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# APPROXIMATE SOLUTION FOR A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY STABLE PROCESSES

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**ABSTRACT.** In this paper we consider a general class of stochastic differential equations driven by  $\alpha$ -stable processes with Lipschitz drift coefficients and non-Lipschitz diffusion coefficients. An Euler-Maruyama approximate solution is proved whenever the diffusion coefficient is Hölder continuous with exponent  $\beta < \frac{\alpha}{2}$  with  $\alpha \in (1, 2)$ . We obtain a strong rate of convergence of order  $n^{-\beta^2}$ . Our proposed method is new in this context and is based on a truncation method by separating the large and small jumps of the  $\alpha$  stable process in the Lévy-Itô decomposition. Along the paper we give some numerical simulations of stochastic models that match our results, namely some stable driven Ornstein-Uhlenbeck, Cox-Ingersoll-Ross and Lotka-Volterra type processes.

**AMS Subject Classification:** 60G17, 60G52, 60H10, 60H20, 65C30, 65C20, 60E07.

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## 1. INTRODUCTION

Models such as Stochastic Differential Equations (SDEs) are a natural choice for modelling the time evolution of dynamic systems and have a wide range of applications in many disciplines such as population dynamics, epidemiology, finance and biology. Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a complete filtered probability space satisfying the usual conditions. Moreover,  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  is rich enough to contain other processes considered in this paper, which aims to study the following stochastic differential equation (SDE) driven by  $\alpha$ -stable processes,  $\alpha \in (0, 2)$  :

$$\begin{cases} dX_t = f(X_t)dt + \phi(X_{t-})dZ_t, & t \in [0, T] \\ X_0 = x \in \mathbb{R}, \end{cases} \quad (1)$$

where  $x_0$  is a starting point and  $T > 0$  is a time horizon, under certain conditions on the functions  $f$  and  $g$ . In the theory of stochastic processes and its applications, it is important to find existence conditions for solutions of the SDE. There are different notions of existence and uniqueness. A weak solution is a solution that satisfies the SDE (1). A strong solution is a predictable process  $(X_t, \mathcal{F}_t^Z)$  that satisfies the SDE (1). By definition, a weak solution that is predictable is a strong solution. Pathwise uniqueness holds if any two solutions are indistinguishable. Some results are available on the existence under various conditions of the coefficients of the above SDE. The existence and uniqueness of solutions to the above SDE under Lipschitz or linear growth conditions are well-known results in stochastic calculus as well as for general Lévy processes  $Z$ , see e.g. Rong (2006); Fu and Li (2010); Protter (1991). For example, assuming that the drift and diffusion coefficients are continuous and have at most linear growth, then there is weak existence in  $L^p$  for any  $p \in (0, \alpha)$  of SDE (1), see Proposition 2 in Fournier (2013) for  $\alpha \in (0, 2)$  and  $\alpha \neq 1$ . This weak existence is contained in Rong (2006).

For non-Lipschitz coefficients and considering  $\alpha$ -Stable driven SDE, some results have been studied before. In Bass et al. (2004); Bass (2004); Komatsu (1982), when  $\alpha \in (1, 2)$  and  $Z$  is symmetric with  $\alpha \in (1, 2)$ , it was shown that if  $f = 0$  and  $\phi$  is Hölder continuous of order  $\beta \in (0, \alpha_0)$  with  $\alpha_0 = \alpha^{-1} \wedge 1$ , there is a weak solutions (see Theorem 1.1 in Bass et al. (2004)). If  $\phi$  has a modulus of continuity  $\rho$  satisfying  $\int_{0+} \frac{1}{\rho(x)^\alpha} = \infty$ , then there is a strong solution and the solution is pathwise unique. This condition is the exact analogue of the Yamada–Watanabe condition for SDE driven by a Brownian motion. If the drift term  $f$  is Hölder continuous of order  $\beta > \frac{2-\alpha}{2}$  and  $\alpha \in [1, 2)$  with  $\beta \in (0, 1)$  and  $g$  is Lipschitz continuous and bounded, then, the existence and uniqueness of a strong solution is proved in Priola (2012), see also Mikulevičius and Xu (2018), for non-degenerate symmetric stable processes  $Z$  with values in the higher dimensional space  $\mathbb{R}^d$ ,  $d \geq 1$ . The proof is based on the associated integro-differential operators of Kolmogorov type. In Pamen and Taguchi (2017), when  $\phi$  is bounded and the drift function on the form  $f = F(t, x)$  is a bounded  $\beta$ -Hölder continuous in space and  $\eta$ -Hölder in time with  $\eta \in [1/2, 1]$  and  $\beta \in (0, 1)$ , the pathwise uniqueness was proved by applying Gronwall’s lemma and using the elliptic version of the Kolmogorov equation and the regularity of its solution.

Recently, in Athreya et al. (2020), they established strong existence and uniqueness whenever  $\phi = 1$  and the drift function  $f$  belongs to the Besov Hölder space  $C^\beta$  for  $\beta > \frac{1}{2} - \frac{\alpha}{2}$  whenever  $Z$  is symmetric with  $\alpha \in (1, 2)$ .

Suppose that  $\phi$  is non-decreasing and Hölder continuous with index  $1 - \frac{1}{\alpha}$  and  $f$  is the sum of a Lipschitz continuous function and a non-increasing function. It is shown in Li and Mytnik (2011); Fu and Li (2010) that both

the strong existence and path uniqueness hold if  $\alpha \in (1, 2)$  and  $Z$  has only positive jumps. This latter result has been extended in Fournier (2013). The technique in Fournier (2013) is based on the Ito formula for jump processes and some technical integrals (see Lemmas 7 and 9 in Fournier (2013)) and an approximate sequence iteration method.

It is possible to appreciate the basics of how to simulate SDEs numerically with only a background knowledge of the Euler-Maruyama scheme, which is a well-known method for approximating any solution. In general, the convergence of numerical methods is studied under the assumption that the coefficient of the SDE is globally Lipschitz continuous. However, this assumption is not always satisfied for SDEs used in practice (e.g. mathematical finance, optimal control problems and filtering), so the global Lipschitz-based idea is not immediately applicable. This leads to the study of the Euler-Maruyama approximation for SDEs with irregular coefficients. Some results are now available on the approximate solutions of SDEs driven by a stable process as well as on the speed of convergence of the approximation. The Itô-Tanaka technique has traditionally been used to obtain convergence rates for Euler schemes, which has important applications in stochastic financial theory.

In Pamen and Taguchi (2017), when  $\phi = 1$  and the drift function  $f = F(t, x)$  is a bounded  $\beta$ -Hölder continuous in space with  $\beta \in (0, 1)$  and,  $\eta$ -Hölder in time with  $\eta \in [1/2, 1]$ , a strong approximate solution is proved and the rate of convergence for the Euler-Maruyama approximation is given. More precisely, using the Itô-Tanaka trick method and assuming that there is a strong solution with  $\alpha + \beta > 2$ , the rate of strong convergence was derived in the form  $n^{-1}$  when  $p \geq \frac{2}{\beta}$  and in the form  $n^{-p\beta/2}$  for  $d$ -dimensional truncated symmetric  $\alpha$ -stable process when  $d \geq 2$  for  $\alpha \in (1, 2)$  and  $1 \leq p\beta < 2$  or  $p \in [1, 2]$ . Similarly, Mikulevičius and Xu (2018) provides a strong approximate solution and a rate of convergence of order  $n^{-p\beta/\alpha}$  when  $p \in (0, \alpha)$  and  $Z$  is a symmetric stable process of index  $\alpha \in (1, 2)$  whenever  $f$  is  $\beta$ -Hölder continuous with  $\beta > 1 - \frac{\alpha}{2}$  and  $\phi$  is a bounded Lipschitz continuous function. See also Wu and Hao (2023) for a weak convergence rate under similar conditions.

In this paper, we consider the following SDE:

$$X_t = x_0 + \int_0^t f(X_s)ds + \int_0^t \phi(X_{s-})dZ_s, \quad t \in [0, T], \quad (2)$$

where  $Z$  is a  $\alpha$ -stable process with  $\alpha \in (1, 2)$ , not necessarily symmetric, under the following assumptions:

**(A1)** The function  $\phi$  is Hölder continuous with exponent  $\beta \in (0, \frac{\alpha}{2})$  that is

$$|\phi(y) - \phi(z)| \leq c_\phi |y - z|^\beta \quad c_\phi > 0 \quad y, z \in \mathbb{R}.$$

**(A2)**  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous  $L$ -Lipchitz function that is :

$$|f(x) - f(y)| \leq L|y - z| \quad L > 0 \quad y, z \in \mathbb{R}.$$

We investigate the existence of a consistent Euler-Maruyama approximate solution defined as

$$\tilde{X}_t^n = x_0 + \int_0^t f(\tilde{X}_{\pi_s^n}^n) ds + \int_0^t \phi(\tilde{X}_{\pi_s^n}^n) dZ_s, \quad (3)$$

where  $\pi_t^n = \max\{t_k, t_k \leq t\}$  with the size step  $h = \Delta_n = n^{-\delta}$  for some  $\delta > 0$ ;  $\tilde{X}_t^n = X_{t_k}^n$  for all  $k = 0, \dots, n-1$  and

$$\begin{cases} X_{t_{k+1}}^n = X_{t_k}^n + f(X_{t_k}^n)\Delta_n + \phi(X_{t_k}^n)(Z_{t_{k+1}} - Z_{t_k}), \\ X_0^n = x. \end{cases}$$

While this process is pleasant to handle from a theoretical point of view, it is easy for simulation purposes at points  $t_k$  such that  $t_{k+1} - t_k = \Delta_n$  and for a practical point of view, we therefore prefer its linearly interpolated version between points  $t_k$  and  $t_{k+1}$ .

Compared to the above mentioned previous works, we have established a new result on the approximate solution as well as the rate of convergence for the SDE (2). Moreover, our proof methodology is new and not based on the Itô-Tanaka trick method, but on the Lévy-Itô decomposition together with a truncation method for stable integrals developed in Manou-Abi (2015); Joulin and Mawaki Manou-Abi (2015). Our result extends the aforementioned work on approximate solution to the case of Lipschitz drift function and Hölder continuous diffusion coefficients. This paper is organised as follows. In Section 2, we discuss the primary analytical tools, introduce notation and preliminary results on weak existence for SDE (2). In Section 3, we first give some preliminaries moment bound results of a solution of the SDE (2). Then we present our main theorem, which extends the result of Mikulevičius and Xu (2018); Pamen and Taguchi (2017); Wu and Hao (2023) for Lipschitz drift function, Hölder diffusion function for a given  $\alpha$ -stable process, not necessarily symmetric with index  $\alpha > 1$  and  $\alpha < 2$ . Namely, if  $\phi$  is  $\beta$ -Hölder continuous for any  $2\beta < \alpha$ , there exists an approximate solution of (2) in  $L^{2\beta}$  space and the rate of convergence is of order  $n^{-\beta^2}$ . In section 4 we consider stochastic models as well as numerical simulations of their solutions that match with our results. The last section is devoted to the proofs of the main results.

## 2. PRELIMINARIES

In this part, we recall the definition of Lévy processes and introduce the  $\alpha$ -stable process as a member of the class of Lévy processes. We also introduce some notations that will be useful throughout the paper. We follow the presentation in Ken-Iti (1999); Applebaum (2009). We consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  equipped with the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  and satisfying the usual conditions as above.

**Definition 2.1.** A real stochastic process  $(Z_t)_{t \in [0, T]}$  defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is said to be a Lévy process if  $Z_0 = 0$  almost surely (a.s.) with the following properties:

1.  $Z$  is stochastically continuous i.e., for any  $\epsilon > 0$  and  $t, s \geq 0$ ,

$$\lim_{s \rightarrow 0} \mathbb{P}(|Z_{t+s} - Z_t| > \epsilon) = 0.$$

2.  $Z$  has stationary increments, i.e., for  $0 \leq s \leq t$ , the distribution of  $Z_t - Z_s$  does not depend on  $s$ , namely has the same distribution as  $Z_{t-s}$ .
3.  $Z$  has independent increments, i.e., for  $0 \leq s \leq t$ ,  $Z_t - Z_s$  is independent of  $\mathcal{F}_s$ .
4. The paths of  $Z$  are  $\mathbb{P}$ -a.s. right-continuous in  $t \geq 0$  and has left limits in  $t > 0$  (càdlàg).

Note that the last point above can be dropped, since one can prove that there is always a càdlàg modification of the stable process  $Z$ . Set  $\Delta Z_t = Z_t - Z_{t-}$ . The jumps part of the stochastic process  $Z$  can be described by its Poisson random measure (jump measure of  $Z$  on the interval  $[0, t]$ ), defined as

$$\mu(t, A) = \sum_{0 \leq s \leq t} \mathbb{I}_A(\Delta Z_s), \quad A \in \mathcal{B}(\mathbb{R}^*),$$

the number of jumps of  $Z$  on the interval  $[0, t]$  whose size lies in the set  $A$  bounded below. For such  $A$ , the process  $\mu(\cdot, A)$  is a Poisson process and the Lévy measure  $\nu$  is defined by  $\nu(A) := \mathbb{E}(\mu(1, A))$ . In the following, we denote the compensated random martingale measure by

$$\tilde{\mu}(t, A) = \mu(t, A) - t\nu(A).$$

A special class of Lévy processes that can contain infinitely many jumps is the following class of  $\alpha$ -stable processes. We now follow the presentation in Cattiaux and S. Manou-Abi (2014); Manou-Abi (2015).

**Definition 2.2.** A real stochastic process  $(Z_t)_{t \in [0, T]}$  defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is said to be an  $\alpha$ -stable process with characteristics  $(b, c_+, c_-)$ , if it is a Lévy process with the following characteristic function :

$$\varphi_{Z_t}(u) = \exp t \left( iub + \int_{-\infty}^{+\infty} (e^{iuy} - 1 - iuy \mathbb{I}_{|y| \leq 1}) \nu(dy) \right), \quad t \in [0, T],$$

where  $b$  stands for the drift parameter of  $Z$  and  $\nu$  the Lévy measure defined on  $\mathbb{R} \setminus \{0\}$  by

$$\nu(dx) := \frac{dx}{|x|^{\alpha+1}} (c_+ 1_{\{x>0\}} + c_- 1_{\{x<0\}}).$$

The parameters  $c_+, c_-$  above are non-negative with furthermore  $c_+ + c_- > 0$  and  $c_+ = c_-$  when  $\alpha = 1$ . The process is said to be symmetric if  $c_+ = c_- :=$

c. It is said to be strictly  $\alpha$ -stable if  $b = 0$ . In the case  $\alpha \in (1, 2)$ , the drift parameter is given by

$$b := - \int_{|y|>1} y \nu(dy) = - \frac{(c_+ - c_-)}{\alpha - 1} \quad \text{when } \alpha \neq 1.$$

An  $\alpha$ -stable process is closely related to the notion of self-similar process. The process  $Z$  is said to be strictly  $\alpha$ -stable if we have the self-similarity property

$$k^{-1/\alpha} (Z(kt))_{t \in [0, T]} \stackrel{d}{=} (Z(t))_{t \in [0, T]},$$

where  $k > 0$  and the equality  $\stackrel{d}{=}$  is understood in the sense of finite dimensional distributions. Note that  $\alpha$ -stable processes are interesting due to the self-similarity property and the fact that the Lévy measure and the Lévy-Itô decomposition are almost completely explicit for the one dimensional case. In order to control the jump size of  $Z$  and the momentum behaviour of a stable stochastic integral, we introduce the following truncation method developed in our previous paper Manou-Abi (2015) and also in Joulin and Mawaki Manou-Abi (2015). As an Lévy process,  $Z$  is a semimartingale with respect to  $\mathcal{F}_t^Z := \sigma(Z_s : s \in [0, t])$ ,  $t \in [0, T]$ , provided it satisfies the usual hypothesis, i.e. completeness and right-continuity. The Lévy-Itô decomposition is given by

$$Z_t = b_R t + \int_0^t \int_{|x| \leq R} x (\mu - \sigma)(ds, dx) + \int_0^t \int_{|x| > R} x \mu(ds, dx), \quad t \in [0, T],$$

where  $R$  is some arbitrary positive truncation level (classically chosen to be 1) and  $\mu$  is a Poisson random measure on  $[0, T] \times \mathbb{R}$  with intensity  $\sigma(dt, dx) = dt \otimes \nu(dx)$ . Here  $b_R$  is the drift parameter given by

$$b_R := b + \int_{1 < |x| \leq R} x \nu(dx).$$

Denote respectively by

$$Z^{R-} = \int_0^t \int_{|x| \leq R} x (\mu - \sigma)(ds, dx) \quad \text{and} \quad Z^{R+} = \int_0^t \int_{|x| > R} x \mu(ds, dx).$$

The first has a compactly supported Lévy measure and is a square-integrable martingale with infinitely many jumps bounded by  $R$  on each compact time interval, while the second is an integrable compound Poisson process with jumps larger than  $R$ . Let  $K$  be a  $(\mathcal{F}_t^Z)_{t \in [0, T]}$ -predictable process belonging to  $L^2(\Omega \times [0, T])$  that is

$$\int_0^T \mathbb{E}[K_t^2] dt < +\infty.$$

Then the stable integral  $X_T := \int_0^T K_s dZ_s$  is well-defined as a stochastic integral with respect to the Lévy-Itô decomposition of  $Z$ , that is  $X = A^R +$

$X^{R-} + X^{R+}$  where for  $t \in [0, T]$ :

$$A_t^R := b_R \int_0^t K_s ds, \quad X_t^{R-} := \int_0^t K_s dZ_s^{R-} \quad \text{and} \quad X_t^{R+} := \int_0^t K_s dZ_s^{R+}.$$

It is also known in Rosinski and Woyczynski (1986), that if  $Z$  is a symmetric stable process and  $K$  is a  $\mathcal{F}^Z$ -predictable process belonging to  $L^\alpha(\Omega \times [0, T])$ , i.e.,

$$\int_0^T |K_t|^\alpha dt < \infty \quad a.s.$$

then the stable integral  $X_T := \int_0^T K_s dZ_s$  is also well defined. The following elementary and classical Gronwall inequality will be useful to make some estimates in this paper.

**Lemma 2.1.** *Let  $C_1, C_2 > 0$  and  $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a function such that for all  $t \in [0, T]$ ,*

$$H(t) \leq C_1 + C_2 \int_0^t H(r) dr.$$

*Then*

$$H(t) \leq C_1 e^{C_2 t}, \quad t \in [0, T].$$

A simpler proof can be done in the following way.

*Proof.* From the assumption (2.1), we deduce that

$$\frac{d}{dt} \left( e^{-K_2 t} \int_0^t \phi(r) dr \right) \leq K_1 e^{at} e^{-K_2 t}.$$

Integrating this inequality yields to

$$e^{-K_2 t} \int_0^t \phi(r) dr \leq \frac{K_1 e^{at}}{K_2} (1 - e^{-K_2 t}),$$

so that

$$\phi(t) \leq K_1 e^{at} + K_2 \int_0^t \phi(r) dr \leq K_1 e^{at} e^{K_2 t} \leq K_1 e^{(a+K_2)T}, \quad \forall t \in [0, T].$$

□

Let us recall a weak existence result from Fournier (2013), Proposition 2. Let  $f$  and  $\phi$  be continuous functions with at most linear growth:

**(C1)** The function  $\phi$  is Hölder continuous with exponent  $\beta \in (0, \frac{\alpha}{2})$  that is

$$|\phi(x)|^p \leq L_\phi (1 + |x|^p) \quad L_\phi > 0 \quad x \in \mathbb{R}.$$

**(C2)**  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous  $L$ -Lipchitz function that is :

$$|f(x)|^p \leq L(1 + |x|^p) \quad L > 0 \quad x \in \mathbb{R}.$$

Typical examples of conditions **(C1)** and **(C2)** are Lipschitz and Hölder continuous functions.



**Lemma 2.2** (Fournier (2013)). *Let  $Z$  be an  $\alpha$ -stable process with index  $\alpha \in (0, 2)$  and  $\alpha \neq 1$ . Let  $f$  and  $\phi$  be continuous functions with at most linear growth. There exists a weak existence for the SDE (2) such that for any  $p \in (0, \alpha)$  and any  $T > 0$ , we have*

$$\mathbb{E} \sup_{t \in [0, T]} |X_t|^p < \infty. \quad (4)$$

In the remainder of this paper we will use the following convention for constants. Throughout the paper,  $C$  and  $C_*$  denote a positive boundary constant whose value may change from line to line. Constants denoted by  $C_{\beta, L}$ ,  $C_{\beta, L, T}$ ,  $C_{\beta, \alpha}$ ,  $C_{\phi, \beta, \alpha}$ ,  $C_\phi$ ,  $C_{\beta, \alpha, \phi, b}$  are all positive and their exact values are not important. The dependency of constants on parameters is given in parentheses if necessary, e.g. we will often use the following constants

$$C_1(\alpha) = \frac{c_+ + c_-}{\alpha}, \quad C_2(\alpha) = \frac{c_+ + c_-}{2 - \alpha}, \quad C(\alpha, b) = 8b^2 + 8C_1(\alpha)C_2(\alpha),$$

$$C(\beta, L) = \max(2^{2\beta}L, 2^{2\beta}|f(0)|^{2\beta}), \quad C(\phi) = \max(2\phi(0)^2, 2c_\phi^2).$$

### 3. MAIN RESULTS

In this section, we aim to state the main theorems of this paper. We assume that  $\alpha > 1$  and we give some preliminary results about the stochastic continuity of any weak solution  $X_t$  in  $L^p$  for the SDE (2) whenever  $p = 2\beta \in (0, \alpha)$  under the conditions that  $f$  is a  $L$ -Lipchitz function and  $\phi$  is Hölder continuous with exponent  $\beta \in (0, \frac{\alpha}{2})$  that is

**(A1)** The function  $\phi$  is Hölder continuous with exponent  $\beta \in (0, \frac{\alpha}{2})$  that is

$$|\phi(y) - \phi(z)| \leq c_\phi |x - y|^\beta \quad c_\phi > 0 \quad x \in \mathbb{R}.$$

**(A2)**  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous  $L$ -Lipchitz function that is :

$$|f(y) - f(z)| \leq L|x - y| \quad L > 0 \quad x \in \mathbb{R}.$$

Secondly, we prove the consistency of an approximate solution (3) as well as the underlined rate of convergence in  $L^{2\beta}$  space for the SDE (2).

Recall that  $L^p$ -boundedness property for any  $p \in (0, \alpha)$  of the stochastic integral process  $\int_0^t \phi(X_s) dZ_s$  is well known in the literature whenever  $Z$  is a symmetric stable process or when  $\phi$  is non-negative, see Hongwei (2010) Lemma 2.4. or Mikulevičius and Xu (2018)) whenever the integrand process  $(\phi(X_t))_t \in L^\alpha(\omega \times [0, T])$ .

We revisit the proof of the  $L^p$  boundedness property (with  $p$  on the form  $p = 2\beta < \alpha$ ) for a weak solution as in Fournier (2013), see also Lemma 2.2. But here we give a different proof, which is a new methodology that we will describe in the following lines.

**Lemma 3.1.** *Let  $Z$  be an  $\alpha$ -stable process with index  $\alpha \in (0, 2)$  and consider a weak solution  $(X_t)_{t \in [0, T]}$  of the SDE (2) starting at point  $x_0$ . Under the assumptions **(A1)** and **(A2)**, for any  $1 \leq 2\beta < \alpha$ , there exists a positive constant  $C$  depending on  $L, \beta, \alpha, c_\phi, b, c_+, c_-$  and  $T$  such that*

$$\sup_{t \in [0, T]} \mathbb{E}|X_t|^{2\beta} \leq C(|x_0|^{2\beta} + 1). \quad (5)$$

**Remark 3.1.** *This result is applicable (according to the methodology of the proof) if we replace the assumption **(A2)** by  $f$  being a sum of a Lipschitz function and a negative function. If  $\phi$  is a bounded function, the result holds again.*

For the sake of simplicity, we give in the proof as follows.

*Proof.* We have for  $\beta \geq \frac{1}{2}$ ,

$$\begin{aligned} \mathbb{E}|X_t|^{2\beta} &\leq 3^{2\beta-1}|x_0|^{2\beta} + 3^{2\beta-1}\mathbb{E}\left|\int_0^t f(X_s)ds\right|^{2\beta} \\ &\quad + 3^{2\beta-1}\mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta} \\ &\leq 3^{2\beta-1}|x|^{2\beta} + 3^{2\beta-1}\mathbb{E}\left|\int_0^t f(X(s))ds\right|^{2\beta} \\ &\quad + 3^{2\beta-1}\mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta} \\ &= 3^{2\beta-1}\left(|x_0|^{2\beta} + J_1(t, \beta) + J_2(t, \beta)\right), \end{aligned}$$

where

$$J_1(t, \beta) = \mathbb{E}\left|\int_0^t |f(X_s)|ds\right|^{2\beta} \quad \text{and} \quad J_2(t, \beta) = \mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta}.$$

Note that from the Lipschitz condition **(A2)** we can derive the following linear growth condition:

$$|f(x)|^{2\beta} \leq C_{\beta, L}(1 + |x|^{2\beta}). \quad (6)$$

Using Hölder inequality we have:

$$J_1(t, \beta) \leq t \int_0^t \mathbb{E}|f(X_s)|^{2\beta} ds \leq C_{\beta, L, T} \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds. \quad (7)$$

Note that

$$J_2(t, \beta) = \mathbb{E}\left|\int_0^t \phi(X_s)dZ_s\right|^{2\beta} = 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P}\left(\left|\int_0^t \phi(X_s)dZ_s\right| \geq x\right) dx.$$

Let us first establish a tail bound for the stochastic integral  $\int_0^t \phi(X_s)dZ_s$  appearing in  $J_2(t, \beta)$  if  $\phi$  satisfies the assumption **(A1)**. We have

$$\begin{aligned}
\mathbb{P}\left(\left|\int_0^t \phi(X_s) dZ_s\right| \geq x\right) &\leq \mathbb{P}\left(\left|\int_0^t b_R \phi(X_s) ds\right| \geq \frac{x}{2}\right) \\
&\quad + \mathbb{P}\left(\left|\int_0^t \int_{|y| \leq R} y \phi(X_s) \tilde{\mu}(ds, dy)\right| \geq \frac{x}{2}\right) \\
&\quad + \mathbb{P}\left(\left|\int_0^t \int_{|y| \leq R} y \phi(X_s) \mu(ds, dy)\right| > 0\right) \\
&:= R_1(x) + R_2(x) + R_3(x).
\end{aligned}$$

By Chebychev's inequality and Cauchy-Schwarz inequality, we have:

$$R_1(x) \leq \frac{4b_R^2}{x^2} \mathbb{E} \left| \int_0^t \phi(X_s) ds \right|^2 \leq \frac{4b_R^2}{x^2} t \int_0^t \mathbb{E}(\phi(X_s)^2) ds. \quad (8)$$

But

$$\begin{aligned}
b_R^2 &\leq 2b^2 + 2\nu\{y \in \mathbb{R} : 1 < |y| \leq R\} \int_{1 < |y| \leq R} y^2 \nu(dy) \\
&\leq 2b^2 + 2\nu\{y \in \mathbb{R} : |y| > 1\} \int_{|y| \leq R} y^2 \nu(dy) \\
&\leq 2b^2 + 2C_1(\alpha)R^2 \int_{|y| \leq R} \nu(dy) \\
&\leq 2b^2 + 2C_1(\alpha)C_2(\alpha)R^{2-\alpha},
\end{aligned}$$

so that

$$R_1(x) \leq t \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_0^t \mathbb{E}\phi^2(X_s) ds.$$

From the Hölder condition, we have the following linear growth condition:

$$|\phi(x)|^2 \leq C(\phi) (1 + |x|^{2\beta}). \quad (9)$$

Therefore,

$$R_1(x) \leq tC(\phi) \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds. \quad (10)$$

From Chebychev's inequality and the isometry formula for Poisson stochastic integrals and the Hölder condition, we have

$$\begin{aligned} R_2(x) &\leq \frac{4}{x^2} \mathbb{E} \left( \int_0^t \int_{|y| \leq R} y \phi(X_s) \tilde{\mu}(ds, dy) \right)^2 \\ &\leq \frac{4}{x^2} \int_{|y| \leq R} y^2 \nu(dy) \int_0^t \mathbb{E} \phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \int_0^t \mathbb{E} \phi(X_s)^2 ds \end{aligned}$$

and,

$$R_2(x) \leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} C(\phi) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}). \quad (11)$$

Let us now deal with  $R_3(x)$ . Note that

$$N_t = \int_0^t \int_{|y| > R} y \phi(X_s) \mu(ds, dy)$$

is a compound Poisson stochastic integral. Since the first jump time  $T_1^R$  of the Poisson process  $\mu(\{y \in \mathbb{R} : |y| > R\} \times [0, t])$  on the set  $\{y \in \mathbb{R} : |y| > R\}$  is exponentially distributed with parameter  $\nu(\{y \in \mathbb{R} : |y| > R\})$ , see e.g. Ken-Iti (1999), Theorem 21.3. We have

$$\begin{aligned} \mathbb{P}(N_t > 0) &= 1 - \mathbb{P}(N_t = 0) \\ &\leq 1 - \mathbb{P}(T_1^R > t) \\ &= 1 - \exp -t\nu(\{y \in \mathbb{R} : |y| > R\}) \end{aligned}$$

and,

$$R_3(x) \leq t\nu(\{y \in \mathbb{R} : |y| > R\}) \leq tC_1(\alpha) R^{-\alpha}. \quad (12)$$

Combining the three estimates above in (10), (11) and (12) we get

$$\begin{aligned} \mathbb{P} \left( \left| \int_0^t \phi(X_s) dZ_s \right| \geq x \right) &\leq tC(\phi) \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \\ &\quad + \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} C(\phi) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds + tC_1(\alpha)R^{-\alpha}. \end{aligned}$$

Set

$$R = x \geq 1.$$

Thus for all  $x \geq 1$ , we have

$$\begin{aligned} \mathbb{P} \left( \left| \int_0^t \phi(X_s) dZ_s \right| \geq x \right) &\leq tC_1(\alpha)x^{-\alpha} + 4C_2(\alpha)x^{-\alpha}C(\phi) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \\ &\quad + x^{-\alpha}tC(\phi)C(\alpha, b) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds. \end{aligned}$$

We have

$$\begin{aligned} \mathbb{E} \left| \int_0^t \phi(X_s) dZ_s \right|^{2\beta} &= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left( \left| \int_0^t \phi(X_s) dZ_s \right| \geq x \right), \\ &\leq 1 + 2\beta \int_1^{+\infty} x^{2\beta-1} \mathbb{P} \left( \left| \int_0^t \phi(X_s) dZ_s \right| \geq x \right) dx \\ &\leq 1 + \frac{tC_1(\alpha)}{\alpha - 2\beta} + \frac{1}{\alpha - 2\beta} C(\phi) \left( tC(\alpha, b) + 4C_2(\alpha) \right) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds, \end{aligned}$$

since

$$\int_1^{+\infty} x^{2\beta-1-\alpha} dx = \frac{1}{\alpha - 2\beta}.$$

Thus,

$$J_2(t, \beta) \leq 3^{2\beta-1} \left( C_{\beta, \alpha} + C_{\beta, \alpha, \phi, b} \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \right). \quad (13)$$

From (7) and (13), we have

$$\mathbb{E}|X_t|^{2\beta} \leq 3^{2\beta-1} \left( |x_0|^{2\beta} + C_{\beta, \alpha, g} + C_{\beta, \alpha, g, \phi, b, L} \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \right)$$

and using Gronwall's lemma we get the desired result.  $\square$

Now, before announcing the main result, let us note the following result.

**Proposition 3.1.** *Let  $Z$  be a  $\alpha$ -stable process with index  $\alpha \in (0, 2)$  and consider a weak solution of the SDE (2) starting at point  $x_0$ . Under the assumptions **(A1)** and **(A2)**, for any  $1 \leq 2\beta < \alpha$  and  $\Delta_n = n^{-\delta}$  with  $\delta \geq \frac{1}{2}$ , there exists a positive constant  $C$  depending on  $L, \beta, \alpha, c_\phi, b, c_+,$  and  $c_-$  such that for  $n \geq 1$ ,*

$$\mathbb{E} \sup_{|u-t| \leq \Delta_n} |X_u - X_t|^{2\beta} \leq C \left( n^{-2\delta\beta} + n^{-(2\beta^2+\delta)} + n^{-(2\beta^2-2\beta+\delta)} + n^{-(2\beta^2-2\beta+2\delta)} \right) \quad (14)$$

**Remark 3.2.** *This result holds (according to the methodology of the proof) if we replace the assumption **(A2)** by  $f$  which is a sum of a Lipschitz function and a negative function. It is also valid if we replace the assumption **(A1)** by  $\phi$  being a continuous bounded function. The relation (14) is new. In particular, we have for  $n \geq 1$ ,*

$$\mathbb{E} \sup_{|u-t| \leq \Delta_n} |X_u - X_t|^{2\beta} \leq Cn^{-\beta^2}.$$

**3.1. Error and rate of the approximation.** We aim here to study the consistency of the above mentioned continuous time version of the Euler-Maruyama approximation of the SDE (2) given by

$$\tilde{X}_t^n = x_0 + \int_0^t f(\tilde{X}_{\pi_s^n}^n) ds + \int_0^t \phi(\tilde{X}_{\pi_s^n}^n) dZ_s.$$

We now have the following result.

**Theorem 3.2.** *Let  $Z$  be a  $\alpha$ -stable process with index  $\alpha \in (1, 2)$  and consider a weak solution  $(X_t)_{t \in [0, T]}$  of the SDE (2) starting at point  $x_0$ . Under the assumptions **(A1)** and **(A2)**, for any  $2\beta < \alpha$  and  $\Delta_n = n^{-\delta}$  with  $\delta \geq 1$ , there exists a positive constant  $C$  depending on  $L, \beta, \alpha, c_\phi, b, c_+, c_-$  such that*

$$\mathbb{E} \sup_{t \in [0, T]} |\tilde{X}_t^n - X_t|^{2\beta} \leq Cn^{-\beta^2}. \quad (15)$$

This approximation is highly dependent on the method used, and we need the approximation in  $L^p$  space with  $p \geq 1$  of the form  $p = 2\beta < \alpha$ .

**Remark 3.3.** *Note that this result extends the work of Mikulevičius and Xu (2018); Wu and Hao (2023) in the case of Lipschitz drift function and Hölder continuous diffusion coefficient. Moreover, we are not restricted to symmetric stable processes and positive stable processes. Finally, the methodology of proof is new.*

Note that Proposition 3.1 ensures that, the approximate paths of a weak solution for the SDE (2) are stochastically continuous whenever the time discretisation frequency  $\Delta_n = n^{-\delta}$  is small enough under the above conditions in Theorem 3.2. Hence according to (Peszat and Zabczyk (2007), Prop. 3.21),  $X$  has a predictable modification. This ensure the existence of a strong approximate solution of the SDE (2).

**Comment 3.1.** *Let  $Z$  be a  $\alpha$  stable process with index  $\alpha \in (1, 2)$ . Under the assumptions **(A1)** and **(A2)**, recall that there exists a weak solution of SDE (2), since from (6) and (9), the functions  $f$  and  $\phi$  are continuous and have at most linear growth. From Proposition 3.1 it can now be seen that the weak approximate solution of the SDE (2) has a predictable version. Following Fournier (2013), the sequence  $(\tilde{X}_t^n)_{n \geq 1}$  defined in (3) is tight in Skorokhod space, and by continuity we have*

$$\left( \tilde{X}_t^n, \int_0^t f(\tilde{X}_{\pi_s^n}^n) ds, \int_0^t \phi(\tilde{X}_{\pi_s^n}^n) dZ_s \right)$$

converges by law to

$$\left( X_t, \int_0^t f(X_s) ds, \int_0^t \phi(X_s) dZ_s \right).$$

and we conclude that there exists a strong solution of the SDE (2). This extends the work of Bass et al. (2004) for not necessarily symmetric stable processes with  $\alpha \in (\sqrt{2}, 2)$  when  $f = 0$ .

## 4. PROOFS OF THE MAIN THEOREMS

### 4.1. Proof of Proposition 3.1.

*Proof.* Consider  $t, u \in [0, T]$  such that  $u - t \leq \frac{1}{n^\delta}$  for some  $\delta > 0$ . We have

$$\begin{aligned} X_u - X_t &= \int_0^u f(X_s)ds - \int_0^t f(X_s)ds \\ &\quad + \int_0^u \phi(X_s)dZ_s - \int_0^t \phi(X_s)dZ_s. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}|X_u - X_t|^{2\beta} &\leq 2^{2\beta} \mathbb{E} \left| \int_t^u f(X_s)ds \right|^{2\beta} \\ &\quad + 2^{2\beta} \mathbb{E} \left| \int_t^u \phi(X_s)dZ_s \right|^{2\beta} \\ &:= I_1(t, \beta) + I_2(t, \beta) \end{aligned}$$

Using the Hölder inequality and the assumption **(A2)**, we have the following estimate

$$\begin{aligned} I_2(t, \beta) &\leq 2^{2\beta} n^{-\delta(2\beta-1)} \int_t^u \mathbb{E}|f(X_s)|^{2\beta} ds \\ &\leq 2^{2\beta} n^{-\delta(2\beta-1)} C \int_t^u (1 + \mathbb{E}|X_s|^{2\beta}) ds \\ &\leq C_\beta n^{-2\beta\delta}, \end{aligned}$$

since  $1 + \mathbb{E}|X_s|^{2\beta}$  is finite, by Lemma 3.1.

Now let us consider the estimation of  $I_2(t, \beta)$ . Imitating the previous argument, we have

$$\begin{aligned} \mathbb{P} \left( \left| \int_t^u \phi(X_s)dZ_s \right| \geq x \right) &\leq \mathbb{P} \left( \left| \int_t^{t+\frac{1}{n}} b_R \phi(X_s)ds \right| \geq \frac{x}{2} \right) \\ &\quad + \mathbb{P} \left( \left| \int_t^u \int_{|y| \leq R} y \phi(X_s) \tilde{\mu}(ds, dy) \right| \geq \frac{x}{2} \right) \\ &\quad + \mathbb{P} \left( \left| \int_t^u \int_{|y| \leq R} y \phi(X_s) \mu(ds, dy) \right| > 0 \right) \\ &:= R_1(x, n) + R_2(x, n) + R_3(x, n). \end{aligned}$$

Using the Chebychev inequality and the Cauchy-Schwarz inequality we have :

$$R_1(x, n) \leq \frac{4b_R^2}{x^2} \mathbb{E} \left| \int_t^u \phi(X_s)ds \right|^2 \leq \frac{4b_R^2}{x^2} n^{-\delta} \int_t^u \mathbb{E}(\phi(X_s))^2 ds. \quad (16)$$

$$R_1(x, n) \leq n^{-\delta} C(\phi) \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_t^u (1 + \mathbb{E}|X_s|^{2\beta}) ds \quad (17)$$

$$\leq n^{-2\delta} C(\phi) \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right), \quad (18)$$

since  $1 + \mathbb{E}|X_s|^{2\beta}$  is bounded. Next, we use Chebychev's inequality and the isometry formula for Poisson stochastic integrals and Hölder condition to get

$$\begin{aligned} R_2(x, n) &\leq \frac{4}{x^2} \mathbb{E} \left| \int_t^u \int_{|y| \leq R} y \phi(X_s) \tilde{\mu}(ds, dy) \right|^2 \\ &\leq \frac{4}{x^2} \int_{|y| \leq R} y^2 \nu(dy) \int_t^u \mathbb{E} \phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \int_t^u \mathbb{E} \phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} C(\phi) \int_t^u (1 + \mathbb{E}|X_s|^{2\beta}) ds \\ &\leq n^{-\delta} C(\phi) \frac{4C_2(\alpha) R^{2-\alpha}}{x^2}. \end{aligned}$$

Let us now consider the compound Poisson stochastic integral appearing in  $R_3(x, h)$ . Note that

$$N_{t,u} = \int_t^u \int_{|y| > R} y \phi(X_s) \mu(ds, dy)$$

have the same law as

$$N_{u-t} = \int_0^{u-t} \int_{|y| > R} y \phi(X_s) \mu(ds, dy),$$

so that we have

$$\mathbb{P}(N_{u-t} > 0) \leq n^{-\delta} C_1(\alpha) R^{-\alpha}.$$

By combining the three estimates, we obtain

$$\begin{aligned} \mathbb{P} \left( \left| \int_t^u \phi(X_s) dZ_s \right| \geq x \right) &\leq n^{-\delta} C_1(\alpha) R^{-\alpha} + n^{-\delta} C(\phi) \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \\ &\quad + n^{-2\delta} C(\phi) \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right). \end{aligned}$$

If we choose the truncation level

$$R = xn^\beta \geq 1$$

and rearrange the terms, we get for all  $x \geq n^{-\beta}$  :



$$\begin{aligned} \mathbb{P} \left( \left| \int_t^u \phi(X_s) dZ_s \right| \geq x \right) &\leq C_1(\alpha) x^{-\alpha} n^{-\beta\alpha-\delta} + C(\phi) 4C_2(\alpha) x^{-\alpha} n^{-\alpha\beta+2\beta-\delta} \\ &\quad + C(\phi) C(\alpha, b) x^{-\alpha} n^{-\alpha\beta+2\beta-2\delta} \end{aligned}$$

From the fact that,

$$\begin{aligned} \mathbb{E} \left| \int_t^u \phi(X_s) dZ_s \right|^{2\beta} &= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left( \left| \int_t^u \phi(X_s) dZ_s \right| \geq x \right) dx, \\ &\leq n^{-2\beta^2} + 2\beta \int_{n^{-\beta}}^{+\infty} x^{2\beta-1} \mathbb{P} \left( \left| \int_t^u \phi(X_s) dZ_s \right| \geq x \right) dx. \end{aligned}$$

and since

$$\begin{aligned} n^{-\alpha\beta-\delta} \int_{n^{-\beta}}^{+\infty} x^{2\beta-1-\alpha} dx &= \frac{n^{-2\beta^2-\delta}}{\alpha-2\beta} \quad \text{and} \quad n^{-\alpha\beta+2\beta-\delta} \int_{n^{-\beta}}^{+\infty} x^{2\beta-1-\alpha} dx = \frac{n^{-2\beta^2+2\beta-\delta}}{\alpha-2\beta} \\ n^{-\alpha\beta+2\beta-2\delta} \int_{n^{-\beta}}^{+\infty} x^{2\beta-1-\alpha} dx &= \frac{n^{-2\beta^2+2\beta-2\delta}}{\alpha-2\beta} \end{aligned}$$

we have

$$\mathbb{E} \left| \int_t^u \phi(X_s) dZ_s \right|^{2\beta} \leq C \left( n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta} \right).$$

Finally,

$$\mathbb{E} \sup_{t \in [0, T]; |u-t| \leq \Delta_n} |X_u - X_t|^{2\beta} \leq C \left( n^{-2\delta\beta} + n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta} \right),$$

and since  $2\beta\delta > 0$ ,  $2\beta^2 > 0$ ,  $2\beta^2 + \delta > 0$ ,  $2\beta^2 - 2\beta + \delta > 0$  and  $2\beta^2 - 2\beta + 2\delta > 0$  we can conclude.  $\square$

#### 4.2. Proof of Theorem 3.2.

*Proof.* Set

$$V_t = \sup_{0 \leq s \leq t} |\tilde{X}_s^n - X_s|.$$

To establish the convergence of  $\tilde{X}^n$  to  $X$ , let's write the explicit expression for  $Y_t^n = \tilde{X}_t^n - X_t$ .

$$\tilde{X}_t^n - X_t = \int_0^t \left[ f(\tilde{X}_{\pi_s^n}^n) - f(X_s) \right] ds + \int_0^t \left[ \phi(\tilde{X}_{\pi_s^n}^n) - \phi(X_s) \right] dZ_s,$$

and

$$\tilde{X}_t^n - X_t = \int_0^t \left[ f(\tilde{X}_{\pi_s^n}^n) - f(X_s) \right] ds + \int_0^t \left[ \phi(\tilde{X}_{\pi_s^n}^n) - \phi(X_{\pi_s^n}) \right] dZ_s + \int_0^t \left[ \phi(X_{\pi_s^n}) - \phi(X_s) \right] dZ_s.$$

We have

$$\begin{aligned} \mathbb{E}|Y_t^n|^{2\beta} &\leq (3t)^{2\beta-1} \int_0^t \mathbb{E} \left| f(\tilde{X}_{\pi_s^n}) - f(X(s)) \right|^{2\beta} ds + 3^{2\beta-1} \mathbb{E} \left| \int_0^t (\phi(\tilde{X}_{\pi_s^n}) - \phi(X_{\pi_s^n})) dZ_s \right|^{2\beta} \\ &\quad + 3^{2\beta-1} \mathbb{E} \left| \int_0^t [\phi(X_{\pi_s^n}) - \phi(X_s)] dZ_s \right|^{2\beta} := P_1(t, \beta) + 3^{2\beta-1} P_2(t, \beta) + 3^{2\beta-1} P_3(t, \beta). \end{aligned}$$

But

$$\begin{aligned} \mathbb{E}|f(\tilde{X}_{\pi_s^n}) - f(X_s)|^{2\beta} &\leq 2^{2\beta-1} \mathbb{E}|f(\tilde{X}_{\pi_s^n}) - f(X_{\pi_s^n})|^{2\beta} \\ &\quad + 2^{2\beta-1} |f(X_{\pi_s^n}) - f(X_s)|^{2\beta} \\ &\leq 2^{2\beta-1} \mathbb{E}|f(\tilde{X}_{\pi_s^n}) - f(X_{\pi_s^n})|^{2\beta} \\ &\quad + 2^{2\beta-1} \mathbb{E}|f(X_{\pi_s^n}) - f(X_s)|^{2\beta}, \end{aligned}$$

so that,

$$\begin{aligned} P_1(t, \beta) &\leq (6t)^{2\beta-1} \mathbb{E} \left( \int_0^t |f(\tilde{X}_{\pi_s^n}) - f(X_{\pi_s^n})|^{2\beta} ds \right) \\ &\quad + (6t)^{2\beta-1} \mathbb{E} \left( \int_0^t |f(X_{\pi_s^n}) - f(X_s)|^{2\beta} ds \right) \\ &\leq C_{\beta, L, T} \left( \mathbb{E} \left( \int_0^t |\tilde{X}_{\pi_s^n} - X_{\pi_s^n}|^{2\beta} ds \right) + \mathbb{E} \left( \int_0^t |X_{\pi_s^n} - X_s|^{2\beta} ds \right) \right) \\ &\leq C_{\beta, L, T} \int_0^t \mathbb{E} V_s^{2\beta} ds + C_{\beta, L, T} \int_0^t \mathbb{E} |X_{\pi_s^n} - X_s|^{2\beta} ds \\ &\leq C_{\beta, L, T} \int_0^t \mathbb{E} V_s^{2\beta} ds + Cn^{-\beta^2}, \end{aligned}$$

where in the last inequality, we apply Proposition 3.1 since  $s - \pi_s^n \leq n^\delta$ . Secondly, we have

$$\begin{aligned} P_2(t, \beta) &= \mathbb{E} \left| \int_0^t (\phi(\tilde{X}_{\pi_s^n}) - \phi(X_{\pi_s^n})) dZ_s \right|^{2\beta} \\ &= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left( \left| \int_0^t \phi(\tilde{X}_{\pi_s^n}) - \phi(X_{\pi_s^n}) dZ_s \right| \geq x \right) dx. \end{aligned}$$

However, based on previous estimates, we have

$$\mathbb{P} \left( \left| \int_0^t \phi(\tilde{X}_{\pi_s^n}) - \phi(X_{\pi_s^n}) dZ_s \right| \geq x \right) dx \leq R_1^*(x, n) + R_2^*(x, n) + R_3^*(x, n),$$

where

$$R_1^*(x, n) \leq \frac{4b_R^2}{x^2} \sup_{k=0, \dots, n-1} \mathbb{E} \left| \int_0^t \mathbb{I}_{[t_k, t_{k+1}[}(s) \phi(\tilde{X}_{\pi_s^n}) - \phi(X_{\pi_s^n}) ds \right|^{2\beta}$$

$$R_1^*(x, n) \leq \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) c_\phi^2 n^{-2\delta} \mathbb{E}V_{\pi_s^n}^{2\beta},$$

$$R_2^*(x, n) \leq \frac{4C_2(\nu)R^{2-\alpha}}{x^2} \sup_{k=0, \dots, n-1} \mathbb{E} \left( \int_0^t \mathbb{I}_{[t_k, t_{k+1}[}(s) |\phi(\tilde{X}_{\pi_s^n}^n) - \phi(X_{\pi_s^n})|^2 ds \right)$$

$$\leq \frac{4C_2(\nu)R^{2-\alpha}}{x^2} c_\phi^2 n^{-\delta} \mathbb{E}V_{\pi_s^n}^{2\beta}$$

and,

$$R_3^*(x, n) \leq n^{-\delta} C_1(\alpha) R^{-\alpha}.$$

Note that  $\mathbb{E}|\tilde{X}_{\pi_s^n}^n - X_{\pi_s^n}|^{2\beta} \leq 2^{2\beta} \mathbb{E}|\tilde{X}_t^n|^{2\beta} + 2^{2\beta} \mathbb{E}|X_t|^{2\beta} < \infty$  because we can show by induction on  $k$  and using the proof arguments of Lemma 3.1 that  $\sup_{n \geq 1} \sup_{t \in [t_k, t_{k+1}[} \mathbb{E}|\tilde{X}_t^n|^{2\beta}$ .

Now, choosing the truncation level  $R = xn^\beta \geq 1$ , we have for all  $x \geq n^{-\beta}$ :

$$\mathbb{P} \left( \left| \int_0^t \phi(X_{\pi_s^n}) - \phi(X_{\pi_s^n}) dZ_s \right| \geq x \right) \leq C_1(\alpha) x^{-\alpha} n^{-\beta\alpha - \delta} + c_\phi^2 4C_2(\alpha) x^{-\alpha} n^{\beta^2 - \alpha\beta + 2\beta - \delta}$$

$$+ c_\phi^2 C(\alpha, b) x^{-\alpha} n^{\beta^2 - \alpha\beta + 2\beta - 2\delta},$$

so that

$$P_2(t, \beta) = \mathbb{E} \left| \int_0^t [\phi(X_{\pi_s^n}) - \phi(X_{\pi_s^n})] dZ_s \right|^{2\beta}$$

$$= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left( \left| \int_0^t \phi(X_{\pi_s^n}) - \phi(X_{\pi_s^n}) dZ_s \right| \geq x \right)$$

$$\leq C \left( n^{-2\beta^2} + n^{-2\beta^2 - \delta} + n^{-\beta^2 + 2\beta - \delta} + n^{-\beta^2 + 2\beta - 2\delta} \right) \leq Cn^{-\beta^2}.$$

Now let us look at  $P_3(t, \beta)$ . We have

$$\mathbb{P} \left( \left| \int_0^t \phi(X_{\pi_s^n}) - \phi(X_s) dZ_s \right| \geq x \right) \leq R_1^{**}(x, n) + R_2^{**}(x, n) + R_3^{**}(x, n),$$

where

$$R_1^{**}(x, n) \leq \frac{4b_R^2}{x^2} \sup_{k=0, \dots, n-1} \mathbb{E} \left| \int_0^t \mathbb{I}_{[t_k, t_{k+1}[}(s) \phi(X_{\pi_s^n}) - \phi(X_s) ds \right|^\beta$$

$$R_1^{**}(x, n) \leq \left( \frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) c_\phi^2 n^{-2\delta} \sup_{|u-t| \leq \Delta_n} \mathbb{E}|X_u - X_t|^{2\beta},$$

$$\begin{aligned}
R_2^{**}(x, n) &\leq \frac{4C_2(\nu)R^{2-\alpha}}{x^2} \sup_{k=0, \dots, n-1} \int_0^t \mathbb{I}_{[t_k, t_{k+1}[}(s) \mathbb{E}|\phi(X_{\pi_s^n}) - \phi(X_s)|^2 ds \\
&\leq \frac{4C_2(\nu)R^{2-\alpha}}{x^2} c_\phi^2 \sup_{k=0, \dots, n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}|X_{\pi_s^n} - X_s|^{2\beta} ds \\
&\leq \frac{4C_2(\nu)R^{2-\alpha}}{x^2} c_\phi^2 n^{-\delta} \sup_{|u-t| \leq \Delta_n} \mathbb{E}|X_u - X_t|^{2\beta},
\end{aligned}$$

and,

$$R_3^{**}(x, n) \leq n^{-\delta} C_1(\alpha) R^{-\alpha}.$$

Again, choosing the truncation level  $R = xn^\beta \geq 1$ , we have for all  $x \geq n^{-\beta}$ :

$$\begin{aligned}
\mathbb{P} \left( \left| \int_0^t \phi(X_{\pi_s^n}) - \phi(X_s) dZ_s \right| \geq x \right) &\leq C_1(\alpha) x^{-\alpha} n^{-\beta\alpha-\delta} + c_\phi^2 4C_2(\alpha) x^{-\alpha} n^{-\alpha\beta+2\beta-\delta} \\
&\quad + c_\phi^2 C(\alpha, b) x^{-\alpha} n^{-\alpha\beta+2\beta-2\delta},
\end{aligned}$$

so that

$$\begin{aligned}
P_3(t, \beta) &= \mathbb{E} \left| \int_0^t [\phi(X_{\pi_s^n}) - \phi(X_s)] dZ_s \right|^{2\beta} \\
&= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left( \left| \int_0^t \phi(X_{\pi_s^n}) - \phi(X_s) dZ_s \right| \geq x \right) \\
&\leq C \left( + n^{-2\beta^2} + n^{-2\beta^2-\delta} + n^{-2\beta^2+2\beta-\delta} + n^{-2\beta^2+2\beta-2\delta} \right).
\end{aligned}$$

Finally, there exist positive constants  $C$  (which depend on all the model parameters) and  $C_{\beta, L}$  such that

$$\mathbb{E}V_t^{2\beta} \leq C \int_0^t V_s^{2\beta} ds + C n^{-\beta^2}$$

and we use the Gronwall Lemma to get the desired result.  $\square$

## 5. EXAMPLES

In this section we will illustrate the usefulness of the above approximate solution results of the SDE (2) that we established in the previous section. Let us first recall the random walk approximation method for the simulation of  $\alpha$ -stable processes, due to the self-similarity property and the property of stationary independent increments. Of course, other methods are available, such as the series approximation of the Lévy process, see Janicki et al. (1997) for more details. The following proposition, based on the algorithm of Chambers et al. (1976) for  $\alpha$ -stable distributions, allows to simulate  $\alpha$ -stable processes.

**Proposition 5.1** (Discretized trajectory for a strictly  $\alpha$ -stable process). *Let  $Z = (Z_t)_{t \in [0, T]}$  be a strictly- $\alpha$  stable process.*

**Step 1** *Simulate  $n$  independent uniformly distributed random variables  $\Phi$  on  $[-\pi/2, \pi/2]$  and  $n$  independent identically distributed random variables  $W$  as exponential with parameter 1.*

**Step 2** *Compute  $\Delta Z_i$  for  $i = 1, \dots, n$  as follows.*

(1) *If  $\alpha \neq 1$*

$$\Delta Z_i = \sigma \left( \frac{T}{n} \right)^{1/\alpha} \frac{\sin(\alpha(\Phi - \phi_0))}{\cos(\Phi)^{1/\alpha}} \left( \frac{\cos(\Phi - \alpha(\Phi - \phi_0))}{W} \right)^{\frac{1-\alpha}{\alpha}}.$$

(2) *If  $\alpha = 1$*

$$\Delta Z_i = \sigma \left( \frac{T}{n} \right)^{1/\alpha} \frac{2}{\pi} \left( \left( \frac{\pi}{2} + \beta\Phi \right) \tan(\Phi) - \beta \log \left( \frac{\frac{1}{2}\pi W \cos(\Phi)}{\frac{1}{2}\pi + \beta\Phi} \right) \right)$$

where

$$\phi_0 = -\frac{\beta\pi}{2} \frac{1 - |1 - \alpha|}{\alpha}.$$

**Step 3** *The discretised trajectory of  $Z$  is given by*

$$Z(t_i) = \sum_{k=1}^i \Delta Z_k.$$

From a practical point of view, we consider the linearly interpolated version between times  $t_i$  and  $t_{i+1}$  for graphical representations. Figure 1 shows typical example paths for some strict and symmetric  $\alpha$  processes. When  $\alpha$  is closed to 2 and  $\beta$  is closed to 0, the sample paths resemble those of Brownian motion. For  $\alpha = 1$  and  $\beta = 0$ , the sample paths correspond to those of the Cauchy process. We can see the existence of jumps in the simulation, for example when  $\alpha$  is not close to 2.

We now consider the following SDE models.

**Example 5.1.** *(Stable driven OU process) Consider the following Ornstein-Uhlenbeck (OU) process driven by a stable process  $(Z_t)_{t \geq 0}$  with parameter  $\theta$  and  $\rho > 0$  is defined by*

$$dX_t = -\theta X_t dt + \rho dZ_t, \quad X_0 = x_0 \in \mathbb{R}. \quad (19)$$

and has the integral representation

$$X_t(\omega) = e^{-\theta t} \left( x_0 + \rho \int_0^t e^{\theta s} dZ_s(\omega) \right), \quad \text{for } t \in [0, T]. \quad (20)$$

Since  $f(x) = \theta x$  and  $\phi(x) = \rho$  satisfy the Lipschitz and boundedness conditions, there is clearly an approximation of the stable driven OU process with step size  $n^{-\delta}$  and  $\delta \geq 1$ . We give in figure 2 (a) a path approximation

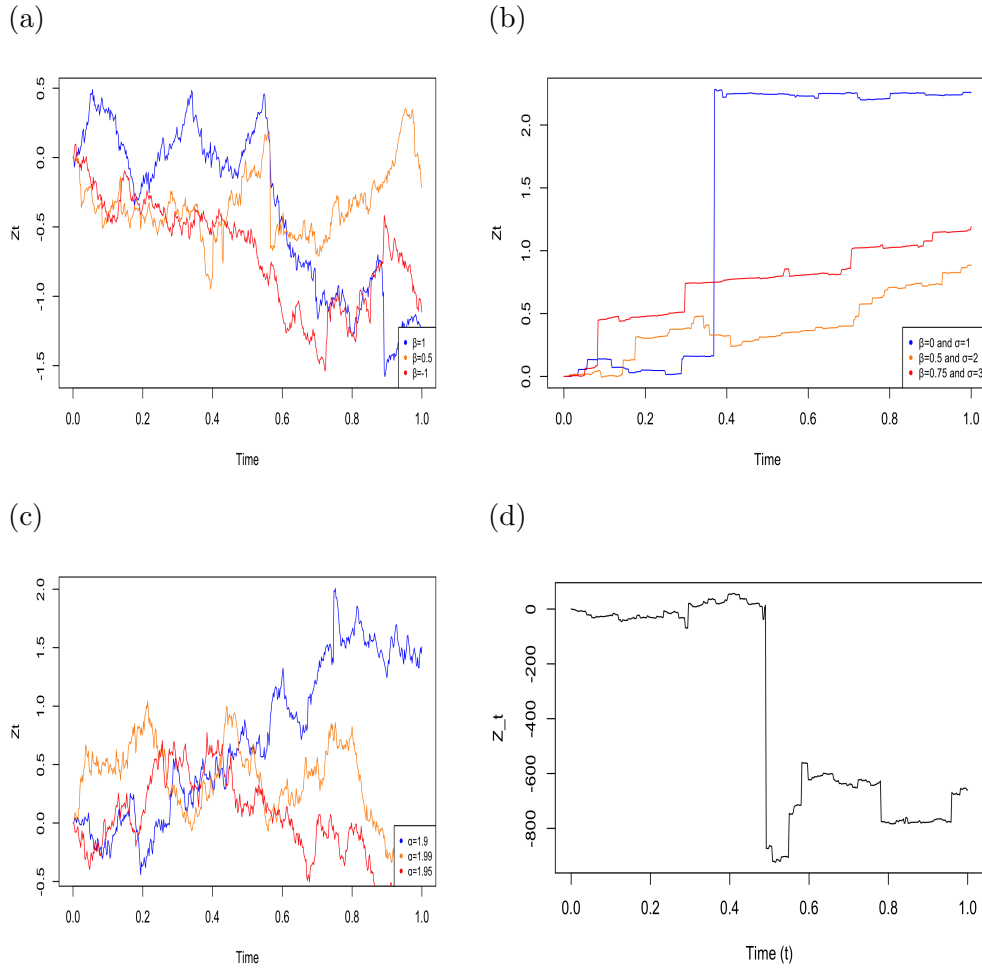


FIGURE 1. Paths of a standard strictly 1.7 stable process with different skewness (a), a 0.7 stable process with different scale and skewness (b), the symmetric and standard  $\alpha$  stable process with different stability index (c) and the case of the standard 1 stable process (d)..

solution for  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\rho = 1$  with  $n = 1000$  sample size and  $\delta = 1$ . Another classic example is given by the following stable driven Cox-Ingersoll-Ross type models.

**Example 5.2.** (Stable driven Cox–Ingersoll–Ross type process)  
 A stable driven Cox–Ingersoll–Ross type process can be defined as follows

$$dX_t = (\lambda - \theta X_t)dt + \rho |X_t|^q dZ_t, \quad X_0 \geq 0,$$

where  $q \in [\frac{1}{2}, 1)$ ,  $\lambda, \rho, \theta$  are real constants and  $Z$  is a symmetric stable process.

When  $q = \alpha^{-1}$  and  $Z$  has only positive jumps, it is called SCIR model in Bayraktar and Clément (2023); Yang (2017)

According to Fournier (2013), there is a weak existence solution. Note that since  $f(x) = \lambda - \theta x$  is a Lipschitz function, and  $\phi$  is Hölder  $\phi(x) = |x|^q$  continuous, then there exists an approximate solution with step size  $n^{-\delta}$  and  $\delta \geq 1$ . In particular, one can choose  $q = \alpha - 1$  and  $\alpha \in [1.5, 2)$  to obtain in Figure 2 (b) a strong approximate path solution for  $\lambda = 0.2$ ,  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\rho = 1$  and  $q = \alpha - 1$  with  $n = 1000$  sample size. In figure 2 (c) we give an approximate path solution for  $\lambda = 0.2$ ,  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\rho = 1$  and  $q = 1 - \alpha^{-1}$  with  $n = 1000$  sample size and  $\delta = 1$ .

We present the following stable driven Lotka-Volterra models.

**Example 5.3.** (Stable driven Lotka-Volterra type process)

A stable driven Lotka-Volterra type process can be defined as follows

$$dX_t = X_t(\lambda - \theta X_t)dt + \rho|X_t|^q dZ_t, \quad X_0 \geq 0,$$

where  $q \in [\frac{1}{2}, \frac{\alpha}{2})$ ,  $\lambda$ ,  $\rho$ ,  $\theta$  are real constants and  $Z$  is a symmetric stable process.

When  $q = 1$ , this model is proposed in Zhang et al. (2017) when  $Z$  is a positive stable process. Note that  $f(x) = x(\lambda - \theta x) = \lambda x - \theta x^2$  is the sum of a Lipschitz function and a negative function if  $\theta > 0$ . Since  $\phi$  is Hölder continuous, there is an approximate solution. We give in Figure 2 (d) an approximate path solution for  $\lambda = 0.2$ ,  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\rho = 1$  and  $q = 1/2$  with  $n = 1000$  sample size and  $\delta = 1$ .

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The authors declare that they have no competing interests.

#### DECLARATION STATEMENT

The author certify that the submission is an original work and is not under review at any other publication.

#### DATA AVAILABILITY STATEMENTS

Data sharing not applicable to this article as no datasets were generated or analysed during the current study. The R codes for the simulation study are available from the author on reasonable request.

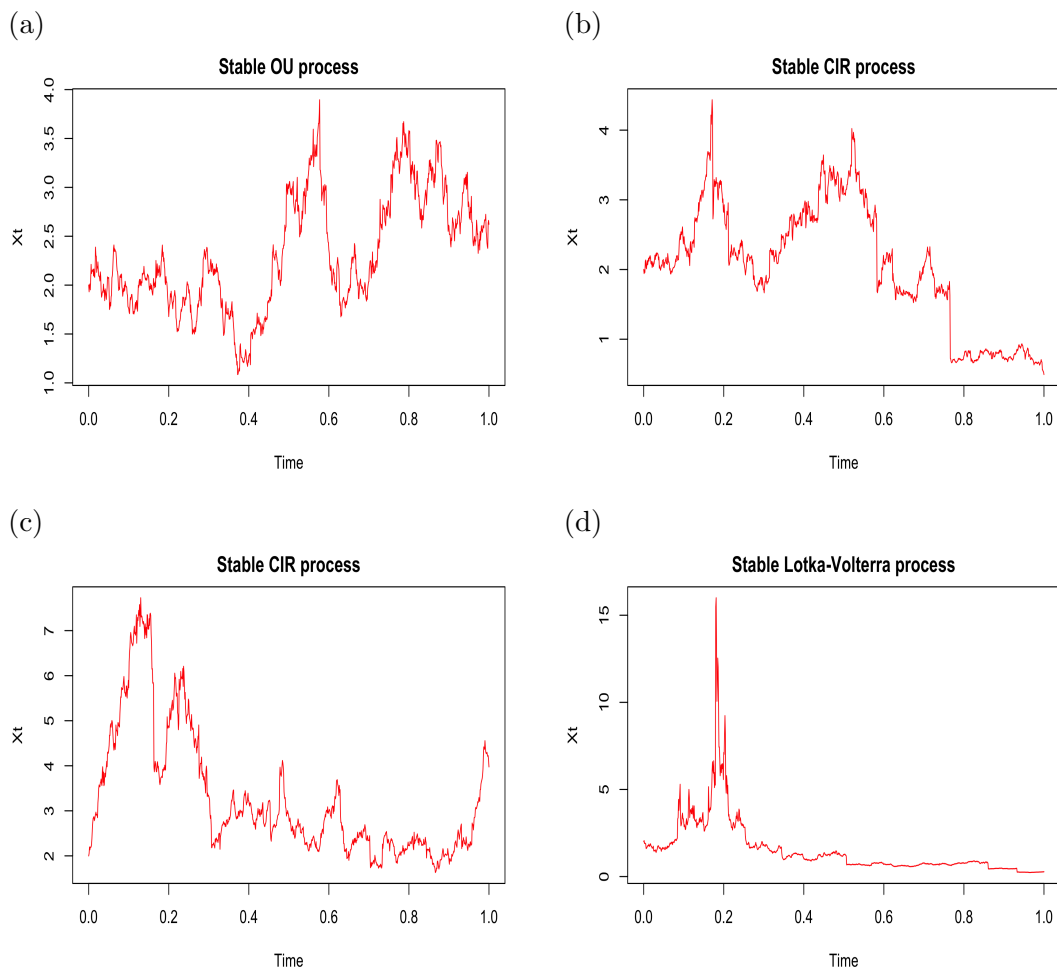


FIGURE 2. An approximate solution path of a standard symmetric stable driven OU process with sample size  $n = 1000$ ,  $\rho = 1$ ,  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$  (a); of an standard symmetric stable driven CIR type process with sample size  $n = 1000$ ,  $\rho = 1$ ,  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\lambda = 0.2$  and  $q = \alpha^{-1}$  (b); a standard symmetric stable driven CIR type process with sample size  $n = 1000$ ,  $\rho = 1$ ,  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\lambda = 0.2$  and  $q = 1 - \alpha^{-1}$  (c) and the case of a standard symmetric stable driven Lotka-Volterra type process with  $\lambda = 0.2$ ,  $\theta = 1.2$ ,  $\alpha = 1.7$ ,  $\beta = 0$ ,  $\rho = 1$ ,  $q = 1/2$  and  $n = 1000$  (d).

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