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Laure Cardoulis, Michel Cristofol

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Abstract

In this article we consider the inverse problem of simultaneously determining three coefficients of a FitzHugh-Nagumo type system defined in a bounded domain. We use a Carleman estimate to establish Hölder estimates for these coefficients by a finite number of measurements of only one component of the system.

1 Statement of the problem

1.1 Introduction

We consider a generalized FitzHugh-Nagumo system stated in $Q = \Omega \times (0,T)$, where Ω is a bounded domain of $\mathbb{R}^3$ and $\Sigma = \partial\Omega \times (0,T)$,

$$\begin{cases} \partial_t u = \Delta u - \alpha u^3 + \delta w + g \ \text{ in } \ Q, \\ \partial_t w = \beta u - \gamma w + h \ \text{ in } \ Q, \\ u(x,0) = a_1, \ w(x,0) = a_2 \ \text{in } \ \Omega, \\ u = a_3 \ \text{on } \ \Sigma. \end{cases} \tag{1.1}$$

Here, the spatial functions $\alpha(x), \beta(x), \gamma(x), \delta(x)$ are defined on Ω and we assume Dirichlet boundary condition for u. The system (1.1) under consideration can be used to describe situations from the field of pharmacology (e.g. [11], [20]), because parabolic PDE's and ODE's are usually associated in therapy modelling, e.g. chimioterapy against a tumor, also in electric wave propagation in the heart (e.g. [17], [23]).

Note that the nonlinearity u^3 in the system (1.1) is chosen to facilitate the formulation of the inverse problem and our results could be adapted easily to another form of nonlinearity similar to the original FitzHug-Nagumo one i.e. in the form $k_1 u(u-k_2)(1-u)$. A lot of work has been done in the domain of computational cardiology (e.g. [8], [9], [10], [24]) but none of it addresses the problem of stability in the reconstruction of the parameters of interest (which depends on the observations).

On the other hand very little research has addressed the problem of the stable reconstruction of parameters in such models (e.g. [1], [22], [3]) and it has already used the methodology of Carleman

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estimates initiated by Bukgheim and Klibanov [5] which allows for the establishment of stability inequalities between the reconstructed coefficients and the associated observations. This method has proven to be efficient but requires in the case of the parabolic operator the knowledge of the components on all the set Ω at one fixed time θ, see also (e.g. [12], [13], [2], [16]) for parabolic systems. Consequently, in all the papers cited previously the stability inequalities contain in their right hand sides the norm of the two components involved in the system on all the set Ω at one time θ. In this paper we improve the existing results in the area of inverse problems for systems similar to system (1.1) by simultaneously reconstructing the three coupling coefficients without needing to observe the second component. More precisely, our goal is to recover the following set of coefficients $\alpha(x), \beta(x)$ and $\gamma(x)$ by observation data of $|\partial_\nu u_i|$ on $\Gamma \times (0,T)$ where Γ is some arbitrary subdomain of $\partial\Omega$ and of u_i on Ω for one time $\theta \in (0,T)$ where the observed data for $u_i, i = A, B$, come from two suitable changes in the initial values and without needing to observe the second component w. Such result, involving the observation of only one component for the reconstruction of coefficients in systems of partial differential equations was already obtained for a strongly coupled system in quantitative thermoacoustic equations [14]. However, it is the first time to our knowledge that such a thing is achieved for a PDE/ODE coupled system. This kind of system induces a real difficulty because we have no diffusion terms for the component w and the first equation in (1.1) can be viewed as a reaction-diffusion equation with memory term and this case may lead to difficulties in the theory of control and inverse problems [3].
Furthermore, the stability estimate involving the coefficients we want to recover and the observation data immediately yields the uniqueness of these coefficients and can be used for numerical reconstruction.
This article is organized as follows: sections 1.2-1.4 lay out the main hypotheses and notations and describe our main tools. In section 2 we prove our result (Theorem 1.1).

1.2 Setting and hypotheses

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider a system involving a parabolic equation strongly linked with a differential equation as follows

$$\begin{cases} \partial_t u = \Delta u - \alpha u^3 + \delta w + g \ \text{ in } \ Q, \\ \partial_t w = \beta u - \gamma w + h \ \text{ in } \ Q, \end{cases} \tag{1.2}$$

with g and h two functions in $L^2(Q)$. We attach to (1.2) the following initial and boundary conditions:

$$u(x,0) = a_1, \ w(x,0) = a_2 \text{ in } \ \Omega,$$

and

$$u = a_3 \text{ on } \ \Sigma.$$

Note that in view to recover three coefficients using the observation of only one component of the system (1.1) we will consider two sets of initial conditions to get more information.
We will consider the admissible set of coefficients

$$\rho \equiv (\alpha, \beta, \gamma, \delta) \in (\Lambda(M_0))^3 \times (\Lambda(M_0) \cap \Lambda(C_0)) \tag{1.3}$$

with

$$\Lambda(M_0) = \{f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} < M_0\} \text{ and } \Lambda(C_0) = \{f \in L^\infty(\Omega), |f| \geq C_0 \text{ a. e. in } \Omega\}$$

for some positive constants M_0 and C_0.
The method of Carleman estimates requires solutions that are sufficiently regular. Indeed the Buckgheim-Klibanov method implies several time differentiations of system (1.1). We assume in the following that (u, w) belongs to $\mathcal{H} = (H^3(\Omega \times (0,T))^2$ satisfying the a-priori bound

$$\|(u, w)\|_{\mathcal{H}} \leq M \text{ for given } M > 0.$$

Note that $(u, w) \in (W^{3,\infty}(\Omega \times (0,T))^2$ since $\Omega \subset \mathbb{R}^3$. For a detailed proof of the regularity, we can refer to [1], [3] or [25].

Remark 1.1. *Note that we have restrained the problem to a subdomain Ω of $\mathbb{R}^3$ which is the main field for desired applications. Furthermore such choice allows us to write in a simpler way the space $\mathcal{H}$. Note that the results could be equally valid for a subdomain Ω of $\mathbb{R}^n$ without any condition of the dimension n but with more restrained regularity conditions for the solutions of* (1.1).

1.3 Weight functions and Carleman estimate

Since our strategy or reconstruction uses Carleman estimates, we need to define special weight functions. Following [26] we carry out classical regular weight functions as follows:
let $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and denote $d(x) = |x - x_0|^2$ for $x \in \Omega$ such that

$$d > 0 \text{ in } \Omega, \quad |\nabla d| > 0 \text{ in } \overline{\Omega}. \tag{1.4}$$

Moreover we define $\Gamma = \{x \in \partial\Omega, < x - x_0, \nu(x) > \geq 0\}$. Here $< ., . >$ denotes the usual scalar product in $\mathbb{R}^3$ and $\nu(x)$ is the outwards unit normal vector to $\partial\Omega$ at x. Now we consider weight functions in the form, for $\lambda > 0$, $t \in (0,T)$,

$$\psi(x,t) = d(x) - \left(t - \frac{T}{2}\right)^2 + M_1 \text{ where } M_1 > (T/2)^2,$$

$$\text{and } \phi(x,t) = e^{\lambda\psi(x,t)}.$$

It could be underlined that this choice of weight functions will play an important role in our proof and will allow us to require less observations in the reconstruction of our three coefficients. On the other hand the non regular weight functions in [19] facilitate the use of Carleman estimates in the resolution of inverse problems but require to keep the observations of all the components involved for a single time $\theta \in (0,T)$.
As underlined above, in view of the Carleman inequalities for the parabolic operators with regular weights, we need to use a cut-off function in time. This cut-off function will induce additive terms coming from the commutator between the evolution operator and this cut-off function. Note that a recent result in [18] avoids the use of such cut-off functions but restricts the domain of reconstruction of the coefficients.

Proposition 1.1. *There exist $T > 0$ and $\epsilon > 0$ such that (1.4) holds and*

$$d_1 < d_0 < d_2 \tag{1.5}$$

where

$$d_0 = \inf_{\overline{\Omega}} \phi(\cdot, \theta), \qquad d_1 = \sup_{\overline{\Omega} \times ([0,2\epsilon] \cup [T-2\epsilon, T])} \phi, \qquad d_2 = \sup_{\overline{\Omega}} \phi(\cdot, \theta), \qquad \theta = \frac{T}{2}.$$

Proof. First we define $\beta_0 = \inf_{x \in \overline{\Omega}} \psi(x, \theta) = \inf_{x \in \overline{\Omega}} |x - x_0|^2 + M_1$ and $\beta_1 > 0$ by

$$\beta_1^2 = \sup_{x \in \overline{\Omega}} |x - x_0|^2 - \inf_{x \in \overline{\Omega}} |x - x_0|^2.$$

Then, we consider T sufficiently large such that $\beta_2^2 = (T - \theta)^2 - \beta_1^2 > 0$. With these definitions, we have $(T - \theta)^2 = \beta_1^2 + \beta_2^2$ so we get for all $x \in \overline{\Omega}$,

$$\psi(x, T) = |x - x_0|^2 + M_1 - \sup_{x \in \overline{\Omega}} |x - x_0|^2 + \inf_{x \in \overline{\Omega}} |x - x_0|^2 - \beta_2^2 \leq \beta_0 - \beta_2^2.$$

As $\psi(x, 0) = \psi(x, T)$, we deduce that there exists $\epsilon > 0$ such that $\epsilon < \frac{T}{4}$ and

$$\text{for all } x \in \overline{\Omega} \text{ and } t \in ([0, 2\epsilon] \cup [T - 2\epsilon, T]), \quad \psi(x, t) < \beta_0$$

and this ends the proof of Proposition 1.1. □

These two estimates will be fruitful in Section 2 to solve our inverse problem. Now we recall a Carleman estimate for a parabolic system with Dirichlet boundary conditions as system (1.7). Let $s > 0$ and denote

$$I(u) = \int_Q \left(\frac{1}{s\phi} (|\partial_t u|^2 + |\Delta u|^2) + s\phi \, |\nabla u|^2 + s^3 \phi^3 |u|^2 \right) e^{2s\phi} \, dx \, dt.$$

Proposition 1.2 ([26, Theorem 2.1, (2)])**.** *There exist a value of $\lambda > 0$ and positive constants s_0 and C such that*

$$I(u) \leq C \| e^{s\phi} f \|^2_{L^2(Q)} + Cs \int_{\Gamma \times (0,T)} |\partial_\nu u|^2 e^{2s\phi} \, d\sigma \, dt, \tag{1.6}$$

for all $s > s_0$, and all $u \in H^1(0, T, L^2(\Omega)) \cap L^2(0, T, H^2(\Omega))$ satisfying

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega, \\ u(\cdot, 0) = u(\cdot, T) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{1.7}$$

In the following parts, C will be a generic positive constant. When needed, we will specify its dependency with respect to the different parameters. Let us remark that this Carleman inequality uses also λ as a second large parameter. We will consider λ fixed in all the rest of the article such that Proposition 1.2 holds, λ sufficiently large.
Let now η be a C^∞ cut-off function satisfying $0 \leq \eta \leq 1$ and

$$\eta(t) = \begin{cases} 0 \text{ if } t \in [0, \epsilon] \cup [T - \epsilon, T], \\ 1 \text{ if } t \in [2\epsilon, T - 2\epsilon], \end{cases} \tag{1.8}$$

with ϵ defined in Proposition 1.1.

1.4 Main result

We obtain the stable reconstruction of the three coupling coefficients (α, β, γ) using only the observations of the component u and without the knowledge of w for the system (1.1). We suppose that the sets of coefficients ρ and $\tilde{\rho}$ verify assumption (1.3) with $\rho = (\alpha, \beta, \gamma, \delta)$ and $\tilde{\rho} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \delta)$.

Theorem 1.1. *Let $V_A = (u_A, w_A)$ be a solution of (1.1) associated with the set of conditions $A = (a_1, a_2, a_3, g, h)$ and the coefficients $\rho = (\alpha, \beta, \gamma, \delta)$, $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$ be a solution of (1.1) associated with the same conditions A and the coefficients $\tilde{\rho} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \delta)$, $V_B = (u_B, w_B)$ be a solution of (1.1) associated with the conditions $B = (b_1, b_2, b_3, g, h)$ and the coefficients ρ, $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$ be a solution of (1.1) associated with the conditions B and the coefficients $\tilde{\rho}$.*
Assume that there exist positive constants $C_1 > 0$, $C_2 > 0$ such that

$$|\tilde{u}_A(\cdot, \theta)| \geq C_1 \text{ and } |\tilde{u}_A(\cdot, \theta)\tilde{w}_B(\cdot, \theta) - \tilde{u}_B(\cdot, \theta)\tilde{w}_A(\cdot, \theta)| \geq C_2 \text{ in } \Omega. \tag{1.9}$$

Then there exist constants $\kappa \in (0, 1)$ and $K > 0$ such that

$$\|\alpha - \tilde{\alpha}\|^2_{L^2(\Omega)} + \|\beta - \tilde{\beta}\|^2_{L^2(\Omega)} + \|\gamma - \tilde{\gamma}\|^2_{L^2(\Omega)} \leq K \Big(\|(u_A - \tilde{u}_A)(\cdot, \theta)\|^2_{H^2(\Omega)} + \|(u_B - \tilde{u}_B)(\cdot, \theta)\|^2_{H^2(\Omega)}$$
$$+ \sum_{k=0}^{2} \|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))\|^2_{L^2(\Gamma\times(0,T))} + \|\partial_\nu(\partial_t^k(u_B - \tilde{u}_B))\|^2_{L^2(\Gamma\times(0,T))} \Big)^{\kappa}.$$

The constants K and κ are depending on $M, T, M_0, M_1, C_0, C_1, C_2, x_0, \epsilon$.

Remark 1.2. *Conditions similar to (1.9) are classical in the process of coefficients reconstruction (see e.g. [2], [7], [12], [14], [26], [27]). It is usual to verify that they hold for numerous situations and the control theory could help us. The goal is to prove that for any coefficient sufficiently regular in an admissible set, there exists a control such that the associated solutions satisfy (1.9). Several papers have already used such a strategy and we can refer among others to [7] for parabolic inverse problem, [2] for systems of parabolic equations and [14] for parabolic-hyperbolic systems. Concerning our assumption (1.9) we refer to [4] where controllability results are obtained for a FitzHugh-Nagumo equation in the case where $\tilde{\alpha} = 1$, $\delta = -1$, $\tilde{\beta}$ is a positive constant, $\tilde{\gamma}$ is a nonnegative constant, $g \in L^2(Q)$ and $h = 0$, $u(\cdot, 0) = 0$, $u = 0$ on Σ. More precisely, we refer to Theorem 1.5 in [4], which is stated in the case where $w(\cdot, 0) = 0$ but Remark 2.3 in [4] allows the case of a nonzero initial value for w. The authors prove that for any target $u_d \in L^r(Q)$ with $r \geq 4$, they can construct a sequence of controls (g_n) and associated states (u_n) that converge to u_d strongly in $L^r(Q)$.*
Therefore we choose a target $u_d \in C(Q)$ such that $u_d > C_1$ in Q with C_1 some positive constant. Thus we have $\tilde{u}_A(x, \theta) \geq C_1$ for $\tilde{u}_A$ close to u_d in $L^r(Q)$ with $A = (0, a_2, 0, g_A, 0)$. Define $\hat{u}_d(x, t) = e^{\mu t} u_d(x, t)$ for $(x, t) \in Q$ with μ some real and consider $\tilde{u}_B$ close to $\hat{u}_d$ in $L^r(Q)$ with $B = (0, b_2, 0, g_B, 0)$. Denote

$$w_d(x, t) = a_2(x)e^{-\tilde{\gamma}t} + \int_0^t \tilde{\beta} u_d(x, s) e^{-\tilde{\gamma}(t-s)}\, ds, \ \hat{w}_d(x, t) = a_2(x)e^{-\tilde{\gamma}t} + \int_0^t \tilde{\beta} \hat{u}_d(x, s) e^{-\tilde{\gamma}(t-s)}\, ds.$$

Since $h = 0$, both second components $\tilde{w}_A$ and $\tilde{w}_B$ satisfy

$$\tilde{w}_A(x, t) = a_2(x)e^{-\tilde{\gamma}t} + \int_0^t \tilde{\beta} \tilde{u}_A(x, s) e^{-\tilde{\gamma}(t-s)}\, ds \text{ and } \tilde{w}_B(x, t) = b_2(x)e^{-\tilde{\gamma}t} + \int_0^t \tilde{\beta} \tilde{u}_B(x, s) e^{-\tilde{\gamma}(t-s)}\, ds.$$

Note that u_A (resp. w_A, u_B, w_B) is close to u_d (resp. w_d, $\hat{u}_d$, $\hat{w}_d$) for all (x,t) in Q. Then

$$u_d(x,t)\hat{w}_d(x,t)-\hat{u}_d(x,t)w_d(x,t) = u_d(x,t)e^{-\tilde{\gamma}t}(b_2(x)-e^{\mu t}a_2(x))+u_d(x,t)\int_0^t \tilde{\beta}e^{-\tilde{\gamma}(t-s)}u_d(x,s)(e^{\mu s}-e^{\mu t})\,ds.$$

We can choose a_2, b_2, μ such that $|u_d(x,t)\hat{w}_d(x,t)-\hat{u}_d(x,t)w_d(x,t)| \geq C_2$ in Q with C_2 some positive constant in order to satisfy assumption (1.9). The fact that g_A and g_B are not necessarily the same does not induce any difficulty for the inverse problem (the steps 1 to 5 of the proof of Theorem 1.1 are valid with one fixed function g for each initial condition and only the sixth step uses the two sets of conditions A and B; note that the function g has been eliminated in the crucial system (2.4) that is used in the sixth step).

Remark 1.3. *We can obtain a similar result for the three coefficients β, γ, δ by substituting the hypothesis $|\tilde{u}_A(\theta)| \geq C_1$ by the hypothesis $|\tilde{w}_A(\theta)| \geq C_1$ in Ω. This theorem could also be expanded for the reconstruction of the four coefficients $\alpha, \beta, \gamma, \delta$ by substituting the hypothesis $|\tilde{u}_A(\theta)| \geq C_1$ by $|(\tilde{u}_A(\theta))^3\tilde{w}_B(\theta)-(\tilde{u}_B(\theta))^3\tilde{w}_A(\theta)| \geq C_1$ in Ω and we still obtain a Hölder stability result for these four coefficients with the same observations of u_A and u_B described above. Note that we could also get the same results in the case of an infinite guide of the type $\mathbb{R}\times\omega$ with ω a domain of $\mathbb{R}^{n-1}$ instead of a bounded domain Ω following ideas developed in [15], [6] and [7]. Lastly, notice that due to the use of regular weight functions, we obtain a Hölder and not Lipschitz stability result.*

2 Solving the inverse problem

2.1 Preliminary lemmas

In this section, we recall and adapt some known results. These lemma will be used, associated with the Carleman estimate (1.6) applied to the systems (2.4), (2.5) and (2.6), to eliminate the observation terms coming from the component w. First we recall the following lemma based on an idea from Klibanov-Timonov ([21]).

Lemma 2.1. *There exists a positive constant C such that*

$$\int_Q e^{2s\phi}\phi\eta^2\left(\int_{T/2}^t f(\xi)\,d\xi\right)^2 dx\,dt \leq \frac{C}{s}\left(e^{2sd_1}+\int_Q e^{2s\phi}\eta^2 f^2\,dx\,dt\right),$$

for all $s>0$ and $f\in L^2(0,T,L^2(\Omega))$.

Proof. By the Cauchy-Schwarz inequality, we have

$$\int_Q \phi\eta^2 e^{2s\phi}\left(\int_{T/2}^t f(x,\xi)\,d\xi\right)^2 dx\,dt \leq \int_Q \phi\eta^2 e^{2s\phi}|t-\frac{T}{2}|\left|\int_{T/2}^t f(x,\xi)^2\,d\xi\right| dx\,dt$$

$$\leq \int_\Omega\int_0^{T/2}\phi\eta^2 e^{2s\phi}(\frac{T}{2}-t)\left|\int_{T/2}^t f(x,\xi)^2 d\xi\right| dxdt + \int_\Omega\int_{T/2}^T \phi\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^t f(x,\xi)^2 d\xi\right) dxdt. \tag{2.1}$$

Note that

$$\partial_t(e^{2s\phi}) = -4s\lambda(t-\frac{T}{2})\phi e^{2s\phi}.$$

For the second integral of the right hand side of (2.1), since $\eta(T)=0$, by integration by parts we have

$$\begin{aligned}
&\int_\Omega \int_{T/2}^T \phi\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^t f(x,\xi)^2 d\xi\right)\ dx\ dt \\
&= -\frac{1}{4s\lambda}\int_\Omega \int_{T/2}^T \eta^2\partial_t(e^{2s\phi})(\int_{T/2}^t f(x,\xi)^2 d\xi)\ dx\ dt \\
&= -\frac{1}{4s\lambda}\int_\Omega \left[\eta^2 e^{2s\phi}(\int_{T/2}^t f(x,\xi)^2 d\xi)\right]_{t=T/2}^{t=T} dx + \frac{1}{4s\lambda}\int_\Omega\int_{T/2}^T e^{2s\phi}\eta^2 f^2\ dx\ dt \\
&\quad + \frac{1}{2s\lambda}\int_\Omega\int_{T/2}^T e^{2s\phi}\eta\partial_t\eta(\int_{T/2}^t f(x,\xi)^2 d\xi)\ dx\ dt \\
&= \frac{1}{2s\lambda}\int_\Omega\int_{T/2}^T e^{2s\phi}\eta\partial_t\eta(\int_{T/2}^t f(x,\xi)^2 d\xi)\ dx\ dt + \frac{1}{4s\lambda}\int_\Omega\int_{T/2}^T e^{2s\phi}\eta^2 f^2\ dx\ dt. \qquad (2.2)
\end{aligned}$$

The first integral of (2.2) is bounded above by $\frac{C}{s}e^{2sd_1}$ due to the derivative of η. Therefore

$$\int_\Omega \int_{T/2}^T \phi\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^t f(x,\xi)^2 d\xi\right)\ dx\ dt \le \frac{C}{s}\left(e^{2sd_1} + \int_Q e^{2s\phi}\eta^2 f^2\ dx\ dt\right).$$

We obtain a similar result for the first integral of (2.1) and this concludes the proof of Lemma 2.1. □

Then we recall the following usual Lemma (see e.g. [7]) adapted from [15, Lemma 4.2].

Lemma 2.2. *There exist positive constants s_1 and C such that*

$$\int_\Omega e^{2s\phi(x,\theta)}|f(x,\theta)|^2\ dx \le Cs\int_Q e^{2s\phi}|f|^2\ dx\ dt + \frac{C}{s}\int_Q e^{2s\phi}|\partial_t f|^2\ dx\ dt$$

for all $s \ge s_1$ and $f \in H^1(0,T;L^2(\Omega))$.

Proof. Remind that η is defined by (1.8). Consider any $w \in H^1(0,T;L^2(\Omega))$. We have

$$\begin{aligned}
\int_\Omega |w(x,\theta)|^2\ dx &= \int_\Omega |\eta(\theta)w(x,\theta)|^2\ dx \\
&= \int_\Omega\int_0^\theta \partial_t(\eta^2(t)|w(x,t)|^2)\ dt\ dx \\
&= 2\int_0^\theta\int_\Omega \eta^2(t)w(x,t)\partial_t w(x,t)\ dx\ dt \\
&\quad + 2\int_0^\theta\int_\Omega \eta(t)\eta'(t)|w(x,t)|^2\ dx\ dt.
\end{aligned}$$

As $0 \leq \eta \leq 1$, using Young's inequality, it comes that for any $s > 0$,

$$\int_\Omega |w(x,\theta)|^2 \, dx \leq C(s+1) \int_Q |w|^2 \, dx \, dt + \frac{C}{s} \int_Q |\partial_t w|^2 \, dx \, dt.$$

Then we can conclude replacing w by $e^{s\phi} f$. □

2.2 Proof of Theorem 1.1

We decompose the proof in several steps.
• First step: In this step, we write the systems satisfied by the solutions of (1.1) introducing the cut-off function $\eta(t)$ defined in (1.8).
Denote $V = (u_A, w_A) = V_A$, $\tilde{V} = (\tilde{u}_A, \tilde{w}_A) = \tilde{V}_A$. For simplicity we omit the underscore A until the sixth step and we write

$$U = u - \tilde{u}, \ W = w - \tilde{w}, \ a = \alpha - \tilde{\alpha}. \ b = \beta - \tilde{\beta}, \ c = \gamma - \tilde{\gamma}.$$

We define for $i = 0, 1, 2$ (with $y_0 = y$ and $z_0 = z$)

$$y = \eta U, \ z = \eta W, \ y_i = \partial_t^i y, \ z_i = \partial_t^i z.$$

Then we get directly by difference

$$\begin{cases} \partial_t U = \Delta U - \alpha U(u^2 + u\tilde{u} + \tilde{u}^2) + \delta W - a\tilde{u}^3 \ \text{ in } \ Q, \\ \partial_t W = \beta U - \gamma W + b\tilde{u} - c\tilde{w} \ \text{ in } \ Q, \\ U = 0 \text{ on } \Sigma \end{cases} \tag{2.3}$$

and after multiplying by η we obtain

$$\begin{cases} \partial_t y = \Delta y - \alpha y H + \delta z - a\eta\tilde{u}^3 + \partial_t \eta U \ \text{ in } \ Q, \\ \partial_t z = \beta y - \gamma z + b\eta\tilde{u} - c\eta\tilde{w} + \partial_t \eta W \ \text{ in } \ Q, \end{cases} \tag{2.4}$$

with

$$H = u^2 + u\tilde{u} + \tilde{u}^2.$$

Then by taking the time derivative of (2.4)

$$\begin{cases} \partial_t y_1 = \Delta y_1 - \alpha y_1 H + \delta z_1 - a\partial_t(\eta\tilde{u}^3) + \partial_t(\partial_t \eta U) - \alpha y \partial_t H \ \text{ in } \ Q, \\ \partial_t z_1 = \beta y_1 - \gamma z_1 + b\partial_t(\eta\tilde{u}) - c\partial_t(\eta\tilde{w}) + \partial_t(\partial_t \eta W) \ \text{ in } \ Q, \end{cases} \tag{2.5}$$

$$\begin{cases} \partial_t y_2 = \Delta y_2 - \alpha y_2 H + \delta z_2 - a\partial_t^2(\eta\tilde{u}^3) + \partial_t^2(\partial_t \eta U) - \alpha y_1 \partial_t H - \alpha \partial_t(y \partial_t H) \ \text{ in } \ Q, \\ \partial_t z_2 = \beta y_2 - \gamma z_2 + b\partial_t^2(\eta\tilde{u}) - c\partial_t^2(\eta\tilde{w}) + \partial_t^2(\partial_t \eta W) \ \text{ in } \ Q, \end{cases} \tag{2.6}$$

• In the second step we estimate $I(y) + I(y_1) + I(y_2)$ thanks to the Carleman inequalities applied to the first equation in each previous system.
Using (1.6) we have for s sufficiently large,

$$I(y) + I(y_1) + I(y_2) \leq C \int_Q e^{2s\phi}(z^2 + z_1^2 + z_2^2 + a^2) \, dx \, dt + Ce^{2sd_1} + Cs \int_{\Gamma \times (0,T)} e^{2s\phi} \sum_{i=0}^{2} |\partial_\nu y_i|^2 \, d\sigma \, dt.$$

Since $z_i = \partial_t^i z$, using (2.4) and (2.5), we get for s large enough

$$I(y) + I(y_1) + I(y_2) \leq C \int_Q e^{2s\phi} z^2 \ dx \ dt + C \int_Q e^{2s\phi}(a^2 + b^2 + c^2) \ dx \ dt$$

$$+Ce^{2sd_1} + Cs \int_{\Gamma\times(0,T)} e^{2s\phi} \sum_{i=0}^{2} |\partial_\nu y_i|^2 \ d\sigma \ dt. \tag{2.7}$$

Moreover by Lemma 2.1, since $\phi \geq 1$ and $\eta(\theta) = 1$ we have

$$\int_Q e^{2s\phi} z^2 \ dx \ dt \leq \int_Q \phi e^{2s\phi} z^2 \ dx \ dt = \int_Q \phi e^{2s\phi} \eta^2 W^2 \ dx \ dt = \int_Q \phi e^{2s\phi} \eta^2 \left| \int_\theta^t \partial_t W + W(\cdot,\theta) \right|^2 \ dx \ dt$$

$$\leq \frac{C}{s} \int_Q e^{2s\phi} \eta^2 |\partial_t W|^2 \ dx \ dt + C \int_Q \phi e^{2s\phi} \eta^2 |W(\cdot,\theta)|^2 \ dx \ dt + \frac{C}{s} e^{2sd_1}$$

$$\leq \frac{C}{s} \int_Q e^{2s\phi} \eta^2 |\partial_t W|^2 \ dx \ dt + C \int_Q \phi e^{2s\phi} |z(\cdot,\theta)|^2 \ dx \ dt + \frac{C}{s} e^{2sd_1}. \tag{2.8}$$

From (2.3) we have

$$\int_Q e^{2s\phi} \eta^2 |\partial_t W|^2 \ dx \ dt \leq C \int_Q e^{2s\phi}(y^2 + z^2 + b^2 + c^2) \ dx \ dt. \tag{2.9}$$

Using now the first equation in (2.4) applied for $t = \theta$ we can write

$$\delta z(\cdot,\theta) = \partial_t y(\cdot,\theta) - \Delta y(\cdot,\theta) - \alpha y(\cdot,\theta) H(\cdot,\theta) - a\tilde{u}^3(\cdot,\theta)$$

in Ω. Then, since δ belongs to the admissible set of coefficients $\Lambda(C_0)$, multiplying by $\phi e^{2s\phi}$ and integrating on Q , we get

$$\int_Q \phi e^{2s\phi} |z(\cdot,\theta)|^2 \ dx \ dt \leq C \int_Q e^{2s\phi}(|\partial_t y(\cdot,\theta)|^2 + |\Delta y(\cdot,\theta)|^2 + |y(\cdot,\theta)|^2 + a^2) \ dx \ dt. \tag{2.10}$$

Thus for s sufficiently large, from (2.8)-(2.10)

$$\int_Q e^{2s\phi} z^2 \ dx \ dt \leq \frac{C}{s} \int_Q e^{2s\phi} y^2 \ dx \ dt + \frac{C}{s} \int_Q e^{2s\phi}(b^2 + c^2) \ dx \ dt + \frac{C}{s} e^{2sd_1}$$

$$+C \int_Q e^{2s\phi}(|\partial_t y(\cdot,\theta)|^2 + |\Delta y(\cdot,\theta)|^2 + |y(\cdot,\theta)|^2 + a^2) \ dx \ dt. \tag{2.11}$$

Moreover by Lemma 2.2,

$$\int_Q e^{2s\phi} |\partial_t y(\cdot,\theta)|^2 \ dx \ dt \leq C \int_\Omega e^{2s\phi(x,\theta)} |\partial_t y(x,\theta)|^2 \ dx \leq Cs \int_Q e^{2s\phi} y_1^2 \ dx \ dt + \frac{C}{s} \int_Q e^{2s\phi} y_2^2 \ dx \ dt$$

so

$$\int_Q e^{2s\phi} |\partial_t y(\cdot,\theta)|^2 \ dx \ dt \leq \frac{C}{s^2}(I(y_1) + I(y_2)). \tag{2.12}$$

Combinining (2.7), (2.11), (2.12) we obtain for s sufficiently large

$$I(y) + I(y_1) + I(y_2) \leq C \int_Q e^{2s\phi}(a^2 + b^2 + c^2)\ dx\ dt + Ce^{2sd_1} + Cse^{2sd_2}F_0(\theta) \tag{2.13}$$

with $F_0(\theta) = \|y(\theta)\|^2_{H^2(\Omega)} + \int_{\Gamma\times(0,T)} \sum_{i=0}^2 |\partial_\nu y_i|^2\ d\sigma\ dt$.
• Third step: Here we summarize the inequalities we have obtained in the second step and that will be used in the next steps. From (2.12) and (2.13) we obtain

$$\int_\Omega e^{2s\phi(x,\theta)}|\partial_t y(x,\theta)|^2\ dx \leq \frac{C}{s^2}\int_Q e^{2s\phi}(a^2 + b^2 + c^2)\ dx\ dt + \frac{C}{s^2}e^{2sd_1} + \frac{C}{s}e^{2sd_2}F_0(\theta). \tag{2.14}$$

Then from (2.11) and (2.13) we get

$$\int_Q e^{2s\phi}z^2\ dx\ dt \leq C\int_Q e^{2s\phi}(a^2 + b^2 + c^2)\ dx\ dt + \frac{C}{s}e^{2sd_1} + Ce^{2sd_2}F_0(\theta). \tag{2.15}$$

Moreover by Lemma 2.2

$$\int_\Omega e^{2s\phi(x,\theta)}|z(x,\theta)|^2\ dx \leq Cs\int_Q e^{2s\phi}z^2\ dx\ dt + \frac{C}{s}\int_Q e^{2s\phi}|\partial_t z|^2\ dx\ dt.$$

Using the second equation in (2.4) we obtain

$$\int_\Omega e^{2s\phi(x,\theta)}|z(x,\theta)|^2\ dx \leq Cs\int_Q e^{2s\phi}z^2\ dx\ dt + \frac{C}{s}\int_Q e^{2s\phi}(y^2 + z^2 + b^2 + c^2)\ dx\ dt + \frac{C}{s}e^{2sd_1}$$

so by (2.13) and (2.15) we deduce that for large s

$$\int_\Omega e^{2s\phi(x,\theta)}|z(x,\theta)|^2\ dx \leq Cs\int_Q e^{2s\phi}(a^2 + b^2 + c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_0(\theta). \tag{2.16}$$

• In the fourth step we estimate $\int_\Omega e^{2s\phi(x,\theta)}|\partial_t z(x,\theta)|^2\ dx$.
By Lemma 2.2 we have

$$\int_\Omega e^{2s\phi(x,\theta)}|\partial_t z(x,\theta)|^2\ dx \leq Cs\int_Q e^{2s\phi}|\partial_t z|^2\ dx\ dt + \frac{C}{s}\int_Q e^{2s\phi}|\partial_t^2 z|^2$$

$$\leq Cs\int_Q e^{2s\phi}(y^2 + z^2 + y_1^2 + z_1^2 + b^2 + c^2)\ dx\ dt + Cse^{2sd_1}.$$

Since $z_1 = \partial_t z$, from (2.4) we deduce

$$\int_\Omega e^{2s\phi(x,\theta)}|\partial_t z(x,\theta)|^2\ dx \leq Cs\int_Q e^{2s\phi}(y^2 + z^2 + y_1^2 + b^2 + c^2)\ dx\ dt + Cse^{2sd_1}.$$

From (2.15) we get

$$\int_\Omega e^{2s\phi(x,\theta)}|\partial_t z(x,\theta)|^2\ dx \leq \frac{C}{s^2}(I(y)+I(y_1)) + Cs\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_0(\theta).$$

Finally from (2.13) obtained in the second step we get

$$\int_\Omega e^{2s\phi(x,\theta)}|\partial_t z(x,\theta)|^2\ dx \le Cs\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_0(\theta). \tag{2.17}$$

• Fifth step: Now we estimate $\int_\Omega e^{2s\phi(\cdot,\theta)}a^2\ dx$.
Multiplying the first equation in (2.4) by $e^{2s\phi}$, evaluating at $t=\theta$ and integrating over Ω we get

$$\int_\Omega e^{2s\phi(\cdot,\theta)}a^2(\tilde{u}(\cdot,\theta))^6\ dx \le Ce^{2sd_2}\|y(\cdot,\theta)\|^2_{H^2(\Omega)} + C\int_\Omega e^{2s\phi(\cdot,\theta)}(|\partial_t y(\cdot,\theta)|^2+|z(\cdot,\theta)|^2)\ dx.$$

Using (2.14) and (2.16), we get

$$\int_\Omega e^{2s\phi(\cdot,\theta)}a^2(\tilde{u}(\cdot,\theta))^6\ dx \le Cs\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_0(\theta).$$

Using now the first hypothesis in (1.9) we get

$$\int_\Omega e^{2s\phi(\cdot,\theta)}a^2\ dx \le Cs\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_0(\theta). \tag{2.18}$$

• Sixth step: Now we estimate $\int_\Omega e^{2s\phi(\cdot,\theta)}(b^2+c^2)\ dx$.
We use the two sets of initial conditions and consider V_A, $\tilde{V}_A$, V_B and $\tilde{V}_B$. Using (2.4) for z_A and z_B we can suppress c and obtain

$$b\eta(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A) = \tilde{w}_B\partial_t z_A-\tilde{w}_A\partial_t z_B-(\beta y_A-\gamma z_A+\partial_t\eta W_A)\tilde{w}_B+(\beta y_B-\gamma z_B+\partial_t\eta W_B)\tilde{w}_A. \tag{2.19}$$

Evaluating (2.19) at $t=\theta$ and using now the second hypothesis in (1.9), we get

$$\int_\Omega e^{2s\phi(\cdot,\theta)}b^2\ dx \le C\int_\Omega e^{2s\phi(x,\theta)}\left(|\partial_t z_A(x,\theta)|^2+|\partial_t z_B(x,\theta)|^2+|y_A(x,\theta)|^2\right.$$

$$\left.+|z_A(x,\theta)|^2+|y_B(x,\theta)|^2+|z_B(x,\theta)|^2\right)\ dx.$$

So using (2.16) and (2.17) obtained in the fourth step for either $z_A(\cdot,\theta)$ or $z_B(\cdot,\theta)$, we obtain

$$\int_\Omega e^{2s\phi(\cdot,\theta)}b^2\ dx \le Cs\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_1(\theta)$$

with $F_1(\theta) = \|y_A(\cdot,\theta)\|^2_{H^2(\Omega)} + \|y_B(\cdot,\theta)\|^2_{H^2(\Omega)} + \int_{\Gamma\times(0,T)}\sum_{i=0}^2(|\partial_\nu\partial_t^i y_A|^2+|\partial_\nu\partial_t^i y_B|^2)\ d\sigma\ dt$.
Similarly the above estimate holds for c instead of b on the left-hand side so

$$\int_\Omega e^{2s\phi(\cdot,\theta)}(b^2+c^2)\ dx \le Cs\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_1(\theta). \tag{2.20}$$

• Seventh and last step: From (2.18) and (2.20) we get

$$\int_\Omega e^{2s\phi(\cdot,\theta)}(a^2+b^2+c^2)\ dx \le Cs\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt + Cse^{2sd_1} + Cse^{2sd_2}F_1(\theta). \tag{2.21}$$

Now, as for example in [15, 18], we prove that $s\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt$ in the right-hand side of (2.21) can be absorbed by $\int_\Omega e^{2s\phi(\cdot,\theta)}(a^2+b^2+c^2)\ dx$. Indeed,

$$s\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt=\int_\Omega (a^2+b^2+c^2)e^{2s\phi(\cdot,\theta)}\left(\int_0^T se^{2s(\phi-\phi(\cdot,\theta))}\ dt\right)\ dx.$$

But $\phi-\phi(\cdot,\theta)=-e^{\lambda(d+M_1)}(1-e^{-\lambda(t-\theta)^2})$ and there exists a positive constant C such that $\phi-\phi(\cdot,\theta)\leq -C(1-e^{-\lambda(t-\theta)^2})$. Therefore $\int_0^T se^{2s(\phi-\phi(\cdot,\theta))}\ dt\leq\int_0^T se^{-2sC(1-e^{-\lambda(t-\theta)^2})}\ dt$. By the Lebesgue convergence theorem, we deduce that $\int_0^T se^{-2sC(1-e^{-\lambda(t-\theta)^2})}\ dt\to 0$ as $s\to\infty$ and so we get that for s sufficiently large we can absorb $s\int_Q e^{2s\phi}(a^2+b^2+c^2)\ dx\ dt$ on the right-hand side of (2.21) by the left-hand side. Thus for s sufficiently large we can write

$$\int_\Omega e^{2s\phi(\cdot,\theta)}(a^2+b^2+c^2)\ dx\leq Cse^{2sd_1}+Cse^{2sd_2}F_1(\theta).$$

Now from Proposition 1.1 we have

$$e^{2sd_0}\int_\Omega (a^2+b^2+c^2)\ dx\leq\int_\Omega e^{2s\phi(\cdot,\theta)}(a^2+b^2+c^2)\ dx\leq Cse^{2sd_1}+Cse^{2sd_2}F_1(\theta),$$

and finally

$$\int_\Omega (a^2+b^2+c^2)\ dx\leq Cse^{2s(d_1-d_0)}+Cse^{2s(d_2-d_0)}F_1(\theta),\tag{2.22}$$

Recall from (1.5) that $d_1-d_0<0$ and that $d_2-d_0>0$. Then we can minimize the right-hand side of (2.22) by a good choice of the parameter s and we obtain the main stability inequality of Theorem 1.1.

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