begin{aligned} \|\theta_m\|^2_{L^2(0,T)} &\leq ce^{c\frac{1}{r}\sqrt{\mu_m}} B(T,r) := ce^{c\frac{(\lambda_N)^{1/4}}{\varsigma}\sqrt{\mu_m}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right) \\ &\leq ce^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right) \end{aligned}$$

because $\sqrt{\mu_m} \leq \sqrt{2}(\lambda_N)^{1/4}\ \forall m \in \{1, \cdot\cdot, 2N\}$. Therefore, by (3.10) we have

$$\sup_{k=1,\cdot\cdot,N} \left(\left\|\sigma_k^0\right\|^2_{L^2(0,T)} + \left\|\sigma_k^1\right\|^2_{L^2(0,T)} \right) \leq 2ce^{2\sqrt{\lambda_N}T} e^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right) . \tag{3.11}$$

Our control given by $g(x,t) := \sum_{j=1,\cdot\cdot,N} \left[\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)\right] \Phi_j(x)$ where

$$\alpha_j := \frac{\int_0^1 \left(-u_1 + \sqrt{\lambda_j} u_0\right) \Phi_j}{\int_\omega |\Phi_j|^2} = \frac{-b_j + \sqrt{\lambda_j} a_j}{\int_\omega |\Phi_j|^2} \text{ and } \beta_j := \frac{\int_0^1 \left(-u_1 - \sqrt{\lambda_j} u_0\right) \Phi_j}{\int_\omega |\Phi_j|^2} = \frac{-b_j - \sqrt{\lambda_j} a_j}{\int_\omega |\Phi_j|^2} ,$$

satisfies

$$\sum_{j=1,\cdot\cdot,N} \left(\alpha_j^2 + \beta_j^2\right) = 2 \sum_{j=1,\cdot\cdot,N} \frac{\left(\lambda_j a_j^2 + b_j^2\right)}{\left(\int_\omega |\Phi_j|^2\right)^2} \leq \frac{2(1+\lambda_N)}{\left(\inf_{j=1,\cdot\cdot,N} \int_\omega |\Phi_j|^2\right)^2} \sum_{j=1,\cdot\cdot,N} \left(a_j^2 + b_j^2\right) . \tag{3.12}$$

Combining the above estimates (3.11) and (3.12), there is some universal constant c independent of T such that for any $N \geq 1$

$$\begin{aligned} \|g\|^2_{L^2((0,1)\times(0,T))} &= \sum_{j=1,\cdot\cdot,N} \int_0^T \left|\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)\right|^2 dt \\ &\leq 2 \sum_{j=1,\cdot\cdot,N} \left(\alpha_j^2 + \beta_j^2\right) \sup_{k=1,\cdot\cdot,N} \left(\left\|\sigma_k^0\right\|^2_{L^2(0,T)} + \left\|\sigma_k^1\right\|^2_{L^2(0,T)} \right) \\ &\leq \frac{8(1+\lambda_N)}{\left(\inf_{j=1,\cdot\cdot,N} \int_\omega |\Phi_j|^2\right)^2} ce^{2\sqrt{\lambda_N}T} e^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right) \sum_{j=1,\cdot\cdot,N} \left(a_j^2 + b_j^2\right) . \end{aligned} \tag{3.13}$$

Recall that the bound

$$\|g\|^2_{L^2((0,1)\times(0,T))} := \sum_{j=1,\cdot\cdot,N} \int_0^T \left|\alpha_j \sigma_j^0(t) + \beta_j \sigma_j^1(t)\right|^2 dt \leq K \|(u_0,u_1)\|^2_{(L^2(0,1))^2} := K \sum_{j=1,\cdot\cdot,N} \left(a_j^2 + b_j^2\right)$$

will imply that the solution φ of (2.1) satisfies

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)} + \|\partial_t \varphi(\cdot,T)\|^2_{L^2(0,1)} \leq K \int_0^T \int_\omega |\varphi|^2 .$$

Now our aim is to bound the quantity

$$\frac{8(1+\lambda_N)}{\left(\inf_{j=1,\cdot\cdot,N} \int_\omega |\Phi_j|^2\right)^2} ce^{2\sqrt{\lambda_N}T} e^{\frac{c\sqrt{2}}{\varsigma}\sqrt{\lambda_N}} B\left(T, \varsigma(\lambda_N)^{-1/4}\right)$$

appearing in (3.13) in order to get the cost K.

First, by Proposition 2.1, $\frac{1}{\left(\inf_{j=1,\cdot\cdot,N}\int_\omega|\Phi_j|^2\right)^2}\leq\frac{1}{\rho^2(2-\alpha)^2}$. Next, recall that $\varsigma=\min\left(\frac{\gamma(2-\alpha)}{2\sqrt{2}},\frac{2\sqrt{\lambda_1}}{1+\sqrt{2}}\right)$ and since $\alpha\in[0,2)$ with (1.2), we have that $\underline{c}\gamma(2-\alpha)\leq\varsigma\leq\frac{1}{\underline{c}}$ where $\underline{c}$ is a positive constant independent on $\alpha\in[0,2)$. Finally, the estimate of $B(T,r)$ in Theorem 3.1

$$B(T,r)=\left\{\begin{array}{ll}\left(\frac{1}{T}+\frac{1}{T^2r^2}\right)e^{c\frac{1}{Tr^2}} & \text{if } T\leq\frac{1}{r^2}\\ cr^2 & \text{if } T\geq\frac{1}{r^2}\end{array}\right.\leq\left\{\begin{array}{ll}\left(1+\frac{1}{c}\right)\frac{1}{T}e^{2c\frac{1}{Tr^2}} & \text{if } T\leq\frac{1}{r^2}\\ cr^2 & \text{if } T\geq\frac{1}{r^2}\end{array}\right.\leq\left((1+\frac{1}{c})\frac{1}{T}+cr^2\right)e^{2c\frac{1}{Tr^2}}$$

leads to the bound

$$\begin{aligned}B(T,\varsigma(\lambda_N)^{-1/4}) &\leq \left((1+\frac{1}{c})\frac{1}{T}+c\frac{\varsigma^2}{\sqrt{\lambda_N}}\right)e^{2c\frac{\sqrt{\lambda_N}}{T\varsigma^2}}\\ &\leq C(1+\frac{1}{T})e^{C\frac{\sqrt{\lambda_N}}{T(2-\alpha)^2}}\end{aligned}$$

for some $C>0$ independent on $N>0$, $\alpha\in[0,2)$ and $T>0$. Therefore, by (3.13) one can conclude that

$$\|g\|^2_{L^2((0,1)\times(0,T))}\leq\frac{C(1+\lambda_N)}{\rho^2(2-\alpha)^2}e^{C\sqrt{\lambda_N}T}e^{C\frac{\sqrt{\lambda_N}}{\gamma(2-\alpha)}}C(1+\frac{1}{T})e^{C\frac{\sqrt{\lambda_N}}{T\gamma^2(2-\alpha)^2}}\sum_{j=1,\cdot\cdot,N}\left(a_j^2+b_j^2\right)\ ,$$

which gives, using $\frac{1}{\gamma(2-\alpha)}\leq T+\frac{1}{T\gamma^2(2-\alpha)^2}$ that

$$\|g\|^2_{L^2((0,1)\times(0,T))}\leq\frac{C(1+\lambda_N)}{\rho^2(2-\alpha)^2}\left(1+\frac{1}{T}\right)e^{C\sqrt{\lambda_N}\left(T+\frac{1}{T\gamma^2(2-\alpha)^2}\right)}\sum_{j=1,\cdot\cdot,N}\left(a_j^2+b_j^2\right)\ .$$

By the cost estimate in Lemma 3.1, we obtain that for any φ solution of (2.1) and any $N\geq1$

$$\|\varphi(\cdot,T)\|^2_{L^2(0,1)}+\|\partial_t\varphi(\cdot,T)\|^2_{L^2(0,1)}\leq\frac{C(1+\lambda_N)}{\rho^2(2-\alpha)^2}\left(1+\frac{1}{T}\right)e^{C\sqrt{\lambda_N}\left(T+\frac{1}{T\gamma^2(2-\alpha)^2}\right)}\int_0^T\int_\omega|\varphi|^2\ ,$$

where $C>0$ does not depend on (N,T,α). This completes the proof of Proposition 2.3.

4 Elliptic observation by Carleman techniques (proof of Proposition 2.2)

In this section, we shall prove Proposition 2.2. Let $0<a<b<1$ and $\Omega=(a,b)\times(0,T)$. We set $(x,t)=(x_1,x_2)\in\Omega$, and for $\alpha\in[0,2)$, introduce

$$Q=-\partial_t^2-\mathcal{P}=-\nabla\cdot(A(x_1,x_2)\nabla\cdot),\quad A(x_1,x_2)=\begin{pmatrix}x_1^\alpha & 0\\ 0 & 1\end{pmatrix},\quad \nabla=\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix}\ .$$

Note that there exists $C_0>0$ such that

$$\|A\|_{W^{3,\infty}(\Omega)}\leq C_0,\quad A(x_1,x_2)\xi\cdot\xi\geq\frac{1}{C_0}|\xi|^2,\quad\forall\xi\in\mathbb{R}^2,\forall(x_1,x_2)\in\Omega\ ,\tag{4.1}$$

where $C_0>0$ is independent on $\alpha\in[0,2)$. We set

$$v=e^{\tau\phi}\chi z$$

where $\tau>0$, $z\in H^2(\Omega)$, $\chi(x_1,x_2)=\chi_1(x_1)\chi_2(x_2)$ with

$$\left\{\begin{array}{l}\chi_1\in C_0^\infty(a,b),\quad 0\leq\chi_1\leq1,\quad \chi_1=1\text{ on }\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)\\ \chi_2\in C^\infty(0,T),\quad 0\leq\chi_2\leq1,\quad \chi_2=1\text{ on }\left(0,\frac{T}{3}\right)\text{ and }\chi_2=0\text{ on }\left(\frac{2T}{3},T\right)\end{array}\right.$$

and we shall consider weight functions $\phi \in C^\infty(\overline{\Omega})$ of the form

$$\phi(x_1, x_2) = e^{\lambda\psi(x_1,x_2)}, \quad \lambda > 0, \quad \psi \in C^\infty(\overline{\Omega}), \quad \nabla\psi \neq 0 \text{ on } \overline{\Omega}\ . \tag{4.2}$$

Here, we give explicitely ψ as follows

$$\psi(x_1, x_2) = -\left(x_1 - x_0\right)^{2k} - \beta^{2k}\left(x_2 + 1\right)^{2k} \tag{4.3}$$

where $x_0 = \frac{a+b}{2}$, $\beta = \frac{2}{3}\left(\frac{b-a}{T+4}\right)$ and k =max$(\ln 2/\ln\left(\left(4T+12\right)/\left(3T+12\right)\right); \ln 2/\ln\left(3/2\right))$.

We set

$$Q_\phi = e^{\tau\phi} Q e^{-\tau\phi}\ .$$

We have $Q_\phi v = \mathcal{S}v + \mathcal{A}v + \mathcal{R}v$ with

$$\mathcal{S}v = -\nabla\cdot\left(A\nabla v\right) - \tau^2 A\nabla\phi\cdot\nabla\phi v, \quad \mathcal{A}v = 2\tau A\nabla\phi\cdot\nabla v + 2\tau\nabla\cdot\left(A\nabla\phi\right)v, \quad \mathcal{R}v = -\tau\nabla\cdot\left(A\nabla\phi\right)v\ ,$$

which gives $\|Q_\phi v - \mathcal{R}v\|^2_{L^2(\Omega)} = \|\mathcal{S}v\|^2_{L^2(\Omega)} + \|\mathcal{A}v\|^2_{L^2(\Omega)} + 2(\mathcal{S}v, \mathcal{A}v)_{L^2(\Omega)}$. Note that $0 \leq \|Q_\phi v - \mathcal{R}v\|^2_{L^2(\Omega)} - 2(\mathcal{S}v, \mathcal{A}v)_{L^2(\Omega)}$ implies

$$(\mathcal{S}v, \mathcal{A}v)_{L^2(\Omega)} \leq \|Q_\phi v\|^2_{L^2(\Omega)} + \|\mathcal{R}v\|^2_{L^2(\Omega)}\ . \tag{4.4}$$

Now we compute $(\mathcal{S}v, \mathcal{A}v)_{L^2(\Omega)}$: By integration by parts, one has with standard summation notations and $A = (A_{ij})_{1\leq i,j\leq 2}$,

$$\begin{aligned}
(\mathcal{S}v, \mathcal{A}v)_{L^2(\Omega)} &= 2\tau\int_\Omega A\nabla^2 v A\nabla\phi\cdot\nabla v + 2\tau\int_\Omega A\nabla^2\phi A\nabla v\cdot\nabla v \\
&\quad + 2\tau\int_\Omega A_{ij}\partial_{x_i} v\partial_{x_\ell} v\partial_{x_j} A_{k\ell}\partial_{x_k}\phi \\
&\quad + 2\tau\int_\Omega \left(A\nabla v\cdot\nabla v\right)\nabla\cdot\left(A\nabla\phi\right) + 2\tau\int_\Omega A\nabla v\cdot\nabla\left(\nabla\cdot\left(A\nabla\phi\right)\right)v \\
&\quad + \tau^3\int_\Omega \left[A\nabla\phi\cdot\nabla\left(A\nabla\phi\cdot\nabla\phi\right) - \left(A\nabla\phi\cdot\nabla\phi\right)\left(\nabla\cdot\left(A\nabla\phi\right)\right)\right]|v|^2 \\
&\quad + 2\tau\int_{\partial\Omega}\left(A\nabla v\cdot n\right)\left(A\nabla\phi\cdot\nabla v + \left(\nabla\cdot\left(A\nabla\phi\right)\right)v\right) - \tau^3\int_{\partial\Omega}\left(A\nabla\phi\cdot\nabla\phi\right)\left(A\nabla\phi\cdot n\right)|v|^2\ .
\end{aligned}$$

But by one integration by parts

$$\begin{aligned}
\int_\Omega A\nabla^2 v A\nabla\phi\cdot\nabla v &= \frac{1}{2}\int_{\partial\Omega}\left(A\nabla v\cdot\nabla v\right)\left(A\nabla\phi\cdot n\right) - \frac{1}{2}\int_\Omega \partial_{x_\ell} A_{ij}\partial_{x_j} v\partial_{x_i} v A_{k\ell}\partial_{x_k}\phi \\
&\quad -\frac{1}{2}\int_\Omega \left(A\nabla v\cdot\nabla v\right)\nabla\cdot\left(A\nabla\phi\right)\ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\mathcal{S}v, \mathcal{A}v)_{L^2(\Omega)} &= 2\tau\int_\Omega A\nabla^2\phi A\nabla v\cdot\nabla v + \tau\int_\Omega \left(A\nabla v\cdot\nabla v\right)\nabla\cdot\left(A\nabla\phi\right) \\
&\quad + \tau^3\int_\Omega \left[A\nabla\phi\cdot\nabla\left(A\nabla\phi\cdot\nabla\phi\right) - \left(A\nabla\phi\cdot\nabla\phi\right)\left(\nabla\cdot\left(A\nabla\phi\right)\right)\right]|v|^2 \\
&\quad + R_1 + R_2
\end{aligned}$$

with

$$\begin{aligned}
R_1 &= 2\tau\int_\Omega A_{ij}\partial_{x_i} v\partial_{x_\ell} v\partial_{x_j} A_{k\ell}\partial_{x_k}\phi - \tau\int_\Omega \partial_{x_\ell} A_{ij}\partial_{x_j} v\partial_{x_i} v A_{k\ell}\partial_{x_k}\phi \\
&\quad + 2\tau\int_\Omega A\nabla v\cdot\nabla\left(\nabla\cdot\left(A\nabla\phi\right)\right)v\ , \\
R_2 &= -2\tau\int_{\partial\Omega}\left(A\nabla v\cdot n\right)\left(A\nabla\phi\cdot\nabla v\right) + \tau\int_{\partial\Omega}\left(A\nabla v\cdot\nabla v\right)\left(A\nabla\phi\cdot n\right) \\
&\quad - 2\tau\int_{\partial\Omega}\left(A\nabla v\cdot n\right)\left(\nabla\cdot\left(A\nabla\phi\right)\right)v - \tau^3\int_{\partial\Omega}\left(A\nabla\phi\cdot\nabla\phi\right)\left(A\nabla\phi\cdot n\right)|v|^2\ ,
\end{aligned}$$

where n is the outward normal vector to $\partial\Omega$.

Notice that from the form of A and ϕ given by (4.1) and (4.2), we have the existence of $C_1 > 0$ independent on $\alpha \in [0,2)$ such that for $\tau > 0$ sufficiently large

$$|R_1| \leq C_1 \left((\tau^{1/2}\lambda^2 + \tau\lambda) \left\| \phi^{1/2} \nabla v \right\|^2_{L^2(\Omega)} + \tau^{3/2}\lambda^4 \left\| \phi^{1/2} v \right\|^2_{L^2(\Omega)} \right) .$$

Note also that from the form of A and ϕ given by (4.1) and (4.2), we have

$$A\nabla^2\phi A\nabla v \cdot \nabla v = \lambda^2 \phi (A\nabla\psi \cdot \nabla v)^2 + \lambda\phi A\nabla^2\psi A\nabla v \cdot \nabla v \geq -C_2\lambda\phi|\nabla v|^2 ,$$

and

$$\begin{aligned}
\tau \int_\Omega (A\nabla v \cdot \nabla v)\nabla \cdot (A\nabla\phi) &= \tau \int_\Omega (A\nabla v \cdot \nabla v)\phi(\lambda\nabla \cdot (A\nabla\psi) + \lambda^2 A\nabla\psi \cdot \nabla\psi)) \\
&\geq C_2\tau\lambda^2 \left\| \phi^{1/2}\nabla v \right\|^2_{L^2(\Omega)} - C_3\tau\lambda \left\| \phi^{1/2}\nabla v \right\|^2_{L^2(\Omega)} \\
&\geq C_4\tau\lambda^2 \left\| \phi^{1/2}\nabla v \right\|^2_{L^2(\Omega)} ,
\end{aligned}$$

for $\lambda > 0$ chosen sufficiently large (independently on $\alpha \in [0,2)$, and where the constants $C_2, C_3, C_4 > 0$ are independent on $\alpha \in [0,2)$. Arguing in the same way, there exist constants $C_5 > 0$ and $\lambda_0 > 0$ such that for all $\alpha \in [0,2)$ and for all $\lambda > \lambda_0$,

$$\tau^3 \int_\Omega \left[A\nabla\phi \cdot \nabla \left(A\nabla\phi \cdot \nabla\phi \right) - \left(A\nabla\phi \cdot \nabla\phi \right) \left(\nabla \cdot (A\nabla\phi) \right) \right] |v|^2 \geq C_5\tau^3\lambda^4 \left\| \phi^{3/2} v \right\|^2_{L^2(\Omega)} .$$

Summing up, (4.4) becomes

$$\begin{aligned}
&C_5\tau^3\lambda^4 \left\| \phi^{3/2} v \right\|^2_{L^2(\Omega)} + C_4\tau\lambda^2 \left\| \phi^{1/2}\nabla v \right\|^2_{L^2(\Omega)} + R_2 \\
&\qquad \leq C_1 \left((\tau^{1/2}\lambda^2 + \tau\lambda) \left\| \phi^{1/2}\nabla v \right\|^2_{L^2(\Omega)} + \tau^{3/2}\lambda^4 \left\| \phi^{1/2} v \right\|^2_{L^2(\Omega)} \right) + \|Q_\phi v\|^2_{L^2(\Omega)} + \|\mathcal{R}v\|^2_{L^2(\Omega)} ,
\end{aligned}$$

where the constants are independent on $\alpha \in [0,2)$. Fixing $\lambda > \lambda_0$ large, and then taking $\tau > \tau_0$ sufficiently large (constants may depend on λ from now), we obtain the existence of $C_6 > 0$ such that

$$C_6\tau^3 \|v\|^2_{L^2(\Omega)} + C_6\tau \|\nabla v\|^2_{L^2(\Omega)} + R_2 \leq \|Q_\phi v\|^2_{L^2(\Omega)} + \|\mathcal{R}v\|^2_{L^2(\Omega)} .$$

Next, one can see that from the form of A and ϕ, there is $C_7 > 0$ such that for all $\alpha \in [0,2)$,

$$\|\mathcal{R}v\|^2_{L^2(\Omega)} \leq C_7\tau^2 \|v\|^2_{L^2(\Omega)} .$$

Therefore, taking $\tau > 0$ sufficiently large yields the existence of $C_8 > 0$ such that

$$C_8 \left(\tau^3 \|v\|^2_{L^2(\Omega)} + \tau \|\nabla v\|^2_{L^2(\Omega)} \right) + R_2 \leq \|Q_\phi v\|^2_{L^2(\Omega)} . \tag{4.5}$$

Now we treat the boundary term R_2: Since $v = A\nabla v \cdot n = 0$ on $\partial\Omega \setminus \Gamma$ where $\Gamma = \{(x_1, 0) ; x_1 \in (a,b)\}$, one can deduce that

$$\begin{aligned}
R_2 \quad &= \tau \int_a^b \partial_2\phi \left| \partial_2 v \left(x_1, 0 \right) \right|^2 dx_1 \\
&\quad + 2\tau \int_a^b x_1^\alpha \partial_1\phi \partial_1 v \left(x_1, 0 \right) \partial_2 v \left(x_1, 0 \right) dx_1 - \tau \int_a^b x_1^\alpha \partial_2\phi \left| \partial_1 v \left(x_1, 0 \right) \right|^2 dx_1 \\
&\quad + 2\tau \int_a^b \left(\nabla \cdot (A\nabla\phi) \right) v \left(x_1, 0 \right) \partial_2 v \left(x_1, 0 \right) dx_1 + \tau^3 \int_a^b \left(A\nabla\phi \cdot \nabla\phi \right) \partial_2\phi \left| v \left(x_1, 0 \right) \right|^2 dx_1 .
\end{aligned}$$

which gives the existence of $C_9 > 0$ independent on $\alpha \in [0,2)$ such that for any $\tau > 0$ sufficiently large

$$|R_2| \leq C_9 \left(\tau \left\| \partial_2 v\left(\cdot,0\right)\right\|^2_{L^2(a,b)} + \tau^3 \left\| v\left(\cdot,0\right)\right\|^2_{H^1(a,b)} \right) .$$

Finally, by (4.5) we have for any $\tau > \tau_0$ with $\tau_0 > 1$, the following inequality

$$C_8 \left(\tau^3 \left\| v\right\|^2_{L^2(\Omega)} + \tau \left\| \nabla v\right\|^2_{L^2(\Omega)} \right) \leq \left\| Q_\phi v\right\|^2_{L^2(\Omega)} + C_9 \left(\tau \left\| \partial_2 v\left(\cdot,0\right)\right\|^2_{L^2(a,b)} + \tau^3 \left\| v\left(\cdot,0\right)\right\|^2_{H^1(a,b)} \right) . \qquad (4.6)$$

Let $U = \left(\frac{2a+b}{3}, \frac{a+2b}{3}\right) \times \left(0, \frac{T}{4}\right)$, $W_1 = \left(\left[a, \frac{3a+b}{4}\right] \cup \left[\frac{a+3b}{4}, b\right]\right) \times \left[0, \frac{2T}{3}\right]$, $W_2 = [a,b] \times \left[\frac{T}{3}, \frac{2T}{3}\right]$ and $W = W_1 \cup W_2$. We have $\mathrm{supp}\nabla\chi = W$ and $\chi = 1$ in U.

Coming back to the function z where $v = e^{\tau\phi}\chi z$, $Q_\phi v = e^{\tau\phi} Q\left(\chi z\right) = e^{\tau\phi}\left(\chi Q z + \left[Q,\chi\right] z\right)$ where the bracket $[Q,\chi] = -\partial_t^2\chi - 2(\partial_t\chi)\partial_t - x^\alpha(\partial_x^2\chi) - 2(\partial_x\chi)x^\alpha\partial_x - \alpha(\partial_x\chi)x^{\alpha-1}$ is a differential operator of order one, supported in W, which is away from a neighborhood of the degeneracy $\{x = 0\}$. From (4.6) and taking any τ sufficiently large yields

$$\begin{aligned} \tau^3 \left\| e^{\tau\phi} z\right\|^2_{L^2(U)} + \tau \left\| e^{\tau\phi}\nabla z\right\|^2_{L^2(U)} &\leq C\left(\left\| e^{\tau\phi}\chi Q z\right\|^2_{L^2(\Omega)} + \tau \left\| e^{\tau\phi} z\right\|^2_{L^2(W)} + \left\| e^{\tau\phi}\nabla z\right\|^2_{L^2(W)} \right) \\ &\quad + C\left(\tau \left\| e^{\tau\phi(\cdot,0)}\partial_t z\left(\cdot,0\right)\right\|^2_{L^2(a,b)} + \tau^5 \left\| e^{\tau\phi(\cdot,0)} z\left(\cdot,0\right)\right\|^2_{H^1(a,b)} \right) . \end{aligned}$$

Let $D = \max\limits_{\Omega}\phi$, $D_W = \max\limits_{W}\phi$, $D_0 = \max\limits_{(a,b)}\phi\left(\cdot,0\right)$ and $D_U = \min\limits_{U}\phi$. We have for any $\tau > \tau_0$ sufficiently large

$$\begin{aligned} e^{2\tau D_U}\left(\left\| z\right\|^2_{L^2(U)} + \left\| \nabla z\right\|^2_{L^2(U)} \right) &\leq Ce^{2\tau D} \left\| Qz\right\|^2_{L^2(\Omega)} + Ce^{2\tau D_K}\left(\tau \left\| z\right\|^2_{L^2(W)} + \left\| \nabla z\right\|^2_{L^2(W)} \right) \\ &\quad + Ce^{2\tau D_0}\left(\tau \left\| \partial_t z\left(\cdot,0\right)\right\|^2_{L^2(a,b)} + \tau^5 \left\| z\left(\cdot,0\right)\right\|^2_{H^1(a,b)} \right) . \end{aligned}$$

Our choice of ψ given by (4.3) allows to get $D > D_U$ and $D_0 > D_U > D_K$. Indeed, by a straightforward computation,

$$\begin{cases} \max\limits_{W_1}\psi - \min\limits_{U}\psi &\leq -\left(\frac{b-a}{4}\right)^{2k} - \beta^{2k} + \left(\frac{b-a}{6}\right)^{2k} + \beta^{2k}\left(\frac{T}{4}+1\right)^{2k} \\ &= \beta^{2k}\left(-1 + \left(\frac{3}{8}\left(T+4\right)\right)^{2k}\left(-1 + 2\left(\frac{2}{3}\right)^{2k}\right)\right) < 0 , \\ \max\limits_{W_2}\psi - \min\limits_{U}\psi &\leq -\beta^{2k}\left(\frac{T}{3}+1\right)^{2k} + \left(\frac{b-a}{6}\right)^{2k} + \beta^{2k}\left(\frac{T}{4}+1\right)^{2k} \\ &= \beta^{2k}\left(\frac{1}{3}\left(T+4\right)\right)^{2k}\left(-1 + 2\left(\frac{3T+12}{4T+12}\right)^{2k}\right) < 0 . \end{cases}$$

Using $W \subset \overline{\Omega}$ and optimizing with respect to τ yield the desired interpolation estimate (see e.g. [R] or [LRLeR1, Lemma 5.4, page 189]). This completes the proof of

$$\left\| \varphi\right\|_{H^1\left(\left(\frac{2a+b}{3}, \frac{a+2b}{3}\right)\times\left(0,\frac{T}{4}\right)\right)} \leq c\left\| \varphi\right\|^{1-\delta}_{H^1((a.b)\times(0,T))}\left(\left\| \varphi_0\right\|_{H^1(a,b)} + \left\| \varphi_1\right\|_{L^2(a,b)} \right)^{\delta} ,$$

since $Q\varphi = 0$.

5 Observability estimate for the eigenfunctions (proof of Proposition 2.1)

In this section we aim to prove Proposition 2.1. Given $0 < a < b < 1$, we will use the notation $X \lesssim Y$, or $Y \gtrsim X$ to denote the bound $|X| \leq cY$ for some constant $c > 0$ only dependent on (a,b).

Cannarsa, Martinez and Vancostenoble proved (see [CMV4] proposition 2.15 at page 10) that

$$\forall \alpha \in [1,2) \qquad \forall j \geq 1 \qquad ||\Phi_j||^2_{L^2(a,b)} \gtrsim 2-\alpha \ .$$

In this section, we extend this result to $\alpha \in [0,2)$. To this end, we focus on the case $\alpha \in [0,1)$ and apply the following observability estimate.

Proposition 5.1 .- *For all $\sigma \in \mathbb{R}$, for all $\alpha \in [0,1)$, for all $\vartheta \in D(\mathcal{P})$*

$$\sigma^2 \left\|\vartheta\right\|^2_{L^2(0,1)} + \left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} \lesssim \left(\left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + \left(1+\sigma^2\right)\left\|\vartheta\right\|^2_{L^2(a,b)}\right) \ .$$

Since $\Phi_j \in D(\mathcal{P})$ is the normalized eigenfunctions of $\mathcal{P}$ associated with an eigenvalue λ_j, $j \in \mathbb{N}^*$. Applying Proposition 5.1 with $\vartheta = \Phi_j$ and $\sigma^2 = \lambda_j$, we obtain

$$\frac{\lambda_j}{1+\lambda_j} \lesssim \left\|\Phi_j\right\|^2_{L^2(a,b)} \ .$$

Using $\frac{\lambda_1}{1+\lambda_1} \leq \frac{\lambda_j}{1+\lambda_j}$ and (1.2), one can deduce that

$$\forall \alpha \in [0,1) \qquad \forall j \geq 1 \qquad \left\|\Phi_j\right\|^2_{L^2(a,b)} \gtrsim 1 \geq \frac{1}{2}\left(2-\alpha\right) \ .$$

This completes the proof of Proposition 2.1.

Now, we prove Proposition 5.1. Before proceeding to the proof we need two lemmas.

Lemma 5.1 .- *There exists $C>0$ such that for all $\sigma \in \mathbb{R}$, for all $\alpha \in [0,1)$,*

$$\sigma^2 \left\|\vartheta\right\|^2_{L^2(0,1)} + \left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} \leq C\left(\left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + \left|\vartheta'(1)\right|^2\right) \ ,$$

for all $\vartheta \in D(\mathcal{P})$.

Lemma 5.2 - *There exists $C>0$ such that for all $\sigma \in \mathbb{R}$, for all $\alpha \in [0,1)$,*

$$\left|\vartheta'(1)\right|^2 \leq C\left(\left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + \left\|\vartheta\right\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)}\right) \ ,$$

for all $\vartheta \in D(\mathcal{P})$.

Proof of Proposition 5.1 .- By Lemmata 5.1 and 5.2,

$$\sigma^2 \left\|\vartheta\right\|^2_{L^2(0,1)} + \left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} \lesssim C\left(\left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + \left\|\vartheta\right\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)}\right) \ .$$

Let $\chi \in C_0^\infty(0,1)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $[\frac{3a+b}{4}, \frac{a+3b}{4}]$. We have

$$\begin{aligned}
\left\|\vartheta\right\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)} = \left\|\chi\vartheta\right\|^2_{H^1\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)} &\lesssim \left\|\vartheta\right\|^2_{L^2(a,b)} + \left|\int_0^1 \chi^2 \mathcal{P}\vartheta\vartheta\right| \\
&\lesssim \left\|\vartheta\right\|^2_{L^2(a,b)} + \left|\int_0^1 \chi^2\left(\mathcal{P}-\sigma^2\right)\vartheta\vartheta\right| + \sigma^2\left\|\vartheta\right\|^2_{L^2(a,b)} \\
&\lesssim \left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + \left(1+\sigma^2\right)\left\|\vartheta\right\|^2_{L^2(a,b)}
\end{aligned}$$

by Cauchy-Schwarz. Combining the above estimates ends the proof Proposition 5.1.

Proof of Lemma 5.1 .- Let us consider $\phi(x) = x^{2-\alpha}$ and $v = e^{\phi}\vartheta$. Note that for $\alpha \in [0,1)$, $v \in D(\mathcal{P})$ because $\vartheta \in D(\mathcal{P})$. We set

$$P_\phi = e^{\phi}\mathcal{P}e^{-\phi} - \sigma^2$$

with

$$\mathcal{S} = -\frac{d}{dx}\left(x^\alpha \frac{d}{dx}\right) - (2-\alpha)^2 x^{2-\alpha} - \sigma^2, \quad \mathcal{A} = 2(2-\alpha)x\frac{d}{dx} + (2-\alpha)\ ,$$

in order that $P_\phi v = e^\phi\left(\mathcal{P}-\sigma^2\right)\vartheta$ and $P_\phi v = \mathcal{S}v + \mathcal{A}v$ which gives $\|P_\phi v\|^2_{L^2(0,1)} = \|\mathcal{S}v\|^2_{L^2(0,1)} + \|\mathcal{A}v\|^2_{L^2(0,1)} + 2(\mathcal{S}v, \mathcal{A}v)_{L^2(0,1)}$.

Classical computations lead to

$$\begin{aligned}(\mathcal{S}v, \mathcal{A}v)_{L^2(0,1)} &= (2-\alpha)^2 \left\|x^{\alpha/2}v'\right\|^2_{L^2(0,1)} + (2-\alpha)^4 \left\|x^{(2-\alpha)/2}v\right\|^2_{L^2(0,1)} - (2-\alpha)\left|v'(1)\right|^2 \\ &\quad + (2-\alpha)\lim_{x\to 0^+}\left[x^{1+\alpha}|v'(x)|^2 + x^\alpha v'(x)v(x) + (2-\alpha)^2 x^{3-\alpha}\left|v(x)\right|^2 + \sigma^2 x\left|v(x)\right|^2\right]\ .\end{aligned}$$

The above limit vanishes from the boundary conditions and the regularity of v. Therefore, the fact that $0 \le \|P_\phi v\|^2_{L^2(0,1)} - 2(\mathcal{S}v, \mathcal{A}v)_{L^2(0,1)}$ implies

$$\begin{aligned}2(2-\alpha)^2\left\|x^{\alpha/2}v'\right\|^2_{L^2(0,1)} + 2(2-\alpha)^4\left\|x^{(2-\alpha)/2}v\right\|^2_{L^2(0,1)} &\le \|P_\phi v\|^2_{L^2(0,1)} + 2(2-\alpha)\left|v'(1)\right|^2 \\ &= \left\|e^\phi\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + 2(2-\alpha)\left|\vartheta'(1)\right|^2\ .\end{aligned}$$

Since $x^{\alpha/2}\vartheta' = e^{-\phi}\left(x^{\alpha/2}v' - (2-\alpha)x^{(2-\alpha)/2}v\right)$,

$$\left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} \le 2\left\|x^{\alpha/2}v'\right\|^2_{L^2(0,1)} + 2(2-\alpha)^2\left\|x^{(2-\alpha)/2}v\right\|^2_{L^2(0,1)}\ .$$

Combining the two above inequalities, we get, for $\alpha \in [0,1)$

$$\begin{aligned}\left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} &\le \frac{1}{(2-\alpha)^2}\left(\left\|e^\phi\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + 2(2-\alpha)\left|\vartheta'(1)\right|^2\right) \\ &\lesssim \left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)} + \left|\vartheta'(1)\right|^2\ .\end{aligned}$$

It remains to bound $\sigma^2\|\vartheta\|^2_{L^2(0,1)}$. By Cauchy-Schwarz,

$$\begin{aligned}\sigma^2\|\vartheta\|^2_{L^2(0,1)} &= \int_0^1 \mathcal{P}\vartheta\vartheta - \int_0^1(\mathcal{P}-\sigma^2)\vartheta\vartheta = \left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} - \int_0^1(\mathcal{P}-\sigma^2)\vartheta\vartheta \\ &\le \left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} + \|\vartheta\|_{L^2(0,1)}\left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|_{L^2(0,1)} \\ &\le \left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} + 2\|x\vartheta'\|_{L^2(0,1)}\left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|_{L^2(0,1)} \\ &\lesssim \left\|x^{\alpha/2}\vartheta'\right\|^2_{L^2(0,1)} + \left\|\left(\mathcal{P}-\sigma^2\right)\vartheta\right\|^2_{L^2(0,1)}\end{aligned}$$

where $\|\vartheta\|^2_{L^2(0,1)} \le 4\|x\vartheta'\|^2_{L^2(0,1)}$ comes from one integration by parts. This ends the proof of Lemma 5.1.

Proof of Lemma 5.2 .- Denote $\widetilde{a} = \frac{3a+b}{4}$ and $\widetilde{b} = \frac{a+3b}{4}$ in order that $0 < a < \widetilde{a} < \widetilde{b} < b < 1$. Let us consider $\phi(x) = e^{\lambda\psi}$, with $\lambda > 0$, $\psi \in C^\infty(0,1)$, $\psi' \neq 0$ on $[\widetilde{a},1]$ and $\psi'(1) < 0$. Let $\chi \in C^\infty(0,1)$ such that $0 \le \chi \le 1$, $\chi = 0$ on $[0,\widetilde{a}]$ and $\chi = 1$ on $[\widetilde{b},1]$, and let $v = e^{\tau\phi}\chi\vartheta$ with $\tau > 0$. We set

$$P_\phi = e^{\tau\phi}\mathcal{P}e^{-\tau\phi} - \sigma^2$$

with

$$\mathcal{S} = -\frac{d}{dx}\left(x^\alpha\frac{d}{dx}\right) - \tau^2 x^\alpha\left|\phi'\right|^2 - \sigma^2, \quad \mathcal{A} = 2\tau x^\alpha\phi'\frac{d}{dx} + \tau\left(x^\alpha\phi'\right)'\ ,$$

in order that $P_\phi v = e^{\tau\phi}\left(\mathcal{P}-\sigma^2\right)(\chi\vartheta)$ and $P_\phi v = \mathcal{S}v + \mathcal{A}v$ which gives $\|P_\phi v\|^2_{L^2(0,1)} = \|\mathcal{S}v\|^2_{L^2(0,1)} + \|\mathcal{A}v\|^2_{L^2(0,1)} + 2(\mathcal{S}v, \mathcal{A}v)_{L^2(0,1)}$.

Classical computations lead to

$$(\mathcal{S}v,\mathcal{A}v)_{L^2(0,1)} = 2\tau\int_0^1 x^{2\alpha}\phi''\left|v'\right|^2 + \tau\alpha\int_0^1 x^{2\alpha-1}\phi'\left|v'\right|^2 \\ -\frac{\tau}{2}\int_0^1\left(\mathcal{P}^2\phi\right)|v|^2 + 2\tau^3\int_0^1 x^{2\alpha}\phi''\left(\phi'\right)^2|v|^2 + \alpha\tau^3\int_0^1 x^{2\alpha-1}(\phi')^3|v|^2 - \tau\phi'(1)\left|v'(1)\right|^2 \ .$$

But, using $\phi = e^{\lambda\psi}$ with ψ having a non-vanishing gradient, there exist five constants $C_0, C_1, C_2, C_3, C_4 > 0$ independent on $\alpha \in [0,1)$ such that

$$(\mathcal{S}v,\mathcal{A}v)_{L^2(0,1)} \geq \tau\lambda^2 C_0\int_0^1\phi\left|v'\right|^2 + \tau^3\lambda^4 C_1\int_0^1\phi^3|v|^2 - \tau\lambda C_2\int_0^1\phi\left|v'\right|^2 \\ -\tau\lambda^4 C_3\int_0^1\phi|v|^2 - \tau^3\lambda^3 C_4\int_0^1\phi^3|v|^2 + \tau\left|\phi'(1)\right|\left|v'(1)\right|^2 \ .$$

Therefore, the fact that $0 \leq \|P_\phi v\|^2_{L^2(0,1)} - 2(\mathcal{S}v,\mathcal{A}v)_{L^2(0,1)}$ implies by taking $\lambda > 0$ sufficiently large, and $\tau > 0$ sufficiently large the following inequality

$$\|v\|^2_{H^1(0,1)} + \left|v'(1)\right|^2 \lesssim \|P_\phi v\|^2_{L^2(0,1)} \ .$$

Taking the weights off the integrals and using commutators, we have

$$\left|\vartheta'(1)\right|^2 \lesssim \|P_\phi\vartheta\|^2_{L^2(0,1)} + \|\vartheta\|^2_{H^1(\widetilde{a},\widetilde{b})} \ .$$

This ends the proof of Lemma 5.2.

6 Observability estimate for the degenerate heat equation (proof of Theorem 1.2)

In this section, we prove that the refine observability from measurable set of Theorem 1.2 is a corollary of the spectral Lebeau-Robbiano inequality of Theorem 1.1.

Let $\widetilde{\omega} \Subset \omega$ and $\chi \in C_0^\infty(\omega)$ be such that $0 \leq \chi \leq 1$ and $\chi = 1$ in $\widetilde{\omega}$.

We start with Theorem 3.1 of [BP, page 1142] stating that (i) implies (ii) where:
(i) $\exists C_1 > 0$, $\forall\{a_j\} \in \mathbb{R}$, $\forall\Lambda > 0$

$$\sum_{\lambda_j\leq\Lambda}|a_j|^2 \leq e^{C_1\left(1+\sqrt{\Lambda}\right)}\int_{\widetilde{\omega}}\left|\sum_{\lambda_j\leq\Lambda}a_j\Phi_j\right|^2 \ ;$$

(ii) $\forall t > 0$, $\forall\varepsilon \in (0,2)$, $\forall y_0 \in L^2(0,1)$

$$\left\|e^{-t\mathcal{P}}y_0\right\|_{L^2(0,1)} \leq 4e^{\frac{C_1}{2}}e^{\frac{C_1^2}{2\varepsilon t}}\left\|e^{-t\mathcal{P}}y_0\right\|^{1-\varepsilon/2}_{L^2(\widetilde{\omega})}\|y_0\|^{\varepsilon/2}_{L^2(0,1)} \ .$$

Therefore, by Theorem 1.1 we know that (ii) holds with $C_1 = C\frac{1}{(2-\alpha)^2} > 1$.

By Nash inequality and regularizing effect, we get for some constants $c > 1$ and $\theta \in (0,1)$ independent on (y_0,t) and $\alpha \in [0,2)$

$$\left\|e^{-t\mathcal{P}}y_0\right\|_{L^2(\widetilde{\omega})} \leq c\left(1+\frac{1}{\sqrt{t}}\right)^\theta\left\|e^{-t\mathcal{P}}y_0\right\|^{1-\theta}_{L^1(\omega)}\|y_0\|^\theta_{L^2(0,1)} \ .$$

Therefore, since $1+\frac{1}{\sqrt{t}} \leq 4e^{\frac{C_1^2}{2\varepsilon t}}$, with $C_2 = 16ce^{\frac{C_1}{2}}e^{\frac{C_1^2}{\varepsilon t}} \geq 4e^{\frac{C_1}{2}}e^{\frac{C_1^2}{2\varepsilon t}}\left(c\left(1+\frac{1}{\sqrt{t}}\right)^{\theta}\right)^{1-\varepsilon/2}$ and $\mu = 1-(1-\theta)\left(1-\frac{\varepsilon}{2}\right)$, we obtain

$$\left\|e^{-t\mathcal{P}}y_0\right\|_{L^2(0,1)} \leq C_2\left\|e^{-t\mathcal{P}}y_0\right\|_{L^1(\omega)}^{1-\mu}\|y_0\|_{L^2(0,1)}^{\mu} \ ,$$

which implies by Young inequality

$$\begin{aligned}\left\|e^{-t\mathcal{P}}y_0\right\|_{L^2(0,1)} &\leq s\|y_0\|_{L^2(0,1)} + \frac{1}{s^{\frac{\mu}{1-\mu}}}C_2^{\frac{1}{1-\mu}}\left\|e^{-t\mathcal{P}}y_0\right\|_{L^1(\omega)} \\ &\leq s\|y_0\|_{L^2(0,1)} + \frac{1}{s^{\frac{\mu}{1-\mu}}}\left(16ce^{\frac{C_1}{2}}\right)^{\frac{1}{1-\mu}}e^{\frac{C_1^2}{\varepsilon t}\frac{1}{1-\mu}}\left\|e^{-t\mathcal{P}}y_0\right\|_{L^1(\omega)} \ .\end{aligned}$$

Reproducing the proof of Theorem 1.1 of [PW, page 684], we have for our system that (iii) implies (iv) where:

(iii) $\exists K_1, K_2, \ell > 0$, $\forall s > 0$

$$\left\|e^{-t\mathcal{P}}y_0\right\|_{L^2(0,1)} \leq s\|y_0\|_{L^2(0,1)} + \frac{1}{s^{\ell}}K_1e^{\frac{K_2}{t}}\left\|e^{-t\mathcal{P}}y_0\right\|_{L^1(\omega)} \ ;$$

(iv) $\forall y_0 \in L^2(0,1)$

$$\left\|e^{-T\mathcal{P}}y_0\right\|_{L^2(0,1)} \leq K_3\int_{\omega\times E}\left|e^{-t\mathcal{P}}y_0\right| \ ,$$

with

$$K_3 = c\frac{K_1}{K_2}e^{cK_2} \text{ when } E \subset (0,T) \text{ is a measurable set of positive measure;}$$

$$K_3 = \kappa\frac{K_1}{K_2}e^{\kappa\frac{K_2}{T}} \text{ when } E = (0,T) \text{ for some } \kappa > 0 \text{ independent on } T \ .$$

Therefore, with $K_1 = \left(16ce^{\frac{C_1}{2}}\right)^{\frac{1}{(1-\theta)\left(1-\frac{\varepsilon}{2}\right)}}$ and $K_2 = \frac{1}{\varepsilon(1-\theta)\left(1-\frac{\varepsilon}{2}\right)}C_1^2$ we have $K_3 \leq Ce^{C\frac{1}{(2-\alpha)^4}}$. This completes the proof of Theorem 1.2.

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