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Extension of some edge graph problems: standard, parameterized and approximation complexity

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Abstract

We consider *extension* variants of some edge optimization problems in graphs containing the classical EDGE COVER, MATCHING, and EDGE DOMINATING SET problems and generalizations thereof. Given a graph $G = (V, E)$ and an edge set $U \subseteq E$, it is asked whether there exists an inclusion-wise *minimal* (or *maximal*, respectively) feasible solution E' which satisfies a given property, for instance, being an edge dominating set (or a matching, respectively) and containing the *forced* edge set U (or avoiding any edges from the *forbidden* edge set $E \setminus U$, respectively). We present hardness results for these problems, for restricted instances such as bipartite or planar graphs. We counter-balance these negative results with parameterized complexity results. We also consider the *price of extension*, a natural optimization problem variant of extension problems, leading to some approximation and inapproximability results.

Keywords: Extension problems, Edge Cover, Matching, Edge Domination, NP-completeness, Parameterized Complexity, Approximation

2020 MSC: 68Q17, 68Q25, 68Q27

1. Introduction

We consider *extension problems* related to several classical edge optimization problems in graphs, namely EDGE COVER, MAXIMUM MATCHING and EDGE DOMINATING SET. Informally, in an extension version of an edge optimization problem, one is given a graph $G = (V, E)$ as well as a subset of edges $U \subseteq E$, and the goal is to *extend* U to a

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minimal (or maximal) solution (if possible). More precisely, for a minimization problem like EDGE DOMINATING SET, given $G = (V, E)$ and $U \subseteq E$, we ask if there is some inclusion-wise minimal edge dominating set D in G with $U \subseteq D$. In contrast, for a maximization problem like MAXIMUM MATCHING, on input $G = (V, E)$ and $U \subseteq E$, we ask if there is some inclusion-wise maximal matching M in G with $M \subseteq U$.

The notion of *extension* might sound somewhat counter-intuitive in the case of maximization problems, but can be understood better in a more general setting, considering optimization problems in NPO with an additionally specified set of partial solutions which we call *pre-solutions* (including the set of solutions) and a partial order \preceq on those. The partial order reflects not only the notion of *extension* but also of *minimality* as follows. For a pre-solution U and a solution S , S *extends* U if $U \preceq S$. A solution S is *minimal* if there exists no solution $S' \neq S$ with $S' \preceq S$. The resulting extension problem is formally defined as the task to decide, for a given pre-solution U , if there exists a minimal solution S which extends U . In the cases considered in this paper, we always have the power-set 2^E of the set of edges E of a given graph defining the set of pre-solutions, and \preceq is either \subseteq in the case of minimization problems, and \supseteq in the case of maximization problems. Different orderings are considered in [1], where also a general setting for extension problems is described.

In general, extension variants of combinatorial problems are interesting for efficient enumeration algorithms or for branching algorithms in general, as efficient algorithms for extension problems would allow to cut off search tree branches at an early stage of the branching. In branching algorithms, solutions are often built gradually; the partial or pre-solutions U can then be associated to the nodes of the search tree. A quick “no” to the extension question would avoid diving deeper into a branch where no solution can be found. Also, this could help develop enumeration algorithms with polynomial delay, relating to so-called flashlight algorithms [2]. There are further intimate relations to greedy strategies and (in this way) also to approximation algorithms. More details of these applications are described in [3, 1].

Clearly, the idea of extensions is not restricted to edge optimization problems. Extension versions have been studied for classical optimization problems, for example, the minimal extension of 3-HITTING SET [4], minimal DOMINATING SET [5, 6] or VERTEX COVER [7]. Extension of HITTING SET was also discussed recently in the context of database theory, see [8].

Another area where extension problems show up is linked to Latin squares [9] (and similar combinatorial questions), or also coloring extensions in graphs [10]. In this context, also a connection to graph drawing is worth mentioning considering the problem to lay out a graph after having fixed the positions of certain vertices [11]. The difference of this type of extension problems to the ones considered in this paper lies in the fact that with the mentioned settings, it is asked if there is any valid solution that contains the pre-set part of the solution: is there any way to fill in the cells of a square-of-numbers where certain cells are already pre-filled, so that the whole square is a Latin square, is there any valid coloring that obeys the given assignment of some colors to vertices, is there any way to find a planar drawing of a graph where certain vertices are already positioned, etc.

In contrast, we assume a certain partial order provided on the solution space, and we are interested in finding an extension that is minimal with respect to this ordering. In [3, 1], we provide a *systematic* study of this type of problems, giving quite a number

of different examples of extension problems, based on different partial orders. In [12], we investigated the complexity of extension versions of VERTEX COVER and INDEPENDENT SET, i.e., classical *vertex* graph problems, while this paper focuses on *edge* problems. For extension variants of automata-theoretic problems, see [13].

An extended abstract of this paper appeared in [14]. Since its appearance as a conference paper, some of the results that are proven in this paper have been also obtained independently in the literature. For instance, in [15], it was shown that a problem called MAXIMAL MATCHING EXTENSION is NP-complete. Notice that our results are more general; more specifically, the mentioned problem is called EXT 1-DCPS in this paper. As explained in [15], also the extension complexity results concerning cliques as established in [16] relate to the questions studied in this paper: EXT CLIQUE is clearly equivalent to EXT IS (referring to the extension variant of INDEPENDENT SET), so that EXT MATCHING can be viewed as a variation thereof in the context of edge problems.

Organization of the paper. Aside from general notation, Section 2 contains the formal definitions of the extension problems discussed in this paper. Specifically, we introduce extension problems related to generalized versions of matching, edge cover and edge domination. Also, that section summarizes the main results of the paper. In Section 3 we show that these problems remain NP-complete, even in bipartite graphs of bounded degree and with some constraints on the forced set of edges. Having a planar embedding does not help much either, as we show in Section 4 that these problems remain hard on subcubic bipartite planar graphs. Motivated by these negative results, we study the parameterized complexity of these problems in Section 5 and the approximability of a natural optimization version in Section 6.

2. Definitions

In the following, we collect all definitions that we need in this paper. The graph-theoretic notions that we are using are standard and may be consulted by the reader if doubts arise at any point, but the problem definitions are rather specific to our setting.

Graph definitions. We consider simple undirected finite graphs only, to which we refer to as *graphs*. A graph can be specified by the set V of vertices and the set E of edges; every edge has two endpoints and these two endpoints are called *adjacent*; if v is an endpoint of e , we also say that e and v are *incident* and two edges e and e' are *adjacent* if they share a common endpoint. Let $G = (V, E)$ be a graph and $S \subseteq V$; $N_G(S) = \{v \in V : \exists u \in S, vu \in E\}$ denotes the *neighborhood* of S in G and $N_G[S] = S \cup N_G(S)$ denotes the *closed neighborhood* of S . For singleton sets $S = \{s\}$, we simply write $N_G(s)$ or $N_G[s]$, even omitting G if clear from context. The cardinality of $N_G(s)$ is called *degree* of s , denoted $d_G(s)$. Vertices of degree zero are called *isolates*. If there is an upper-bound on the degree of all vertices, we speak of *subcubic graphs*. For a subset S of edges, $V(S)$ denotes the vertices incident to S . A vertex set S *induces* the graph $G[S]$ with vertex set S and $e \in E$ being an edge in $G[S]$ if and only if both endpoints of e are in S . If $S \subseteq E$ is an edge set, then S induces the graph $G[V(S)]$, while $G_S = (V, S)$ denotes the *partial subgraph induced by S* ; in particular, $G_{\bar{S}} = (V, E \setminus S)$ with $\bar{S} = E \setminus S$. If G is a graph, we also refer by $V(G)$ to its set of vertices and by $E(G)$ to its set of edges. A vertex set S of the graph $G = (V, E)$ is *independent* if the induced graph $G[S]$ constitutes of isolated

vertices, or equivalently, if S is a set of pairwise non-adjacent vertices. An edge set S is called an *edge cover* if for each vertex $v \in V$, there is an edge $e \in S$ that is incident to v , i.e., that has v as one of its endpoints, and it is a *matching* if S is a set of pairwise non-adjacent edges. An edge set S is *minimal* (or *maximal*, respectively) with respect to a graph property Π if S satisfies the graph property and any proper subset $S' \subset S$ of S (or any proper superset $S' \supset S$ of S , respectively) does not satisfy Π . A graph $G = (L \cup R, E)$ is called *bipartite* if its vertex set decomposes into two independent sets L and R . The *line graph* $L(G) = (V', E')$ of a graph $G = (V, E)$ is a simple graph where each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common vertex in G . Hence, it is exactly the intersection graph of the edges of G . It is well known the class of line graphs is a subclass of *claw-free* graphs (i.e., not possessing the *claw* $K_{1,3}$, a star with three rays, as an induced subgraph).

Problem definitions. Let $G = (V, E)$ be a graph where the minimum degree is at least $r \geq 1$. We assume r is a fixed constant (but all results given here hold even if r depends on the graph). An *r -degree constrained partial subgraph* is defined by an edge subset $S \subseteq E$ such that none of the vertices in V is incident to more than r edges in S . The problem of finding such a set S of size at least k is termed *r -DEGREE CONSTRAINED PARTIAL SUBGRAPH*, abbreviated as *r -DCPS*. An *r -degree edge-cover*¹ is defined as a subset S of edges such that each vertex of G is incident to at least $r \geq 1$ distinct edges $e \in S$, leading to the decision problem *r -DEGREE EDGE-COVER*, or *r -EC* for short, which asks for determining if such a set of size at most k exists in a given graph. For the particular cases of $r = 1$, 1-DCPS corresponds to the famous *MATCHING* problem and 1-EC is also known as the *EDGE COVER* problem. Inclusion-wise minimal r -degree edge-covers are also called *tight r -tolerant edge-covers* in [18].

The optimization problem associated to r -DCPS, denoted *MAX r -DCPS*, consists of finding an edge subset E' of maximum cardinality that is a solution to r -DCPS. *MAX r -DCPS* is known to be solvable in polynomial time even for the edge weighted version (here, we want to maximize the weight of E') [19]. When additionally the constraint r is not uniform and depends on each vertex (i.e., at most $b(v) = r_v$ edges incident to vertex v), *MAX r -DCPS* is usually known as *SIMPLE b -MATCHING* and remains solvable in polynomial time even for the edge-weighted version (Theorem 33.4, Chapter 33 of Volume A in [20]).

A well-studied optimization version of a generalization of r -EC, known as the *MIN LOWER-UPPER-COVER PROBLEM (MINLUCP)*, is the following. Given a graph $G = (V, E)$ and two functions $a, b: V \rightarrow \mathbb{N}$ such that for all $v \in V$, $0 \leq a(v) \leq b(v) \leq d_G(v)$, find a subset $M \subseteq E$ such that the partial subgraph $G_M = (V, M)$ induced by M satisfies $a(v) \leq d_{G_M}(v) \leq b(v)$ (such a solution will be called a *lower-upper-cover*), minimizing its cardinality $|M|$ among all such solutions (if any). Hence, an r -EC solution corresponds to a lower-upper-cover with $a(v) = r$ and $b(v) = d_G(v)$ for every $v \in V$. *MINLUCP* is known to be solvable in polynomial time even for edge-weighted graphs (Theorem 35.2 in Chapter 35 of Volume A in [20]).

¹A different generalization of edge cover was considered in [17], requiring that each connected component induced by the edge cover solution contains at least t edges. Clearly, if every vertex is incident to at least r edges from the cover, then each connected component induced by the edge cover solution contains at least r edges.

We are considering the following *extension problems* associated to r -DCPS and r -EC.

EXT r -DCPS

Input: A graph $G = (V, E)$ and $U \subseteq E$.

Question: Does there exist an edge set $S \subseteq E$ with $S \subseteq U$ such that the partial subgraph G_S has maximum degree at most r and is maximal in G ?

EXT r -EC

Input: A graph $G = (V, E)$ and $U \subseteq E$.

Question: Does there exist an edge set $S \subseteq E$ with $S \supseteq U$ such that the partial subgraph G_S has minimum degree at least r and is minimal in G ?

An r -edge dominating set $S \subseteq E$ of a simple graph $G = (V, E)$ is a set S of edges such that for any edge $e \in E$ of G , at least r edges of S are incident to e (by definition, an edge dominates itself one time). The MINIMUM r -EDGE DOMINATING SET problem (MIN r -EDS for short) consists in finding an r -edge dominating set of minimum size. Notice that there is a feasible solution if and only if $r \leq \min_{xy \in E} (d_G(x) + d_G(y) - 1)$. Obviously, 1-EDS is the classical EDGE DOMINATING SET problem (EDS), which is NP-hard in general graphs (problem [GT2] in [21]). The generalization to r -EDS has been studied in [22, 23] (under the name b -EDS) from an approximation point of view. However, to the best of our knowledge, r -EDS for every $r \geq 2$ was not proved NP-hard so far. As associated extension problem, we formally study the following problem.

EXT r -EDS

Input: Given a simple graph $G = (V, E)$ and $U \subseteq E$.

Question: Is there a subset $S \subseteq E$ such that $U \subseteq S$ and S is a minimal r -edge dominating set?

For an edge extension problem π , $ext_\pi(G, U)$ denotes the set of *extremal* extensions of U (i.e., minimal or maximal depending on the context). For a minimization version, U corresponds to a subset of *forced* edges (i.e., each minimal solution has to contain U) while for a maximization version, $E \setminus U$ corresponds to a subset of *forbidden* edges (i.e., each maximal solution has to contain no edges from $E \setminus U$), while we could call U the set of *permitted* edges. Sometimes, the set $ext_\pi(G, U)$ is empty, which makes the question of the existence of such extensions interesting. Hence, for $\pi \in \{\text{EXT } r\text{-DCPS}, \text{EXT } r\text{-EC}, \text{EXT } r\text{-EDS}\}$, the extension problems ask if $ext_\pi(G, U) \neq \emptyset$. We call $|U|$ the *standard parameter* when considering these problems as parameterized. We may drop the subscript π if clear from context.

Summary of important results

After we have defined the combinatorial problems, we can now summarize our most important results on these problems in the sequence of their presentation. Notice that we can further restrict our instances by putting additional constraints to the pre-solutions, but we refrain from making this explicit in this summary. Also, from our results concerning “edge problems”, we can mostly also derive results for “vertex problems” when restricted to line graphs. We make this more explicit throughout the paper, but not in this summary.

Ext r -DCPS (Theorem 1) NP-complete in bipartite graphs with maximum degree $\max\{3, r+1\}$ for any fixed $r \geq 1$.

Ext r -EC (Theorem 4) NP-complete in bipartite graphs with maximum degree $r+2$ for any fixed $r \geq 1$.

Ext 1-DCPS and Ext 1-EC (Theorems 8 and 7) NP-complete for planar bipartite subcubic graphs.

Ext r -EDS (Theorem 5) NP-complete for planar bipartite graphs of maximum degree $r+2$ for any fixed $r \geq 1$.

These hardness results motivated us to study these extension problems with the toolboxes of parameterized complexity and approximation.

Ext r -DCPS (Theorem 14) FPT (parameterized by the number of forbidden edges collected in \overline{U}) for any fixed $r \geq 1$.

Ext r -DCPS (Proposition 15) FPT with respect to the number of isolated edges in \overline{U} for $r \geq 2$ and on graphs with maximum degree $r+1$; even polynomial-time decidable when $r = 1$.

Ext r -EC (Theorem 10) FPT (with standard parameter) for any fixed $r \geq 1$.

Ext r -EDS (Theorem 16) $W[1]$ -hard (with standard parameter) for any fixed $r \geq 1$, even when restricted to bipartite graphs.

Notice that the parameterization by the size of the set \overline{U} of forbidden edges can be seen as the dual of the standard parameter $|U|$. By cycling through all subsets of U , it is clear that EXT r -DCPS is also FPT with the standard parameter. A similar trivial argument shows that EXT r -EC and EXT r -EDS are FPT with the dual parameter $|\overline{U}|$.

As indicated above, we could also prove some non-approximability results and occasionally also give some approximation algorithms, but these approximation problems will only formally be introduced in Section 6, so that we refrain from explaining them here.

3. Complexity results

The results given in this section are based on a reduction from 2-BALANCED 3-SAT, or (3, B2)-SAT for short. An instance $(\mathcal{C}, \mathcal{X})$ of (3, B2)-SAT is given by a set \mathcal{C} of CNF clauses defined over a set \mathcal{X} of Boolean variables such that each clause has exactly three literals and such that each variable appears exactly twice as a negative and twice as a positive literal in \mathcal{C} . (3, B2)-SAT is NP-hard by [24, Theorem 1].

Theorem 1. *For every fixed $r \geq 1$, EXT r -DCPS is NP-complete in bipartite graphs with maximum degree $\max\{3, r+1\}$, even if \overline{U} is an induced matching for $r \geq 2$, or $G[V(\overline{U})]$ is a collection of paths of length at most two for $r = 1$.*

PROOF. Let us first consider the case when $r = 1$. Consider an instance of (3, B2)-SAT with clauses $\mathcal{C} = \{c_1, \dots, c_m\}$ and variables $\mathcal{X} = \{x_1, \dots, x_n\}$. We build a bipartite graph $G = (V, E)$ of maximum degree three as follows:

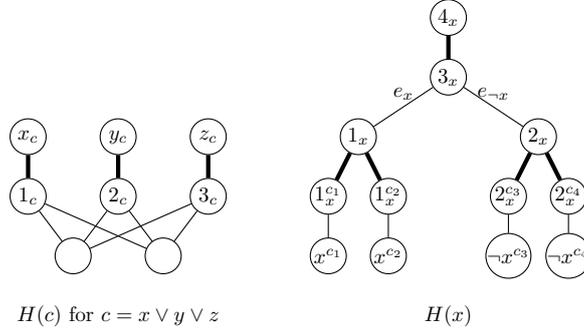


Figure 1: The gadgets $H(c)$ and $H(x)$. Edges not in U are drawn as thicker lines.

- For each clause $c = x \vee y \vee z$, where x, y, z are literals, introduce a subgraph $H(c) = (V_c, E_c)$ with 8 vertices and 9 edges. V_c contains three specified vertices x_c, y_c and z_c corresponding to literals of the clause c . Moreover, $\bar{U}_c = \{x_c 1_c, y_c 2_c, z_c 3_c\}$ is a set of three forbidden edges included in $H(c)$. The gadget $H(c)$ is illustrated in the left part of Figure 1.
- For each variable x , introduce 12 new vertices. They induce the subgraph $H(x) = (V_x, E_x)$ illustrated in Figure 1. The vertex set V_x contains four special vertices $x^{c_1}, x^{c_2}, \neg x^{c_3}$ and $\neg x^{c_4}$, where it is implicitly assumed that variable x appears as a positive literal in clauses c_1, c_2 and as a negative literal in clauses c_3, c_4 . Finally, there are two sets of *free* edges (non-forbidden edges): $F_x = \{e_x\} \cup \{2_x^{c_3} \neg x^{c_3}, 2_x^{c_4} \neg x^{c_4}\}$ and $F_{\neg x} = \{e_{\neg x}\} \cup \{1_x^{c_1} x^{c_1}, 1_x^{c_2} x^{c_2}\}$. Hence, the forbidden edges U_x in $H(x)$ are given by $\bar{U}_x = E_x \setminus (F_x \cup F_{\neg x})$.
- We interconnect $H(x)$ and $H(c)$, where x is a literal of clause c , by adding edge $x_c x^c$ if x appears as a positive literal and edge $x_c \neg x^c$ if x appears as a negative literal. We call these edges *crossing edges*.

We set $U = E \setminus ((\bigcup_{c \in C} \bar{U}_c) \cup (\bigcup_{x \in X} \bar{U}_x))$. This construction is computable within polynomial time and G is a bipartite graph of maximum degree three. Gadget $H(c)$ is designed such that any maximal matching excluding \bar{U} has to find other edges to match each of the literal-vertices $1_c, 2_c, 3_c$, and such that only two of them can be matched within $H(c)$ with an edge in U . This in turn means, at least one literal-vertex needs to be matched by a crossing edge, indicating that the corresponding literal is satisfied by the variable assignment. Gadget $H(x)$ ensures that either x^{c_1} and x^{c_2} or $\neg x^{c_3}$ and $\neg x^{c_4}$ have to be matched within $H(x)$. Consequently, all crossing-edges connecting to $H(x)$ are either to only positive (x^{c_1} and x^{c_2}) or to only negated occurrences ($\neg x^{c_3}$ and $\neg x^{c_4}$) of x ; thus the set of all crossing-edges to the sets $H(x)$ properly corresponds to a valid variable assignment.

We claim that there is a truth assignment of I which satisfies all clauses if and only if there is a maximal matching $S \subseteq U$ of G .

If T is a truth assignment of I which satisfies all clauses, then we add the set of edges $x_c x^c$ and F_x if $T(x) = \text{true}$; otherwise, if $T(x) = \text{false}$, we add the edge $x_c \neg x^c$ and all edges in $F_{\neg x}$. For each clause c , we choose one literal l_c which satisfies the clause; then, we

add two edges saturating vertices $1_c, 2_c$ and 3_c and which are not incident to the edge of \overline{U}_c saturating l_c . For instance, assume it is y ; then, we add two edges saturating vertices 1_c and 3_c and the white vertices in the gadget clause $H(c)$. The resulting matching S is maximal with $S \cap \overline{U} = \emptyset$.

Conversely, assume the existence of a maximal matching S with $S \subseteq U$. Hence, for each variable $x \in X$ exactly one edge between e_x and $e_{\neg x}$ is in S (in order to block edge $3_x 4_x$). If it is $e_x \in S$ (or $e_{\neg x} \in S$, respectively), then $F_x \subset S$ (or $F_{\neg x} \subset S$, respectively). Hence, S does not contain any crossing edges saturating $\neg x^c$ (or x^c , respectively). Now for each clause $c = x \vee y \vee z$, at least one vertex among x_c, y_c, z_c must be adjacent to a crossing edge of S . In conclusion, by setting $T(x) = \text{true}$ if at least one vertex x^{c_1} or x^{c_2} of $H(x)$ is saturated by S and $T(x) = \text{false}$ otherwise, we get a valid assignment T satisfying all clauses.

Assume now $r \geq 2$. The construction is an adaptation of the previous proof. A main building block of our construction is a subgraph, denoted $B_k(v)$ with $k < r$, containing $(kr) + 1$ vertices which are arranged as a tree of depth two with root v such that v has k children within this gadget, and each child w of v has r children. For each child w of v one edge connecting w to a leaf in $B_k(v)$ will be forbidden in our construction, and we will use $F_{B_k(v)}$ to denote a fixed set of k edges in $B_k(v)$ such that each child of v is adjacent to an edge in $F_{B_k(v)}$ and v is not adjacent to an edge in $F_{B_k(v)}$. The left part of Figure 2 gives an illustration of $B_k(v)$. The purpose of this construction is that the root v will connect to other parts of the graph, and the structure of $B_k(v)$ with the forbidden edges will make sure that a maximum r -degree constrained partial subgraph contains all edges between v and its children in $B_k(v)$, and can consequently only contain $r - k$ edges connecting v to a vertex outside $B_k(v)$. Namely, if one edge e would be missing from the edges incident to v in $B_k(v)$ in any maximal edge set E' , say, $e = vv'$, then E' would have to include the forbidden edge incident to v' by maximality.

Consider now an instance I of (3, B_2)-SAT with clauses $\mathcal{C} = \{c_1, \dots, c_m\}$ and variables $\mathcal{X} = \{x_1, \dots, x_n\}$. We build a bipartite graph $G = (V, E)$ of maximum degree $r + 1$, together with a set U of permitted edges (among which a maximal partial subgraph of degree at most r should be chosen) as follows:

- For each clause $c \in \mathcal{C}$, build a clause gadget $H(c) = (V_c, E_c)$ which is a $B_{(r-2)}(c)$ (the root c of $B_{(r-2)}(c)$ has $r - 2$ children). Hence, we denote by $U_c = E_c \setminus F_{B_{(r-2)}(c)}$ the set of permitted edges in $H(c)$.
- For each variable x , introduce $3r$ new vertices which induce the primary subgraph denoted $H'(x) = (V'_x, E'_x)$. The vertex set V'_x contains four special vertices $x, x', \neg x, \neg x'$. The vertices x and $\neg x$ have $r - 2$ distinct vertices in their neighborhoods and x' and $\neg x'$ are connected to r common vertices $v_x^1, v_x^2, \dots, v_x^r$. Also, we connect $x, \neg x$ to $x', \neg x'$, respectively, with two forbidden edges in $H'(x)$. The right part of Figure 2 gives an illustration of $H'(x)$. By adding a component $B_{(r-1)}(y)$ for each vertex $y \in \{v_x^i : 1 \leq i \leq r\}$ and identifying the root of $B_{(r-1)}(y)$ with y , we construct a new subgraph $H(x) = (V_x, E_x)$. We define the set of forbidden edges in $H(x)$ by $F_x = \{xx', \neg x\neg x'\} \cup (\bigcup_{1 \leq i \leq r} F_{B_{(r-1)}(v_x^i)})$ and hence $U_x = E_x \setminus F_x$ denotes the set of permitted edges in $H(x)$.
- We interconnect $H(x)$ and $H(c)$ by adding edge xc if x appears positively in clause c and $\neg xc$ if x appears negatively. These *crossing edges* are always permitted and

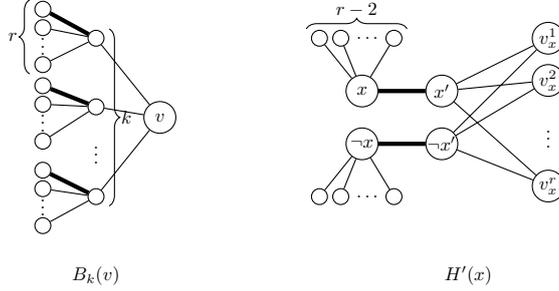


Figure 2: The gadgets $B_k(v)$ and $H'(x)$. Edges from the forbidden subset in $F_{B_k(v)}$ are marked with bold lines of the left side and more generally, edges not in U are marked with bold lines.

collected in the set U_{cross} .

Let $U = (\bigcup_{c \in C} U_c) \cup (\bigcup_{x \in X} U_x) \cup U_{\text{cross}}$ be the global set of permitted edges. This construction is computable in polynomial time, yielding a graph G that is a bipartite graph of maximum degree $r + 1$.

We claim that there is a truth assignment T of I which satisfies all clauses if and only if there is a maximal r -degree constrained partial subgraph $G_S = (V, S)$ of G , where $S \subseteq U$.

If T is a truth assignment of I which satisfies all clauses, a maximal r -degree constrained partial subgraph $G_S = (V, S)$ with $S \subseteq U$ can be constructed as follows:

1. For each variable gadget $H(c)$, by maximality $U_c \subseteq S$.
2. For each variable x , we add edges according to the assignment as follows: If $T(x) = \text{true}$, we add $v_x^i x'$ for all $1 \leq i \leq r$ and the two crossing edges connecting $\neg x$ with their respective clause gadgets. If $T(x) = \text{false}$, we add $v_x^i \neg x'$ for all $1 \leq i \leq r$ and the two crossing edges connecting x with their respective clause gadgets. In both cases, by maximality we add $2(r - 2)$ pendent edges incident to x and $\neg x$ and also all permitted edges in $B_{(r-1)}(y)$ for all $y \in \{v_x^i : 1 \leq i \leq r\}$.
3. At last, for each variable c which has more than one true literal in assignment T , add some arbitrary crossing edges to c such that $d_{G_S}(c) = r$.

The resulting subgraph is a maximal r -degree constrained partial subgraph $G_S = (V, S)$ with $S \subseteq U$.

Conversely, assume the existence of a maximal r -degree partial subgraph $G_S = (V, S)$ with $S \subseteq U$. First, recall that for every gadget $B_k(v)$, we must have $E(B_k(v)) \setminus F_{B_k(v)} \subseteq S$ for $k \in \{r - 1, r - 2\}$. Moreover, for each variable gadget $H(x)$, at least one of the pairs of crossing edges incident to x and $\neg x$ have to be in S (by maximality). Hence, we set $T(x) = \text{true}$ if both crossing edges incident to $\neg x$ are in S and otherwise we set $T(x) = \text{false}$. This assignment is valid and since for each clause c , at most two crossing edges incident to vertex c are in S (G_S is a subgraph with maximum degree r), then T satisfies all clauses of I . \square

In Theorem 1, we showed that, for every fixed $r \geq 2$, EXT r -DCPS is hard even when the set of forbidden edges $E \setminus U$ is an induced matching. In the following, we prove

the same result does not hold when $r = 1$, by reducing this problem to the problem of finding a maximum matching in a weighted graph.

Proposition 2. *EXT 1-DCPS is polynomial-time decidable when the forbidden edges $\bar{U} = E \setminus U$ form a matching.*

PROOF. Let $G = (V, E)$ with $U \subseteq E$ be an instance of EXT 1-DCPS where the partial subgraph $G_{\bar{U}}$ is a collection of disjoint edges and isolated vertices. Construct an edge-weighted graph G' from G as follows. First subdivide each edge in \bar{U} , i.e., for each $uv \in \bar{U}$, add a new vertex s_{uv} and replace edge uv by two new edges us_{uv} and vs_{uv} , each with weight $W = |E| + |V|$. Then assign weights to the remaining edges, by setting the weight for edge $xy \in E$ to 1, $W + 1$ or $2W + 1$ if $|\{x, y\} \cap V(\bar{U})|$ equals, 0, 1 or 2, respectively.

We claim that with this construction, $G = (V, E)$ with $U \subseteq E$ is a yes-instance of EXT 1-DCPS if and only if G' contains a maximum weighted matching of weight at least $2W|\bar{U}|$. To this end, first assume that $G = (V, E)$ with $U \subseteq E$ is a yes-instance. This means that there exists a maximal matching M in G that contains no edge from \bar{U} . Observe that M is also a matching in G' , and we will extend it to a matching M' in G' of weight at least $2W|\bar{U}|$. Since M is maximal in G , it follows that $\{u, v\} \cap V(M) \neq \emptyset$ for each $uv \in \bar{U}$. For each $uv \in \bar{U}$ such that $|\{u, v\} \cap V(M)| = 1$, add to M' the corresponding new edge, i.e., if $u \notin V(M)$, add edge us_{uv} , and otherwise add vs_{uv} . After this, M' contains for each $u \in V(\bar{U})$ one edge e_u in G' that contains u . The summed weight of these edges is at least $W|\{e_u = ux \mid x \notin V(\bar{U})\}| + 2W|\{uv \mid e_u = uv = e_v\}| = W|V(\bar{U})| = 2W|\bar{U}|$.

Conversely, assume that there exists a matching M' in G' of weight at least $2W|\bar{U}|$. We claim that $M = M' \cap E$ is a maximum matching in G with $M \subseteq U$. Since G' contains no edge from \bar{U} it immediately follows that $M \subseteq U$. It remains to show maximality. Assume towards contradiction that there exists an edge $e \in E \setminus M$ such that $M \cup \{e\}$ is still a matching.

If $e \in U$, then e is also an edge in G' . Since M' is maximal, it follows that there is at least one edge $e' \in M'$ such that $e \cap e' \neq \emptyset$. Since $e' \notin M$, e' has to be an added edge involving s_{uv} for some $uv \in \bar{U}$. If e intersects only e' in M' , then $M' \setminus \{e'\} \cup \{e\}$ is also a matching in G' , and its weight is larger than the weight of M' - note that e' is an added edge of weight W , while e is an original edge involving at least one vertex from $V(\bar{U})$, thus of weight at least $W + 1$. This means that $M' \setminus \{e'\} \cup \{e\}$ is a matching in G' of strictly larger weight than M' , a contradiction to the choice of M' . Similarly, if e intersects more edges, then it intersects exactly one more edge $e'' \in M'$, and again e'' contains some added vertex $s_{xy} \neq s_{uv}$. Let, w.l.o.g., $e' = us_{uv}$ and $e'' = xs_{xy}$. Since \bar{U} is a matching, we know that $u \neq x$ and thus, the only edge in G that intersects both e' and e'' in G' is $e = ux$. Since both u and x are in $V(\bar{U})$, e has weight $2W + 1$ in G' , while both e' and e'' have weight W . Again, we see that $M' \setminus \{e', e''\} \cup \{e\}$ is a matching with weight more than M' in G' , contradicting the choice of M' .

If $e \notin U$, then e does not exist in G' . Let $e = xy$ and note that in M' only one of the endpoints xy can be matched with s_{xy} while the other remains unmatched; otherwise, this edge would also be in M . (Note that here again it is important that \bar{U} is a matching.) By the definition of W , for M' to have weight at least $2W|\bar{U}|$, it has to contain at least r edges of weight $2W + 1$ and $t = 2|\bar{U}| - 2r$ edges of weight W for some $r \in \mathbb{N}$. Edges

of weight at least W and $2W + 1$ contain one, respectively two, vertices from $V(\overline{U})$, so all vertices in $V(\overline{U})$ need to be matched in order to reach the claimed weight. So in particular, there cannot be an edge $xy \in \overline{U}$ such that only one of its endpoints is matched in M' . \square

In [12], several results are proposed for the extension variant of the INDEPENDENT SET problem (EXT IS for short) in different classes of graphs like bipartite graphs, planar graphs, chordal graphs, etc. Here, we deduce a new result for a subclass of claw-free graphs.

Corollary 3. *EXT IS is NP-complete restricted to line graphs of subcubic bipartite graphs.*

PROOF. Let $G = (V, E)$ be a subcubic bipartite graph and $L(G) = (V', E')$ be its line graph. It is well known that any matching S of G corresponds to an independent set $S' = L(S)$ of G' and vice versa. In particular, S is a maximal matching of G if and only if $L(S)$ is a maximal independent set. Hence, (G, U) is a yes-instance of EXT 1-DCPS if and only if $(L(G), L(U))$ is a yes-instance of EXT IS. Theorem 1 with $r = 1$ concludes the proof. \square

A reduction from $(3, B2)$ -SAT can also be used to show the following.

Theorem 4. *For every fixed $r \geq 1$, EXT r -EC is NP-complete in bipartite graphs with maximum degree $r + 2$, even if the forced edge set U is a matching.*

PROOF. The proof is based on a reduction from $(3, B2)$ -SAT. A main building block $B_{2r}(v)$ (or $B_{2r+1}(v)$) in our construction is based on a complete bipartite subgraph of $2r$ (or $2r + 1$) vertices where one specified edge between two special vertices v and v' has been deleted. So, $B_{2r}(v) = K_{r,r} - \{vv'\}$ and $B_{2r+1}(v) = K_{r+1,r} - \{vv'\}$. Except for these two vertices v, v' , the other vertices of $B_{2r}(v)$ are not linked to any other vertex in the whole construction, while for $B_{2r+1}(v)$, it is only the case of v (i.e., only v is also linked outside $B_{2r+1}(v)$). Block $B_{2r}(v)$ will appear five times in each *variable gadget* and block $B_{2r+1}(v)$ will correspond to each *clause gadget* (see Figure 3 for an illustration). By construction, all edges of $B_{2r}(v)$ will belong to any r -EC solution (in fact, vertices v and v' still need one more edge to satisfy the minimum degree constraint) and for $B_{2r+1}(c)$, it will be almost the case (except for few edges of $B_{2r+1}(c)$ incident to c , as all neighbors of c in $B_{2r+1}(c)$ have degree $r + 1$, and all edges between $N(c)$ and $N(N(c)) \setminus \{c\}$ have to be in the edge cover; c will need one more incident edge in the edge cover besides (some of) the edges from $B_{2r+1}(c)$).

Now, consider an instance I of $(3, B2)$ -SAT with clauses $\mathcal{C} = \{c_1, \dots, c_m\}$ and variables $\mathcal{X} = \{x_1, \dots, x_n\}$. We build a bipartite graph $G = (V, E)$ of maximum degree $r + 2$, together with a set U of permitted edges as follows:

- For each clause $c \in \mathcal{C}$, we build a clause gadget $B_{2r+1}(c)$ which is a component $K_{r,r-1}$ plus two vertices c, c' . An illustration of $B_{2r+1}(c)$ is given in the left side of Figure 3.
- For each variable $x \in \mathcal{X}$, we construct a subgraph $H(x) = (V_x, E_x)$ as follows: build two P_5 denoted $P = (x, l, m, r, \neg x)$ and $P' = (x', l', m', r', \neg x')$, respectively;

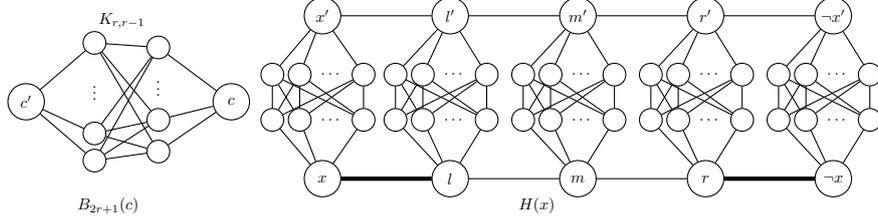


Figure 3: Block $B_{2r+1}(c)$ for clause c is depicted on the left-hand side. The subgraph $H(x) = (V_x, E_x)$ is shown on the right-hand side. Edges of U are drawn in bold.

then between each pair of vertices v, v' of P and P' a block $B_{2r}(v)$ is added for each v on P ; this interconnects v on P with the corresponding vertex v' on P' , as v and v' are special to $B_{2r}(v)$. The variable gadget $H(x) = (V_x, E_x)$ is illustrated to the right of Figure 3.

- We interconnect $H(x)$ and $B_{2r+1}(c)$, where x is a literal of clause c by adding edge xc if x appears positively in c and the edge $\neg xc$ if x appears negated. Such edges will be called *crossing*.

Now, it is easy to see that G is bipartite of maximum degree $r + 2$. Finally, let $U = \{x_i l_i, \neg x_i r_i : 1 \leq i \leq n\}$, picking the corresponding vertices and edges in each $H(x_i)$. Gadget $H(x)$ guarantees that for any minimal r -EC containing U , crossing edges can only be incident to either x or $\neg x$, but not both, meaning that these edges correspond to variable assignments. Block $B_{2r+1}(c)$ guarantees that for any minimal r -EC containing U , there is at least one crossing edge incident to vertex c . By construction, and the role of the crossing edges to $H(x)$, any crossing edge incident to c has to correspond to a variable assignment that satisfies a literal of c .

We claim that there is a truth assignment T of I which satisfies all clauses if and only if G admits a minimal r -EC solution $H = (V, S)$ of G , where $U \subseteq S$.

If T is a truth assignment of I which satisfies all clauses, a minimal r -EC solution $H = (V, S)$ can be constructed as follows:

- For each variable x , if $T(x) = \text{true}$, then

$$\{xc : x \text{ appears positively in } c\} \cup (E_x \setminus \{lm, l'm'\}) \subseteq S,$$

and if $T(x) = \text{false}$, then

$$\{\neg xc : x \text{ appears negatively in } c\} \cup (E_x \setminus \{mr, m'r'\}) \subseteq S.$$

- Since T is a satisfying assignment, we have already added in the previous step $k \geq 1$ crossing edges connected to block $B_{2r+1}(c)$ for each clause c . Then, we arbitrarily delete $k - 1$ edges S_c of $B_{2r+1}(c)$ incident to c , and we add $E(B_{2r+1}(c)) \setminus S_c$ to S .

Conversely, assume that $H = (V, S)$ is a minimal r -EC solution of G containing U . By considering the variable gadget $H(x)$ and in order to keep minimality, S contains either lm or rm but not both, since $\{xl, \neg xr\} \subset S$ by hypothesis and since all edges in

the block $B_{2r}(v)$ for $v \in \{l, m, r\}$ have to be included into the edge cover by our previous observations. Hence, we set $T(x_i) = \text{true}$ if $rm \in S$ and $T(x_i) = \text{false}$ if $lm \in S$. Since H has to get a minimum degree at least r for each vertex and vertex c has a degree $r - 1$ in clause gadget $B_{2r+1}(c)$, then the partial subgraph H contains at least one crossing edge incident to each c and hence T is a valid assignment of I . \square

4. Complexity results for planar graphs

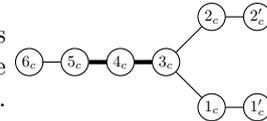
In this section, we turn our attention to planar graphs. Hence, we need a specific problem that can cope with this restriction. All reductions given in this section are hence from 4-BOUNDED PLANAR 3-CONNECTED SAT (or 4P3C3SAT for short), the restriction of EXACT 3-SATISFIABILITY² to clauses in \mathcal{C} over variables in \mathcal{X} where each variable occurs in at most four clauses (at least one time but at most two times negated while it appears one time unnegated and at most three times unnegated) and the associated bipartite graph BP is planar of maximum degree four. (The bipartite graph associated to $(\mathcal{C}, \mathcal{X})$ is the graph $BP = (C \cup X, E(BP))$, where vertex set $C = \{c_1, \dots, c_m\}$ represents the clauses in \mathcal{C} , vertex set $X = \{x_1, \dots, x_n\}$ represents the variables in \mathcal{X} , and $E(BP) = \{c_j x_i : \text{variable } i \text{ occurs in clause } j\}$.) This restriction is also NP-complete [25]; in the following, we always assume that the planar graph comes with an embedding in the plane. This gives us a planar *variable-clause-graph* G , corresponding to the original SAT instance I . The additional technical difficulties come with the embeddings that need to be preserved. Suppose that a variable x_i appears in at most four clauses c_1, c_2, c_3, c_4 of I such that in the induced (embedded) subgraph $G_i = G[\{x_i, c_1, c_2, c_3, c_4\}]$, $c_1 x_i, c_2 x_i, c_3 x_i, c_4 x_i$ is an anti-clockwise ordering of edges around x_i . By looking at G_i and considering how variable x_i appears as a negative or positive literal in the four clauses c_1, c_2, c_3, c_4 in I , the construction should handle the three following cases: (1): $x_i \in c_1, c_2$ and $\neg x_i \in c_3, c_4$; (2): $x_i \in c_1, c_3$ and $\neg x_i \in c_2, c_4$; (3): $x_i \in c_1, c_2, c_3$ and $\neg x_i \in c_4$. All other cases are included in these cases by rotations and / or interchanging x_i with $\neg x_i$.

Theorem 5. *For any $r \geq 1$, EXT r -EDS is NP-complete for planar bipartite graphs of maximum degree $r + 2$.*

PROOF. Consider first $r = 1$, corresponding to EXT EDS. Given an instance $I = (\mathcal{C}, \mathcal{X})$ of 4P3C3SAT with clause set $\mathcal{C} = \{c_1, \dots, c_m\}$ and variable set $\mathcal{X} = \{x_1, \dots, x_n\}$, we build a planar subcubic bipartite graph $H = (V_H, E_H)$ together with a set $U \subseteq E_H$ of forced edges as an instance of EXT EDS.

For each variable x_i , we introduce a corresponding gadget $H(x_i)$ as depicted in Figure 4, the forced edge set U_{x_i} contains $\{m_i r_i, r_i p_i\}$ for case (1), $\{p_i^j r_i^j, r_i^j m_i^j : 1 \leq j \leq 4\}$ for case (2) and $\{p_i^1 p_i^2, p_i^2 p_i^3, p_i^5 p_i^6, p_i^6 p_i^7, m_i^2 f_i\}$ for case (3).

For each clause $c_j \in \mathcal{C}$, we construct a clause gadget $H(c_j)$ as depicted on the right, and a forced edge set U_{c_j} , each clause gadget $H(c_j)$ contains 8 vertices and 7 edges, with $|U_{c_j}| = 2$. Edges in U are drawn in bold.



$H(c)$ for clause $c = \ell_1 \vee \ell_2 \vee \ell_3$

²addressing the problem to decide whether there is a truth assignment setting exactly one literal in each clause to true

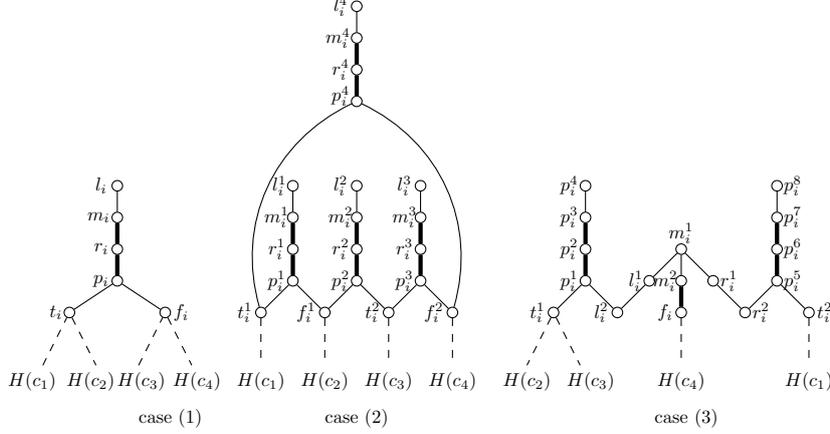
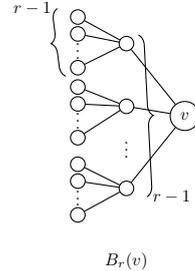


Figure 4: Variable gadgets $H(x_i)$ of Theorem 5. Cases (1), (2), (3) are corresponding to $H(x_i)$, depending on how x_i appears (as a negative or positive literal) in the four clauses (here, case 3 is rotated). Bold edges denote elements of U_{x_i} . Crossing edges are marked by dashed lines.

Moreover, we interconnect with some crossing edges the subgraphs $H(x_i)$ and $H(c_j)$ by linking x_i (or $\neg x_i$) to c_j according to how it appears in the clause. More precisely, each clause gadget $H(c_j)$ is connected to the rest of the graph via two (or one, respectively) crossing edges incident to $2'_{c_j}$ (or to $1'_{c_j}$, respectively). We also set the forced edge set $U = (\bigcup_{x_i \in \mathcal{X}} U_{x_i}) \cup (\bigcup_{c_j \in \mathcal{C}} U_{c_j})$. This construction is built in polynomial time, giving a planar subcubic bipartite graph.

Note that by minimality, for any edge of U , there exists at least one private edge to dominate it. So, let S be a minimal edge dominating set with $S \supseteq U$, then for each clause gadget $H(c)$, at least one of the crossing edges incident to it is in S . Further, for each variable x , let c_t^x (or c_f^x , respectively) be the set of crossing edges incident to t_i (or to f_i , respectively), $\{t_i^1, t_i^2\}$ (or to $\{f_i^1, f_i^2\}$, respectively), and $\{t_i^1, t_i^2\}$ (or to f_i , respectively) for the cases 1, 2 and 3 of $H(x)$, respectively, then by minimality of S , at most one of $(S \cap c_t^x)$ or $(S \cap c_f^x)$ is non-empty. Therefore, it can be easily checked that I has a satisfying assignment T if and only if H has a minimal edge dominating set containing U .

For $r \geq 2$, we start with the instance $I = (H, U)$ given in the construction above for $r = 1$. Recall that $H = (V_H, E_H)$ is a bipartite graph with bipartition $V_H = L \cup R$, while $U \subseteq E_H$ is a subset of forced edges. Now, for each vertex v of the left part L , we add the gadget $B_r(v)$ depicted to the right. Denote by H' the resulting bipartite graph and consider $I' = (H', U)$ as an instance of EXT r -EDS.



Let $B = \bigcup_{v \in L} B_r(v)$ be the added edges from H to H' . Note that any r -EDS S' of H' must contain B . Moreover, S' is a minimal r -EDS of H' if and only if $S' \setminus B$ is a minimal edge dominating set of H . \square

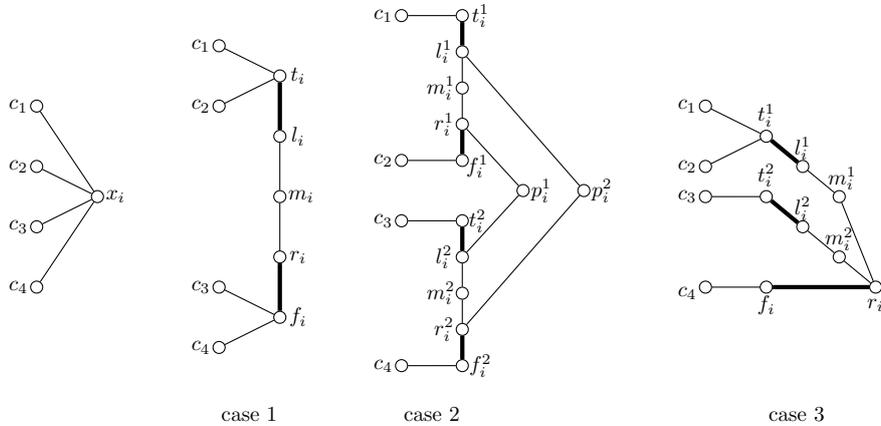


Figure 5: Construction for EXT EC (planar). On the left: A variable x_i appearing in four clauses c_1, c_2, c_3, c_4 in I . On the right, cases 1, 2, 3: The gadgets $H(x_i)$ in the constructed instance depend on how x_i appears (negated or non-negated) in the four clauses. Bold edges denote elements of U .

Remark 1. Reconsidering the previous construction that reduces the case when $r > 1$ to the case when $r = 1$, and using the NP-hardness of EDS in bipartite graphs [26, 27], we deduce NP-hardness of r -EDS for all $r \geq 1$.

In [28], several results are proposed for the enumeration of minimal dominating sets in line graphs. Here, we strengthen these results by showing that extending a given vertex set to a minimal dominating set (a problem we call EXT DS) in line graphs of a planar bipartite subcubic graphs is already a hard problem.

Corollary 6. EXT DS is NP-complete, even when restricted to line graphs of planar bipartite subcubic graphs.

PROOF. Let $G = (V, E)$ be a bipartite subcubic graph and $L(G) = (V', E')$ be its line graph. It is well known that any edge dominating set S of G corresponds to a dominating set $S' = L(S)$ of G' and vice versa. In particular, S is a minimal edge dominating set of G if and only if $L(S)$ is a minimal dominating set. Hence, (G, U) is a yes-instance of EXT EDS if and only if $(L(G), L(U))$ is a yes-instance of EXT DS. Theorem 5 with $r = 1$ concludes the proof. \square

The two next statements appear to be only strengthening Theorems 1 and 4 in the particular case of $r = 1$, but the proof details behind are different indeed.

Theorem 7. EXT 1-EC is NP-complete for planar bipartite subcubic graphs.

PROOF. The proof is based on a reduction from 4P3C3SAT. We illustrate how these cases are used in the reductions explicitly for EXT 1-EC. While the interconnections of the clause gadgets and the variable gadgets are similar to the non-planar case, the variable gadgets differ according to the cases listed above, see Figure 5.

We start from a variable-clause graph $G = (C \cup X, E)$ corresponding to an instance $I = (C, \mathcal{X})$ of 4P3C3SAT and transform it into a planar bipartite graph $H = (V_H, E_H)$

by replacing every node x_i in G with one of the three variable gadgets $H(x_i)$ which are illustrated in Figure 5. The forced edge set U_i , corresponding to variable gadget $H(x_i)$, contains

- $t_i l_i, r_i f_i$ for case 1,
- $t_i^1 l_i^1, r_i^1 f_i^1, t_i^2 l_i^2, r_i^2 f_i^2$ for case 2 and
- $t_i^1 l_i^1, t_i^2 l_i^2, r_i f_i$ for case 3.

Let $U = \bigcup_{1 \leq i \leq n} U_i$ be the set of forced edges of H . This construction can be undertaken in polynomial time and the final graph H is planar, bipartite and subcubic. We now claim that I is satisfiable if and only if H admits a minimal edge cover containing U .

Suppose T is a truth assignment of I which satisfies all clauses. For each clause c_j , let $h(j)$ be an index such that variable $x_{h(j)}$ satisfies clause c_j for T and let $J = [n] \setminus h([m])$ be the indices unused by the mapping h . We construct a minimal edge cover S of H by considering all possibilities of $H(x_i)$:

- for each variable gadget $H(x_i)$ which complies with “case (1)”, we set:

$$\begin{aligned} S_1 := & \{t_{h(j)} c_j, m_{h(j)} r_{h(j)} : T(x_{h(j)}) = \text{true}, x_{h(j)} \text{ appears positively in } c_j\} \\ & \cup \{f_{h(j)} c_j, m_{h(j)} l_{h(j)} : T(x_{h(j)}) = \text{false}, x_{h(j)} \text{ appears negatively in } c_j\} \\ & \cup \{m_i r_i : i \in J\}. \end{aligned}$$

- for each variable gadget $H(x_i)$ which complies with “case (2)” by assuming $h(j) = k$, we set S_2 to:

$$\begin{aligned} & \{t_k^1 c_j (t_k^2 c_j), m_k^1 r_k^1, r_k^1 p_k^1, m_k^2 r_k^2, r_k^2 p_k^2 : T(x_k) = \text{true} \wedge t_k^1 c_j \in E_H (t_k^2 c_j \in E_H)\} \\ & \cup \{f_k^1 c_j (f_k^2 c_j), m_k^1 l_k^1, l_k^1 p_k^1, m_k^2 l_k^2, l_k^2 p_k^2 : T(x_k) = \text{false} \wedge f_k^1 c_j \in E_H (f_k^2 c_j \in E_H)\} \\ & \cup \{l_i^1 p_i^2, l_i^1 m_i^1, l_i^2 m_i^2, l_i^2 p_i^1 : i \in J\}. \end{aligned}$$

- for each variable gadget $H(x_i)$ which complies with “case (3)” by assuming $h(j) = k$, we set:

$$\begin{aligned} S_3 := & \{t_k^1 c_j (t_k^2 c_j), m_k^1 r_k, m_k^2 r_k : T(x_k) = \text{true} \wedge t_k^1 c_j \in E_H (t_k^2 c_j \in E_H)\} \\ & \cup \{f_k c_j, m_k^1 l_k^1, m_k^2 l_k^2 : T(x_k) = \text{false}\} \\ & \cup \{l_i^1 m_i^1, l_i^2 m_i^2 : i \in J\}. \end{aligned}$$

Finally we set $S := S_1 \cup S_2 \cup S_3 \cup U$. One can easily check that S is a minimal edge cover of H .

Conversely, suppose S is a minimal edge cover of H containing U . By minimality of S , we propose an assignment T of I depending on different types of variable gadgets of H as follows:

- If $H(x_i)$ complies with case (1), in order to cover vertex m_i , the edge cover S either contains $m_i r_i$ or $m_i l_i$ (not both by minimality). This means that we set $T(x_i) = \text{true}$ (or $T(x_i) = \text{false}$, respectively) if $m_i r_i \in S$ (or $m_i l_i \in S$, respectively).

- If $H(x_i)$ complies with case (2), in order to cover vertices $m_i^1, m_i^2, p_i^1, p_i^2$, the edge cover S contains exactly one of edges in the pairs $(l_i^1 m_i^1, r_i^1 m_i^1)$, $(l_i^2 m_i^2, r_i^2 m_i^2)$, $(l_i^1 p_i^2, r_i^2 p_i^2)$ and $(r_i^1 p_i^1, l_i^2 p_i^1)$. Hence, we set
 - $T(x_i) = \text{true}$ if $\{l_i^1 m_i^1, l_i^1 p_i^2, l_i^2 p_i^1, l_i^2 m_i^2\} \cap S = \emptyset$, and
 - $T(x_i) = \text{false}$ if $|\{l_i^1 m_i^1, l_i^1 p_i^2, l_i^2 p_i^1, l_i^2 m_i^2\} \cap S| \geq 1$.
- If $H(x_i)$ complies with case (3), in order to cover vertices m_i^1, m_i^2 , S contains exactly one of edges in the pairs $(r_i m_i^1, l_i^1 m_i^1)$, $(r_i m_i^2, l_i^2 m_i^2)$. This means that we set
 - $T(x_i) = \text{true}$ if $S \cap \{l_i^1 m_i^1, l_i^2 m_i^2\} = \emptyset$ and
 - $T(x_i) = \text{false}$ if $|S \cap \{l_i^1 m_i^1, l_i^2 m_i^2\}| \geq 1$.

We obtain a valid assignment T . Since S covers all vertices of C , T satisfies all clauses of I . \square

Theorem 8. EXT 1-DCPS is NP-complete even for planar bipartite subcubic graphs.

PROOF. The proof is based on a reduction from 4P3C3SAT. For an instance $I = (\mathcal{C}, \mathcal{X})$ of 4P3C3SAT with clause set $\mathcal{C} = \{c_1, \dots, c_m\}$ and variable set $\mathcal{X} = \{x_1, \dots, x_n\}$, we build a planar bipartite subcubic graph $H = (V_H, E_H)$, together with a set $U \subseteq E_H$ of permitted edges as an instance of EXT 1-DCPS as follows.

- For each clause c_j , we introduce a clause gadget $H(c_j)$ together with a permitted edge set U_{c_j} which is already explained in detail in Theorem 1 for $r = 1$.
- For each variable x_i depending on how x_i appears (negated or non-negated) in clauses, we introduce three different gadgets $H(x_i)$ together with a set of permitted edges U_{x_i} , which is depicted in Figure 6.
- We also interconnect $H(x_i)$ to $H(c_j)$, where x_i appears positively or negatively in clause c_j by crossing edges like in the proof of Theorem 1. Let U_{cross} be the set of all crossing edges.

Let $U = (\bigcup_{j=1}^m U_{c_j}) \cup (\bigcup_{i=1}^n U_{x_i}) \cup U_{\text{cross}}$. This construction computes in polynomial time a planar bipartite subcubic graph. We claim that (H, U) is a yes-instance of EXT 1-DCPS if and only if T is a satisfying assignment of I .

Suppose T is a truth assignment of I which satisfies all clauses. We produce a maximal edge matching $S \subseteq U$ as follows: the method of choosing edges from clause gadgets and crossing edges is already explained in Theorem 1 for $r = 1$, so we here just show which edges of each $H(x_i)$ should be in S :

- If $H(x_i)$ complies with case (1), we add $\{m_i l_i^1, h_i^1 f_i^1, h_i^2 f_i^2\}$ if $T(x_i) = \text{true}$ and we add $\{m_i l_i^2, g_i^1 t_i^1, g_i^2 t_i^2\}$ if $T(x_i) = \text{false}$.
- If $H(x_i)$ complies with case (2), we add $\{f_i^1 h_i^1, p_i^1 p_i^3, m_i^2 r_i^2, f_i^2 h_i^2, p_i^5 p_i^7, l_i^1 m_i^1\}$ if $T(x_i) = \text{true}$; if $T(x_i) = \text{false}$, we add $\{p_i^2 p_i^3, t_i^1 g_i^1, l_i^2 m_i^2, p_i^6 p_i^7, t_i^2 g_i^2, m_i^1 r_i^1\}$.
- If $H(x_i)$ complies with case (3), we add $\{l_i^1 m_i^1, l_i^2 m_i^2, l_i^3 m_i^3, f_i h_i\}$ if $T(x_i) = \text{true}$; otherwise, if $T(x_i) = \text{false}$, we add $\bigcup_{1 \leq j \leq 3} \{r_i^j m_i^j, t_i^j g_i^j\}$.

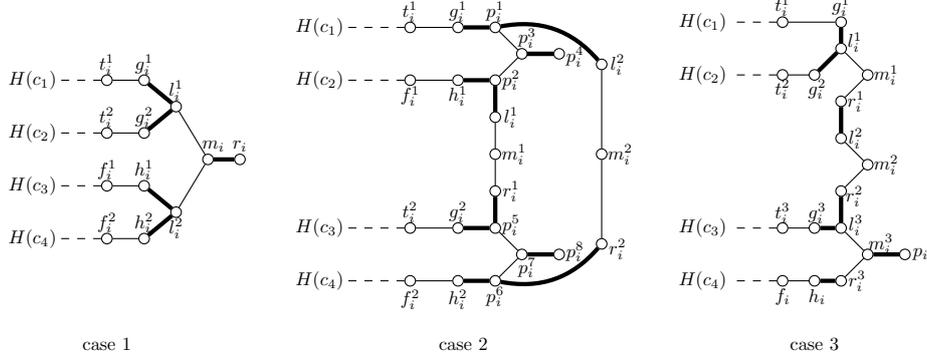


Figure 6: Variable gadgets $H(x_i)$ of Theorem 8. Cases (1), (2), (3) are corresponding to $H(x_i)$, depending on how x_i appears (negated or non-negated) in the four clauses. Edges not in U_{x_i} are drawn bold. Crossing edges are marked with dashed lines.

Conversely, suppose $S \subseteq U$ is a maximal edge matching of H . Because of maximality, for each clause gadget $H(x_j)$ there exists at least one crossing edge in S incident to a vertex of $H(x_j)$. This means that there is an assignment T which satisfies all clauses of I . We now show that T is a valid assignment:

- If $H(x_i)$ complies with case (1), by maximality either $l_i^1 m_i$ or $l_i^2 m_i$ (not both) is in S , hence we set $T(x_i) = \text{true}$ if $l_i^1 m_i \in S$ and $T(x_i) = \text{false}$ if $l_i^2 m_i \in S$.
- If $H(x_i)$ complies with case (2), by maximality either $S_1 = \{p_i^1 p_i^3, m_i^2 r_i^2, p_i^5 p_i^7, l_i^1 m_i^1\}$ or $S_2 = \{p_i^2 p_i^3, l_i^2 m_i^2, p_i^6 p_i^7, m_i^1 r_i^1\}$ (not both) is in S , so we set $T(x_i) = \text{true}$ if $S_1 \in S$ and $T(x_i) = \text{false}$ if $S_2 \in S$.
- If $H(x_i)$ complies with case (3), by maximality either $S_1 = \bigcup_{1 \leq j \leq 3} \{l_i^j m_i^j\}$ or $S_2 = \bigcup_{1 \leq j \leq 3} \{r_i^j m_i^j\}$ (not both) is in S , hence we set $T(x_i) = \text{yes}$ if $S_1 \in S$ and $T(x_i) = \text{false}$ if $S_2 \in S$. \square

5. Parameterized perspective

The next result is quite simple and characterizes the yes-instances of EXT r -EC.

Lemma 9. *There is an r -EC solution $G' = (V, E')$ where $E' \supseteq U$ such that $S_{G'} = \{v \in V(U) : d_{G'}(v) > r\}$ is an independent set of G_U if and only if $\text{ext}(G, U) \neq \emptyset$.*

PROOF. (\Leftarrow) An edge among two vertices $x, y \in U$ of a minimal extension $X \supseteq U$ certifying that $\text{ext}(G, U) \neq \emptyset$ can only exist if x or y is, or both x and y are, incident to at most r edges from X because of minimality.

(\Rightarrow) Let us look into the other direction. Let $G' = (V, E')$ be a partial subgraph of G with $U \subseteq E'$ and $d_{G'}(v) \geq r$ for all $v \in V$. Moreover, assume $S_{G'} = \{v \in V(U) : d_{G'}(v) > r\}$ is an independent set of G_U . Consider any minimal partial subgraph $H = (V, E_H)$ of $G' = (V, E')$ maintaining the property $d_{G'}(v) \geq r$ for all $v \in V$. Since $S_{G'}$ is an independent set of G_U , $U \subseteq E_H$ and therefore, $E_H \in \text{ext}(G, U)$. \square

This structural property can be used to design an FPT-algorithm for EXT r -EC. More precisely, our proposed algorithm lists all $3^{|U|}$ many independent sets of $G[U]$ included in $V(U)$ from an instance $I = (G, U)$ of EXT r -EC. In each case, we produce an equivalent instance of MIN LOWER-UPPER-COVER PROBLEM (MINLUCP), the optimization version of a generalization of r -EC, that can be solved in polynomial time which gives the following result.

Theorem 10. *For every fixed $r \geq 1$, EXT r -EC, with standard parameter, is in FPT; more precisely, the running time can be upper-bounded by $O^*(3^{|U|})$.*

PROOF. Let us establish a relation between the instances of the two problems EXT r -EC and MINLUCP. Let (G, U) be a yes-instance of EXT r -EC, where $G = (V, E)$ is a graph of minimum degree at least r and $U \subseteq E$. So, $\text{ext}(G, U) \neq \emptyset$ which implies by Lemma 9 the existence of a particular independent set S of G_U . We build an instance $(G_{\overline{U}}, a, b)$, $\overline{U} = E \setminus U$, of MINLUCP, where a, b are two non-negative functions defined as follows:

$$a(v) := \begin{cases} r & \text{if } v \in V \setminus V(U) \\ r - d_{G_U}(v) & \text{if } v \in V(U), \end{cases}$$

and

$$b(v) := \begin{cases} d_G(v) & \text{if } v \in (V \setminus V(U)) \cup S \\ r - d_{G_U}(v) & \text{if } v \in V(U) \setminus S. \end{cases}$$

The next property is rather immediate.

Property 11. *If there is a solution of MINLUCP for the instance $(G_{\overline{U}}, a, b)$, then $\text{ext}(G, U) \neq \emptyset$.*

PROOF. Assume that instance $(G_{\overline{U}}, a, b)$ of MINLUCP admits a feasible solution and let $G^* = (V, E^*)$ be an optimal solution. Then, the partial subgraph $H = (V, E^* \cup U)$ satisfies the hypothesis of Lemma 9 (actually, H is already minimal with respect to the property $\forall v \in V, d_H(v) \geq r$). \diamond

Using the outcome given in Property 11, consider the algorithm that lists all possible instances $(G[\overline{U}], a, b)$ for MINLUCP by checking all independent sets of $G[U]$ included in $V(U)$ from an instance $I = (G, U)$ of EXT r -EC. This means that we try different values for function b . Since MINLUCP is solvable in polynomial time [20], then the running time is dominated by the procedure that lists all possible independent sets of $G[U]$, i.e., there are $3^{|U|}$ possibilities: each vertex of each edge in U can be either included or excluded of the independent set, except for taking both endpoints in. \square

For EXT r -DCPS, we can also exploit some structural properties of yes-instances and then use the polynomial solvability of SIMPLE b -MATCHING to show that EXT r -DCPS, with dual parameter, is in FPT. It is sometimes more convenient to think about this problem as follows: Given a graph $G = (V, E)$ and an edge set \overline{U} , the question is if there exists an inclusion-wise maximal edge subset $S \subseteq E$ of G such that the partial subgraph G_S has maximum degree r and avoids \overline{U} , i.e., $S \cap \overline{U} = \emptyset$. Our parameter is $|\overline{U}|$. Assume there is an inclusion-wise maximal partial subgraph $G_{E'} = (V, E')$ of G with maximum degree r such that $E' \cap \overline{U} = \emptyset$. The next property is quite immediate.

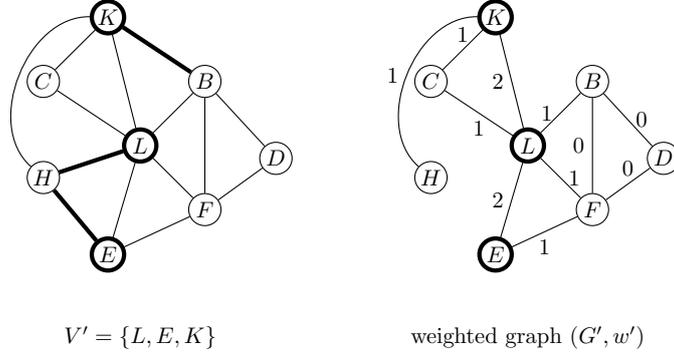


Figure 7: Graph $G = (V, E)$ and forbidden set $\bar{U} \subseteq E$ (edges are marked with bold line). A vertex cover V' of $G_{\bar{U}}$ (vertices with a bold border) is shown on the left side. On the right side, the weighted graph $G' = (V, U)$ built from G and V' is displayed.

Lemma 12. *The set $\{v \in V : v \text{ is incident to } \bar{U} \text{ and to } r \text{ edges in } E'\}$ is a vertex cover of $G_{\bar{U}} = (V, \bar{U})$.*

PROOF. By contradiction, suppose there is an edge $xy \in \bar{U}$ such that x and y are both incident to less than r edges in E' . Then $E' \cup \{e\}$ is a new solution of r -DCPS; this is a contradiction to the maximality of E' . \square

We now introduce some notations useful in the following. For a minimal vertex cover V' of $G_{\bar{U}}$, let (G', w') be the edge-weighted graph of (V, U) defined as follows: for $v \in V$, $d'(v) = 0$ if $v \notin V'$, and $d'(v) = 1$ if $v \in V'$. We define the edge weight w' by: $w'(e) = d'(x) + d'(y)$ for $e = xy \in U$. Figure 7 illustrates the construction of (G', w') from an original graph G and a specified vertex cover V' of $G_{\bar{U}}$.

Theorem 13. *Let $r \geq 1$. There is a maximal r -DCPS set S for G such that $S \cap \bar{U} = \emptyset$ if and only if there is a vertex cover V' of $G_{\bar{U}}$ such that there exists an r -DCPS set S' for the corresponding weighted graph G' such that $w'(S') \geq |V'| \times r$.*

PROOF. Let $G = (V, E)$ be a graph and let $\bar{U} \subseteq E$. Let $G_S = (V, S)$ be any maximal partial subgraph with maximum degree r of G such that $S \cap \bar{U} = \emptyset$ (if any). First observe that since S is an r -DCPS set for G which avoids \bar{U} , it is also an r -DCPS set for G' . From Lemma 12, we know $V' = \{v \in V : v \text{ is incident once to } \bar{U} \text{ and } r \text{ times to } S\}$ is a vertex cover of $G_{\bar{U}}$, and let $V'' \subseteq V'$ be a minimal vertex cover of $G_{\bar{U}}$. Hence, for the graph G'' with edge-weight w'' associated to the minimal vertex cover V'' of $G_{\bar{U}}$, it follows that $w''(S) \geq \sum_{v \in V''} d_{G_S}(v) = |V''| \times r$.

Conversely, assume that V' is a minimal vertex cover of $G_{\bar{U}}$ such that there exists an r -DCPS set S' for G' such that $w'(S') \geq |V'| \times r$. By the definition of the edge-weights w' , it follows that $w'(S') = \sum_{v \in V'} d_{G_{S'}}(v)$. As the subgraph $G_{S'}$ has maximum degree r , the weight $w'(S') \geq |V'| \times r$ is only possible if $d_{G_{S'}}(v) = r$ for all $v \in V'$. Greedily extending S' to an r -DCPS set for the original graph G hence gives a solution for EXT r -DCPS on (G, U) ; observe that the edges in \bar{U} can not be chosen by the greedy procedure, as each edge in \bar{U} is already incident to at least one vertex in V' which already has degree r . \square

Using the characterization given in Theorem 13, the next result is rather straightforward.

Theorem 14. *For every fixed $r \geq 1$, EXT r -DCPS, parameterized by the number of forbidden edges collected in \bar{U} , is in FPT. More precisely, the running time can be upper-bounded by $O^*(2^{|\bar{U}|})$. If the graph $G_{\bar{U}}$ is connected, then this upper bound improves to $O^*(1.4423^{|\bar{U}|})$.*

PROOF. Given an instance (G, U) of EXT r -DCPS, consider an algorithm that lists all minimal vertex covers V' of $G_{\bar{U}}$ and checks if the optimal value of MAX r -DCPS on (G', w') is at least $|V'| \times r$. According to Theorem 13, this procedure suffices to decide if there exists an r -DCPS set S with $S \cap \bar{U} = \emptyset$, i.e., if (G, U) is a yes-instance for EXT r -DCPS.

The running time is dominated by the procedure that lists all minimal vertex covers, as solving MAX r -DCPS can be done in polynomial time by [19]. As the number of edges in a graph is an upper bound on any minimal vertex cover of that graph, it is clear that we can enumerate all minimal vertex covers of $G_{\bar{U}}$ in time $O^*(2^{|\bar{U}|})$ by [29, 30, 31].

Using our approach, the running time estimate cannot be improved; consider for instance the case when $E(G_{\bar{U}})$ is a perfect matching. However, if $G_{\bar{U}}$ is connected, then we know that the number of vertices $|V(G_{\bar{U}})|$ is upper-bounded by $|E(G_{\bar{U}})| - 1$. Hence, the classical results of Moon and Moser [32] on enumerating maximal independent sets show the last claim on the running time. \square

When bounding the degree of the graphs, we can consider an even smaller parameter and obtain feasibility results.

Proposition 15. *For graphs with maximum degree $r + 1$, EXT r -DCPS is polynomial-time decidable when $r = 1$ and is in FPT with respect to the number ι of isolated edges in \bar{U} for $r \geq 2$. More precisely, we can upper-bound the running time by $O^*(2^\iota)$.*

PROOF. Consider the partial subgraph $G' = G_{\bar{U}} = (V, E \setminus U)$, i.e., the graph induced by the forbidden edges. Since the maximum degree of G is $r + 1$, if G' contains a triangle or a path of length at least 3, then the answer is *no*. Therefore, we can suppose that G' is a collection of stars. If one leaf of a star of G' has a degree at most r in G , then this star must be an isolated edge in G' and it is exactly for one of these two endpoints (otherwise, the answer is *no*). Hence, let $P_{G'}$ be the set of the stars which are isolated edges in G' and such that both endpoints are of degree $r + 1$ in G .

• For $r \geq 2$, for the set of stars of G' with more than one edge, leaves and center are clearly determined and for each single edge of G' not in $P_{G'}$, the endpoint with degree less than r is chosen as a center. Now, for each star in $P_{G'}$ we have to determine one of the endpoints as a center and the other one as a leaf. This determines the claimed running time. We can now build the set L of leaves for all stars of G' . Let $E'_L = \{uv \in E : u \in L\}$ and $G'_L = (V, E'_L)$. We check for all possible labelings, if there is a label which satisfies two following conditions the answer is *yes*, else the answer is *no*.

1. for each $v \in L$, $d_{G'_L}(v) = r + 1$.
2. for each vertex $v \in V \setminus L$, $d_{G'_L}(v) \leq r$.

• For $r = 1$, we construct a new graph H by omitting all sets of vertices $\{u'_i, u_i, v_i, w_i, w'_i\}$ for the stars $[u_i, v_i, w_i]$ of G' (with center v_i) where u'_i, w'_i are neighbors of the leaves u_i, w_i (without v_i). Notice at each time u'_i, w'_i have to be disjoint from one star to another one, otherwise the answer is *no*. Now, H is a collection of paths (maybe trivial) and cycles where the forbidden edges induce a matching. Remove from H all cycles and the paths where both end edges are in U . Now, H is a collection of paths where at least one of end edges is forbidden. For all of these paths, start from one side and satisfy the maximality by assigning the first possible edge to a forbidden edge; if there is a path that does not satisfy the maximality, the answer is *no*, else the answer is *yes*. \square

Remark 2. For graphs with maximum degree $r + 1$, EXT r -DCPS with $r \geq 2$ is parameterized equivalent to SAT with respect to the number of isolated edges in $E \setminus U$ and variables, respectively.

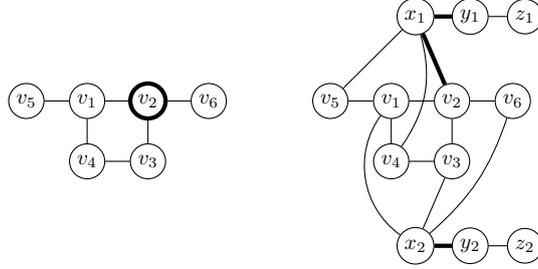


Figure 8: (G, U) as an instance of EXT VC is shown on the left, with $V_1 = \{v_2, v_4, v_5\}$ and $V_2 = \{v_1, v_3, v_6\}$ and $U = \{v_2\}$. The constructed instance (G', U') of EXT EDS is shown on the right. The vertices and edges of U and U' are marked with bold lines.

Theorem 16. For any $r \geq 1$, EXT r -EDS (with standard parameter) is $W[1]$ -hard, even when restricted to bipartite graphs.

PROOF. We only consider $r = 1$. For $r \geq 2$, we can use the gadget $B_r(v)$ as in Theorem 5. The hardness result comes from a reduction from EXT VC on bipartite graphs, the extension version of VERTEX COVER; see [12]. Let $I = (G, U)$ be an instance of EXT VC, where $G = (V, E)$ is a bipartite graph with partition (V_1, V_2) of V and $U \subseteq V$, the question of VERTEX COVER is to decide if G has a minimal vertex cover S with $U \subseteq S$. We build an instance $I' = (G', U')$ of EXT EDS as follows. Let us first construct a new graph $G' = (V', E')$ with $V' = V \cup \{x_i, y_i, z_i : i = 1, 2\}$ and

$$E' = E \cup \bigcup_{i=1,2} (\{x_i y_i, y_i z_i\} \cup \{v x_i : v \in V_i\}).$$

G' is bipartite with partition into $V'_1 = V_1 \cup \{x_2, y_1, z_2\}$ and $V'_2 = V_2 \cup \{x_1, y_2, z_1\}$. Set $U' = \{u x_1 : u \in U \cap V_1\} \cup \{u x_2 : u \in U \cap V_2\} \cup \{x_1 y_1, x_2 y_2\}$ so, $|U'| = |U| + 2$. This construction is illustrated in Figure 8. We claim that (G', U') is a *yes*-instance of EXT EDS if and only if (G, U) is a *yes*-instance of EXT VC.

Suppose (G, U) is a *yes*-instance for EXT VC; so there exists a minimal vertex cover S for G with $U \subseteq S$. The set $S' = \{v x_1 : v \in V_1 \cap S\} \cup \{v x_2 : v \in V_2 \cap S\} \cup \{x_1 y_1, x_2 y_2\}$ is an

edge dominating set of G' which includes U' because S contains U . Since S is minimal, S' is minimal, too; observe that private edges of a vertex $v \in S \cap V_1$ (i.e. an edge vu with $u \notin S \cap V_1$) translate to private edges of $vx_1 \in S'$, analogously for $x \in S \cap V_2$. By construction, $y_i z_i$ is a private edge for $x_i y_i$, $i = 1, 2$.

Conversely, suppose S' is a minimal edge dominating set of G' containing U' . Since S' is minimal, then for each $e \in S'$ there is a private edge set $S_e \subseteq E'$, $S_e \neq \emptyset$, which is dominated only by e . Moreover, we have, for $i \in \{1, 2\}$:

$$\forall v \in V_i ((vx_i \in S') \iff (\forall u \in V_{3-i}(vu \notin S' \cap E)) ,$$

since S' is minimal and $\{x_1 y_1, x_2 y_2\} \subseteq U'$. We now show how to safely modify S' such that $S' \cap E = \emptyset$. If it is not already the case, there is some edge, w.l.o.g., $e = uv \in S' \cap E$ with $u \in V_1$ and $v \in V_2$. In particular from the above observations, we deduce $u \notin U$, $v \notin U$ and $S_e \subseteq E$. Modify S' by the following procedure.

- If the private solution set $S_e \setminus \{e\}$ contains some edges incident to u and some edges incident to v , then $e \in S'$ will be replaced by ux_1 and vx_2 ;
- if every edge in the private solution S_e is adjacent to u , replace e in S' by ux_1 , otherwise if every edge in the private solution S_e is adjacent to v , replace e in S' by vx_2 .

The case distinction is necessary to guarantee that S' stays a minimal edge dominating set after each modification step. We repeat this procedure until $S' \cap E = \emptyset$. At the end of the process, every vertex $v \in V$ covers the same set of edges as vx_1 or vx_2 dominates. Hence, by setting $S = \{v \in V : vx_1 \in S' \text{ or } vx_2 \in S'\}$, we build a minimal vertex cover of G containing U . \square

Remark 3. Note that the procedure of local modifications given in Theorem 16 does not preserve optimality, but only inclusion-wise minimality.

6. Price of extension

Considering the possibility that some set U might not be extensible to any minimal solution, one might ask how far U is from an extensible set. This concept, introduced in [3], is called *Price of Extension (PoE)*. A similar approach has already been studied in the past called *the Price of Connectivity* in [33] in the context of connectivity. This notion has been introduced in [33] for MIN VC which is defined as the maximum ratio between the connected vertex cover number and the vertex cover number. Here, the goal of studying PoE is to measure how far efficiently computable extensible subsets of the given presolution U are to U or to the largest possible extensible subsets of U . To formalize this, we define optimization problems corresponding to EXT r -EC and EXT r -EDS. Actually, since we mainly propose negative results, we only focus on $r = 1$ by introducing and considering the following problems:

MAX EXT EC

Input: A graph $G = (V, E)$ and a set of edges $U \subseteq E$.

Solution: Minimal edge cover S of G .

Output: Maximize $|S \cap U|$.

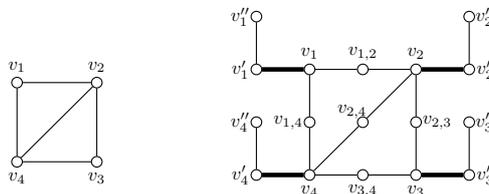


Figure 9: On the left side, an instance of MAX IS and on the right side, the corresponding instance of MAX EXT EC. Bold edges of H are the set of forced edges U .

MAX EXT EDS

Input: A graph $G = (V, E)$ and a set of edges $U \subseteq E$.

Solution: Minimal edge dominating set S of G .

Output: Maximize $|S \cap U|$.

For $\Pi = \text{MAX EXT EC}$ or $\Pi = \text{MAX EXT EDS}$, we denote the value of an optimal solution by $\text{opt}_\Pi(G, U)$. Since for both problems $\text{opt}_\Pi(G, U) \leq |U|$, with equality if and only if (G, U) is a yes-instance of the extension variant, we deduce from our previous results that MAX EXT EC and MAX EXT EDS are NP-hard. In the particular case $U = E$, MAX EXT EDS is exactly the problem called UPPER EDS where the goal is to find the largest minimal edge dominating set; UPPER EDS can be also viewed as UPPER DS in line graphs. In [34], it is shown that UPPER EDS is NP-hard in bipartite graphs. Very recently, an NP-hardness proof for planar graphs of bounded degree, an APX-completeness result for graphs of maximum degree six and a tight $\Omega(n^{\varepsilon-1/2})$ -inapproximability result for general graphs and for any constant $\varepsilon \in (0, \frac{1}{2})$, are given in [35].

The price of extension PoE is defined exactly as the ratio of approximation, i.e., $\frac{\text{apx}}{\text{opt}}$. We say that Π admits a polynomial ρ -PoE if for every instance (G, U) , we can compute a solution S of G in polynomial time which satisfies $\text{PoE}(S) \geq \rho$.

Theorem 17. *For any constant $\varepsilon \in (0, \frac{1}{2})$ and any $\rho \in \Omega(\Delta^{\varepsilon-1})$ and $\rho \in \Omega(n^{\varepsilon-\frac{1}{2}})$, MAX EXT EC does not admit a polynomial ρ -PoE for general graphs of n vertices and maximum degree Δ , unless $\text{P} = \text{NP}$.*

PROOF. The proof is based on a reduction from the MAXIMUM INDEPENDENT SET problem (MAX IS for short). Given a graph $G = (V, E)$ with n vertices and m edges where $V = \{v_1, \dots, v_n\}$, as an instance of MAX IS, we build a bipartite graph $H = (V_H, E_H)$ as follows: for each $v_i \in V$, add a P_3 with edge set $\{v_i v'_i, v'_i v''_i\}$, and for each edge $e = v_i v_j \in E$ with $i < j$, add a middle vertex $v_{i,j}$ and connect v_i to v_j via $v_{i,j}$. Consider $I = (H, U)$ as an instance of MAX EXT EC, where the forced edge subset is given by $U = \{v_i v'_i : 1 \leq i \leq n\}$. Clearly, H is a bipartite graph with $|V_H| = 3n + m$ vertices, $|E_H| = 2(m + n)$ edges and $\Delta(H) = \Delta(G) + 1$. An example of this construction is illustrated in Figure 9.

We claim that there is a solution of size k for MAX EXT EC on (H, U) if and only if G has an independent set of size k . Suppose that S is a maximal independent set of G of size k . For each $e \in E$, let $v^e \in V \setminus S$ be a vertex which covers e . Clearly, $S' = \{v_{i,j} v^e : e = v_i v_j \in E\} \cup \{v'_i v''_i : v_i \in V\} \cup \{v_i v'_i : v_i \in S\}$ is a minimal edge cover

of H with $|S' \cap U| = k$. Conversely, suppose S' is a minimal edge cover of H such that $|S' \cap U| = k$. Let us make some simple observations about every minimal edge cover of H . $\{v'_i v''_i : v_i \in V\}$ is a part of every edge cover since, v''_i for $v_i \in V$ are leaves of H . Moreover, for each $e = v_i v_j \in E$ with $i < j$, at least one edge between $v_i v_{i,j}$ or $v_j v_{i,j}$ belongs to any edge cover of H . Furthermore, if $v_i v_{i,j} \in S'$, by minimality we deduce that $v_i v'_i \notin S'$. Hence, for each $v_i v_j \in E$, at most one of $v_i v'_i, v_j v'_j$ can be in S' . Hence, $S = \{v_i : v_i v'_i \in S'\}$ is an independent set of G with size k .

Using the strong inapproximability results for MAX IS given in [36, 37], observing $\Delta(H) = \Delta(G) + 1$ and $|V_H| \leq 2|V|^2$, we obtain the claimed result. \square

Using result given in [35], an $\Omega(n^{\varepsilon-1/2})$ -inapproximability result can be immediately deduced for MAX EXT EDS, as well. The next result is obtained by a simple approximation preserving reduction from MAX EXT VC to MAX EXT EDS and improves on the mentioned immediate inapproximability result.

Theorem 18. *For any constant $\varepsilon \in (0, 1)$ and any $\rho \in \Omega(n^{\varepsilon-1})$, MAX EXT EDS does not admit a polynomial ρ -PoE for general graphs of n vertices, unless $P = NP$.*

PROOF. we propose a simple approximation preserving reduction from MAX EXT VC to MAX EXT EDS similar to Theorem 16. In [12], it is shown that MAX EXT VC is hard to approximate within ratio $n^{\varepsilon-1}$ for any constant $\varepsilon \in (0; 1)$ for a graph of order n . Let $(G = (V, E), U)$ be an instance of MAX EXT VC, we build an instance $(G' = (V', E'), U')$ of MAX EXT EDS such that $V' = V \cup \{v'\}$ and $E' = E \cup \{uv' : u \in V\}$. Let $U' = \{uv' : u \in U\}$. So G' is a graph with $|V'| = |V| + 1$ and $|U'| = |U|$.

We claim that there is a solution of size k for MAX EXT VC on (G, U) if and only if MAX EXT EDS has a solution of size k on (G', U') . Without loss of generality, suppose that $U \neq \emptyset$. Furthermore, suppose that S is a minimal vertex cover of G such that $|S \cap U| = k$, so $S' = \{uv' : u \in S\}$ is a minimal edge dominating set where $|S' \cap U'| = k$. Since S is a minimal vertex cover, for any $u \in S$, there exists at least one private edge $e_u \in E$ which is only covered by u , hence e_u will be a private edge for any $uv' \in S'$. Conversely, suppose that S' is a minimal edge dominating set of G' containing k edges of U' . In the following, we will show that how we replace any minimal edge dominating set S' of G' with a new minimal solution S'' including just some edges incident to v' such that $S' \cap U' = S'' \cap U''$. The procedure is completely analogous to what we did in Theorem 16, keeping all edges incident to v' in S' and replacing any edges non-incident to v' with some edges incident to v' as follows. For a non-incident edge $e = xy \in S'$, not incident to v' :

- if the private solution set $S_e \setminus \{e\}$ contains some edges incident to x and some edges incident to y , then $e \in S'$ will be replaced by xv' and yv' ;
- if every edge in the private solution S_e is adjacent to x , replace e in S' by xv' , otherwise if every edge in the private solution S_e is adjacent to y , replace e in S' by yv' .

Now, by setting $S = \{u : uv' \in S''\}$, we make a minimal vertex cover of G , such that $|S'' \cap U'| = |S \cap U|$. Since $|V'| = |V| + 1$ and considering the inapproximability result of MAX EXT VC from [12], the proof is complete. \square

In contrast to the last hardness result, we give a simple approximation depending on the maximum degree $\Delta(G)$.

Theorem 19. MAX EXT EDS is $\frac{1}{\Delta(G_U)+1}$ -approximable for instance (G, U) of maximum degree Δ .

PROOF. Let $(G = (V, E), U)$ be an instance of MAX EXT EDS, where the maximum degree of partial subgraph G_U induced by U is bounded by Δ . Compute a maximum matching M of G_U and transform it into a maximal matching M' of G containing M . It is well known that any maximal matching is an edge dominating set. Obviously, $(\Delta(G_U) + 1)|M| \geq |U| \geq \text{opt}_{\text{MAX EXT EDS}}(G, U)$ since G_U is $(\Delta(G_U) + 1)$ -edge colorable. \square

Considering EXT 1-DCPS, we need to adapt the notion of the price of extension because we have to consider subset of forbidden edges (i.e., \bar{U}); more precisely, we want to increase $|U|$ as little as possible. Hence, the optimization problem, called MIN EXT 1-DCPS, is defined as follows:

MIN EXT 1-DCPS

Input: A graph $G = (V, E)$ and a set of edges $U \subseteq E$.

Solution: Maximal matching S of G .

Output: Minimize $|U \cup S|$.

Recall that PoE is meant to measure how far efficiently computable extensible subsets are from the given presolution U or to the largest possible extensible subsets of U . We say that MIN EXT 1-DCPS admits a polynomial ρ -PoE if for every instance (G, U) , we can compute a solution S of G in polynomial time which satisfies $\text{PoE}(S) = \frac{\text{opt}}{\text{opt}} \leq \rho$. In the particular case $U = \emptyset$, MIN EXT 1-DCPS is exactly the well known problem MINIMUM MAXIMAL MATCHING, where the goal is to find the smallest maximal matching. In [38, 39], it is shown that MINIMUM MAXIMAL MATCHING is hard to approximate with a factor better than 2 and 1.18, respectively, assuming Unique Games Conjecture (UGC) and $P \neq NP$, respectively. We complement this bound by showing the following.

Theorem 20. A 2-approximation for MIN EXT 1-DCPS can be computed in polynomial time.

PROOF. Let $(G = (V, E), U)$ be an instance of MIN EXT 1-DCPS. The approximate solution for MIN EXT 1-DCPS consists in outputting a maximal matching M' of G in the following way: we build first a maximum matching in G_U and then we extend it into a maximal matching of the whole graph. Let M^* be an optimal solution of MIN EXT 1-DCPS with value $\text{opt}_{\text{MIN EXT 1-DCPS}}(G, U) = |U \cup M^*|$, i.e., a maximal matching of G containing a minimum number of edges outside of U . For $A \in \{U, \bar{U}\}$, and a matching M , define $M_A = A \cap M$. Now, we claim that M' is a 2-approximation solution for MIN EXT 1-DCPS.

By the decrease of PoE in term of U , and considering that $|M_U^*| \leq |U|$ together with $|M_U^*| \leq |M_U'|$ (M_U' is a maximum matching in G_U), we deduce:

$$\text{PoE} = \frac{|U| + |M_U'|}{|U| + |M_U^*|} \leq \frac{|M_U^*| + |M_U'|}{|M_U^*| + |M_U^*|} \leq \frac{|M'|}{|M^*|} \quad (1)$$

It is well known that $|M_1| \leq 2|M_2|$ for any pair M_1, M_2 of maximal matchings. So in particular, putting this inequality with $M_1 = M'$ and $M_2 = M^*$ in inequality (1), gives the expected result. \square

7. Conclusions

We have undertaken some study on several complexity aspects of extension variants of edge graph problems. Our results should be useful in particular to the (input-sensitive) enumeration algorithms community that has so far not put that much attention on edge graph problems; we are only aware of [40] in this direction. Conversely, output-sensitive enumeration algorithms, e.g., for matchings have been around for more than twenty years [41]. Some thoughts on edge cover enumeration can be found in [42]. Our research might also inspire to revisit exact and / or parameterized algorithms on EDGE DOMINATION; previous papers like [31] or [43] did not focus on special graph classes, where we see some potential for future research.

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