Approximate solution for a class of stochastic differential equation driven by stable processes

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APPROXIMATE SOLUTION FOR A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY STABLE PROCESSES

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Abstract. In this paper we consider a general class of stochastic differential equations driven by α-stable processes, $\alpha \in (1,2)$, with continuous coefficients functions with at most linear growth. An Euler-Maruyama approximate solution is proved in the L^{2p} space for any $1 \leq 2p < \alpha$. We obtain a rate of convergence of order $n^{-\frac{p^2}{2}}$. Our proposed method is new in this context and is based on a level truncation method by separating the large and small jumps of the α stable process along the Lévy-Itô decomposition. Along the paper we give some numerical simulations of stochastic models that agree with our results, namely some stable driven Ornstein-Uhlenbeck, Cox-Ingersoll-Ross and Lotka-Volterra type processes.

AMS Subject Classification: 60G17, 60G52, 60H10, 60H20, 65C30, 65C20, 60E07.
Key words and phrases: Stochastic differential equation; Non-Lipschitz condition, Approximation, Stable process.

1. Introduction

Models such as Stochastic Differential Equations (SDEs) are a natural choice for modelling the time evolution of dynamic systems and have a wide range of applications in many disciplines such as population dynamics, epidemiology, finance and biology. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions. Moreover, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is rich enough to contain other processes considered in this paper, which aims to study the following stochastic differential equation (SDE) driven by α-stable processes, $\alpha \in (0,2)$:

$$\begin{cases} dX_t = f(X_t)dt + \phi(X_{t-})dZ_t, \quad t \in [0,T] \\ X_0 = x \in \mathbb{R}, \end{cases} \tag{1}$$

where x_0 is a starting point and $T > 0$ is a time horizon, under certain conditions on the functions f and ϕ. In the theory of stochastic processes and its applications, it is important to find existence conditions for solutions of the SDE. There are different notions of existence and uniqueness. A weak solution is a solution that satisfies the SDE (1). A strong solution is a predictable process $(X_t, \mathcal{F}_t)$ that satisfies the SDE (1). By definition, a weak solution that is predictable is a strong solution. Pathwise uniqueness holds if any two solutions are indistinguishable. Some results are available on the existence under various conditions of the coefficients of the above SDE. The existence of weak solutions to the above SDE under linear growth conditions are well-known results in stochastic calculus as well as for general Lévy processes Z, see e.g. Rong (2006); Protter (1991). For example, assuming that the drift and diffusion coefficients are continuous and have at most linear growth, then there is weak existence in L^p for any $p \in (0, \alpha)$, see Proposition 2 in Fournier (2013) for $\alpha \in (0, 2)$ and $\alpha \neq 1$. This weak existence is also contained in Rong (2006).

For non-Lipschitz coefficients and considering α-Stable driven SDE, some results have been studied before. In Bass et al. (2004); Bass (2004); Komatsu (1982), when Z is symmetric with $\alpha \in (1, 2)$, it was shown that if $f = 0$ and ϕ is Hölder continuous of order $\beta \in (0, \alpha_0)$ with $\alpha_0 = \alpha^{-1} \wedge 1$, there is a weak solution (see Theorem 1.1 in Bass et al. (2004)). If ϕ has a modulus of continuity ρ satisfying $\int_{0+} \frac{1}{\rho(x)^\alpha} = \infty$, then there is a strong solution and the solution is pathwise unique. This condition is the exact analogue of the Yamada–Watanabe condition for SDE driven by a Brownian motion. If the drift term f is Hölder continuous of order $\beta > \frac{2-\alpha}{2}$ and $\alpha \in [1, 2)$ with $\beta \in (0, 1)$ and g is Lipshitz continuous and bounded, then, the existence and uniqueness of a strong solution is proved in Priola (2012), see also Mikulevičius and Xu (2018), for non-degenerate symmetric stable processes Z with values in the higher dimensional space $\mathbb{R}^d$, $d \geq 1$. The proof is based on the associated integro-differential operators of Kolmogorov type.
In Pamen and Taguchi (2017), when ϕ is bounded and the drift function on the form $f = F(t, x)$ is a bounded β-Hölder continuous in space and η-Hölder in time with $\eta \in [1/2, 1]$ and $\beta \in (0, 1)$, the pathwise uniqueness was proved by applying Gronwall's lemma and using the elliptic version of the Kolmogorov equation and the regularity of its solution.
Recently, in Athreya et al. (2020), they established strong existence and uniqueness whenever $\phi = 1$ and the drift function f belongs to the Besov Hölder space C^β for $\beta > \frac{1}{2} - \frac{\alpha}{2}$ with Z symmetric for $\alpha \in (1, 2)$.

Suppose that ϕ is non-decreasing and Hölder continuous with index $1 - \frac{1}{\alpha}$ and f is the sum of a Lipschitz continuous function and a non-increasing function. It is shown in Li and Mytnik (2011); Fu and Li (2010) that both the strong existence and path uniqueness hold if $\alpha \in (1, 2)$ and Z has only positive jumps. This latter result has been extended in Fournier (2013).

The technique in Fournier (2013) is based on the Itô formula for jump processes and some technical integrals (see Lemmas 7 and 9 in Fournier (2013)) together with an approximate sequence iteration method.

It is possible to appreciate the basics of how to simulate SDEs numerically with only a background knowledge of the Euler-Maruyama scheme, which is a well-known method for approximating any solution. In general, the convergence of numerical methods is studied under the assumption that the coefficients of the SDE is globally Lipschitz continuous. However, this assumption is not always satisfied for SDEs used in practice (e.g. mathematical finance, optimal control problems and filtering), so the global Lipschitz-based idea is not immediately applicable. This leads to the study of the Euler-Maruyama approximation for SDEs with irregular coefficients. Some results are now available on the approximate solutions of SDEs driven by a stable process as well as on the speed of convergence of the approximation. The Itô-Tanaka technique has traditionally been used to obtain convergence rates for Euler schemes, which has important applications in stochastic financial theory.
In Pamen and Taguchi (2017), when $\phi = 1$ and the drift function $f = F(t, x)$ is a bounded β-Hölder continuous in space with $\beta \in (0, 1)$ and, η-Hölder in time with $\eta \in [1/2, 1]$, a strong approximate solution is proved and the rate of convergence for the Euler-Maruyama approximation is given. More precisely, using the Itô-Tanaka trick method and assuming that there is a strong solution with $\alpha + \beta > 2$, the rate of strong convergence was derived in the form n^{-1} when $p \geq \frac{2}{\beta}$ and in the form $n^{-p\beta/2}$ for d-dimensional truncated symmetric α-stable process when $d \geq 2$ for $\alpha \in (1, 2)$ and $1 \leq p\beta < 2$ or $p \in [1, 2]$.

Similarly, Mikulevičius and Xu (2018) find a strong approximate solution and a rate of convergence of order $n^{-p\beta/\alpha}$ when $p \in (0, \alpha/\beta)$ and Z is a symmetric stable process of index $\alpha \in [1, 2)$ whenever f is β-Hölder continuous with $\beta > 1 - \frac{\alpha}{2}$ and ϕ is a bounded Lipshitz continuous function. See also Wu and Hao (2023) for a weak convergence rate under similar conditions.

In this paper, we consider the following SDE:

$$X_t = x_0 + \int_0^t f(X_s)ds + \int_0^t \phi(X_{s-})dZ_s, \quad t \in [0, T], \tag{2}$$

where Z is a α-stable process with $\alpha \in (1, 2)$, not necessarily symmetric, under the following linear growth assumptions for $p > 0$:

(C1)

$$|\phi(x)|^p \leq L_\phi(1 + |x|^p) \qquad L_\phi > 0 \quad x \in \mathbb{R}.$$

(C2)

$$|f(x)|^p \leq L(1 + |x|^p) \qquad L > 0 \quad x \in \mathbb{R}.$$

Particularly, we also consider the case where f and ϕ are continuous Lipchitz and Hölder functions:

(A1) The function ϕ is Hölder continuous with exponent $\beta \in (0, \frac{\alpha}{2})$ that is

$$|\phi(y) - \phi(z)| \leq c_\phi |y - z|^\beta \qquad c_\phi > 0 \quad y, z \in \mathbb{R}.$$

(A2) The function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous L-Lipchitz function that is :

$$|f(x) - f(y)| \leq L|y - z| \qquad L > 0 \quad y, z \in \mathbb{R}.$$

We investigate the existence of a consistent Euler-Maruyama approximate solution, defined as follows.

$$\tilde{X}_t^n = x_0 + \int_0^t f(\tilde{X}_{\pi_s^n}^n) ds + \int_0^t \phi(\tilde{X}_{\pi_s^n}^n) dZ_s, \tag{3}$$

where $\pi_t^n = \max\{t_k, t_k \leq t\}$ with the size step $h = \Delta_n = n^{-\delta}$ for some $\delta > 0$; $\tilde{X}_t^n = X_{t_k}^n$ for all $k = 0, ..., n-1$ and

$$\begin{cases} X_{t_{k+1}}^n = X_{t_k}^n + f(X_{t_k}^n)\Delta_n + \phi(X_{t_k}^n)(Z_{t_{k+1}} - Z_{t_k}). \\ X_0^n = x. \end{cases}$$

While this process is pleasant to handle from a theoretical point of view, it is easy for simulation purposes at points t_k as well as its linearly interpolated version between points t_k and t_{k+1} with $t_{k+1} - t_k = \Delta_n$.

Compared to the above mentioned previous works, we have established a new result on the weak approximate solution as well as the rate of convergence for the SDE (2). Moreover, our proof methodology is new and not based on the Itô-Tanaka trick method, but on the Lévy-Itô decomposition together with a truncation method for stable integrals developed in Manou-Abi (2015); Joulin and Manou-Abi (2015). Our result extends the aforementioned work on approximate solution to the case of Lipschitz drift function and Hölder continuous diffusion coefficients. This paper is organised as follows. In Section 2, we discuss the primary analytical tools, introduce notation and preliminary results on weak existence for SDE (2). In Section 3, we first give some preliminaries moment bound results of a solution of the SDE (2). After that we present our main theorems. Namely, if the functions ϕ and f are continuous and satisfy linear growth conditions then for any $1 \leq 2p < \alpha$, then there exists a weak approximate solution of (2) in L^{2p} space and the rate of convergence is of order $n^{-\frac{p^2}{2}}$. In section 4 we present the proofs of the main results. The last section is devoted to numerical simulations of various stochastic models that are consistent with our results.

2. Preliminaries

In this part, we recall the definition of Lévy processes and introduce the α-stable process as a member of the class of Lévy processes. We also introduce some notations that will be useful throughout the paper. We follow the presentation in Ken-Iti (1999); Applebaum (2009). We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and satisfying the usual conditions as above.

Definition 2.1. *A real stochastic process $(Z_t)_{t\in[0,T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is said to be a Lévy process if $Z_0 = 0$ almost surely (a.s.) with the following properties:*

1. *Z is stochastically continuous i.e., for any $\epsilon > 0$ and $t, s \geq 0$,*
$$\lim_{s\to 0} \mathbb{P}\left(|Z_{t+s} - Z_t| > \epsilon\right) = 0.$$
2. *Z has stationary increments, i.e., for $0 \leq s \leq t$, the distribution of $Z_t - Z_s$ does not depend on s, namely has the same distribution as Z_{t-s}.*
3. *Z has independent increments, i.e., for $0 \leq s \leq t$, $Z_t - Z_s$ is independent of $\mathcal{F}_s$.*
4. *The paths of Z are $\mathbb{P}$-a.s. right-continuous in $t \geq 0$ and has left limits in $t > 0$ (cádlág).*

Note that the last point above can be dropped, since one can prove that there is always a cádlág modification of the stable process Z. Set $\Delta Z_t = Z_t - Z_{t-}$. The jumps part of the stochastic process Z can be described by its Poisson random measure (jump measure of Z on the interval $[0, t]$), defined as

$$\mu(t, A) = \sum_{0\leq s\leq t} \mathbb{I}_A(\Delta Z_s), \quad A \in \mathcal{B}(\mathbb{R}^*),$$

the number of jumps of Z on the interval $[0, t]$ whose size lies in the set A bounded below. For such A, the process $\mu(, A)$ is a Poisson process and the Lévy measure ν is defined by $\nu(A) := \mathbb{E}(\mu(1, A))$. In the following, we denote the compensated random martingale measure by

$$\tilde{\mu}(t, A) = \mu(t, A) - t\nu(A).$$

A special class of Lévy processes that can contain infinitely many jumps is the following class of α-stable processes. We now follow the presentation in Cattiaux and S. Manou-Abi (2014); Manou-Abi (2015).

Definition 2.2. *A real stochastic process $(Z_t)_{t\in[0,T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is said to be an α-stable process with characteristics (b, c_+, c_-), if it is a Lévy process with the following characteristic function :*

$$\varphi_{Z_t}(u) = \exp t\left(iub + \int_{-\infty}^{+\infty} (e^{iuy} - 1 - iuy\mathbb{I}_{|y|\leq 1})\nu(dy)\right), \quad t \in [0, T],$$

where b stands for the drift parameter of Z and ν the Lévy measure defined on $\mathbb{R}\setminus\{0\}$ by

$$\nu(dx) := \frac{dx}{|x|^{\alpha+1}}\,\left(c_+\,1_{\{x>0\}} + c_-\,1_{\{x<0\}}\right).$$

The parameters c_+, c_- above are non-negative with furthermore $c_+ + c_- > 0$ and $c_+ = c_-$ when $\alpha = 1$. The process is said to be symmetric if $c_+ = c_- := c$. It i said to be strictly α-stable if $b = 0$. In the case $\alpha \in (1,2)$, the drift parameter is given by

$$b \;:=\; -\int_{|y|>1} y\,\nu(dy) \;=\; -\frac{(c_+ - c_-)}{\alpha - 1} \quad \text{when } \alpha \neq 1.$$

An α-stable process is closely related to the notion of self-similar process. The process Z is said to be strictly α-stable if we have the self-similarity property

$$k^{-1/\alpha}\,(Z(kt))_{t\in[0,T]} \;\overset{d}{=}\; (Z(t))_{t\in[0,T]},$$

where $k > 0$ and the equality $\overset{d}{=}$ is understood in the sense of finite dimensional distributions. Note that α-stable processes are interesting due to the self-similarity property and the fact that the Lévy measure and the Lévy-Itô decomposition are almost completely explicit for the one dimensional case. In order to control the jump size of Z and the momentum behaviour of a stable stochastic integral, we introduce the following truncation method developed in our previous paper Manou-Abi (2015) and also in Joulin and Manou-Abi (2015). As an Lévy process, Z is a semimartingale with respect to $\mathcal{F}_t^Z := \sigma(Z_s : s \in [0,t])$, $t \in [0,T]$, provided it satisfies the usual hypothesis, i.e. completeness and right-continuity. The Lévy-Itô decomposition is given by

$$Z_t = b_R\,t + \int_0^t\int_{|x|\leq R} x\,(\mu-\sigma)(ds,dx) + \int_0^t\int_{|x|>R} x\,\mu(ds,dx), \quad t\in[0,T],$$

where R is some arbitrary positive truncation level (classically chosen to be 1) and μ is a Poisson random measure on $[0,T]\times\mathbb{R}$ with intensity $\sigma(dt,dx) = dt\otimes\nu(dx)$. Here b_R is the drift parameter given by

$$b_R \;:= b + \int_{1<|x|\leq R} x\,\nu(dx).$$

Denote respectively by

$$Z^{R-} = \int_0^t\int_{|x|\leq R} x\,(\mu-\sigma)(ds,dx) \quad \text{and} \quad Z^{R+} = \int_0^t\int_{|x|>R} x\,\mu(ds,dx).$$

The first has a compactly supported Lévy measure and is a square-integrable martingale with infinitely many jumps bounded by R on each compact time interval, while the second is an integrable compound Poisson process with

jumps larger than R. Let K be a $(\mathcal{F}_t^Z)_{t\in[0,T]}$-predictable process belonging to $L^2(\Omega\times[0,T])$ that is

$$\int_0^T \mathbb{E}[K_t^2]\,dt < +\infty.$$

Then the stable integral $X_T := \int_0^T K_s\,dZ_s$ is well-defined as a stochastic integral with respect to the Lévy-Itô decomposition of Z, that is $X = A^R + X^{R-} + X^{R+}$ where for $t\in[0,T]$:

$$A_t^R := b_R\int_0^t K_s\,ds, \quad X_t^{R-} := \int_0^t K_s\,dZ_s^{R-} \quad \text{and} \quad X_t^{R+} := \int_0^t K_s\,dZ_s^{R+}.$$

It is also known in Rosinski and Woyczynski (1986), that if Z is a symmetric stable process and K is a $\mathcal{F}^Z$-predictable process belonging to $L^\alpha(\Omega\times[0,T])$, i.e.,

$$\int_0^T |K_t|^\alpha dt < \infty \quad a.s.,$$

then the stable integral $X_T := \int_0^T K_s dZ_s$ is also well defined. The following elementary and classical Gronwall inequality will be useful to make some estimates in this paper.

Lemma 2.1. *Let C_1, $C_2 > 0$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that for all $t\in[0,T]$,*

$$H(t) \le C_1 + C_2\int_0^t H(r)dr.$$

Then

$$H(t) \le C_1 e^{C_2 t}, \qquad t\in[0,T].$$

A simpler proof can be done in the following way.

Proof. From the assumption (2.1), we deduce that

$$\frac{d}{dt}\Big(e^{-K_2 t}\int_0^t \phi(r)dr\Big) \le K_1 e^{at}e^{-K_2 t}.$$

Integrating this inequality yields to

$$e^{-K_2 t}\int_0^t \phi(r)dr \le \frac{K_1 e^{at}}{K_2}\,(1-e^{-K_2 t}),$$

so that

$$\phi(t) \le K_1 e^{at} + K_2\int_0^t \phi(r)dr \le K_1 e^{at}e^{K_2 t} \le K_1 e^{(a+K_2)T}, \quad \forall\, t\in[0,T].$$

□

Let us recall a weak existence result from Fournier (2013), Proposition 2.

Lemma 2.2 (Fournier (2013)). *Let Z be a α-stable process with index $\alpha \in (0,2)$ and $\alpha \neq 1$. Let f and ϕ be continuous functions with at most linear growth* **(C1)** *and* **(C2)**. *There is a weak solution for the SDE (2) such that for any $p \in (0,\alpha)$, we have*

$$\mathbb{E} \sup_{t\in[0,T]} |X_t|^p < \infty. \tag{4}$$

Typical examples of conditions **(C1)** and **(C2)** are Lipschitz and Hölder continuous functions.

In the remainder of this paper we will use the following convention for constants: C and C_* denote a positive boundary constant whose value may change from line to line. Constants denoted by $C_{\beta,L}$, $C_{\beta,L,T}$, $C_{\beta,\alpha}$, $C_{\phi,\beta,\alpha}$, C_ϕ, $C_{\beta,\alpha,\phi,b}$ are all positive and their exact values are not important. The dependency of constants on parameters is given in parentheses if necessary, e.g. we will often use the following constants

$$C_1(\alpha) = \frac{c_+ + c_-}{\alpha}, \quad C_2(\alpha) = \frac{c_+ + c_-}{2-\alpha}, \quad C(\alpha, b) = 8b^2 + 8C_1(\alpha)C_2(\alpha),$$

$$C(\beta, L) = \max\left(2^{2\beta}L, 2^{2\beta}|f(0)|^{2\beta}\right), \quad C(\phi) = \max\left(2\phi(0)^2, 2c_\phi^2\right).$$

Sometimes 2β is replaced by p and c_ϕ by L_ϕ.

3. Main Results

In this section we aim to present the main theorems of this paper. We assume $\alpha > 1$ and we first consider a given weak solution of the SDE (2). Recall that L^p-boundedness property for any $p \in (0,\alpha)$ of the stochastic integral process $\int_0^t \phi(X_s)dZ_s$ is also well kwown in the literature whenever Z is a symmetric stable process or when ϕ is non-negative, see Hongwei (2010), Lemma 2.4. or Mikulevičius and Xu (2018) whenever the integrand process $(\phi(X_t))_t \in L^\alpha(\Omega \times [0,T])$. The L^p boundedness property for a weak solution is well known, as in Lemma 2.2. However, we give a different proof in $L^{2p}(\Omega \times [0,T])$ space with $1 \leq p = 2\beta < \alpha$, which is a new methodology that we describe in the following lines.

Lemma 3.1. *Let Z be an α-stable process with index $\alpha \in (1,2)$ and consider a weak solution $(X_t)_{t\in[0,T]}$ of the SDE (2) starting at point x_0. Under the assumptions* **(C1)** *and* **(C2)**, *for any $1 \leq 2p < \alpha$, there exists a positive constant C depending on $L, p, \alpha, L_\phi, b, c_+, c_-$ and T such that*

$$\sup_{t\in[0,T]} \mathbb{E}|X_t|^{2p} \leq C\big(|x_0|^{2p} + 1\big). \tag{5}$$

Lemma 3.2. *Let Z be an α-stable process with index $\alpha \in (1,2)$ and consider a weak solution $(X_t)_{t\in[0,T]}$ of the SDE (2) starting at point x_0. Under the*

assumptions **(A1)** *and* **(A2)**, *for any* $1 \leq 2\beta < \alpha$, *there exists a positive constant* C *depending on* $L, \beta, \alpha, c_\phi, b, c_+, c_-$ *and* T *such that*

$$\sup_{t\in[0,T]} \mathbb{E}|X_t|^{2\beta} \leq C\big(|x_0|^{2\beta} + 1\big). \tag{6}$$

Remark 3.1. *This result holds (according to the methodology of the proof) if we replace the assumption* **(A2)** *by* f *being a sum of a Lipschitz function and a negative function. If* ϕ *is bounded, the result holds again.*

For the sake of simplicity and clarity, we give the proof of Lemma 3.1 as follows.

Proof. Consider $1 \leq 2\beta < \alpha$ and $\alpha \in (1, 2)$. We have

$$\begin{aligned}
\mathbb{E}|X_t|^{2\beta} &\leq 3^{2\beta-1}\,|x_0|^{2\beta} + 3^{2\beta-1}\mathbb{E}\left|\int_0^t f(X_s)ds\right|^{2\beta} \\
&\quad + 3^{2\beta-1}\mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta} \\
&\leq 3^{2\beta-1}\,|x|^{2\beta} + 3^{2\beta-1}\mathbb{E}\left|\int_0^t f(X(s))ds\right|^{2\beta} \\
&\quad + 3^{2\beta-1}\mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta} \\
&= 3^{2\beta-1}\Big(|x_0|^{2\beta} + J_1(t,\beta) + J_2(t,\beta)\Big),
\end{aligned}$$

where

$$J_1(t,\beta) = \mathbb{E}\left|\int_0^t |f(X_s)|ds\right|^{2\beta} \quad \text{and} \quad J_2(t,\beta) = \mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta}.$$

Note that from the Lipschitz condition **(A2)** we can derive the following linear growth condition:

$$|f(x)|^{2\beta} \leq C_{\beta,L}(1 + |x|^{2\beta}). \tag{7}$$

Using Hölder inequality we have:

$$J_1(t,\beta) \leq t\int_0^t \mathbb{E}|f(X_s)|^{2\beta}ds \leq C_{\beta,L,T}\int_0^t (1 + \mathbb{E}|X_s|^{2\beta})ds. \tag{8}$$

Note that

$$J_2(t,\beta) = \mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta} = 2\beta\int_0^{+\infty} x^{2\beta-1}\mathbb{P}\left(\left|\int_0^t \phi(X_s)dZ_s\right| \geq x\right)dx.$$

Let us first establish a tail bound for the stochastic integral $\int_0^t \phi(X_s)dZ_s$ appearing in $J_2(t,\beta)$ if ϕ satisfies the assumption **(A1)**. We have

$$\begin{aligned}\mathbb{P}\left(\left|\int_0^t \phi(X_s)dZ_s\right| \geq x\right) &\leq \mathbb{P}\left(\left|\int_0^t b_R\phi(X_s)ds\right| \geq \frac{x}{2}\right) \\ &+ \mathbb{P}\left(\left|\int_0^t \int_{|y|\leq R} y\,\phi(X_s)\tilde{\mu}(ds,dy)\right| \geq \frac{x}{2}\right) \\ &+ \mathbb{P}\left(\left|\int_0^t \int_{|y|\leq R} y\,\phi(X_s)\mu(ds,dy)\right| \neq 0\right) \\ &:= R_1(x) + R_2(x) + R_3(x).\end{aligned}$$

By Chebychev's inequality and Cauchy-Schwarz inequality, we have:

$$R_1(x) \leq \frac{4b_R^2}{x^2}\mathbb{E}\left|\int_0^t \phi(X_s)ds\right|^2 \leq \frac{4b_R^2}{x^2}t\int_0^t \mathbb{E}(\phi(X_s)^2 ds. \tag{9}$$

But

$$\begin{aligned}b_R^2 &\leq 2b^2 + 2\nu\{y\in\mathbb{R}:\,1<|y|\leq R\}\int_{1<|y|\leq R} y^2\nu(dy) \\ &\leq 2b^2 + 2\nu\{y\in\mathbb{R}:\,|y|>1\}\int_{|y|\leq R} y^2\nu(dy) \\ &\leq 2b^2 + 2C_1(\alpha)R^2\int_{|y|\leq R}\nu(dy) \\ &\leq 2b^2 + 2C_1(\alpha)C_2(\alpha)R^{2-\alpha},\end{aligned}$$

so that

$$R_1(x) \leq t\left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\right)\int_0^t \mathbb{E}\phi^2(X_s)ds.$$

From the Hölder condition, we have the following linear growth condition:

$$|\phi(x)|^2 \leq C(\phi)\,(1+|x|^{2\beta}). \tag{10}$$

Therefore,

$$R_1(x) \leq tC(\phi)\left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\right)\int_0^t (1+\mathbb{E}|X_s|^{2\beta})ds. \tag{11}$$

From Chebychev's inequality and the isometry formula for Poisson stochastic integrals and the Hölder condition, we have

$$\begin{aligned} R_2(x) &\leq \frac{4}{x^2}\mathbb{E}\Big(\int_0^t \int_{|y|\leq R} y\,\phi(X_s)\,\tilde{\mu}(ds,dy)\Big)^2 \\ &\leq \frac{4}{x^2}\int_{|y|\leq R} y^2\nu(dy)\int_0^t \mathbb{E}\phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha)\,R^{2-\alpha}}{x^2}\int_0^t \mathbb{E}\phi(X_s)^2 ds \end{aligned}$$

and,

$$R_2(x) \leq \frac{4C_2(\alpha)\,R^{2-\alpha}}{x^2}C(\phi)\int_0^t \big(1+\mathbb{E}|X_s|^{2\beta}\big). \tag{12}$$

Let us now deal with $R_3(x)$ in the spirit of Joulin and Manou-Abi (2015). Note that

$$N_t^{\phi} = \int_0^t \int_{|y|>R} y\,\phi(X_s)\,\mu(ds,dy)$$

is a compound Poisson stochastic integral. Denote T_1^R the first jump time of the Poisson process $\mu\big(\{y\in\mathbb{R}:|y|>R\}\times[0,t]\big)$ on the set $\{y\in\mathbb{R}:|y|>R\}$. Since T_1^R is exponentially distributed with parameter $\nu\big(\{y\in\mathbb{R}:|y|>R\}\big)$, see e.g. Ken-Iti (1999), Theorem 21.3; we have

$$\begin{aligned} &\mathbb{P}(N_t^{\phi}\neq 0) \leq 1-\mathbb{P}(T_1^R>t) \\ &= 1-\exp -t\nu\big(\{y\in\mathbb{R}:|y|>R\}\big) \end{aligned}$$

and,

$$R_3(x) \leq t\nu\big(\{y\in\mathbb{R}:|y|>R\}\big) \leq tC_1(\alpha)\,R^{-\alpha}. \tag{13}$$

Combining the three estimates above in (11), (12) and (13) we get

$$\begin{aligned} \mathbb{P}\left(\left|\int_0^t \phi(X_s)dZ_s\right|\geq x\right) &\leq tC(\phi)\Big(\frac{8b^2}{x^2}+\frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\Big)\int_0^t(1+\mathbb{E}|X_s|^{2\beta})ds \\ &\quad+\frac{4C_2(\alpha)\,R^{2-\alpha}}{x^2}C(\phi)\int_0^t\big(1+\mathbb{E}|X_s|^{2\beta}\big)ds+tC_1(\alpha)R^{-\alpha}. \end{aligned}$$

Set

$$R = x \geq 1.$$

Thus for all $x\geq 1$, we have

$$\begin{aligned} \mathbb{P}\left(\left|\int_0^t \phi(X_s)dZ_s\right|\geq x\right) &\leq tC_1(\alpha)x^{-\alpha}+4C_2(\alpha)\,x^{-\alpha}C(\phi)\int_0^t\big(1+\mathbb{E}|X_s|^{2\beta}\big)ds \\ &\quad+x^{-\alpha}t\,C(\phi)\,C(\alpha,b)\int_0^t\big(1+\mathbb{E}|X_s|^{2\beta}\big)ds. \end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta} &= 2\beta\int_0^{+\infty} x^{2\beta-1}\mathbb{P}\left(\left|\int_0^t \phi(X_s)dZ_s\right| \geq x\right),\\ &\leq 1+2\beta\int_1^{+\infty} x^{2\beta-1}\mathbb{P}\left(\left|\int_0^t \phi(X_s)dZ_s\right| \geq x\right)dx\\ &\leq 1+\frac{tC_1(\alpha)}{\alpha-2\beta}+\frac{1}{\alpha-2\beta}C(\phi)\left(tC(\alpha,b)+4C_2(\alpha)\right)\int_0^t\left(1+\mathbb{E}|X_s|^{2\beta}\right)ds,\end{aligned}$$

since

$$\int_1^{+\infty} x^{2\beta-1-\alpha}dx = \frac{1}{\alpha-2\beta} \quad \text{for} 2\beta<\alpha.$$

Thus,

$$J_2(t,\beta) \leq 3^{2\beta-1}\Big(C_{\beta,\alpha}+C_{\beta,\alpha,\phi,b}\int_0^t\left(1+\mathbb{E}|X_s|^{2\beta}\right)ds\Big). \tag{14}$$

From (8) and (14), we have

$$\mathbb{E}|X_t|^{2\beta} \leq 3^{2\beta-1}\Big(|x_0|^{2\beta}+C_{\beta,\alpha,g}+C_{\beta,\alpha,g,\phi,b,L}\int_0^t(1+\mathbb{E}|X_s|^{2\beta})ds\Big)$$

and using Gronwall's lemma we get the desired result. □

To obtain the proof of Lemma 3.1, it suffices to replace (7) by $|f(x)|^{2p} \leq C_{p,L}(1+|x|^{2p})$ and (10) by $|\phi(x)|^2 \leq C(\phi)(1+|x|^{2p})$ with $p=2\beta$ together with the conditions (**C1**) and (**C2**).

Now, before announcing the main result, let us quote the following result.

Proposition 3.1. *Let Z be a α-stable process with index $\alpha \in (0,2)$ and consider a weak solution of the SDE (2) starting at point x_0. Under the assumptions* (**A1**) *and* (**A2**)*, for any $1 \leq 2\beta < \alpha$ and $\Delta_n = n^{-\delta}$ with $\delta \geq \frac{1}{2}$, there exists a positive constant C depending on $L, \beta, \alpha, c_\phi, b, c_+, c_-$ and T such that for $n \geq 1$,*

$$\sup_{|u-t|\leq\Delta_n}\mathbb{E}\,|X_u-X_t|^{2\beta} \leq C\Big(n^{-2\beta^2}+n^{-(2\beta^2+\delta)}+n^{-(2\beta^2-2\beta+\delta)}+n^{-(2\beta^2-2\beta+2\delta)}\Big) \tag{15}$$

Remark 3.2. *This result holds (according to the methodology of the proof) if we replace the assumption* **(A2)** *by f being a sum of a Lipschitz function and a negative function. It is also valid if ϕ is bounded. The relation (16) is new. In particular, for $\delta \geq 1$ we have*

$$\sup_{|u-t|\leq\Delta_n}\mathbb{E}\,|X_u-X_t|^{2\beta} \leq Cn^{-2\beta^2}.$$

Proposition 3.2. *Let Z be a α-stable process with index $\alpha \in (0,2)$ and consider a weak solution of the SDE (2) starting at point x_0. Under the assumptions* (**C1**) *and* (**C2**)*, for any $1 \leq 2p < \alpha$ and $\Delta_n = n^{-\delta}$ with $\delta \geq \frac{1}{2}$,*

there exists a positive constant C depending on $L, \beta, \alpha, L_\phi, b, c_+, c_-$ and T such that for $n \geq 1$,

$$\sup_{|u-t|\leq\Delta_n} \mathbb{E}\,|X_u - X_t|^{2p} \leq C\Big(n^{-\frac{p^2}{2}} + n^{-(\frac{p^2}{2}+\delta)} + n^{-(\frac{p^2}{2}-p+\delta)} + n^{-(\frac{p^2}{2}-p+2\delta)}\Big). \tag{16}$$

Notice that here again for $\delta \geq 1$ we have

$$\sup_{|u-t|\leq\Delta_n} \mathbb{E}\,|X_u - X_t|^{2\beta} \leq Cn^{-\frac{p^2}{2}}.$$

For the sake of simplicity, we only give the proof of Proposition 3.1.

3.1. **Error and rate of the approximation.** We aim here to study the consistency of the above mentioned continuous time version of the Euler-Maruyama approximation of the SDE (2) given by

$$\tilde{X}_t^n = x_0 + \int_0^t f(\tilde{X}_{\pi_s^n}^n)ds + \int_0^t \phi(\tilde{X}_{\pi_s^n}^n)dZ_s.$$

We are now going to establish the main results.

Theorem 3.3. *Let Z be a α-stable process with index $\alpha \in (1,2)$ and consider a weak solution $(X_t)_{t\in[0,T]}$ of the SDE (2) starting at point x_0. Under the assumptions* **(A1)** *and* **(A2)**, *for any $1 \leq 2\beta < \alpha$ and $\Delta_n = n^{-\delta}$ with $\delta \geq \frac{1}{2}$, there exists a positive constant C depending on $L, \beta, \alpha, c_\phi, b, c_+, c_-, T$ such that*

$$\mathbb{E}\sup_{t\in[0,T]} |\tilde{X}_t^n - X_t|^{2\beta}) \leq C\Big(n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta}\Big). \tag{17}$$

Theorem 3.4. *Let Z be a α-stable process with index $\alpha \in (1,2)$ and consider a weak solution $(X_t)_{t\in[0,T]}$ of the SDE (2) starting at point x_0. Under the assumptions* **(C1)** *and* **(C2)**, *for any $1 \leq 2p < \alpha$ and $\Delta_n = n^{-\delta}$ with $\delta \geq \frac{1}{2}$, there exists a positive constant C depending on $L, p, \alpha, L_\phi, b, c_+, c_-, T$ such that*

$$\mathbb{E}\sup_{t\in[0,T]} |\tilde{X}_t^n - X_t|^{2p}) \leq C\Big(n^{-\frac{p^2}{2}} + n^{-\frac{p^2}{2}-\delta} + n^{-\frac{p^2}{2}+p-\delta} + n^{\frac{p^2}{2}+p-2\delta}\Big). \tag{18}$$

This weak approximation is highly dependent on the method used, and we actually need to calculate the rate of convergence in L^{2p} space with $1 \leq 2p < \alpha$. In particular, if $\delta \geq 1$ we have

$$\mathbb{E}\sup_{t\in[0,T]} |\tilde{X}_t^n - X_t|^{2p}) \leq Cn^{-\frac{p^2}{2}}. \tag{19}$$

Note that, if the stable process is symmetric, the drift function is β-Hölder with $\beta > 1 - \frac{\alpha}{2}$ and ϕ is a bounded Lipschitz function, the authors in Mikulevičius and Xu (2018) find a similar speed of convergence of the form

$n^{-p\beta/\alpha}$ with $\delta = 1$ and $\alpha \in [1, 2)$ for dimension $d \geq 1$ and any $p < \alpha$. See also similar results in Pamen and Taguchi (2017) for the dimension $d \geq 2$.

Thus, Theorems 3.3 and 3.4 extends the work of Mikulevičius and Xu (2018); Wu and Hao (2023); Pamen and Taguchi (2017) in the case of Lipschitz drift function and Hölder continuous diffusion coefficient. We are in dimension 1, and are not restricted to symmetric stable processes. Finally, the methodology of proof is new and not based on Itô-Tanaka trick method.

Note also that Proposition 3.1 or Proposition 3.2 ensures that, the approximate paths of a weak solution for the SDE (2) are stochastically continuous under the above conditions in Theorem 3.3. Hence according to (Peszat and Zabczyk (2007), Prop. 3.21), there is a predictable modification of the SDE (2).

4. Proofs of the Main Results

4.1. **Proof of Proposition 3.1.**

Proof. Consider $t, u \in [0, T]$ such that $u - t \leq \frac{1}{n^\delta}$ for some $\delta \geq \frac{1}{2}$. We have

$$\begin{aligned} X_u - X_t = &\int_0^u f(X_s)ds - \int_0^t f(X_s)ds \\ &+ \int_0^u \phi(X_s)dZ_s - \int_0^t \phi(X_s)dZ_s. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}|X_u - X_t|^{2\beta} &\leq 2^{2\beta}\mathbb{E}\left|\int_t^u f(X_s)ds\right|^{2\beta} \\ &\quad + 2^{2\beta}\mathbb{E}\left|\int_t^u \phi(X_s)dZ_s\right|^{2\beta} \\ &:= I_1(t, \beta) + I_2(t, \beta) \end{aligned}$$

Using the Hölder inequality and the assumption **(A2)**, we have the following estimate

$$\begin{aligned} I_1(t, \beta) &\leq 2^{2\beta}\, n^{-\delta(2\beta-1)} \int_t^u \mathbb{E}|f(X_s)|^{2\beta}ds \\ &\leq C_{\beta,L}\, n^{-\delta(2\beta-1)} \int_t^u (1 + \mathbb{E}|X_s|^{2\beta})ds \\ &\leq C_{\beta,L} n^{-2\beta\delta}, \end{aligned}$$

since $1 + \mathbb{E}|X_s|^{2\beta}$ is finite, by Lemma 3.2.

Now let us consider the estimation of $I_2(t, \beta)$. Imitating the previous argument, we have

$$\begin{aligned}\mathbb{P}\left(\left|\int_t^u \phi(X_s)dZ_s\right| \geq x\right) &\leq \mathbb{P}\left(\left|\int_t^u b_R\phi(X_s)ds\right| \geq \frac{x}{2}\right) \\ &+ \mathbb{P}\left(\left|\int_t^u \int_{|y|\leq R} y\phi(X_s)\tilde{\mu}(ds,dy)\right| \geq \frac{x}{2}\right) \\ &+ \mathbb{P}\left(\left|\int_t^u \int_{|y|\leq R} y\,\phi(X_s)\mu(ds,dy)\right| \neq 0\right) \\ &:= R_1(x,n) + R_2(x,n) + R_3(x,n).\end{aligned}$$

Using the Chebychev and Cauchy-Schwarz inequalities, we have

$$R_1(x,n) \leq \frac{4b_R^2}{x^2}\mathbb{E}\left|\int_t^u \phi(X_s)ds\right|^2 \leq \frac{4b_R^2}{x^2}n^{-\delta}\int_t^u \mathbb{E}(\phi(X_s)^2 ds. \tag{20}$$

$$R_1(x,n) \leq n^{-\delta}C(\phi)\left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\right)\int_t^u (1+\mathbb{E}|X_s|^{2\beta})ds \tag{21}$$

$$\leq n^{-2\delta}C(\phi)\left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\right), \tag{22}$$

since $1+\mathbb{E}|X_s|^{2\beta}$ is finite. Next, we use Chebychev's inequality and the isometry formula for Poisson stochastic integrals and Hölder condition to get

$$\begin{aligned}R_2(x,n) &\leq \frac{4}{x^2}\mathbb{E}\left|\int_t^u \int_{|y|\leq R} y\,\phi(X_s)\,\tilde{\mu}(ds,dy)\right|^2 \\ &\leq \frac{4}{x^2}\int_{|y|\leq R} y^2\nu(dy)\int_t^u \mathbb{E}\phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha)\,R^{2-\alpha}}{x^2}\int_t^u \mathbb{E}\phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha)\,R^{2-\alpha}}{x^2}C(\phi)\int_t^u \left(1+\mathbb{E}|X_s|^{2\beta}\right)ds \\ &\leq n^{-\delta}\,C(\phi)\frac{4C_2(\alpha)\,R^{2-\alpha}}{x^2}.\end{aligned}$$

Let us now consider the compound Poisson stochastic integral appearing in $R_3(x,n)$. Note that

$$N_{t,u}^{\phi,R} = \int_t^u \int_{|y|>R} y\phi(X_s)\,\mu(ds,dy)$$

have the same law as

$$N_{u-t}^{\phi,R} = \int_0^{u-t} \int_{|y|>R} y\,\phi(X_s)\,\mu(ds,dy),$$

so that as above, we have

$$\mathbb{P}(N_{u-t}^{\phi,R} > 0) \leq (u-t)C_1(\alpha)\, R^{-\alpha} \leq n^{-\delta} C_1(\alpha)\, R^{-\alpha}.$$

By combining the three estimates, we obtain

$$\begin{aligned}\mathbb{P}\left(\left|\int_t^u \phi(X_s)dZ_s\right| \geq x\right) &\leq n^{-\delta} C_1(\alpha)\, R^{-\alpha} + n^{-\delta}\, C(\phi)\frac{4C_2(\alpha)\, R^{2-\alpha}}{x^2} \\ &\quad + n^{-2\delta} C(\phi)\left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\right).\end{aligned}$$

If we choose the truncation level

$$R = xn^{\beta} \geq 1,$$

and rearrange the terms, we get for all $x \geq n^{-\beta}$:

$$\begin{aligned}\mathbb{P}\left(\left|\int_t^u \phi(X_s)dZ_s\right| \geq x\right) &\leq C_1(\alpha)\, x^{-\alpha} n^{-\beta\alpha-\delta} + C(\phi) 4C_2(\alpha)\, x^{-\alpha} n^{-\alpha\beta+2\beta-\delta} \\ &\quad + C(\phi)\, C(\alpha, b) x^{-\alpha} n^{-\alpha\beta+2\beta-2\delta}.\end{aligned}$$

From the fact that,

$$\begin{aligned}\mathbb{E}\left|\int_t^u \phi(X_s)dZ_s\right|^{2\beta} &= 2\beta \int_0^{+\infty} x^{2\beta-1}\mathbb{P}\left(\left|\int_t^u \phi(X_s)dZ_s\right| \geq x\right) dx, \\ &\leq n^{-2\beta^2} + 2\beta \int_{n^{-\beta}}^{+\infty} x^{2\beta-1}\mathbb{P}\left(\left|\int_t^u \phi(X_s)dZ_s\right| \geq x\right) dx.\end{aligned}$$

and since

$$\begin{cases} n^{-\alpha\beta-\delta}\int_{n^{-\beta}}^{+\infty} x^{2\beta-1-\alpha}dx & = \frac{n^{-2\beta^2-\delta}}{\alpha-2\beta} \\ n^{-\alpha\beta+2\beta-\delta}\int_{n^{-\beta}}^{+\infty} x^{2\beta-1-\alpha}dx & = \frac{n^{-2\beta^2+2\beta-\delta}}{\alpha-2\beta} \\ n^{-\alpha\beta+2\beta-2\delta}\int_{n^{-\beta}}^{+\infty} x^{2\beta-1-\alpha}dx & = \frac{n^{-2\beta^2+2\beta-2\delta}}{\alpha-2\beta}, \end{cases}$$

we have

$$\mathbb{E}\left|\int_t^u \phi(X_s)dZ_s\right|^{2\beta} \leq C\Big(n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta}\Big).$$

Finally,

$$\sup_{t\in[0,T];|u-t|\leq\Delta_n} \mathbb{E}\,|X_u - X_t|^{2\beta} \leq C\Big(n^{-2\delta\beta} + n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta}\Big),$$

and since $2\beta\delta > 0$, $2\beta^2 > 0$, $2\beta^2+\delta > 0$ $2\beta^2-2\beta+\delta > 0$ and $2\beta^2-2\beta+2\delta > 0$ we can conclude and see that this bound tends to 0 for all $\delta \geq 1/2$. □

4.2. **Proof of Theorem 3.3.**

Proof. Set

$$V_t = \sup_{0\le s\le t} |\tilde{X}^n_s - X_s|.$$

To establish the convergence of $\tilde{X}^n$ to X, let's write the explicit expression for $Y^n_t = \tilde{X}^k_t - X_t$.

$$\tilde{X}^n_t - X_t = \int_0^t \Big[f(\tilde{X}^n_{\pi^n_s}) - f(X_s)\Big]\, ds + \int_0^t \Big[\phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_s)\Big]\, dZ_s,$$

and

$$\tilde{X}^n_t - X_t = \int_0^t \Big[f(\tilde{X}^n_{\pi^n_s}) - f(X_s)\Big]\, ds + \int_0^t \Big[\phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_{\pi^n_s})\Big]\, dZ_s + \int_0^t \big[\phi(X_{\pi^n_s}) - \phi(X_s)\big]\, dZ_s.$$

We have

$$\begin{aligned}\mathbb{E}|Y^n_t|^{2\beta} &\le (3t)^{2\beta-1}\int_0^t \mathbb{E}\left|f(\tilde{X}^n_{\pi^n_s}) - f(X(s))\right|^{2\beta} ds + 3^{2\beta-1}\mathbb{E}\left|\int_0^t \big(\phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_{\pi^n_s})\big) dZ_s\right|^{2\beta} \\ &+ 3^{2\beta-1}\mathbb{E}\left|\int_0^t \big[\phi(X_{\pi^n_s}) - \phi(X_s)\big]\, dZ_s\right|^{2\beta} := P_1(t,\beta) + 3^{2\beta-1}\, P_2(t,\beta) + 3^{2\beta-1}\, P_3(t,\beta).\end{aligned}$$

But

$$\begin{aligned}\mathbb{E}|f(\tilde{X}^n_{\pi^n_s}) - f(X_s)|^{2\beta} &\le 2^{2\beta-1}\mathbb{E}|f(\tilde{X}^n_{\pi^n_s}) - f(X_{\pi^n_s})|^{2\beta} \\ &+ 2^{2\beta-1}|f(X_{\pi^n_s}) - f(X_s)|^{2\beta} \\ &\le 2^{2\beta-1}\mathbb{E}|f(\tilde{X}^n_{\pi^n_s}) - f(X_{\pi^n_s})|^{2\beta} \\ &+ 2^{2\beta-1}\mathbb{E}|f(X_{\pi^n_s}) - f(X_s)|^{2\beta},\end{aligned}$$

so that

$$\begin{aligned}P_1(t,\beta) &\le (6t)^{2\beta-1}\mathbb{E}\left(\int_0^t |f(\tilde{X}^n_{\pi^n_s}) - f(X_{\pi^n_s})|^{2\beta} ds\right) \\ &+ (6t)^{2\beta-1}\mathbb{E}\left(\int_0^t |f(X_{\pi^n_s}) - f(X_s)|^{2\beta} ds\right) \\ &\le C_{\beta,L,T}\left(\mathbb{E}\left(\int_0^t |\tilde{X}^n_{\pi^n_s} - X_{\pi^n_s}|^{2\beta} ds\right) + \mathbb{E}\left(\int_0^t |X_{\pi^n_s} - X_s|^{2\beta} ds\right)\right) \\ &\le C_{\beta,L,T}\int_0^t \mathbb{E}V_s^{2\beta} ds + C_{\beta,L,T}\int_0^t \mathbb{E}|X_{\pi^n_s} - X_s|^{2\beta} ds \\ &\le C_{\beta,L,T}\int_0^t \mathbb{E}V_s^{2\beta} ds + C\Big(n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-\beta^2+2\beta-\delta} + n^{-\beta^2+2\beta-2\delta}\Big),\end{aligned}$$

where in the last inequality, we apply Proposition 3.1 since $s - \pi^n_s \le n^{-\delta}$ with $\delta \ge \frac{1}{2}$. But one could also directly apply the remark of Proposition 3.1.

Secondly, we have

$$P_2(t,\beta) = \mathbb{E}\left|\int_0^t \big(\phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_{\pi^n_s})\big)dZ_s\right|^{2\beta}$$
$$= 2\beta\int_0^{+\infty} x^{2\beta-1}\mathbb{P}\Big(\left|\int_0^t \big(\phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_{\pi^n_s})\big)dZ_s\right| \geq x\Big)dx.$$

However, based on previous estimates, we have

$$\mathbb{P}\Big(\left|\int_0^t \phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_{\pi^n_s})dZ_s\right| \geq x\Big) \leq R_1^*(x,n) + R_2^*(x,n) + R_3^*(x,n),$$

where

$$R_1^*(x,n) \leq \frac{4b_R^2}{x^2}\sup_{k=0,..,n-1}\mathbb{E}\left|\int_0^t \mathbb{I}_{[t_k,t_{k+1}[}(s)\phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_{\pi^n_s})ds\right|^{2\beta}$$

$$R_1^*(x,n) \leq \Big(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\Big)c_\phi^2 n^{-2\delta}\mathbb{E}V^{2\beta}_{\pi^n_s},$$

$$R_2^*(x,n), \leq \frac{4C_2(\alpha)R^{2-\alpha}}{x^2}\sup_{k=0,..,n-1}\mathbb{E}\Big(\int_0^t \mathbb{I}_{[t_k,t_{k+1}[}(s)|\phi(\tilde{X}^n_{\pi^n_s}) - \phi(X_{\pi^n_s})|^2 ds\Big)$$
$$\leq \frac{4C_2(\alpha)R^{2-\alpha}}{x^2}c_\phi^2 n^{-\delta}\mathbb{E}V^{2\beta}_{\pi^n_s}$$

and,

$$R_3^*(x,n) \leq n^{-\delta}C_1(\alpha)R^{-\alpha}.$$

Note that it is not difficult to show by induction on k; using the proof arguments of Lemma 3.2 that

$$\sup_{n\geq 1}\sup_{t\in[t_k,t_{k+1}]}\mathbb{E}|\tilde{X}^n_t|^{2\beta} < \infty,$$

so that since $\mathbb{E}|X_t|^{2\beta} < \infty$, we have

$$\mathbb{E}|\tilde{X}^n_{\pi^n_s} - X_{\pi^n_s}|^{2\beta} \leq 2^{2\beta}\mathbb{E}|\tilde{X}^n_t|^{2\beta} + 2^{2\beta}\mathbb{E}|X_t|^{2\beta} < \infty.$$

Now, choosing the truncation level $R = xn^\beta \geq 1$, we have for all $x \geq n^{-\beta}$:

$$\mathbb{P}\left(\left|\int_0^t \phi(X_{\pi^n_s}) - \phi(X_{\pi^n_s})dZ_s\right| \geq x\right) \leq C_1(\alpha)\,x^{-\alpha}n^{-\beta\alpha-\delta} + c_\phi^2 4C_2(\alpha)\,x^{-\alpha}n^{\beta^2-\alpha\beta+2\beta-\delta}$$
$$+ c_\phi^2\,C(\alpha,b)x^{-\alpha}n^{\beta^2-\alpha\beta+2\beta-2\delta},$$

so that

$$P_2(t,\beta) = \mathbb{E}\left|\int_0^t \left[\phi(X_{\pi_s^n}) - \phi(X_{\pi_s^n})\right] dZ_s\right|^{2\beta}$$
$$= 2\beta \int_0^{+\infty} x^{2\beta-1}\mathbb{P}\left(\left|\int_0^t \phi(X_{\pi_s^n}) - \phi(X_{\pi_s^n}) dZ_s\right| \geq x\right)$$
$$\leq C\Big(n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-\beta^2+2\beta-\delta} + n^{-\beta^2+2\beta-2\delta}\Big).$$

Now let us look at $P_3(t,\beta)$. We have

$$\mathbb{P}\left(\left|\int_0^t \phi(X_{\pi_s^n}) - \phi(X_s) dZ_s\right| \geq x\right) \leq R_1^{**}(x,n) + R_2^{**}(x,n) + R_3^{**}(x,n),$$

where

$$R_1^{**}(x,n) \leq \frac{4b_R^2}{x^2} \sup_{k=0,..,n-1} \mathbb{E}\left|\int_0^t \mathbb{I}_{[t_k,t_{k+1}[}(s)\phi(X_{\pi_s^n}) - \phi(X_s) ds\right|^{\beta}$$

$$R_1^{**}(x,n) \leq \Big(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2}\Big) c_\phi^2 n^{-2\delta} \sup_{|u-t|\leq\Delta_n} \mathbb{E}|X_u - X_t|^{2\beta},$$

$$R_2^{**}(x,n), \leq \frac{4C_2(\nu)R^{2-\alpha}}{x^2} \sup_{k=0,..,n-1} \int_0^t \mathbb{I}_{[t_k,t_{k+1}[}(s)\mathbb{E}|\phi(X_{\pi_s^n}) - \phi(X_s)|^2 ds$$
$$\leq \frac{4C_2(\alpha)R^{2-\alpha}}{x^2} c_\phi^2 n^{-\delta} \sup_{|u-t|\leq\Delta_n} \mathbb{E}|X_u - X_t|^{2\beta},$$

and

$$R_3^{**}(x,n) \leq n^{-\delta} C_1(\alpha) R^{-\alpha}.$$

Again, choosing the truncation level $R = xn^\beta \geq 1$, we have for all $x \geq n^{-\beta}$:

$$\mathbb{P}\left(\left|\int_0^t \phi(X_{\pi_s^n}) - \phi(X_s) dZ_s\right| \geq x\right) \leq C_1(\alpha)\, x^{-\alpha} n^{-\beta\alpha-\delta} + c_\phi^2 4C_2(\alpha)\, x^{-\alpha} n^{-\alpha\beta+2\beta-\delta}$$
$$+ c_\phi^2\, C(\alpha,b) x^{-\alpha} n^{-\alpha\beta+2\beta-2\delta},$$

so that

$$P_3(t,\beta) = \mathbb{E}\left|\int_0^t \left[\phi(X_{\pi_s^n}) - \phi(X_s)\right] dZ_s\right|^{2\beta}$$
$$= 2\beta \int_0^{+\infty} x^{2\beta-1}\mathbb{P}\left(\left|\int_0^t \phi(X_{\pi_s^n}) - \phi(X_s) dZ_s\right| \geq x\right)$$
$$\leq C\Big(n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta}\Big).$$

Finally, there exist positive constants C (which depend on all the model parameters) such that

$$\mathbb{E}V_t^{2\beta} \leq C_{\beta,L,T} \int_0^t V_s^{2\beta} ds + C\left(n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta}\right),$$

and we use Gronwall's lemma to get the desired result. □

5. Examples

In this section, we will illustrate the usefulness of the above approximate solution results of the SDE (2) that we established in the previous section. Let us first recall the random walk approximation method for the simulation of α-stable processes, due to the self-similarity property and the property of stationary independent increments. Of course, other methods are available, such as the series approximation of the Lévy process, see Janicki et al. (1997) for more details. The following proposition, based on the algorithm of Chambers et al. (1976) for α-stable distributions, allows to simulate α-stable processes.

Proposition 5.1 (Discretised trajectory for a strictly α-stable process)**.** *Let $Z = (Z_t)_{t\in[0,T]}$ be a strictly-α stable process.*

Step 1 *Simulate n independent uniformly distributed random variables Φ on $[-\pi/2, \pi/2]$ and n independent identically distributed random variables W as exponential with parameter 1.*

Step 2 *Compute ΔZ_i for $i = 1, ...n$ as follows.*

(1) *If $\alpha \neq 1$*

$$\Delta Z_i = \sigma \left(\frac{T}{n}\right)^{1/\alpha} \frac{\sin(\alpha(\Phi - \phi_0))}{\cos(\Phi)^{1/\alpha}} \left(\frac{\cos(\Phi - \alpha(\Phi - \phi_0))}{W}\right)^{\frac{1-\alpha}{\alpha}}.$$

(2) *If $\alpha = 1$*

$$\Delta Z_i = \sigma \left(\frac{T}{n}\right)^{1/\alpha} \frac{2}{\pi} \left(\left(\frac{\pi}{2} + \beta\Phi\right)\tan(\Phi) - \beta \log\left(\frac{\frac{1}{2}\pi W \cos(\Phi)}{\frac{1}{2}\pi + \beta\Phi}\right)\right)$$

where

$$\phi_0 = -\frac{\beta\pi}{2}\frac{1 - |1-\alpha|}{\alpha}.$$

Step 3 *The discretised trajectory of Z is given by*

$$Z(t_i) = \sum_{k=1}^{i} \Delta Z_k.$$

From a practical point of view, we consider the linearly interpolated version between times t_i and t_{i+1} for graphical representations. Figure 1 shows typical example paths for some strict and symmetric α processes. When α is closed to 2 and β is closed to 0, the sample paths resemble those of

Brownian motion. For $\alpha = 1$ and $\beta = 0$, the sample paths correspond to those of the Cauchy process. We can see the existence of jumps in the simulation, for example when α is not close to 2.

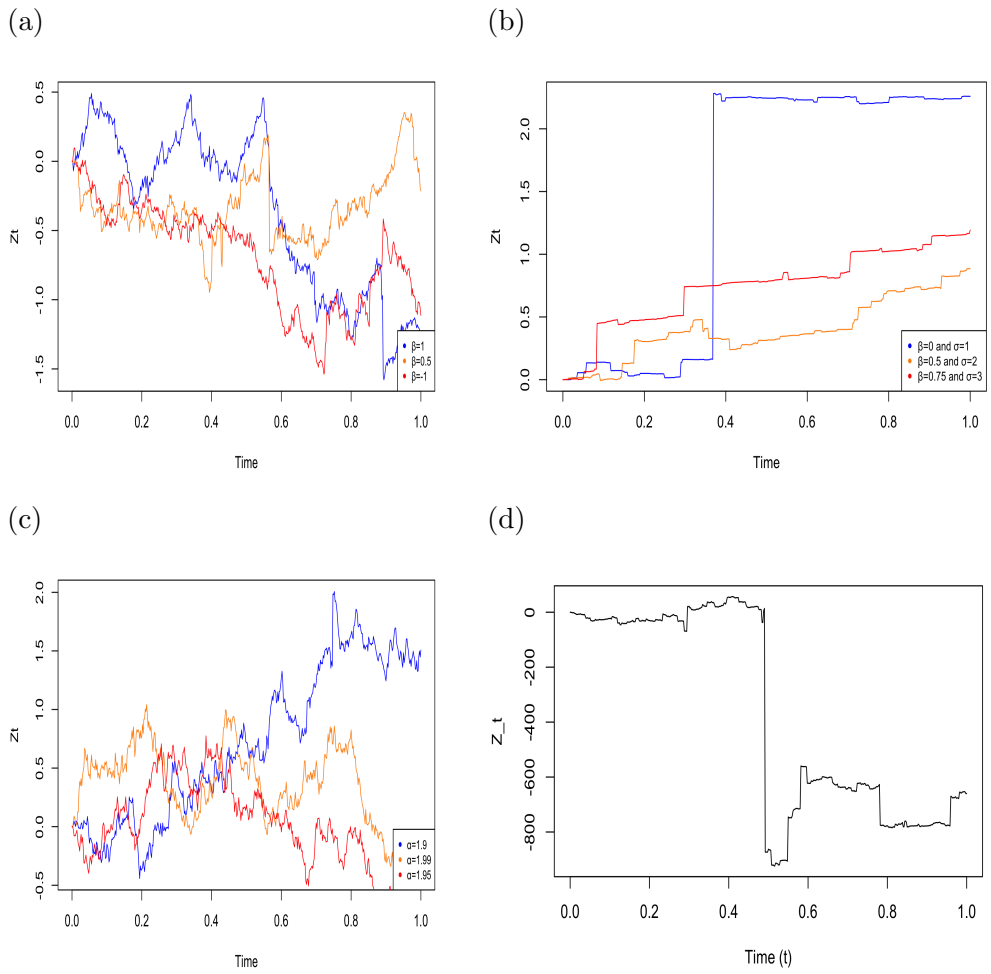

FIGURE 1. Paths of a standard strictly 1.7 stable process with different skewness (a), a 0.7 stable process with different scale and skewness (b), the symmetric and standard α stable process with different stability index (c) and the case of the standard 1 stable process (d)..

We now consider the following stable driven SDE models.

Example 5.1. *(Stable driven OU process) Consider the following Ornstein-Uhlenbeck (OU) process driven by a stable process* $(Z_t)_{t\geq 0}$ *with parameter* θ

and $\rho > 0$ is defined by

$$dX_t = -\theta X_t dt + \rho dZ_t, \quad X_0 = x_0 \in \mathbb{R}. \tag{23}$$

and has the integral representation

$$X_t(\omega) = e^{-\theta t}\Big(x_0 + \rho \int_0^t e^{\theta s} dZ_s(\omega)\Big), \quad \textit{for } t \in [0, T].$$

Since $f(x) = \theta x$ and $\phi(x) = \rho$ satisfy the Lipschitz and boundedness conditions, there is clearly an approximation of the stable driven OU process with step size $n^{-\delta}$ for $\delta \geq 1$. We give in Figure 2 (a) a path approximation solution for $\theta = 1.2$, $\alpha = 1.7$, $\rho = 1$ with $n = 1000$ sample size and $\delta = 1$. Another classic example is given by the following stable driven Cox-Ingersoll-Ross type models.

Example 5.2. *(Stable driven Cox–Ingersoll–Ross type process)*
A stable driven Cox-Ingersoll-Ross type process can be defined as follows

$$dX_t = (\lambda - \theta X_t)dt + \rho |X_t|^\beta dZ_t, \quad X_0 \geq 0,$$

where $\beta \in [\frac{1}{2}, 1)$ λ, ρ, θ are real constants and Z is, for example, a symmetric stable process.

According to Fournier (2013), there is a weak existence solution. When $\beta = \alpha^{-1}$ and Z has only positive jumps, it is called SCIR model in Bayraktar and Clément (2023); Yang (2017). Note that since $f(x) = \lambda - \theta x$ is a Lipschitz function, and ϕ is positive and Hölder $\phi(x) = |x|^\beta$ continuous function, then there exists a weak approximate solution with step size n^{-1}. In particular, one can choose $\beta = \alpha - 1$ and $\alpha \in [1.5, 2)$ to obtain in Figure 2 (b) an approximate path solution for $\lambda = 0.2$, $\theta = 1.2$, $\alpha = 1.7$, $\rho = 1$ and $\beta = \alpha - 1$ with $n = 1000$ sample size. In Figure 2 (c) we give an approximate path solution for $\lambda = 0.2$, $\theta = 1.2$, $\alpha = 1.7$, $\rho = 1$ and $\beta = 1 - \alpha^{-1}$ with $n = 1000$.

We also introduce the following stable driven Lotka-Volterra type model.

Example 5.3. *(Stable driven Lotka–Volterra type process)*
A stable driven Lotka-Volterra type process can be defined as follows

$$dX_t = X_t(\lambda - \theta X_t)dt + \rho |X_t|^\beta dZ_t, \quad X_0 \geq 0,$$

where $\beta \in [\frac{1}{2}, \frac{\alpha}{2})$, λ, ρ, θ are real constants and Z is, for example, a symmetric stable process.

Note that $f(x) = x(\lambda - \theta x) = \lambda x - \theta x^2$ is the sum of a Lipschitz function and a negative function if $\theta > 0$. Since $\phi(x) = |x|^\beta$ is Hölder continuous, there is a weak approximate solution.
When $\beta = 1$, this model is proposed in Zhang et al. (2017) for positive stable process Z.
We give in Figure 2 (d) an approximate path solution for $\lambda = 0.2$, $\theta = 1.2$ $\alpha = 1.7$, $\beta = 1/2$, $\rho = 1$ and $n = 1000$.

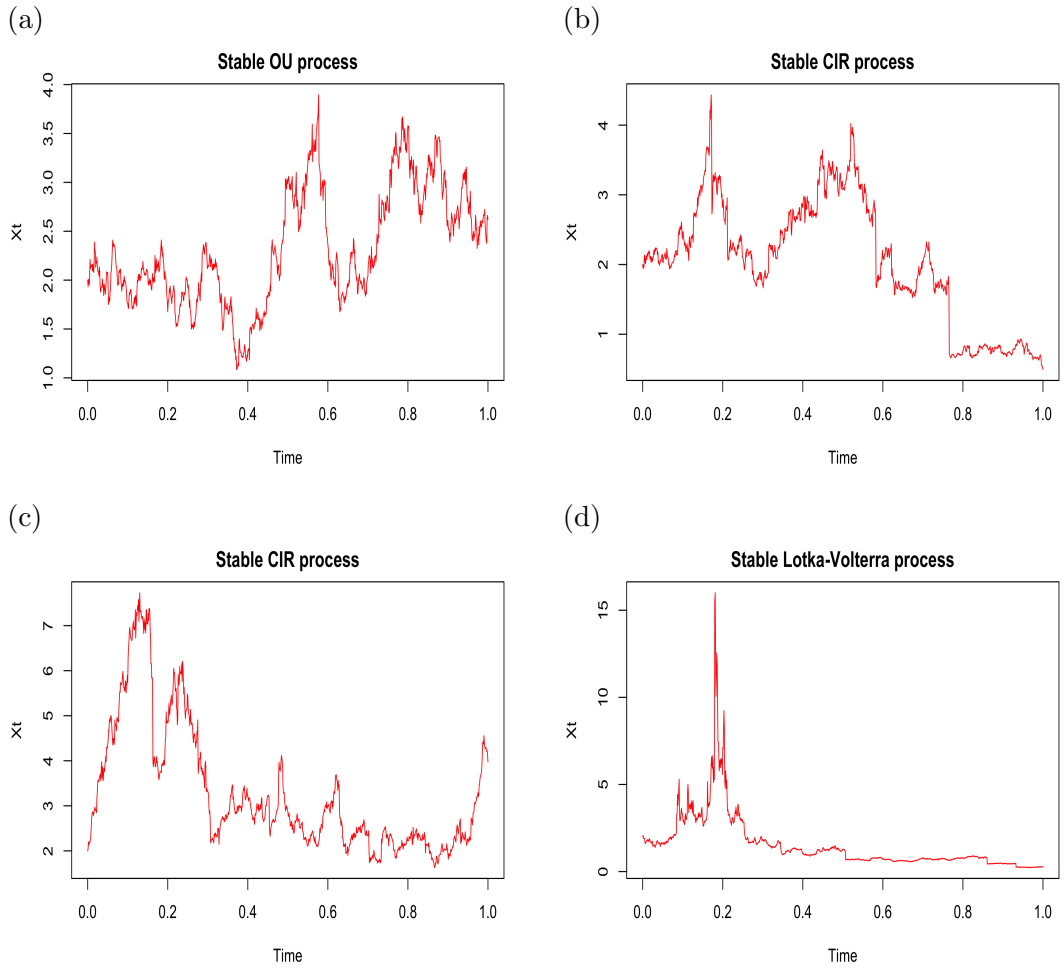

FIGURE 2. An approximate solution path of a standard symmetric stable driven OU process with sample size $n = 1000$, $\rho = 1$, $\theta = 1.2$, $\alpha = 1.7$, (a); of an standard symmetric stable driven CIR type process with sample size $n = 1000$, $\rho = 1$, $\theta = 1.2$, $\alpha = 1.7$, $\lambda = 0.2$ and $\beta = \alpha^{-1}$ (b); a standard symmetric stable driven CIR type process with sample size $n = 1000$, $\rho = 1$, $\theta = 1.2$, $\alpha = 1.7$, $\lambda = 0.2$ and $\beta = 1 - \alpha^{-1}$ (c) and the case of a standard symmetric stable driven Lotka-Volterra type process with $\lambda = 0.2$, $\theta = 1.2$, $\alpha = 1.7$, $\rho = 1$, $\beta = 1/2$ and $n = 1000$ (d).

FUNDING

No funding was obtained for this study.

Competing Interest

The authors declare that they have no competing interests.

Declaration statement

The author certify that the submission is an original work and is not under review at any other publication.

Acknowledgements

Authors show their gratitude to the reviewers because their comments improved the paper.

Data availability statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study. The R codes for the simulation study will be available on the Github page `https://github.com/smanouab`.

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