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Scale-relativistic correction to the muon $g - 2$ and its hadronic contribution

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Abstract

The anomalous magnetic moment (AMM) of the muon $a_\mu = (g - 2)/2$ is one of the most precisely measured quantities in physics. Its experimental value shows, in 2023, a 5.2σ discrepancy $\delta a_\mu = (249 \pm 48) \times 10^{-11}$ with its theoretical value calculated in the standard model framework, using a data-driven (R ratio) dispersive method to calculate the Hadron Vacuum Polarization (HVP) contribution. Meanwhile, lattice QCD numerical calculations of this contribution (L) have also yielded a significant discrepancy with respect to the data-driven value (R), reaching $\sim 5\sigma$ for the ratio of their best determinations (in reduced windows), $a_{\mu L}^{\text{HVP}}/a_{\mu R}^{\text{HVP}} = 1.0257 \pm 0.0052$. We suggest here a common solution to these two problems.

In standard quantum mechanics, mass ratios and inverse Compton length ratios are identical. This is no longer the case in the special scale-relativity (SSR) framework, in which the Planck length-scale is identified with a lower limit scale, invariant under dilations and replacing the zero point. Consequently a generalized form of Compton relation holds in this theory.

Regarding the HVP contribution to the muon $g - 2$, the lattice QCD calculation is performed in position *space-time* while the data-driven result is calculated in terms of energy-momentum. The lattice QCD result should therefore be corrected for SSR effects. We estimate this correction to amount to a factor $\rho_{\text{SSR}} = 1.022 \pm 0.002$, which is fully compatible with the observed excess. Once corrected, the lattice and R ratio HVP contributions agree within uncertainties.

As regards the muon $g - 2$ theoretical calculation, it involves a mass-dependent contribution which comes from two-loop vacuum polarization insertions due to electron-positron pairs and depends on the electron to muon mass ratio $x = m_e/m_\mu$. Using the renormalization group approach, we show that, in this relation, $\ln x$ logarithmic terms depend on mass, while linear x terms are expected to actually depend on inverse Compton lengths. By defining the constant $\mathbb{C}_0 = \ln(m_{\mathbb{P}}/m_0)$ in terms of the Planck mass $m_{\mathbb{P}}$ and of a reference mass m_0 , the resulting scale-relativistic correction writes $\delta a_\mu = -\alpha^2 (x \ln^3 x)/(8 \mathbb{C}_0^2)$, where α is the fine structure constant. For $m_0 = m_\mu$, the numerical values of this correction, $\delta a_\mu = (230 \pm 15) \times 10^{-11}$, would fully account for the observed experiment-theory difference.

1 Introduction

The Dirac equation predicts a muon magnetic moment, $M_\mu = g_\mu(e/2m_\mu)S$, with gyromagnetic ratio $g_\mu = 2$ and spin operator S . Quantum loop effects lead to a small calculable deviation, parameterized by the anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ [1].

That quantity can be accurately measured and precisely predicted within the framework of the Standard Model (SM). Hence, comparison of experiment and theory tests the SM at its quantum loop level.

While the theoretical and experimental values of the electron anomalous magnetic moment agree within uncertainties [2, 3],

$$a_e(\text{th}) = 0.00115965218178(77), \quad a_e(\text{exp}) = 0.00115965218091(26), \quad (1)$$

on the contrary, the muon anomalous magnetic moment exhibits since the E821 Brookhaven experiment [4] about 20 years ago, a difference of $(268 \pm 72) \times 10^{-11}$ [1, 5, 6], statistically significant at the 3.7σ level:

$$a_\mu(\text{th}) = 116591823(36) \times 10^{-11}, \quad a_\mu(\text{exp}) = 116592091(63) \times 10^{-11}. \quad (2)$$

This effect constitutes one of the main discrepancies between the SM theoretical predictions and experiments. It is all the more puzzling that it is not found in the high energy realm of today's particle physics frontier (≈ 10 TeV), but instead at the atomic-nuclear scale (≈ 100 MeV) which was up to now thought to be fully understood.

In 2021, the first results (run 1) of the Fermilab National Accelerator Laboratory (FNAL) Muon $g-2$ E989 experiment [7] have been found to be in excellent agreement with the previous BNL E821 measurement. They obtained $a_\mu(\text{FNAL}) = 116592040(54) \times 10^{-11}$, leading to a combined BNL+FNAL average result:

$$a_\mu(\text{exp}) = 116592061(41) \times 10^{-11}, \quad (3)$$

increasing to 4.2 standard deviations the tension between experiment [7] and theory [8] (White Paper from the Muon $g-2$ Theory Initiative [WP]), $\delta a_\mu = (251 \pm 59) \times 10^{-11}$. This difference has been obtained by deriving the Hadron Vacuum Polarization (HVP) contribution, which dominates the uncertainty of the Standard Model (SM) prediction, from a data-driven approach using the experimental $e^+e^- \rightarrow \text{hadrons}$ R -ratio through a dispersion relation. In 2023 the new result from the FNAL E989 runs 2&3 has yielded a two times more precise world average value

$$a_\mu(\text{exp}) = 116592059(22) \times 10^{-11}. \quad (4)$$

The resulting difference with the theoretical SM prediction, $\delta a_\mu^R = (249 \pm 48) \times 10^{-11}$, now reaches the 5.2σ level using the R ratio, seemingly pointing to possible new physics beyond the standard model.

However, in the course of these studies, a second statistically highly significant discrepancy has emerged. A numerical calculation of the HVP contribution to the muon $g-2$ from lattice QCD by Borsanyi et al [BMW collaboration] [9] has reached for the first time a level of precision similar to the data-driven method. They have found $a_\mu^{\text{HVP-L}} = (7075 \pm 55) \times 10^{-11}$, which differed by 2.1σ from the R -ratio WP recommended value $a_\mu^{\text{HVP-R}} = (6931 \pm 40) \times 10^{-11}$.

Although the cause of this difference between the data driven and lattice QCD SM prediction remained misunderstood, a recent new method has allowed to point out more precisely its origin. The window observable method [10] amounts to restrict the numerical integration to a subinterval which essentially removes the regions of strong statistical fluctuations, large finite-volume effects and large lattice artefacts [11]. In addition, it is possible to evaluate a_μ^{win} also using the R -ratio, so that a more precise view of the origin of the discrepancy can be acquired by comparing the two determinations.

The window results have confirmed and reinforced the existence of the HVP discrepancy. Davies et al [12] have used one-sided windows of various widths contributing to increasing percentages of the whole result and have found $(a_{\mu L}^w - a_{\mu R}^w)/a_{\mu R}^w = 0.016 \pm 0.007$, 0.020 ± 0.005 and 0.021 ± 0.0075 in windows covering respectively 14%, 43% and 70% of the whole difference, thus reaching an almost 4σ effect in the second window.

Other authors have used a two-sided window covering $(0.4 - 1.0)$ fm contributing for 33% of the result. An excellent self-consistency has been found between these various lattice calculations [13, 11, 14], therefore validating the initial BMW claim [9]. Their average results for R [15, 9, 16] and L [9, 14, 17, 18] determinations also lead to a similar statistical significance, with a relative difference 0.0285 ± 0.0072 . The effect reaches $\sim 5\sigma$ using their best determinations yielding $(a_{\mu L}^w - a_{\mu R}^w)/a_{\mu R}^w = 0.0257 \pm 0.0052$ [Aubin et al /CL/KNT19 vs. RBC/UKQCD23 in Fig.11 of ref.[14)]. Alexandrou et al (ETMC22, [17]) even quote a 6.1σ effect and conclude that this striking low energy anomaly definitely needs to be understood.

On the experimental side, although some difficulties have also appeared (see e.g. [13]) which may point to an underestimation of the experimental errors in the determination of the R ratio, some authors have studied the deviation which should be imposed to the data in order to cancel the observed differences in a_μ values [19]. They have found that shifts in the e^+e^- to hadrons cross section needed to bridge it are excluded above $s > 0.7$ GeV at the 95% C.L., while prospects for Δa_μ originating below that energy are deemed improbable given the required increases in the hadronic cross section. This conclusion would also apply to the $L - R$ difference, which is of the same order of size.

The overall conclusion seems now to be that the (R) and (L) HVP contributions to the muon $g - 2$ cannot be reconciled, neither from the data-driven R -ratio side, nor from the numerical calculation lattice side, so that one is led to conclude that a new genuine effect has appeared, linked to the muon $g - 2$ discrepancy but different from it and now reaching the same highly significant statistical level.

2 Special Scale-Relativity

We suggest here that both the muon $g - 2$ discrepancy and the new HVP problem can be explained in the special scale-relativity (SSR) framework. This is achieved by accounting for the correction to the relation between mass ratios and Compton length ratios which occurs in this theory, where one introduces a generalized Compton relation based on the true nature of the Planck length as being a limit-length invariant under scale transformations.

We have long ago argued [20, 21, 22, 23] that the geometry of space-time has no reason to remain Minkowskian (i.e. classical) in the quantum domain, at scales smaller than

the de Broglie scale, owing to the fundamentally quantum nature of all existing objects at these scales. We have shown that the quantum space-time [21], on the contrary, is expected to be scale-dependent, i.e. fractal in a general meaning [23, 24] in the quantum realm.

Moreover, the principle of relativity, which was up to now applied, in current theories of relativity, to origin, orientation and motion transformations of the coordinate system [25], has been extended to apply also to scale transformations of the measurement resolutions, which are added to the variables that characterize its (relative) state [21, 22, 23, 24].

The scale-relativity theory is the general framework built from this first principle [23, 24] including the construction of new scale laws of log-Lorentzian form [22]; a geometric foundation of quantum mechanics [26] and of gauge fields [27] in terms of a nondifferentiable and therefore fractal space-time continuum whose geodesics define wave functions; and the suggestion of the possibility of a new macroscopic quantum-type mechanics (based on a constant different from \hbar) relevant to chaotic systems beyond their horizon of predictability [23, 28, 29, 30, 31, 32], in particular to turbulence in velocity-space [33].

Here we are concerned with only the log-Lorentzian scale laws aspect of the theory, specifically, with the fact that inverse length-scales and mass-scales, which are identical in standard quantum mechanics (QM), become different in the SSR framework.

We have mathematically proved [22], [23, Chapt. 6], [24, Chapt. 4.4] that the general solution to the special relativity problem (i.e., find the linear laws of transformations which come under the principle of relativity) is, as well for motion as for scales, the Lorentz transformation. This proof is based on only two axioms, internality of the composition law and reflection invariance, which are both expressions of the only principle of relativity.

We know since Poincaré and Einstein that the special relativity law of composition of two velocities u and v writes $w = (u + v)/(1 + uv/c^2)$. In the same way, the general law of composition of length-scale ratios, $r \rightarrow r' = \rho \times r$, writes in special scale-relativity theory [22] $\mathbb{W} = (\mathbb{U} + \mathbb{V})/(1 + \mathbb{U}\mathbb{V}/\mathbb{C}^2)$, where $\mathbb{U} = \ln(r/\lambda)$, $\mathbb{V} = \ln \rho$ and $\mathbb{W} = \ln(r'/\lambda)$ and \mathbb{C} is a constant whose meaning will be specified in the following.

Another relevant result of special (motion)-relativity is that the Galilean relation of proportionality between velocity and momentum, $p/mc = v/c$, becomes

$$\frac{p}{mc} = \frac{v/c}{\sqrt{1 - (v/c)^2}}. \quad (5)$$

In the same way as velocity characterizes the *state of motion* of the reference system, we consider in SSR that the length and time measurement resolutions characterize its *state of scale* [21, 22, 23, 24]. The difference is that motion transformations constitute an additive group, while scale transformations are a multiplicative group. However, using the Gell-Mann-Levy method, one can show [23, 24] that the natural variables for describing length-scales and their transformations are the logarithm of a scale ratio, $\mathbb{V} = \ln(r/\lambda)$, so that one recovers an additive group in terms of these variables.

In analogy with the various levels of theories of motion-relativity, one can first define a ‘‘Galilean scale-relativity’’ (GSR) framework, which just corresponds to the usual laws of dilation and contraction, expressed by the direct product $r'/\lambda = \rho \times r/\lambda$. The standard de Broglie law which relates momentum and length-scales in quantum mechanics, $p = \hbar/\lambda$, can be established from Noether’s theorem in this framework. This is similar to the obtention, in motion-relativity theory, of the standard relation $p = mv$ from uniformity

of space. It may be also written as a direct equality $\ln(p/p_0) = \ln(\lambda_0/\lambda)$, with $p_0\lambda_0 = \hbar$. The generalization of this relation in SSR involves a log-Lorentz factor [22, 23, 24]:

$$\ln \frac{p}{p_0} = \frac{\ln(\lambda_0/\lambda)}{\sqrt{1 - \ln(\lambda_0/\lambda)^2/\mathbb{C}^2}}. \quad (6)$$

The usual GSR law is clearly recovered in the limit $\mathbb{C} \rightarrow \infty$. The meaning of this constant can be clarified by expressing it also in terms of the reference scale λ_0 :

$$\mathbb{C}_0 = \ln \frac{\lambda_0}{\lambda_{\mathbb{P}}}. \quad (7)$$

This introduces a minimal scale $\lambda_{\mathbb{P}}$ which is invariant under dilations and contractions, unreachable and unpassable, whatever the scale λ_0 which has been taken as reference [22, 23]. Moreover, momentum-energy now tends to infinity when the length-time scale tends to this limit, which therefore plays the role of the zero scale interval of the standard theory. This remarkable property has naturally led us [22] to identify it with the Planck length-scale,

$$\lambda_{\mathbb{P}} = \sqrt{\frac{\hbar G}{c^3}}. \quad (8)$$

Let us now analyse the current situation of the muon $g - 2$ in the light of the SSR framework. The fact that there is no strong effect on the electron but only on the muon points toward a manifestation of the mass-scale increase by a factor ≈ 200 between the electron and the muon, which is the only difference between these two particles (apart from their consequently different lifetimes).

In order to apply the SSR framework to the muon $g - 2$ problem, we need to specify the nature of two fundamental scales. The first is the scale at which occurs the transition between Galilean scale relativity (GSR) and Lorentzian special scale relativity (SSR). A natural identification of this transition is with the Compton scale of the electron λ_e [22, 23, 24]. Indeed, physics changes drastically at scales smaller than λ_e , in a way that is directly related to our purpose. (1) As remarked by Landau [34], the very nature of positions and lengths is fundamentally changed at these scales, because of the inevitable production of electron-positron pairs in the process of measuring the coordinates of an electron. This formation of new particles in a way which cannot be detected by the process itself renders meaningless the measurement of the electron coordinates. (2) The various physical quantities, in particular masses and charges, become explicitly dependent on scale below λ_e (running), a behaviour that is currently accounted for in terms of vacuum polarization and radiative corrections and well described by the renormalization group equations.

The second fundamental scale that should be known is the reference scale defining the constant \mathbb{C}_0 . It is naturally given by the Compton length of the particle studied [22]. When the particle considered is the electron, both scales coincide. But here we are concerned with the muon, so that the SSR constant is given by the Compton length of the muon, i.e. $\mathbb{C}_\mu = \ln(\lambda_\mu/\lambda_{\mathbb{P}}) \approx \ln(m_{\mathbb{P}}/m_\mu) = 46.196$. This is a new configuration, in which the electron Compton length is larger than the reference length. This is justified by the fact that the e^+e^- pairs considered here are virtual, i.e. they are interpreted as being part of the set of fractals geodesics which constitutes the muon as a whole.

A possible intervention of SSR corrections can therefore be expected, since the ratio of the muon and electron Compton lengths is slightly different from their inverse mass ratio in this framework. Taking $m = m_0$ as reference scale, the SSR generalized Compton relation [22, 23, 24] writes:

$$\ln \frac{\lambda_\mu}{\lambda_e} = \frac{\ln(m_e/m_\mu)}{\sqrt{1 + \ln^2(m_e/m_\mu)/\mathbb{C}_0^2}}. \quad (9)$$

Therefore, the length-scale ratio becomes $y = \lambda_\mu/\lambda_e = x^{1/\sqrt{1+(\ln x/\mathbb{C}_0)^2}} \approx 1/200$ when $\mathbb{C}_0 = \mathbb{C}_\mu$ instead of the mass ratio $x = m_e/m_\mu \approx 1/207$. We recover as expected the standard Compton relation $\lambda_\mu = \hbar/m_\mu c$ in the limit $\mathbb{C}_0 \rightarrow \infty$, i.e. $\lambda_{\mathbb{P}} \approx 0$.

3 SSR correction to the HVP contribution to muon $g - 2$

Let us first apply the SSR theory to the discrepancy of the muon $g - 2$ HVP contribution between the data-driven (R) and lattice QCD (L) approaches. Its applicability to this problem is straightforward owing to the fact that the lattice QCD calculation is performed in space-time while the R -ratio measurements and the contribution to the muon $g - 2$ derived from it are performed in terms of energy-momentum. In the SSR view, the two calculations are not identical and there must therefore exist a transformation between them, so that the lattice result should be corrected.

As regard the light-by-light contribution, the data-driven and lattice QCD results are in excellent agreement [9]: this comes in support of our interpretation, since there is no mass scale directly involved in that case, so that we do not expect any SSR effect.

Actually, the lattice QCD numerical calculation is made in position space while there is no possible direct measurement of length and time intervals at the muon scale and below. The smallest scale where direct length measurements have been performed is the Bohr atomic scale, while, from the Landau argument recalled above, they become impossible below the Compton electron scale. The lattice calculation therefore relies on the assumption of the universality of the standard Compton relation $\lambda \times m = \hbar$, although it is a mere postulate of quantum mechanics which cannot be directly established at high energies. In the scale-relativity theory, on the contrary, one derives it (and more generally the Einstein-de Broglie relations) from first principles and space-time symmetries [26, 24] as recalled hereabove. The passage to Lorentzian-form laws is then a natural generalization in both scale and motion cases and can be directly deduced from the principle of relativity itself in its special version [22, 23, 24].

We have estimated the SSR correction to the lattice QCD results from the fact, emphasized by BMW, that a relative error of the lattice spacing a propagates into about twice as large a relative error on a_μ [9]. As long as the SSR correction remains small enough it can be treated as such an error, so that one expects $\delta a_\mu = 2a_\mu \delta a/a$.

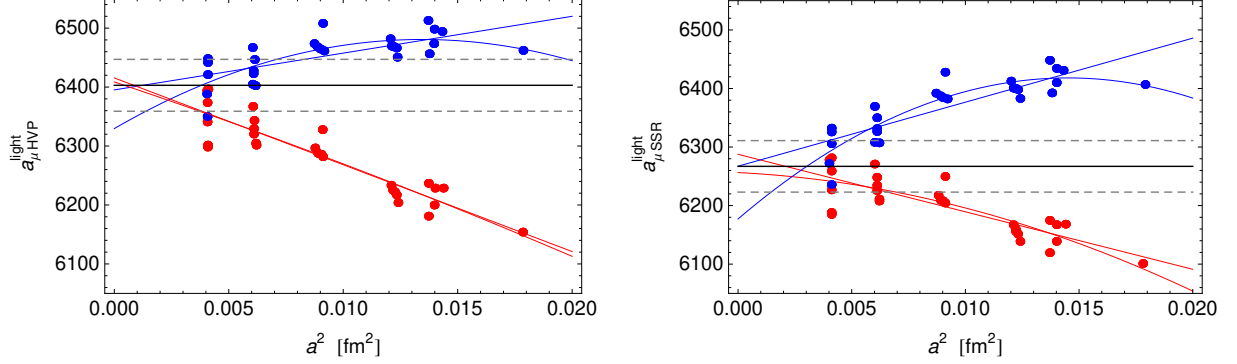


Figure 1: SSR correction to lattice QCD calculation of the HVP contribution of light quarks to muon $g-2$. The left figure shows the original BMW result (Fig. 26 of [9], where we have omitted the error bars on the individual points). The right figure shows the same points corrected for the SSR effect following Eq. 10. The blue and red curves are linear and quadratic fits of these points allowing to extrapolate the continuum limit, using the same method as BMW. The resulting corrected continuum limit is $a_\mu \approx 6270$ (down dashed horizontal line), smaller than the BMW value 6403 (up dashed line) by a factor $\approx 1/1.022$ and in good agreement with the corresponding data-driven R -ratio estimate.

We therefore predict an effect given, in terms of the variable a^2 , by:

$$\frac{\delta a_\mu}{a_\mu} = 2 \left(-1 + \left(\frac{\hbar/\sqrt{a^2}}{m_\mu c} \right)^{1-1/\sqrt{1+\ln^2\left(\frac{\hbar/\sqrt{a^2}}{m_\mu c}\right)/\mathbb{C}_\mu^2}} \right). \quad (10)$$

(Recall that, when the mass-energy is expressed in MeV and the length in fm, the conversion constant is $\hbar c = 197.327$ MeV fm). In order to estimate the SSR correction, we have considered the individual calculation points of BMW Fig. 26 [9], which exemplifies the fit method of continuum extrapolation in the case of the light quark contribution $a_\mu^{\text{HVP,light}}$. We have first verified that we recover the BMW result just using linear (a^2) and quadratic (a^2, a^4) fits on the two series of points. We obtain $a_\mu = 6400 \pm 39$ (omitting the 10^{-11} factor here and in what follows) from a weighted average of these four fits, which yields a good estimation of the BMW result (itself obtained from 258048 fits [9]), $a_\mu = 6403 \pm 44$.

Then we have applied the SSR correction to each of the individual points and used the same fitting procedure on the corrected points. The result is shown in Fig. 1. We find $a_\mu^{\text{SSR}} = 6267 \pm 44$ (where we have kept the BMW error), i.e. a difference of ≈ 130 with the BMW result. This means that the SSR effect on the lattice QCD determination of the HVP contribution to the muon $g-2$ amounts to a factor $\rho_{\text{SSR}} = 1.0217$.

Actually this value of the correction should be considered up to now as just an estimate, since it relies on an approximative relation between the deviations on a and on a_μ . We intend to improve the situation in future works. The precise value of the correction depends also on the slopes of the fits and on the smallest lattice spacing and may therefore vary from a lattice calculation to another. For example, by applying the same method to the window ETMC22 results [17], for which the smallest lattice spacing $a^2 = 0.0035$ is smaller than the BMW one ($a^2 = 0.004$ fm²), we find a slightly larger effect $\rho_{\text{SSR}} = 2065/2017 = 1.024$. Using RBC/UKQCD23 window data [14] we obtain

$\rho_{\text{SSR}} = 2065/2025 = 1.020$. This leads to an estimated correction of $\rho_{\text{SSR}} = 1.022 \pm 0.002$ for the presently available lattice QCD calculations.

This points to a potential problem: the SSR correction is expected to increase in a divergent way when $a \rightarrow 0$. However, when looking at the relative contributions from the various energies in the R -ratio version of the HVP calculation, one finds that the contributions become negligible beyond the b quark, in agreement with the lattice QCD calculations taking account only of the u , d , s and c quarks. Then there is no reason to consider lattice spacings smaller than $a^2 \approx 0.0035$, corresponding to energies larger than ≈ 3 GeV. Moreover our present estimation relies on the hypothesis of a small relative deviation on lattice spacing, and it is therefore expected to become irrelevant at high energies. It will be possible to clarify this point by integrating the SSR correction directly in the numerical lattice QCD calculation.

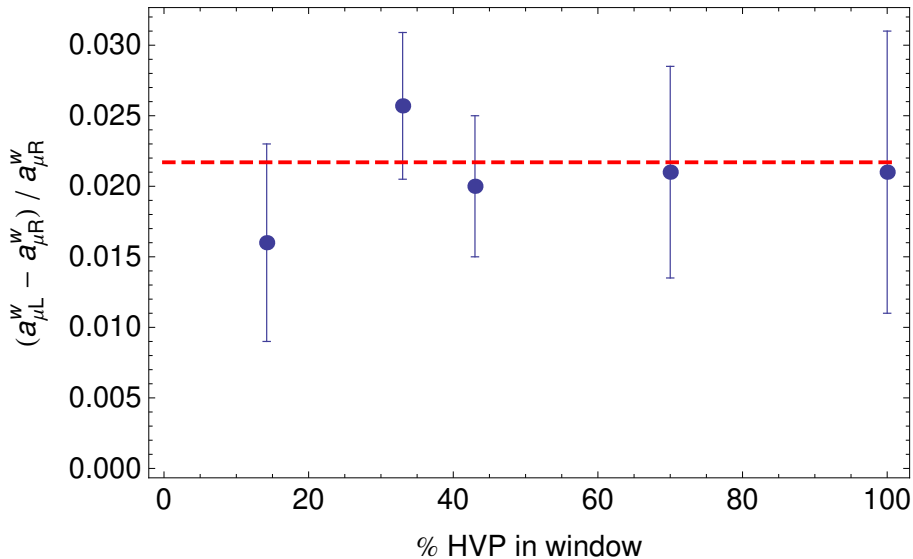


Figure 2: Fractional differences between determination of the HVP contribution a_{μ}^w to the muon $g-2$, from the lattice (L) and from data-driven dispersion relation (R), with windows of various widths. It is compared to its theoretical estimation in the SSR theory (red dashed horizontal line), $\rho_{\text{SSR}} - 1 \approx 0.022$. The abscissa yields the fraction of the contribution for each window. The fractions 14%, 43% and 70% correspond to one-sided windows [12]. The 100% full result is the BMW estimate [9]. The fraction 33% gives the best result for the two-sided window 0.4 – 1 fm, using current most precise R value (quoted as Aubin et al 2019/CN/KNT in [14] and L value [14]).

The observed factor from the BMW calculation compared to the WP recommended value is $\rho_{\text{obs}} = (7075 \pm 55)/(6931 \pm 40) = 1.021 \pm 0.010$ and is therefore in full agreement with the SSR theoretical prediction. The one-sided window values of Davies et al [12] are of the same order (see Fig. 2), while the two-sided window yields, using mean R and L values, $\rho_{\text{obs}} = (2362 \pm 11)/(2295 \pm 13) = 1.0285 \pm 0.0072$, within one σ of our theoretical prediction. The best R and L values (cf. [14] Fig.11) yield a compatible and even more precise result reaching $\sim 5\sigma$ significance, $\rho_{\text{obs}} = 1.0257 \pm 0.0052$. All these estimates are in full agreement with the theoretical SSR prediction.

The one-sided window results are compatible with a constant ratio between the L and R values of the HVP effect, independent of the window size [12]. Such a constancy

is also predicted in the SSR interpretation of the effect. This allows us to estimate the expected full HVP contribution from the best determined ratio. One finds $a_\mu^{\text{HVP}} = 1.0257 \times 6931 \times 10^{-11} = 7109 \times 10^{-11}$, slightly larger than the BMW result. This leads after SSR correction to $a_\mu^{\text{HVP,L}} = (6924 \pm 57) \times 10^{-11}$, which agrees with the R result $a_\mu^{\text{HVP,R}} = (6931 \pm 40) \times 10^{-11}$ [8] (see Fig. 4).

We conclude that, once corrected for the SSR effects, the lattice QCD and data-driven estimates of the HVP contribution to the muon $g - 2$ are expected to agree with each other within uncertainties.

4 SSR correction to muon $g - 2$

Let us now remind how one can apply the SSR theory to the full muon $g - 2$ itself [35]. Since it implies corrections to the Compton relation between mass-scales and length-scales, one is led to look for a possible SSR effect only among the mass-dependent terms of the muon $g - 2$ calculation [5]. There is no such contribution to 1-loop level, i.e. in Schwinger's original $\alpha/2\pi$ correction to the magnetic moment [5]. To 2-loop, the main mass-dependent and electron-dependent term comes from the Feynman diagram shown in Fig. 3, which corresponds to the insertion of a closed lepton loop (e^+e^- pair).

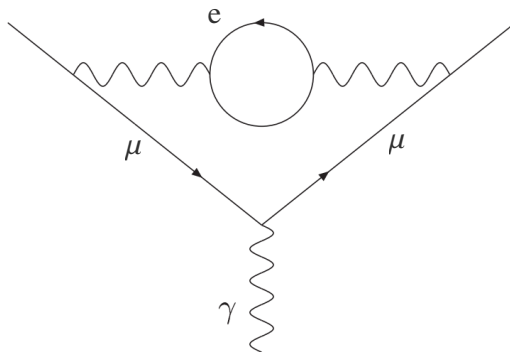


Figure 3: Feynman diagram for the leading mass-dependent term, involving e^+e^- pairs, of the fourth-order radiative correction to the scattering of a muon by an electromagnetic field, which plays a central role in the SSR correction to the theoretical calculation of the muon $g - 2$.

Let us first recall how the standard calculation of this contribution is performed without SSR correction. It has been calculated in 1949 by Karplus and Kroll (KK) [36]. It reads in position space (we denote by X the position coordinate so as not to confuse it with $x = m_e/m_\mu$ in what follows):

$$\bar{M}^{IIe} = -\frac{e\alpha^2}{4\hbar c} \int d^4 X_0 d^4 X_1 d^4 X_2 A_\mu^e(X_0) \bar{D}_F^{(2)}(X_2 - X_1) \bar{\psi}(X_1) \gamma_\nu S_F(X_0 - X_1) \gamma_\mu S_F(X_2 - X_0) \gamma_\nu \psi(X_2). \quad (11)$$

The bar on \bar{M}^{IIe} and \bar{D}_F indicates that the renormalization terms have been removed. The Feynman function D_F describes the properties of a virtual photon as modified by its interaction with the electron-positron field. The leading term is obviously

$$D_F^{(1)}(X) = -\frac{2i}{(2\pi)^4} \int e^{-ipX} \frac{d^4 p}{p^2}. \quad (12)$$

The radiative corrections to this function arise from the ability of the virtual photon to create pairs. The first term $D_F^{(2)}$ is simply due to the creation and annihilation of one pair, as can be seen in Fig. ???. After removal of the renormalization term and performing its Fourier transform, it reads

$$\bar{D}_F^{(2)}(p) = \frac{\alpha}{2\pi} \int_0^1 dv \frac{2v^2(1-v^2/3)}{p^2(1-v^2) + 4\kappa_e^2}, \quad (13)$$

where κ_e is the electron mass term. Feynman's $S_F(X)$ function describes the propagation of the particle for which the magnetic moment is computed (by looking at its elastic scattering by an electromagnetic field). This particle is an electron in the original KK calculation and a muon in the case considered here:

$$S_F(X, \kappa_\mu) = -\frac{2i}{(2\pi)^4} \int d^4p e^{-ipX} \frac{i\gamma p - \kappa_\mu}{p^2 + \kappa_\mu^2}, \quad (14)$$

where the mass term κ_μ is now the muon one.

The subsequent calculations are continued by KK in momentum space. The momentum p_1 is used to denote the momentum of the final state and p_2 the momentum of the initial state:

$$\psi(X) = \int e^{ip_2 X} \psi(p_2) d^4p_2, \quad (i\gamma p_2 + \kappa_\mu)\psi(p_2) = 0, \quad (15)$$

$$\bar{\psi}(X) = \int e^{ip_1 X} \bar{\psi}(p_1) d^4p_1, \quad \bar{\psi}(p_1)(i\gamma p_1 + \kappa_\mu) = 0. \quad (16)$$

Then KK find, for the contribution considered, the final expression:

$$\begin{aligned} \bar{M}^{IIe} = & -\frac{2ie\alpha^2}{\hbar c} \int d^4p_1 d^4p_2 A_\mu^e(p_1 - p_2) \int d^4k \int_0^1 dv \frac{2v^2(1-v^2/3)}{k^2(1-v^2) + 4\kappa_e^2} \\ & \times \bar{\psi}(p_1) \gamma_\nu \frac{i\gamma(p_1 - k) - \kappa_\mu}{(p_1 - k)^2 + \kappa_\mu^2} \gamma_\mu \frac{i\gamma(p_2 - k) - \kappa_\mu}{(p_2 - k)^2 - \kappa_\mu^2} \gamma_\nu \psi(p_2), \end{aligned} \quad (17)$$

in which we have now clearly identified the muon mass term κ_μ and the electron-positron pair mass term $2\kappa_e$.

The resulting muon vacuum polarization insertion was explicitly computed in the late 1950's [37, 38]. It is given by the double integral:

$$a_\mu^{(4)} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 du u^2(1-u) \int_0^1 dv \frac{v^2(1-v^2/3)}{u^2(1-v^2) + 4x^2(1-u)}, \quad (18)$$

where $x = \kappa_e/\kappa_\mu = m_e/m_\mu$ is the electron to muon mass ratio in the SM framework. An exact integration has been performed by Elend in 1966 [39] and the final expression has been written in compact form by Passera [40] as:

$$\begin{aligned} a_\mu^{(4)} = & \left(\frac{\alpha}{\pi}\right)^2 \left(-\frac{25}{36} - \frac{1}{3} \ln x + x^2(4 + 3 \ln x) + x^4 \left[\frac{\pi^2}{3} - 2 \ln x \ln \left(\frac{1}{x} - x \right) - \text{Li}_2(x^2) \right] \right. \\ & \left. + \frac{x}{2} (1 - 5x^2) \left[\frac{\pi^2}{2} - \ln x \ln \left(\frac{1-x}{1+x} \right) - \text{Li}_2(x) + \text{Li}_2(-x) \right] \right). \end{aligned} \quad (19)$$

A complete expansion of this expression has been given by Li et al. [41], for both cases $x > 1$ and $x < 1$. Keeping only the leading terms, it yields in terms of $\ln x$ and x :

$$a_\mu^{(4)} = \left(\frac{\alpha}{\pi}\right)^2 \left(-\frac{25}{36} - \frac{1}{3} \ln x + \frac{\pi^2}{4} x + \mathcal{O}(x^2 \ln x)\right). \quad (20)$$

In the SSR framework, this formula can no longer be correct. Indeed, the muon Compton length-scale and mass scale, when they are referenced to the electron scale, are no longer strictly inverse quantities. One should therefore make the difference between $x = m_e/m_\mu$ and $y = \lambda_\mu/\lambda_e$, whose relationship is given in Eq. (9).

An analysis of the way the above mass dependent contribution to the muon $g - 2$ is obtained shows that it depends on *both* mass and Compton length. The original KK work starts from position space then ends the calculation in momentum space through the standard assumption of Fourier transform between position and momentum representations. This calculation relies on the previous work of Dyson [42], who explicitly specifies that the mass term κ in Feynman's $S_F(X)$ function (Eq. 14) is given by the reciprocal Compton length, not the mass itself. On the other hand, we also know that masses enter as such in the $g - 2$ calculation, in particular through the threshold $2m_e$ for pair creation.

However, the two contributions are not separated in the KK calculation of the electron $g - 2$, nor in the muon calculation of Suura-Wichmann [37] and Peterman [38] derived from it. We therefore need to use another approach where the two contributions are separated. The renormalization group approach provides us with such a separation. It has been shown by Lautrup and de Raphael [43] that the equation for the contribution to the muon $g - 2$ from electron vacuum polarization insertions takes the form of a Callan-Symanzik equation

$$\left(m_e \frac{\partial}{\partial m_e} + \beta(\alpha) \frac{\partial}{\partial \alpha}\right) a_\mu \left(\frac{m_e}{m_\mu}, \alpha\right) = R \left(\frac{m_e}{m_\mu}\right), \quad (21)$$

where the right hand side of this equation is found to be vanishing as $x = m_e/m_\mu$ instead of the naive expectation x^2 from Eq. (18). In this expression, no difference is yet made between the mass ratio and inverse Compton length ratio. However, the leading logarithmic term of Eq. (20), $-\frac{1}{3} \ln x$, is now provided by the left hand side of this equation [5], while the linear term $\frac{\pi^2}{4} x$ comes from its right hand side.

In the l.h.s. of Eq. (21), $\beta(\alpha)$ is the standard QED β -function, $\beta(\alpha) = 2\alpha^2/3\pi$ to one-loop. The meaning of this equation is that the origin of the leading logarithm term $-\frac{1}{3} \ln x$ is just charge screening of the electromagnetic charge [5] and that it comes from the mere running QED coupling at muon scale,

$$\alpha_\mu = \alpha \left(1 + \frac{2\alpha}{3\pi} \ln \frac{m_\mu}{m_e}\right). \quad (22)$$

This contribution is therefore generated by the electron-muon mass ratio $x = m_e/m_\mu$.

On the contrary, if one now considers the r.h.s. of Eq. (21), a detailed analysis by Lautrup and de Raphael [43] shows that it finds its origin in the $S_F(X)$ functions in which the mass terms are actually defined as inverse Compton lengths [42]. We are therefore led to differentiate between $\ln x$ logarithmic contributions which depend on mass, and x linear contributions which we assume to actually depend on inverse Compton lengths.

Therefore the Callan-Symanzik equation for the muon $g-2$ can now be written (keeping only the leading terms) as

$$\left(x \frac{\partial}{\partial x} + \beta(\alpha) \frac{\partial}{\partial \alpha}\right) a_\mu(x, \alpha) = \left(\frac{\alpha}{2}\right)^2 y, \quad (23)$$

where $y = \lambda_\mu/\lambda_e$ while $x = m_e/m_\mu$.

The solution to order α^2 of this equation finally yields a correction to the muon $g-2$, $\delta a_\mu^{\text{SSR}} = (y-x)\alpha^2/4$, which is well approximated by

$$\delta a_\mu^{\text{SSR}} = -\alpha^2 \frac{x \ln^3 x}{8 \mathbb{C}_\mu^2} = 230 \times 10^{-11}. \quad (24)$$

Assuming that higher order power contributions x^k should also be corrected and replaced by y^k , one finds that the 2-loop x^2 power terms of Eq. (19), coming from e^+e^- pairs, yield a small correction $\delta a_\mu = 4 \times 10^{-11}$, while $x^2 \ln x$ terms yield -20×10^{-11} . Hadron loops contribute also by $\sim (m_\mu/m)^2$ terms, yielding a SSR correction $\approx +17 \times 10^{-11}$, so that these possible higher order effects cancel each other and are anyway smaller than current theoretical uncertainties (43×10^{-11}). Three-loop effects as well as tau lepton loops yield corrections smaller than 1×10^{-11} .

We therefore estimate the special scale-relativistic correction to $\delta a_\mu = (230 \pm 16) \times 10^{-11}$, yielding a theoretically estimated (SM+SSR) muon $g-2$

$$a_\mu(\text{th}) = 116592040(43) \times 10^{-11}, \quad (25)$$

in excellent agreement with the present experimental value $a_\mu(\text{exp}) = 116592061(41) \times 10^{-11}$ [7].

We finally find an SSR correction:

$$\delta a_\mu^{\text{SSR}} = (230 \pm 15) \times 10^{-11}, \quad (26)$$

where the error bar is an estimation of the effects of the possible inclusion of higher order mass terms. This special scale-relativity correction agrees within uncertainties with the observed difference between the experimental and theoretical muon $g-2$, $\delta a_\mu = (249 \pm 48) \times 10^{-11}$ (using the data-driven HVP contribution). It is also now in agreement with the SM prediction using lattice QCD for the HVP contribution once the SSR correction of this contribution is also taken into account. The future decrease by a new factor 2 of the experimental measurement of a_μ at FNAL when all runs will be taken into account is not expected to change this conclusion (see Fig. 4).

5 Conclusion

We have shown in the present paper that the two highly significant discrepancies affecting the muon anomalous magnetic moment (a 5.2σ difference between its theoretical and experimental values and another 5σ difference between the lattice QCD calculation of the HVP contribution to $g-2$ and its data-driven estimation derived from the R ratio) can be both explained in the framework of the special theory of scale-relativity (Fig. 4).

It will be possible to test these proposals with improved precision in the next years, since the Muon $g-2$ Fermilab experiment E989 aims to finally reduce the experimental

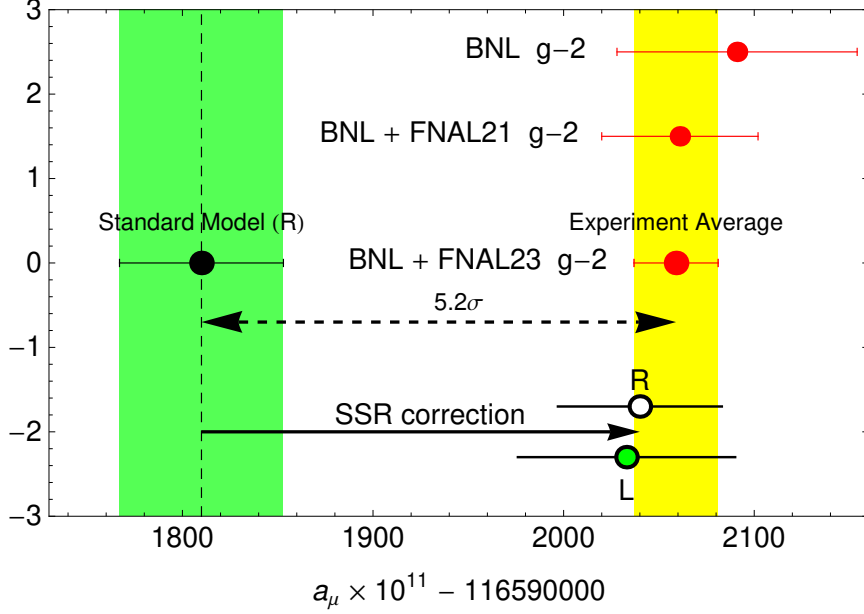


Figure 4: Illustration of the present state of the muon $g-2$ discrepancies and of their solution in the SSR framework. The green band to the left stands for the SM prediction and its error bar in the case when the HVP contribution is derived from the R ratio experimental measurement. The yellow band to the right is today's (2023) state of experimental measurements from BNL and FNAL runs 1, 2 and 3. The difference between them reaches a 5.2σ statistical significance (which may be slightly overevaluated due to possible under estimations of experimental uncertainties on the R ratio). The black arrow is the theoretically predicted SSR correction $\delta a_\mu^{\text{SSR}} = (230 \pm 15) \times 10^{-11}$. The open point is the SM-R prediction corrected for special scale-relativity effects. The green point is the SM-L prediction where the HVP contribution comes from windowed lattice QCD numerical integration, doubly corrected for SSR effects (correction to the lattice length-scale and correction to closed electron loop insertion).

uncertainty on the experimental value of the muon $g-2$ by a factor of four [44]. If this goal is attained (experimental error $\approx 13 \times 10^{-11}$), the main uncertainty in the comparison between theory and experiment will come from the hadronic theoretical contribution, for which improvements are also expected [45].

Provided the experimental result and SM theoretical prediction are confirmed in the future, one can reversely consider the two effects of muon $g-2$ (theory-experiment) difference and HVP contribution (L lattice QCD - R ratio) difference as tests of the special scale-relativity theory. These tests are both positively satisfied at the 5σ level in terms of the most precise data and numerical calculations. From the observed effect, one can derive the value of the constant \mathbb{C}_0 ,

$$\mathbb{C}_0 = \sqrt{\frac{-\alpha^2 x \ln x^3}{8 \delta a_\mu}} = 44.3_{-3.7}^{+5.0}. \quad (27)$$

The value of \mathbb{C}_μ lies within 0.4σ of this theoretical prediction and \mathbb{C}_e at 1.4σ . Expressed in terms of scale ratio, one finds $\rho_{\mathbb{P}0} = e^{\mathbb{C}_0} = 2 \times 10^{19 \pm 2}$, which clearly points toward the Planck mass-scale $m_{\mathbb{P}} = 1.22 \times 10^{19}$ GeV. This brings direct support to our deduction according to which the new length-scale, naturally introduced in SSR as being invariant

under scale transformations of the reference system, must be identified with the Planck length-scale [22]. Such a length-scale, which is unreachable, uncrossable toward smaller scales, invariant under dilatations and contractions (including the Lorentz length contraction and time dilatation), plays the same role for resolution-scales as played by the velocity of light for velocities, since c is also unreachable by massive particles, unsurpassable and invariant under motion transformations of the reference system.

Moreover, the theory could be newly put to the test in the future by performing in position space SM numerical calculations of other quantities of particle physics expected to be perturbed by SSR effects and comparing them to their data-driven counterpart derived in momentum space.

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