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# FIXED-POINT STATISTICS FROM SPECTRAL MEASURES ON TENSOR ENVELOPE CATEGORIES

ARTHUR FOREY, JAVIER FRESÁN, AND EMMANUEL KOWALSKI

ABSTRACT. We prove some old and new convergence statements for fixed-points statistics and characters of the symmetric groups using tensor envelope categories, such as the Deligne–Knop category of representations of the “symmetric group”  $S_t$  for an indeterminate  $t$ . We also discuss some arithmetic speculations related to Chebotarev’s density theorem.

## 1. SPECTRAL MEASURES IN MONOIDAL CATEGORIES

Spectral measures associated to operators on Hilbert spaces are key tools in functional analysis and its applications, for instance to quantum mechanics and ergodic theory. Recall that a continuous normal linear operator  $u: E \rightarrow E$  has a compact spectrum and that, for each vector  $x \in E$ , there exists a unique bounded positive Radon measure  $\mu_x$  on  $\mathbf{C}$ , supported on the spectrum of  $u$ , such that the equality

$$\int_{\mathbf{C}} f d\mu_x = \langle x | f(u)x \rangle$$

holds for all continuous functions  $f: \mathbf{C} \rightarrow \mathbf{C}$ . This measure is called the *spectral measure of  $u$  relative to  $x$* . In particular, we then have

$$\int_{\mathbf{C}} z^a \bar{z}^b d\mu_x(z) = \langle x | u^a (u^*)^b(x) \rangle$$

for all non-negative integers  $a$  and  $b$ . In this paper, we consider an analogue of this last relation for objects in symmetric monoidal categories, in the sense of [9, Def. 2.1.1, Def. 8.1.12].

**Definition 1.1** (Spectral measure). Let  $\mathcal{C}$  be a symmetric monoidal category with an endofunctor  $D$ , and let  $i$  be a complex-valued invariant of  $\mathcal{C}$ , by which we mean a map from isomorphism classes of objects of  $\mathcal{C}$  to  $\mathbf{C}$ . Let  $M$  be an object of  $\mathcal{C}$ . A positive measure  $\mu$  on  $\mathbf{C}$  is called a *spectral measure* of  $M$  relative to  $i$  if the equality

$$\int_{\mathbf{C}} z^a \bar{z}^b d\mu(z) = i(M^{\otimes a} \otimes D(M)^{\otimes b})$$

holds for all non-negative integers  $a$  and  $b$ .

We will think of  $D$  as a duality functor on  $\mathcal{C}$ , although no extra condition is required for this general definition. The basic motivation is provided by the following example.

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**Example 1.2.** Let  $r \geq 1$  be an integer, and let  $G \subset \mathbf{GL}_r(\mathbf{C})$  be a compact group with probability Haar measure  $\nu$ . Let  $\mathcal{C}$  be the category of finite-dimensional continuous complex representations of  $G$ , and  $D$  the contragredient endofunctor of  $\mathcal{C}$ . By representation theory of compact groups, the direct image of  $\nu$  by the trace  $\mu = \mathrm{Tr}_*(\nu)$  is a spectral measure of the “tautological” object of  $\mathcal{C}$  corresponding to the inclusion of  $G$  in  $\mathbf{GL}_r(\mathbf{C})$ , relative to the invariant  $i(\rho) = \dim(\rho^G)$  for a representation  $\rho$ . In number theory, measures of this type are often called “Sato–Tate measures”, the original example being that of  $\mathbf{SU}_2(\mathbf{C})$ .

Our first main result is a new proof of a statement which goes back to the very early studies of probability theory through the analysis of card games and the like (see the historical paper of Takács [20] for references). Interestingly, the tensor categories that will arise in the proof are the categories of representations of the “symmetric group”  $S_t$  of Deligne [7], in the version given by Knop [15], which is more amenable to generalizations<sup>1</sup>.

**Theorem 1.3** (“Problème des rencontres”; Montmort [5]; N. Bernoulli I; de Moivre [4]). *Let  $(X_n)_{n \geq 1}$  be a sequence of random variables with  $X_n$  a uniformly chosen random permutation in the symmetric group  $S_n$ . The sequence  $(|\mathrm{Fix}(X_n)|)_{n \geq 1}$ , where  $\mathrm{Fix}(\sigma)$  denotes the set of fixed points of  $\sigma \in S_n$ , converges in law to a Poisson distribution with parameter 1.*

Recall that the Poisson distribution with parameter  $\lambda$  is the measure  $\varpi_\lambda$  supported on non-negative integers with

$$\varpi_\lambda(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!}$$

for all integers  $k \geq 0$ . The meaning of the statement (and how it was originally proved) is therefore that the formula

$$\lim_{n \rightarrow +\infty} \frac{1}{n!} |\{\sigma \in S_n \text{ with } |\mathrm{Fix}(\sigma)| = k\}| = \frac{1}{e} \frac{1}{k!}$$

holds for all integers  $k \geq 0$ . Neither categories nor spectral measures appear in the statement; the link is that the limit Poisson distribution will arise as the spectral measure of a suitable object in the Deligne–Knop category  $\mathrm{Rep}(S_t)$  for an indeterminate  $t$ . From that point of view, an interesting feature is that we show how the Poisson distribution (maybe the most natural measure on non-negative integers) is an analogue of a Sato–Tate measure. In Remark 3.8, we will see a similar statement for the complex gaussian distribution, and in Section 5, we will discuss some arithmetic speculations related to Chebotarev’s Density Theorem.

Our second main result is the following new result.

**Theorem 1.4.** *Let  $m \geq 1$  be an integer and let  $\lambda$  be a partition of  $m$  with parts*

$$\lambda_1 \geq \lambda_2 \geq \dots$$

*For  $n \geq m + \lambda_1$ , let  $\pi_{\lambda,n}$  be the representation of  $S_n$  corresponding to the partition*

$$\lambda^{(n)} = (n - m, \lambda_1, \lambda_2, \dots),$$

*and let  $\chi_{\lambda,n}: S_n \rightarrow \mathbf{C}$  be its character.*

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<sup>1</sup> Another, rather different, construction of this category has recently been given by Harman and Snowden [12, §15]; we have not investigated in depth how this construction interacts with our results.

The sequence of measures  $(\chi_{\lambda,n}(X_n))$ , where as before  $X_n$  is uniformly distributed on  $S_n$ , converges in law as  $n \rightarrow +\infty$ , and its limit is a spectral measure of the simple object  $x_{\lambda,t}$  of  $\mathcal{C}_t = \text{Rep}(S_t)$  associated to  $\lambda$ , relative to the invariant  $i$  defined by

$$i(M) = \dim_{\mathbf{C}(t)} \text{Hom}(1_{\mathcal{C}_t}, M).$$

This second result has a corollary which relates it to the theory of FI-modules of Church, Ellenberg and Farb [?cef].

**Corollary 1.5.** *Let  $M = (M_n)_{n \geq 1}$  be a finitely-generated FI-module over  $\mathbf{C}$ . For  $n \geq 1$ , let  $\xi_n$  be the character of  $M_n$  as an  $S_n$ -representation. The sequence  $(\xi_n(X_n))$  converges in law. and its limit is a combination of spectral measures for  $\mathcal{C}_t$  relative to the invariant  $i$ .*

*Proof.* A fundamental result of the theory of FI-modules [?cef, Prop.3.3.3] implies that there exists a polynomial  $Q \in \mathbf{C}[(T_\lambda)_\lambda]$  (in indeterminates parameterized by all partitions of all integers  $m \geq 1$ ) such that

$$\xi_n = Q((\chi_{\lambda,n})_\lambda),$$

for all  $n$  large enough, and hence the corollary follows immediately from Theorem 1.4.  $\square$

**Remark 1.6.** We refer to the introduction of [?cef] for a list of examples of (finitely-generated) FI-modules arising in algebra, geometry and topology.

Finally, we note that this paper is excerpted from a longer work in progress [10], whose status and evolution are however unpredictable. We hope that the simple example of an application of spectral measures to classical problems will motivate the reader's interest in this notion.

## 2. EXISTENCE AND UNIQUENESS OF SPECTRAL MEASURES

From now on, we only consider  $k$ -linear tensor categories, for some field  $k$ , in the sense of [6, 1.2]. We always use the duality functor of such a category as the endofunctor  $D$  in Definition 1.1. An object  $M$  is called *self-dual* if there exists an isomorphism  $M \simeq D(M)$ .

**Proposition 2.1** (Spectral measures for self-dual objects). *Let  $\mathcal{C}$  be a tensor category in which every object is self-dual, and let  $i$  be an  $\mathbf{R}$ -valued additive invariant of  $\mathcal{C}$ . If*

$$(2.1) \quad 2i(M \otimes N) \leq i(M \otimes M) + i(N \otimes N)$$

*holds for all objects  $M$  and  $N$  of  $\mathcal{C}$ , then every object of  $\mathcal{C}$  admits a spectral measure relative to  $i$  which is supported on  $\mathbf{R}$ .*

*Proof.* By the solution of the Hamburger moment problem (see, e.g., [19, Th.3.8]), a sequence  $(\mu_a)_{a \geq 0}$  of real numbers is the sequence of moments of a positive Borel measure  $\mu$  on  $\mathbf{R}$  if and only if the inequality

$$(2.2) \quad \sum_{1 \leq a, b \leq A} \alpha_a \alpha_b \mu_{a+b} \geq 0$$

holds for all integers  $A \geq 1$  and all real numbers  $\alpha_a$ . Therefore,  $M$  admits a spectral measure relative to  $i$  if and only if the values  $\mu_a = i(M^{\otimes a})$  satisfy this condition.

We first consider the case where  $\alpha_a$  are integers. Setting

$$P = \bigoplus_{\alpha_a \geq 0} \alpha_a M^{\otimes a} \quad \text{and} \quad N = \bigoplus_{\alpha_a \leq -1} (-\alpha_a) M^{\otimes a},$$

we then get

$$\sum_{1 \leq a, b \leq A} \alpha_a \alpha_b \mu_{a+b} = i(P \otimes P) + i(N \otimes N) - 2i(P \otimes N),$$

and hence the assumption (2.1) implies the inequality

$$\sum_{1 \leq a, b \leq A} \alpha_a \alpha_b \mu_{a+b} \geq 0.$$

This extends to  $\mathbf{Q}$  by homogeneity, and to  $\mathbf{R}$  by continuity.  $\square$

If not all objects are self-dual (as often happens), the situation is more subtle, because the moment problem on  $\mathbf{C}$  is more challenging than on  $\mathbf{R}$ : the analogue of the positivity condition above is not sufficient to ensure the existence of a positive measure on  $\mathbf{C}$  with given moments. However, under an extra growth condition, one obtains both existence and uniqueness of the spectral measure.

**Proposition 2.2** (Spectral measures for general objects). *Let  $\mathcal{C}$  be a tensor category. Let  $i$  be a  $\mathbf{C}$ -valued additive invariant of  $\mathcal{C}$ . Suppose that the inequality*

$$(2.3) \quad i(M \otimes D(N)) + i(D(M) \otimes N) \leq i(M \otimes D(M)) + i(N \otimes D(N))$$

*holds for all objects  $M$  and  $N$  of  $\mathcal{C}$ . Let  $M$  be an object of  $\mathcal{C}$  satisfying the Carleman condition*

$$(2.4) \quad \sum_{a \geq 1} i((M \otimes D(M))^{\otimes a})^{-1/(2a)} = +\infty.$$

*Then there exists a unique spectral measure for  $M$  relative to  $i$ .*

*Proof.* This follows as above (*mutatis mutandis* using complexification) using the fact (due to Nussbaum) that the Carleman condition (2.4) combined with the analogue of (2.2), is a sufficient condition for the existence *and* uniqueness of a measure on  $\mathbf{C}$  with given moments; see for instance [19, Th. 15.11].  $\square$

**Remark 2.3.** The Carleman condition holds in particular if there exist  $c \geq 0$  and  $r \geq 0$  such that the inequality  $i((M \otimes D(M))^{\otimes n}) \leq cr^n$  holds for all non-negative integers  $n$ . This is a frequent occurrence, but it corresponds to measures with compact support (compare with Deligne’s “subexponential growth theorem”, see [9, Th. 9.11.4]).

**Definition 2.4** (Positive and complete invariants). An additive invariant  $i$  on  $\mathcal{C}$  is called a *positive invariant* if it satisfies (2.3) for all objects  $M$  and  $N$  of  $\mathcal{C}$ . If, in addition, all objects of  $\mathcal{C}$  admit a spectral measure, then we say that  $i$  is a *complete positive invariant*.

The following result gives a usable criterion to check that certain invariants are positive.

**Proposition 2.5.** *Let  $\mathcal{C}$  be an essentially small tensor category. Let  $\widehat{\mathcal{C}}$  be a set of objects of  $\mathcal{C}$  such that every object of  $\mathcal{C}$  is isomorphic to a finite direct sum of objects from  $\widehat{\mathcal{C}}$ . Let  $i$  be an additive invariant of  $\mathcal{C}$ . Then  $i$  is positive if the bilinear form*

$$b(n, m) = \sum_{V, W \in \widehat{\mathcal{C}}} n_V m_W i(V \otimes D(W))$$

on  $\mathbf{Z}^{\widehat{\mathcal{C}}}$  is positive, i.e.,  $b(n, n) \geq 0$  for all functions  $n: \widehat{\mathcal{C}} \rightarrow \mathbf{Z}$  with finite support.

*Proof.* Let  $M$  and  $N$  be objects of  $\mathcal{C}$ , and represent them as direct sums

$$M = \bigoplus_{V \in \widehat{\mathcal{C}}} m_V V, \quad N = \bigoplus_{W \in \widehat{\mathcal{C}}} n_W W$$

with only finitely many non-zero integers  $m_V, n_W$ . By additivity, we obtain the formulas

$$\begin{aligned} i(M \otimes D(N)) + i(D(M) \otimes N) &= \sum_{V, W} m_V n_W i(V \otimes D(W)) + \sum_{V, W} m_V n_W i(D(V) \otimes W), \\ i(M \otimes D(M)) + i(N \otimes D(N)) &= \sum_{V, W} m_V m_W i(V \otimes D(W)) + \sum_{V, W} n_V n_W i(V \otimes D(W)) \end{aligned}$$

so that we get

$$\left( i(M \otimes D(M)) + i(N \otimes D(N)) \right) - \left( i(M \otimes D(N)) + i(D(M) \otimes N) \right) = b(m - n, m - n),$$

and the result then follows from Proposition 2.2.  $\square$

As a special case, we deduce:

**Corollary 2.6.** *Let  $k$  be a field. Let  $\mathcal{C}$  be any essentially small  $k$ -linear semisimple tensor category with unit object  $1_{\mathcal{C}}$ . The formula*

$$i(M) = \dim_k \text{Hom}(1_{\mathcal{C}}, M)$$

*defines a positive invariant on  $\mathcal{C}$ .*

*Proof.* We apply Proposition 2.5 to the set  $\widehat{\mathcal{C}}$  of isomorphism classes of simple objects of  $\mathcal{C}$ . Then  $i(V \otimes D(W)) = 0$  for  $V$  and  $W$  in  $\widehat{\mathcal{C}}$ , unless  $V$  is equal to  $W$ , so that the bilinear form  $b$  in the statement is diagonal in the canonical basis of  $\mathbf{Z}^{\widehat{\mathcal{C}}}$ , with diagonal coefficients equal to  $i(V \otimes D(V)) = \dim_k \text{Hom}(1_{\mathcal{C}}, V \otimes D(V)) \geq 0$ .  $\square$

**Remark 2.7.** (1) We emphasize that it is essential to impose the positivity of the spectral measure in Definition 1.1: it is known (due to Boas and Pólya, independently; see, e.g., [2]) that *any* sequence of complex numbers is the sequence of moments of infinitely many complex measures on  $\mathbf{R}$ .

(2) In general, spectral measures are not uniquely determined given the object of interest, and only their moments are unambiguously known (see the conclusion of Remark 3.8 for a simple example where the spectral measure is not unique). One can hope that (at least for certain categories or invariants) that some additional condition or property might “fix” the spectral measure.

### 3. TENSOR ENVELOPES AND FIXED-POINT STATISTICS

Let  $t \in \mathbf{C}$  be a complex number. Deligne [7, Th. 2.18] defined a rigid semisimple  $\mathbf{C}$ -linear monoidal category  $\text{Rep}(S_t)$ , which “interpolates” the categories of finite-dimensional representations of the symmetric groups  $S_n$  when  $n \geq 0$  is an integer. Knop [15] discovered an alternative approach to constructing new rigid symmetric monoidal categories which is *a priori* independent of ideas of interpolating other categories; this leads to many more examples, and happens to recover in a special case the categories of Deligne.

The input data in Knop’s construction is a base category  $\mathcal{A}$  satisfying some regularity conditions, a ring  $k$  and a degree function  $\delta$  which associates to every surjective morphism  $e$  in  $\mathcal{A}$  an element  $\delta(e)$  of  $k$ , again subject to some conditions. The resulting category is denoted  $\mathcal{T}(\mathcal{A}, \delta)$  by Knop, and is called the *tensor envelope* of  $\mathcal{A}$  with respect to  $\delta$ . For the moment, it is sufficient for us to recall that every object  $x$  of  $\mathcal{A}$  defines an object  $[x]$  of  $\mathcal{T}(\mathcal{A}, \delta)$ , which is always self-dual, and that the  $k$ -linear space of morphisms from  $[x]$  to  $[y]$  admits as a basis the set of all *relations* from  $x$  to  $y$ , i.e., the set of all subobjects of the product  $x \times y$ . In Appendix A, to give some context, we spell out the construction of  $\mathcal{T}(\mathcal{A}, \delta)$  in the special case relevant to Theorem 1.3, namely when  $\mathcal{A}$  is the *opposite* of the category of finite sets.

*Outline of proof of Theorem 1.3.* Thanks to the method of moments, it suffices to prove the convergence of the sequence  $(\mathbf{E}(|\text{Fix}(X_n)|^k))_{n \geq 1}$  to the corresponding Poissonian moment for all integers  $k \geq 0$  (see, e.g., [1, Th. 30.2, Th. 30.1]).

The first observation is that  $|\text{Fix}(X_n)| = \chi_n(X_n)$ , where  $\chi_n$  is the character of the “standard” permutation representation  $\text{Std}_n$  of  $S_n$  acting on  $\mathbf{C}^n$ . By basic representation theory of finite groups, we then get the expression

$$\mathbf{E}(|\text{Fix}(X_n)|^k) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_n(\sigma)^k = \dim_{\mathbf{C}} \text{Hom}_{S_n}(1_n, \text{Std}_n^{\otimes k}),$$

where  $1_n$  is the trivial one-dimensional representation of  $S_n$ .

We will then appeal to the Deligne–Knop category  $\mathcal{C}_t = \text{Rep}(S_t)$  for  $t$  the “indeterminate” in the field  $\mathbf{C}(t)$ . It is known that  $\mathcal{C}_t$  is a semisimple  $\mathbf{C}(t)$ -linear tensor category. Moreover, if we “specialize” the indeterminate  $t$  to an integer  $n \in \mathbf{N}$ , then we recover (in a way made precise below) the category of finite-dimensional complex representations of  $S_n$ .

Let  $1_t$  (resp.  $\text{Std}_t$ ) denote the unit object of  $\text{Rep}(S_t)$  (resp. the natural object with “dimension”  $t$ ). By Corollary 2.6, the assignment  $i(M) = \dim_{\mathbf{C}(t)} \text{Hom}_{\mathcal{C}_t}(1_t, M)$  defines a positive invariant on  $\mathcal{C}_t$ .

The object  $\text{Std}_t$  is self-dual. For  $k$  and  $n \in \mathbf{N}$ , we will show in Lemma 3.3 below that the inequality

$$\dim_{\mathbf{C}} \text{Hom}_{S_n}(1_n, \text{Std}_n^{\otimes k}) \leq \dim_{\mathbf{C}(t)} \text{Hom}_{S_t}(1_t, \text{Std}_t^{\otimes k}) = i(\text{Std}_t^{\otimes k})$$

holds, with equality if  $n \geq k$ . Thus we deduce the limit

$$\lim_{n \rightarrow +\infty} \mathbf{E}(|\text{Fix}(X_n)|^k) = i(\text{Std}_t^{\otimes k}),$$

since the left-hand side is constant and equal to the right-hand side for all large enough  $n$ .

Moreover, we will see that  $\text{Std}_t$  admits a spectral measure with respect to  $i^2$ , so this formula implies *a priori* that  $(|\text{Fix}(X_n)|)_{n \geq 1}$  converges in law to that measure for the object  $\text{Std}_t$ , as soon as we know that its moments are small enough to ensure uniqueness of the spectral measure.

We will also see that  $i(\text{Std}_t^{\otimes k})$  is the number of partitions of a set with  $k$  elements<sup>3</sup> (see Lemma 3.2 below). By the so-called Dobiński’s Formula (see, e.g., [18]), this is known to

<sup>2</sup> In fact, one can show similarly that  $i$  is a complete positive invariant.

<sup>3</sup> Not to be mistaken with the partition number  $p(k)$ .

be the  $k$ -th moment of the Poisson distribution with parameter 1, and to be small enough, which concludes the proof.  $\square$

**Remark 3.1.** (1) In comparison with other proofs, this abstract argument has the advantage of explaining (to some extent) “where the Poisson distribution comes from”.

(2) It is natural to ask if similar ideas can be used for some other related asymptotic problems (e.g., for counting 2-cycles, etc, in random permutations, or more ambitiously, for counting the number of cycles in a random permutation).

(3) To the best of our knowledge, the fact that the first moments of  $\text{Fix}(X_n)$  coincide with those of the Poisson distribution first appears in the work of Diaconis–Shashahani [8, Th. 7].

We now fill up the outline above. For  $t$  either an indeterminate or a complex number, we denote by  $\mathcal{C}_t$  Knop’s  $k$ -linear tensor envelope of the category  $\mathbf{Set}^{opp}$  opposite of the category of finite sets, relative to the degree function  $\delta(e: x \rightarrow y) = t^{|y-e(x)|}$  (see [15, §2]). If  $t$  is an indeterminate, we use the base field  $k = \mathbf{C}(t)$  in the construction, otherwise  $k = \mathbf{C}$ . This is a symmetric rigid  $k$ -linear monoidal category [15, §3].

For any objects  $x$  and  $y$  of  $\mathcal{C}_t$ , we denote by  $\mathcal{N}_t(x, y)$  the tensor radical of  $\mathcal{C}_t$  [15, §4.1]; the quotient category  $\overline{\mathcal{C}}_t = \mathcal{C}_t / \mathcal{N}_t$  is semisimple and abelian by [15, Th. 6.1], and is hence a tensor category.<sup>4</sup> For integer values  $t = n$ , this quotient category is equivalent (as tensor category) to the category of representations of  $S_n$ , an equivalence being given by a functor that maps an object of the form  $[x]$ , for  $x$  a finite set, to the permutation representation of  $S_n$  on the set  $\{1, \dots, n\}^x$  (see [15, Th. 9.8, Example 1, p. 606]). In particular, this functor sends the object  $[\{1\}]$  to the standard permutation representation  $\text{Std}_n$  on  $\mathbf{C}^n$ .

**Lemma 3.2.** *Let  $t$  be an indeterminate. The category  $\mathcal{C}_t$  contains an object  $\text{Std}_t$  such that, for any integer  $k \geq 0$ , the space  $\text{Hom}_{S_t}(1_t, \text{Std}_t^{\otimes k})$  has dimension equal to the number of set partitions of a finite set with  $k$  elements. In particular,  $i(\text{Std}_t^k) \leq k^k$ , and therefore  $\text{Std}_t$  admits a unique spectral measure relative to  $i$ .*

*Proof.* We consider  $\text{Std}_t = [\{1\}]$  as above. By construction, a basis of the vector space  $\text{Hom}_{S_t}(1_t, \text{Std}_t^{\otimes k})$  is given by all “relations” from  $\emptyset$  (because  $1_t = [\emptyset]$ ) to  $\text{Std}_t^{\otimes k}$ . Since  $\text{Std}_t^{\otimes k} = \{[1, \dots, k]\}$  in  $\mathcal{C}_t$ , this basis is the set of all subobjects of  $\{1, \dots, k\}$  in the base category  $\mathbf{Set}^{opp}$ , i.e., the set of all isomorphism classes of quotients of the finite set  $\{1, \dots, k\}$ . But this set can be identified with the set of all partitions of  $\{1, \dots, k\}$ .  $\square$

**Lemma 3.3.** *Let  $t$  be an indeterminate and  $n$  an integer. For each integer  $k \geq 0$ , we have*

$$\dim_{\mathbf{C}} \text{Hom}_{S_n}(1_n, \text{Std}_n^{\otimes k}) \leq \dim_{\mathbf{C}(t)} \text{Hom}_{S_t}(1_t, \text{Std}_t^{\otimes k})$$

*holds, with equality if  $n \geq k$ .*

*Proof.* We denote by  $\mathcal{C}_n$  the  $\mathbf{C}$ -linear tensor envelope of  $\mathbf{Set}^{opp}$  for the degree function  $\delta_n$  defined as  $\delta_n(e: x \rightarrow y) = n^{|y-e(x)|}$ .

Let  $x$  and  $y$  be arbitrary finite sets; they define the objects  $[x]$  and  $[y]$  of  $\mathcal{C}_t$  as well as of  $\mathcal{C}_n$ , and the corresponding spaces  $\text{Hom}_{\mathcal{C}_t}([x], [y])$  and  $\text{Hom}_{\mathcal{C}_n}([x], [y])$  have the same basis

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<sup>4</sup> Knop uses a different definition of “tensor category” (namely rigid, symmetric, monoidal category).



(as  $\mathbf{C}(t)$  or  $\mathbf{C}$ -vector spaces, respectively), namely the “relations” from  $x$  to  $y$  in  $\mathbf{Set}^{opp}$ . Thus we have a specialization isomorphism

$$\mathrm{Hom}_{\mathcal{C}_t}([x], [y]) \otimes_{\mathbf{C}(t)} \mathbf{C} \rightarrow \mathrm{Hom}_{\mathcal{C}_n}([x], [y]).$$

Now recall that  $\mathcal{C}_n$  is *not* equivalent to the category of representations of  $S_n$ , but only its quotient  $\overline{\mathcal{C}}_n$ , as described above. By definition, the objects of  $\overline{\mathcal{C}}_n$  are the same as those of  $\mathcal{C}_n$ , and the morphisms between representations  $V_x$  and  $V_y$  of  $S_n$  associated to  $[x]$  and  $[y]$  by the equivalence of categories, are given by

$$\mathrm{Hom}_{S_n}(V_x, V_y) = \mathrm{Hom}_{\overline{\mathcal{C}}_n}([x], [y]) = \mathrm{Hom}_{\mathcal{C}_n}([x], [y]) / \mathcal{N}([x], [y]).$$

Therefore, we obtain an inequality

$$\begin{aligned} \dim_{\mathbf{C}(t)} \mathrm{Hom}_{\mathcal{C}_t}([x], [y]) &= \dim_{\mathbf{C}} \mathrm{Hom}_{\mathcal{C}_n}([x], [y]) \\ &\geq \dim_{\mathbf{C}} (\mathrm{Hom}_{\mathcal{C}_n}([x], [y]) / \mathcal{N}([x], [y])) = \dim_{\mathbf{C}} \mathrm{Hom}_{S_n}(V_x, V_y). \end{aligned}$$

Besides, the proof of the previous lemma indicates that for  $x = \emptyset$  and  $y = \{1, \dots, k\}$ , we have  $[x] = 1_t$  and  $[y] = \mathrm{Std}_t^{\otimes k}$  in the category  $\mathcal{C}_t$ , and  $V_x = 1_n$  and  $V_y = \mathrm{Std}_n^{\otimes k}$  as representations of  $S_n$ , whence the desired inequality.

It remains to see when equality holds. By the above, this is the case if and only if  $\mathcal{N}(1_n, \mathrm{Std}_n^{\otimes k}) = \{0\}$ . By a result of Knop [15, Cor. 8.5], this holds if certain invariants  $\omega_e$  in  $\mathbf{C}$  are non-zero for all indecomposable surjective morphisms  $u \rightarrow v$  (in the base category  $\mathbf{Set}^{opp}$ ) such that  $u$  is a subquotient of  $1_t \otimes \mathrm{Std}_t^{\otimes k} = \mathrm{Std}_t^{\otimes k}$ . By [15, Example 1, p. 596], for such morphisms, we have  $\omega_e = n - |v|$ ; since indecomposable surjective morphisms  $u \rightarrow v$  in  $\mathbf{Set}^{opp}$  are injective maps of sets  $v \hookrightarrow u$  with  $|v| = |u| - 1$ , and  $u$  is a subquotient of  $\mathrm{Std}_n^{\otimes k} = [\{1, \dots, k\}]$ , we have  $|v| \leq k - 1$ , hence  $\omega_e = n - |v| \geq 1$  if  $k \leq n$ .  $\square$

Knop’s approach constructs many more instances of tensor categories, and the principles above are then applicable. As an example, we recover a result of Fulman (proved in his 1997 unpublished thesis) which appears in a paper of Fulman and Stanton [11, Th. 4.1].

**Proposition 3.4** (Fulman). *Let  $E$  be a finite field and let  $(X_n)$  be a sequence of random variables with  $X_n$  uniformly distributed in  $\mathbf{GL}_n(E)$ . The sequence  $(|\mathrm{Fix}(X_n)|)_{n \geq 1}$ , where  $\mathrm{Fix}(g)$  is the 1-eigenspace of  $g \in \mathbf{GL}_n(E)$ , converges in law as  $n \rightarrow +\infty$ . For  $k \geq 0$ , the  $k$ -th moment of the limiting distribution is equal to the number of vector subspaces of  $E^k$ .*

*Moreover, the  $k$ -th moment of  $|\mathrm{Fix}(X_n)|$  is equal to the limiting moment for  $n \geq k$ .*

*Proof.* We argue as in the proof of Theorem 1.3, using instead the base category  $\mathbf{Vec}(E)$  of finite-dimensional  $E$ -vector spaces and the degree function  $\delta(e: U \rightarrow V) = t^{\dim_E(\ker(e))}$  for a surjective  $E$ -linear map to construct Knop’s category  $\mathcal{C}_t$ . We use as before the unit object  $1_t = [\{0\}]$  and the standard object  $\mathrm{Std}_t = [E]$ , which is self-dual.

Specializing to  $t = |E|^n$  for some integer  $n \geq 1$ , the quotient  $\overline{\mathcal{C}}_{|E|^n}$  is naturally equivalent to the category of finite-dimensional complex representations of  $\mathbf{GL}_n(E)$  (see [15, Example 5, p. 606]). We obtain

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{GL}_n(E)}(1_n, \mathrm{Std}_n^{\otimes k}) \leq \dim_{\mathbf{C}(t)} \mathrm{Hom}_{\mathcal{C}_t}(1_t, \mathrm{Std}_t^{\otimes k}),$$

where  $\mathrm{Std}_n$  is the  $|E|^n$ -dimensional permutation representation of  $\mathbf{GL}_n(E)$  associated to its natural action on  $E^n$ . As before, there is equality if the numerical invariants  $\omega_e$  are non-zero

for indecomposable surjective  $E$ -linear maps  $e: U \rightarrow V$  where  $U$  is a subquotient of  $\text{Std}_n^{\otimes k}$  in  $\mathcal{C}_{|E|^n}$ . We have  $\omega(e) = |E|^n - |V|$ , and hence there is equality if  $n \geq k$  (note that in  $\mathcal{C}_{|E|^n}$ , the tensor product is defined using the *direct sum* of finite-dimensional  $E$ -vector spaces).

We then note, on the one hand, that for all  $n \geq 1$ , the function  $g \mapsto |\text{Fix}(g)|$  is the character of the standard representation, and on the other hand that, by Knop's construction, the dimension

$$\dim_{\mathbf{C}(t)} \text{Hom}_{\mathcal{C}_t}(1_t, \text{Std}_t^{\otimes k})$$

is the number of subspaces of  $E^k$ . Thus,  $\mathbf{E}(|\text{Fix}(X_n)|^k)$  converges to this number. To conclude, we need however to apply Lemma 3.5 below, since in this case the size of the moments do not satisfy the Carleman condition, but it is known that

$$|\{\text{subspaces of } E^k\}| \ll |E|^{k(k+1)/4}. \quad \square$$

**Lemma 3.5** (Heath–Brown). *Let  $q \geq 1$  be an integer, and let  $(m_k)_{k \geq 0}$  be a sequence of real numbers such that  $m_k \ll q^{k(k+1)/4}$  for  $k \geq 0$ . Let  $(Z_n)$  be a sequence of random variables such that*

- (1) *For all  $n$ , the support of  $Z_n$  is contained in the set of powers  $q^k$  for  $k \geq 0$ .*
- (2) *For all  $k \geq 0$ , we have  $\mathbf{E}(Z_n^k) \rightarrow m_k$ .*

*Then  $(Z_n)$  converges in law to a random variable  $Z$  supported on powers of  $q$  with moments  $m_k$  for all  $k \geq 0$ .*

*Proof.* This is implicit in [13, Lemmas 17, 18]. More precisely, it follows from standard results in the method of moments that the second assumption implies that any subsequence of  $(Z_n)$  which converges in law has a limit with moments  $m_k$ , and it is elementary from the first assumption that all such limits are supported on powers of  $q$ . Heath–Brown's result (proved in [13] in the case  $q = 4$ , but with immediate generalization) is that there is a unique probability measure on  $\mathbf{R}$  with these two properties. Since moreover the convergence of moments implies uniform integrability (or tightness), this means that the sequence  $(Z_n)$  is relatively compact and has a unique limit point, hence converges. The stated properties of the limit are then clear.  $\square$

**Remark 3.6.** A result of Christiansen [3] (also cited by Fulman and Stanton) shows that the limiting measure of Proposition 3.4, as a measure on  $\mathbf{R}$ , is not characterized by its moments. Thus, some extra condition is necessary to ensure uniqueness, and this is provided by the assumption that the support is restricted to powers of  $q$ .

Considering another example of Knop leads by the same method to a similar result which is new, to the best of our knowledge.

**Proposition 3.7.** *Let  $E$  be a finite field and let  $(X_n)$  be a sequence of random variables with  $X_n$  uniformly distributed in the affine-linear group  $\mathbf{Aff}_n(E)$  of  $E^n$ .*

*The sequence  $(|\text{Fix}(X_n)|)_{n \geq 1}$ , where  $\text{Fix}(g)$  is the set of fixed points of  $g \in \mathbf{Aff}_n(E)$ , converges in law as  $n \rightarrow +\infty$ . For  $k \geq 0$ , the  $k$ -th moment of the limiting distribution is equal to the number of affine subspaces of  $E^{k-1}$ .*

*Moreover, the  $k$ -th moment of  $|\text{Fix}(X_n)|$  is equal to the limiting moment for  $n \geq k$ .*

*Proof.* We argue as above with the base category  $\mathcal{A}$  of (non-empty) affine spaces over  $\mathbf{E}$  (see [15, p. 597, Ex. 6; p. 607, Ex. 7]).  $\square$

**Remark 3.8.** It is also natural to consider the category  $\mathcal{C}_t = \text{Rep}(\mathbf{GL}_t)$  of Deligne and Milne (see [7, §10, Déf. 10.2]), interpolating for  $t \in \mathbf{C}$  the representations of  $\mathbf{GL}_n(\mathbf{C})$ . Indeed, the argument applies rather similarly, and leads to the analogue of Theorem 1.3 in this context, namely: the direct image under the trace of the probability Haar measure on the unitary group  $\mathbf{U}_n(\mathbf{C})$  converges as  $n \rightarrow +\infty$  to a standard complex gaussian (see Diaconis–Shashahani [8], or the paper [17] of Larsen, in the case of the symplectic groups and real gaussians).

First, by Corollary 2.6, the assignment  $i(M) = \dim \text{Hom}_{\mathcal{C}_t}(1_{\mathcal{C}_t}, M)$  defines a positive invariant on  $\mathcal{C}_t$ . One can then show that there is an object  $\text{Std}_t$  corresponding to the standard representation, and for which

$$i(\text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b}) = \dim \text{Hom}(1_{\mathcal{C}_t}, \text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b}) = \begin{cases} 0 & \text{if } a \neq b, \\ a! & \text{if } a = b. \end{cases}$$

More precisely, with the notation of [7, Déf. 10.2], the object  $\text{Std}_t$  corresponds to the pair of finite sets  $(\{1\}, \emptyset)$  and is denoted  $X_0^{\otimes 1}$ ; thus,  $\text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b}$  corresponds to the pair  $(\{1, \dots, a\}, \{1, \dots, b\})$  and the value of  $i(\text{Std}_t^{\otimes a} \otimes \text{D}(\text{Std}_t)^{\otimes b})$  is the dimension of the space

$$\text{Hom}((\emptyset, \emptyset), (\{1, \dots, a\}, \{1, \dots, b\})),$$

which is by definition the number of bijections  $\{1, \dots, b\} \rightarrow \{1, \dots, a\}$ .

These values are the moments

$$\frac{1}{\pi} \int_{\mathbf{C}} z^a \bar{z}^b e^{-|z|^2} dz$$

of a standard complex gaussian random variable, which is therefore the spectral measure associated to  $\text{Std}_t$ . Using a stabilization property of the corresponding invariants for  $\mathbf{GL}_n(\mathbf{C})$  when  $n > a + b$ , one gets convergence as before (see [7, Prop. 10.6]).

This proof is not as satisfactory as that of Theorem 1.3, because Deligne and Milne’s definition of  $\text{Rep}(\mathbf{GL}_t)$  involves some *a priori* knowledge of the representations and linear invariants of  $\mathbf{GL}_n(\mathbf{C})$  (especially stability properties). The argument does show, however, that the convergence to the gaussian can be interpreted in terms of spectral measures, and that the standard gaussian can also be interpreted as a “generalized” Sato–Tate measure. Moreover, it suggests the question: what are the spectral measures for other objects of  $\text{Rep}(\mathbf{GL}_t)$ ?

We note also that since it is known that the third power of a gaussian is not determined by its moments (due to Berg [?berg]), the third tensor power of the standard object of  $\text{Rep}(\mathbf{GL}_t)$  gives an example of an object whose spectral measure is not unique.<sup>5</sup>

#### 4. PROOF OF THEOREM 1.4

Let  $m \geq 1$  be an integer and let  $\lambda$  be a partition of  $m$ . Recall that the statement concerns the limiting behavior of the measures  $(\chi_{\lambda, n}(X_n))$  described in the statement.

The argument will consist of two stages:

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<sup>5</sup> Attention, car Berg parle de gaussiennes réelles, vérifier le cas complexe.

- We prove *a priori* that the sequence of measures  $(\chi_{\lambda,n}(X_n))$ , where as before  $X_n$  is uniformly distributed on  $S_n$ , converges in law as  $n \rightarrow +\infty$  to some measure.
- We compute the moments of the limiting measure and show that they coincide with those of a spectral measure.

**Remark 4.1.** In general, the moments of the limiting measure are too large to deduce the convergence in law directly from the second step. However, one may speculate that it might be possible to do so using the recent results of Sawin and Wood [[?sawin-wood](#), Prop. 6.24].

**Lemma 4.2.** *The measures  $(\chi_{\lambda,n}(X_n))$  converge in law as  $n \rightarrow +\infty$ .*

*Proof.* For  $i \geq 1$ ,  $n \geq 1$  and  $\sigma \in S_n$ , we denote by  $\ell_i(\sigma)$  the number of  $i$ -cycles (fixed points if  $i = 1$ ) in the representation of  $\sigma$  as a product of disjoint cycles. It is known from the theory of symmetric functions that there exists a polynomial  $q_\lambda \in \mathbf{Q}[(L_i)_{i \geq 1}]$  such that, for all  $n$  large enough, the equality

$$\chi_{\lambda,n}(\sigma) = q_\lambda(\ell_1(\sigma), \dots, \ell_i(\sigma), \dots)$$

holds for all  $\sigma \in S_n$  (see for instance [[?macdonald](#), Ex. I.7.14]; these are the so-called “character polynomials”).

Since is also known that the sequences  $(\ell_i(X_n))_{i \geq 1}$  converge in law as  $n \rightarrow +\infty$  to a sequence  $(\varpi_{1/i})_{i \geq 1}$  of *independent* Poisson random variables with parameters  $1/i$  (see, e.g., [8, Th. 7]), the result follows by composition, with the limit being  $q_\lambda(\varpi_1, \dots, \varpi_n, \dots)$ .  $\square$

We now begin the second step, where spectral measures are involved. The goal is to prove that the moments of  $\chi_{\lambda,n}(X_n)$  converge to those of a spectral measure of a simple object of the Deligne–Knop category.

The proof will use Deligne’s construction of the categories  $\mathcal{C}_t$  and  $\mathcal{C}_n$ , since some necessary properties are not explicitly established by Knop (although they could undoubtedly be generalized to more general tensor envelopes). Note that since Theorem 1.4 is new, the fact that Deligne’s definition involves some a priori knowledge of representations of the symmetric groups is not an instance of circular reasoning.

To avoid confusion, we denote by  $\mathbf{Rep}(S_t)$  and  $\mathbf{Rep}(S_n)$  the categories  $\mathcal{C}_t$  and  $\mathcal{C}_n$  according to Deligne’s definition. In addition, when  $A$  is a commutative ring and  $t \in A$ , we use the  $A$ -linear category  $\mathbf{Rep}(S_t, A)$  of Deligne [7, Déf. 2.17]. Further, for any integer  $N \geq 0$ , we consider the full subcategories  $\mathbf{Rep}(S_t)^{(N)}$  and  $\mathbf{Rep}(S_t, A)^{(N)}$  (see [7, § 4.1]). The objects of these subcategories are the direct factors of sums of certain objects  $[U]$  associated to finite sets of size  $\leq N$ ; Deligne proved that the category  $\mathbf{Rep}(S_t)^{(N)}$  is a semisimple abelian category if  $t$  is not an integer between 0 and  $2N - 2$  (see [7, Prop. 5.1]).

**Remark 4.3.** Deligne’s basic generators  $[U]$  and their morphism spaces are introduced in [7, Déf. 2.12]. Note that these  $[U]$  are not the same as the basic objects in Knop’s definition, but the precise relation between them is explained by Knop in [[?knop2](#), Remark 1.2].

Fix the ring  $A$ , the element  $t \in A$  and some integer  $N \geq 0$ . We assume that

$$(4.1) \quad t - k \in A^\times \text{ for } 0 \leq k \leq 2N - 2, \quad N! \in A^\times.$$

Under these assumptions, Deligne [7, Prop. 5.1, Remarque 5.6] has associated to any pair  $(y, \varrho)$  consisting of a finite set  $y$  with  $|y| \leq 2N$  and an irreducible representation  $\varrho$  of the

symmetric group  $S_y$ , an object  $\mathbf{x}_{y,\varrho,A}$  of  $\mathbf{Rep}(S_t, A)^{(N)}$ . (This object is independent, up to isomorphism, of the choice of  $N$ , provided the above conditions hold, hence the value of  $N$  is omitted from the notation.)

These objects are functorial with respect to  $A$  under the natural base-change functor

$$\Gamma_{A,B}: \mathbf{Rep}(S_t, A) \rightarrow \mathbf{Rep}(S_t, B)$$

when  $B$  is an  $A$ -algebra (see [7, Déf. 2.17]), i.e., there are isomorphisms

$$\mathbf{x}_{y,\varrho,B} \simeq \Gamma_{A,B}(\mathbf{x}_{y,\varrho,A}).$$

If  $B$  is a field of characteristic zero, then the full category  $\mathbf{Rep}(S_t, B)$  is a semisimple abelian category and  $\mathbf{x}_{y,\varrho,B}$  is one of its simple objects. In fact, conversely, all simple objects are of this form for a unique pair  $(y, \varrho)$ , up to isomorphism.

From now on, we consider  $N$  as given and we consider the case where  $A$  is the ring defined by

$$A = \mathbf{C}[t] \left[ \left( \frac{1}{t-k} \right)_{0 \leq k \leq 2N-2} \right]$$

for some integer  $N \geq 1$ . This satisfies the assumption (4.1).

Let  $m \geq 1$  be an integer and  $\lambda$  a partition of  $m$ . We then denote  $\mathbf{x}_{\lambda,A} = \mathbf{x}_{y,\varrho,A}$ , where  $y = \{1, \dots, m\}$  and  $\varrho$  is the irreducible representation of  $S_m$  associated to the partition  $\lambda$ . We denote by  $x_{\lambda,t}$  the base change of  $\mathbf{x}_{\lambda,A}$  to  $\mathbf{Rep}(S_t)$ . Furthermore, if  $n > 2N - 2$ , then we denote by  $x_{\lambda,n}$  the base change of  $\mathbf{x}_{\lambda,A}$  to  $\mathbf{Rep}(S_n)$  under the morphism such that  $t \mapsto n$ .

We begin with a lemma generalizing the first step of the proof of Lemma 3.3 (in the sense that it shows that certain hom-spaces have the same dimension in all Deligne–Knop categories  $\mathbf{Rep}(S_t)$ , even when  $t$  is a non-negative integer, provided it is “large enough”).

**Lemma 4.4.** *Let  $\lambda$  be a partition of an integer  $m \geq 1$ . Let  $a \geq 0$  be an integer. For any integer  $n$  large enough, depending only on  $a$  and  $\lambda$ , the equality*

$$\dim \mathrm{Hom}_{\mathbf{Rep}(S_t)}(1_t, x_{\lambda,t}^{\otimes a}) = \dim \mathrm{Hom}_{\mathbf{Rep}(S_n)}(1_n, x_{\lambda,n}^{\otimes a})$$

holds.

*Proof.* We apply the previous remarks with an integer  $N \geq 1$  such that  $N \geq 2a|\lambda|$ . Then  $\mathbf{x}_{\lambda,A}$  is defined in  $\mathbf{Rep}(S_t, A)^{(N)}$ , and furthermore the object  $\mathbf{x}_{\lambda,A}^{\otimes a}$  is also an object of  $\mathbf{Rep}(S_t, A)^{(N)}$  (this follows from the fact that the tensor product of two basic objects  $[U]$  and  $[V]$  is a direct sum of objects  $[W]$  with  $|W| \leq |U| + |V|$ , see [7, § 5.10]). Consequently, by [7, Remarque 5.6], there is a decomposition

$$\mathbf{x}_{\lambda,A}^{\otimes a} \simeq \bigoplus_{|\mu| \leq N} v(\mu) \mathbf{x}_{\mu,A}$$

for some non-negative integers  $v(\mu)$ , where the sum is over partitions of integers  $\leq N$ , and we have used the fact that the ring  $A$  is a principal ideal domain. Assume  $n > 2N - 2$ . Applying base-change to  $\mathbf{C}(t)$  and to  $\mathbf{C}$  with  $t \mapsto n$ , we obtain also decompositions

$$x_{\lambda,t}^{\otimes a} \simeq \bigoplus_{|\mu| \leq N} v(\mu) x_{\mu,t} \quad x_{\lambda,n}^{\otimes a} \simeq \bigoplus_{|\mu| \leq N} v(\mu) x_{\mu,n}.$$

The objects  $\mathbf{x}_{\mu,A}$  have the property that

$$\mathrm{Hom}(\mathbf{x}_{\mu,A}, \mathbf{x}_{\nu,A}) = 0 \text{ if } \mu \neq \nu,$$

whereas  $\text{Hom}(\mathbf{x}_{\mu,A}, \mathbf{x}_{\mu,A}) \simeq A$  (see [7, Remarque 5.6]). Since the unit object of  $\mathbf{Rep}(S_t)$  (resp. of  $\mathbf{Rep}(S_n)$ ) is  $x_{\mu,t}$  (resp.  $x_{\mu,n}$ ) for the partition  $\mu = (m)$  (corresponding to the trivial representation of  $S_m$ ), we therefore deduce from these decompositions the equalities

$$\dim \text{Hom}(1_t, x_{\lambda,t}^{\otimes a}) = v((m)) = \dim \text{Hom}(1_n, x_{\lambda,n}^{\otimes a}),$$

which concludes the proof.  $\square$

**Remark 4.5.** A combinatorial formula for

$$\dim \text{Hom}_{\mathbf{Rep}(S_t)}(1_t, x_{\lambda,t}^{\otimes a})$$

has been obtained (in the generality of tensor envelopes) by Knop, see [?knop3, Cor. 5.4, Ex. 5.6].

We can now conclude the proof. Let  $a \geq 0$  be an integer. By Lemma 4.4, we have the equalities

$$i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}(t)} \text{Hom}_{\mathbf{Rep}(S_t)}(1_t, x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}} \text{Hom}_{\mathbf{Rep}(S_n)}(1_n, x_{\lambda,n}^{\otimes a})$$

for all  $n$  large enough, depending on  $a$  and  $\lambda$ .

Deligne [7, Prop. 6.4] has shown that, provided  $n > 2|\lambda|$ , the functor

$$\mathbf{Rep}(S_n) \rightarrow \mathbf{Rep}(S_n)/\mathcal{N} = \text{Rep}(S_n)$$

maps  $x_{\lambda,n}$  to the representation  $\pi_{\lambda,n}$  of  $S_n$ . Thus, we obtain the lower bound

$$i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}} \text{Hom}_{\mathbf{Rep}(S_n)}(1_n, x_{\lambda,n}^{\otimes a}) \geq \dim_{\mathbf{C}} \text{Hom}_{S_n}(1_n, \pi_{\lambda,n}^{\otimes a}),$$

with equality if and only if  $\mathcal{N}(1_n, x_{\lambda,n}^{\otimes a}) = 0$ . For all  $n$  large enough (depending on  $a$  and  $\lambda$ ), we have  $\mathcal{N}(1_n, x_{\lambda,n}^{\otimes a}) = 0$  (e.g., by Knop's criterion), and hence for such  $n$ , we obtain

$$\int_{\mathbf{R}} x^a \mu_{\lambda}(x) = i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}} \text{Hom}_{S_n}(1_n, \pi_{\lambda,n}^{\otimes a}) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\lambda,n}(\sigma)^a.$$

## 5. ARITHMETIC SPECULATIONS

The distribution of the number of fixed points of random permutations in  $S_n$  for a fixed integer  $n \geq 1$  occurs naturally in number theory as a limiting distribution for the number of zeros modulo a prime number  $p$  of a fixed polynomial with integer coefficients  $f \in \mathbf{Z}[X]$  with Galois group  $S_n$ . Indeed, let  $\varrho_f(p)$  be this number. A special case of Chebotarev's density theorem states in that case<sup>6</sup> that the limit formula

$$\lim_{x \rightarrow +\infty} \frac{1}{\pi(x)} |\{p \leq x \mid \varrho_f(p) = k\}| = \mathbf{P}(|\text{Fix}(X_n)| = k)$$

holds for all integers  $k \geq 0$ , where  $\pi(x)$  denotes the number of primes  $p \leq x$ .

One may ask if a similar framework can give rise to the Poisson distribution, viewed as the number of fixed points of a "random element" of  $S_t$  for an indeterminate  $t$ . Some work of Kowalski and Soundararajan [16, § 2.4] involving *pseudopolynomials* might be related. Indeed, they have formulated the following conjecture:

<sup>6</sup> For an arbitrary polynomial  $f$ , irreducible over  $\mathbf{Q}$ , the corresponding limit would be the number of fixed points of uniformly distributed random elements of the Galois group of the splitting field of  $f$ , viewed as permutations of the roots of  $f$  in  $\mathbf{C}$ .

**Conjecture 5.1** (Kowalski–Soundararajan). *Let  $F(n) = \lfloor en! \rfloor$  for integers  $n \geq 0$ . For any prime  $p$ , let  $\varrho_F(p)$  be the number of integers  $x$  satisfying  $0 \leq x \leq p-1$  and  $F(x) \equiv 0 \pmod{p}$ . Then, for each integer  $k \geq 0$ , the following limit formula holds:*

$$\lim_{x \rightarrow +\infty} \frac{1}{\pi(x)} |\{p \leq x \mid \varrho_F(p) = k\}| = \frac{1}{e} \frac{1}{k!}.$$

Recall that a *pseudopolynomial* in the sense of Hall is a sequence  $(a_n)_{n \geq 0}$  of integers such that  $m - n$  divides  $a_m - a_n$  for all  $m > n$ . Setting  $F(n) = a_n$ , this condition guarantees that the value  $F(x) \pmod{p}$  is well-defined for  $x \in \mathbf{Z}/p\mathbf{Z}$ , independently of the choice of a representative to compute it. Besides the sequences  $(f(n))_n$  of values of a polynomial with integer coefficients  $f \in \mathbf{Z}[X]$ , a standard example is  $F(n) = \lfloor en! \rfloor$  as above.

Numerical evidence in favour of Conjecture 5.1 is quite convincing (loc. cit.). We speculate that, if true, this limiting behaviour might be explained by appealing to the properties of  $S_t$  and some avatar of Chebotarev’s density theorem.

Another tantalizing experimental parallel observation is the following. It results from Deligne’s equidistribution theorem and from work of Katz (see [14, Th. 7.10.6]) that, given a polynomial  $f \in \mathbf{Z}[X]$  of degree  $n \geq 6$  such that the derivative  $f'$  has Galois group  $S_{n-1}$ , the exponential sums

$$W_f(a; p) = \frac{1}{\sqrt{p}} \sum_{x \pmod{p}} \exp\left(2\pi i \frac{af(x)}{p}\right)$$

for  $a \in (\mathbf{Z}/p\mathbf{Z})^\times$  become equidistributed as  $p \rightarrow +\infty$  like the traces of random matrices in a compact group  $K \subset \mathbf{U}_n(\mathbf{C})$  which contains  $\mathbf{SU}_n(\mathbf{C})$ .

By analogy and comparison with the results of Diaconis–Shashahani and Larsen, we are then led to expect the following:

**Conjecture 5.2.** *Let  $F(n) = \lfloor en! \rfloor$  for integers  $n \geq 0$ . For  $p$  prime and  $a \in (\mathbf{Z}/p\mathbf{Z})^\times$ , let*

$$W_F(a; p) = \frac{1}{\sqrt{p}} \sum_{x \pmod{p}} \exp\left(2\pi i \frac{aF(x)}{p}\right).$$

*Then the values  $(W_F(a; p))_{a \in (\mathbf{Z}/p\mathbf{Z})^\times}$  become equidistributed as  $p \rightarrow +\infty$  like a standard complex gaussian, i.e., for any continuous bounded function  $\varphi: \mathbf{C} \rightarrow \mathbf{C}$ , the following holds:*

$$\lim_{p \rightarrow +\infty} \frac{1}{p-1} \sum_{a \in (\mathbf{Z}/p\mathbf{Z})^\times} \varphi(W_F(a; p)) = \frac{1}{\pi} \int_{\mathbf{C}} \varphi(z) e^{-|z|^2} dz.$$

Numerical evidence is again very convincing here. A potential link suggests itself with the representations of  $\mathbf{GL}_t$ , and even more tantalizing is the suggestion of arithmetic connections between  $S_t$  and  $\mathbf{GL}_t$ .

## APPENDIX A. KNOP’S CONSTRUCTION OF THE CATEGORY $\text{Rep}(S_t)$

In this section, we will recall the steps of Knop’s construction of tensor envelopes, specialized to the case which leads to Deligne’s category of representations of  $S_t$ .

Given sets  $X$ ,  $Y$  and  $Z$  with maps  $f: Y \rightarrow X$  and  $g: Y \rightarrow Z$ , we define the gluing  $X \sqcup_Y Z$  to be the quotient of the disjoint union  $X \sqcup Z$  by the equivalence relation such that  $f(y)$  is identified with  $g(y)$  for all  $y \in Y$ .

Recall that a partition of a set  $X$  is defined here<sup>7</sup> to be a set of subsets of  $X$ , pairwise disjoint and with union equal to  $X$ ; we can, and will, identify these with equivalence relations on  $X$ .

Given sets  $X$ ,  $Y$  and  $Z$  and partitions  $\alpha$  of  $X \sqcup Y$  and  $\beta$  of  $Y \sqcup Z$ , one defines a partition  $\beta \odot \alpha$  of  $X \sqcup Z$  as follows:

- the equivalence class of an element  $x \in X$  is the union of the  $\alpha$ -equivalence class of  $x$  and of the set of  $z \in Z$  such that there exists  $y \in Y$  which is  $\alpha$ -equivalent to  $x$  and  $\beta$ -equivalent to  $z$ .
- the equivalence class of an element  $z \in Z$  is the union of the  $\beta$ -equivalence class of  $z$  and of the set of  $x \in X$  such that there exists  $y \in Y$  which is  $\alpha$ -equivalent to  $x$  and  $\beta$ -equivalent to  $z$ .

Using the quotient maps

$$X \rightarrow (X \sqcup Y)/\alpha, \quad Y \rightarrow (Y \sqcup Z)/\beta,$$

we can define the gluing  $(X \sqcup Y)/\alpha \sqcup_Y (Y \sqcup Z)/\beta$ . There is an injective map

$$j: (X \sqcup Z)/\beta \odot \alpha \rightarrow (X \sqcup Y)/\alpha \sqcup_Y (Y \sqcup Z)/\beta,$$

and we define  $\gamma(\alpha, \beta)$  to be the cardinality of the complement of the image of  $j$ . This is the number of equivalence classes of elements of  $Y$  which are not equivalent to either an element of  $X$  or an element of  $Z$ .

We fix a ring  $k$  and an element  $t$  of  $k$ . The category  $\mathcal{C}_t$  is constructed in three steps.

One first defines a  $k$ -linear category  $\mathcal{C}_t^0$ : its objects are finite sets, and the morphism space  $\text{Hom}_{\mathcal{C}_t^0}(X, Y)$  is the free  $k$ -module generated by partitions of the finite set  $X \sqcup Y$ . The composition maps

$$\text{Hom}_{\mathcal{C}_t^0}(Y, Z) \times \text{Hom}_{\mathcal{C}_t^0}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}_t^0}(X, Z)$$

are given by

$$(\beta, \alpha) \mapsto \beta \circ \alpha = t^{\gamma(\alpha, \beta)} \beta \odot \alpha.$$

The associativity of the composition is not obvious, and relates to basic properties of the function  $\gamma$ .

If  $f: X \rightarrow Y$  is a map of finite sets, there is an associated morphism  $Y \rightarrow X$  in  $\mathcal{C}_t^0$  given by the equivalence relation  $\alpha_f$  on  $Y \sqcup X$  with classes  $\{x, f(x)\}$  for  $x \in X$ . This construction gives a contravariant functor from the category of finite sets to the category  $\mathcal{C}_t^0$  (because it is elementary that  $\gamma(\beta, \alpha) = 0$  whenever  $\alpha$  and  $\beta$  are equivalence relations associated to maps); this functor is faithful.

From  $\mathcal{C}_t^0$ , a category  $\mathcal{C}'_t$  is constructed as the category of formal finite direct sums of objects of  $\mathcal{C}_t^0$ , with morphisms given by matrices in the obvious way. Finally, Knop's tensor envelope category  $\mathcal{C}_t$  is defined by “adding images of projectors”: an object is a pair  $(X, p)$  of an object  $X$  of  $\mathcal{C}'_t$  and an endomorphism  $p$  of  $X$  such that  $p \circ p = p$ , and

$$\text{Hom}_{\mathcal{C}_t}((X, p), (Y, q)) = q \circ \text{Hom}_{\mathcal{C}'_t}(X, Y) \circ p \subset \text{Hom}_{\mathcal{C}'_t}(X, Y).$$

The category  $\mathcal{C}_t^0$  admits a monoidal structure ([9, Def. 2.1.1]). The tensor product bifunctor is defined on objects by  $X \otimes Y = X \sqcup Y$  for finite sets  $X$  and  $Y$ . As for morphisms, given

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<sup>7</sup> Contrast with, e.g., Bourbaki's definition, where a partition is a *family* of subsets.



$\alpha \in \text{Hom}_{\mathcal{C}_t^0}(X, X')$  and  $\beta \in \text{Hom}_{\mathcal{C}_t^0}(Y, Y')$ , the tensor product  $\alpha \otimes \beta \in \text{Hom}_{\mathcal{C}_t^0}(X \otimes Y, X' \otimes Y')$  is the equivalence relation on

$$(X \otimes Y) \sqcup (X' \otimes Y') = (X \sqcup Y) \sqcup (X' \sqcup Y')$$

which “coincides” with  $\alpha$  on  $X \sqcup X'$  and with  $\beta$  on  $Y \sqcup Y'$ . The associativity isomorphism  $(X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$  is given by the morphism associated to the obvious identification  $(X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z)$ , and the unit object  $\mathbf{1}$  is the empty set, with the unique morphism  $\mathbf{1} \otimes \mathbf{1} \rightarrow \mathbf{1}$ .

It is then elementary that if  $p$  and  $q$  are projectors, then  $p \otimes q$  is also one, so the rules

$$(X, p) \otimes (Y, q) = (X \otimes Y, p \otimes q)$$

and bilinearity define a monoidal structure on  $\mathcal{C}_t$ . This monoidal structure is clearly symmetric.

The monoidal category  $\mathcal{C}_t^0$  is rigid ([9, Def. 2.10.1]). Indeed, the dual of a finite set  $X$  is defined to be  $D(X) = X$  itself, and the evaluation and coevaluation morphisms

$$\text{ev}_X: D(X) \otimes X \rightarrow \mathbf{1}, \quad \text{coev}_X: \mathbf{1} \rightarrow X \otimes D(X)$$

are both identified with the equivalence relation on  $X \sqcup X$  associated to the identity map on  $X$ . For a morphism  $\alpha \in \text{Hom}_{\mathcal{C}_t^0}(X, Y)$ , the transpose  ${}^t\alpha \in \text{Hom}_{\mathcal{C}_t^0}(Y, X)$  is defined as the composition

$$\begin{aligned} D(Y) = Y = Y \otimes \mathbf{1} &\xrightarrow{\text{id} \otimes \text{coev}} Y \otimes (X \otimes X) \simeq (Y \otimes X) \otimes X \\ &\xrightarrow{(\text{id} \otimes \alpha) \otimes \text{id}} (D(Y) \otimes Y) \otimes X \xrightarrow{\text{ev} \otimes \text{id}} \mathbf{1} \otimes X = X = D(X), \end{aligned}$$

and corresponds to the obvious equivalence relation on  $X \sqcup Y$  which is “the same” as  $\alpha$  on  $Y \sqcup X$ .

The duality functor thus defined extends by linearity to  $\mathcal{C}_t'$ , and finally to  $\mathcal{C}_t$ : we have

$$D(X, p) = (D(X), \text{Id}_{D(X)} - {}^t p)$$

for an object  $(X, p)$  of  $\mathcal{C}_t$ , and  ${}^t(q \circ \alpha \circ p) = {}^t p \circ {}^t \alpha \circ {}^t q$  for  $q \circ \alpha \circ p \in \text{Hom}_{\mathcal{C}_t}((X, p), (Y, q))$ .

Thus  $\mathcal{C}_t$  has the structure of a rigid symmetric monoidal  $k$ -linear category.

Suppose that the ring  $k$  is a field of characteristic 0 and  $t \notin \mathbf{N}$ . Then Knop [15, Th. 6.1, Ex. 1, p. 596] proved that  $\mathcal{C}_t$  is a semisimple tensor category.

Let  $A$  be a fixed finite set and  $G = \text{Aut}(A)$  the corresponding symmetric group. The functor

$$h_A(X) = \text{Hom}_{\text{Set}}(A, X)$$

from the opposite of the category of finite sets to the category  $\text{Set}_G$  of finite sets with a  $G$ -action can be extended to a tensor functor  $T_A: \mathcal{C}_t \rightarrow \text{Rep}_k(G)$  of finite-dimensional  $k$ -representations of  $G$  so that the diagram

$$\begin{array}{ccc} \text{Set}^{opp} & \xrightarrow{h_A} & \text{Set}_G \\ \downarrow & & \downarrow \\ \mathcal{C}_t & \xrightarrow{T_A} & \text{Rep}_k(G) \end{array}$$

commutes [15, Th. 9.4, (9.23)], where the right-hand functor  $\text{Set}_G \rightarrow \text{Rep}_k(G)$  associates to a finite set  $X$  with a  $G$ -action the permutation  $k$ -representation of  $G$  on  $X$ .

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