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APPROXIMATE SOLUTION FOR A CLASS OF STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY STABLE PROCESSES

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ABSTRACT. We consider in this paper, a general class of stochastic differential equations driven by α -stable processes with Lipschitz drift coefficients and non-Lipschitz diffusion coefficients. A strong Euler-Maruyama approximate solution is proved whenever the diffusion coefficient is Hölder continuous with exponent $\beta \in [\frac{1}{2}, \frac{\alpha}{2})$ and $\alpha \in (1, 2)$. We obtain a strong rate of convergence of order $n^{-\beta^2}$. Our proposed method is new in this context and based on a truncation method by separating the big and small jumps of the α -stable process in the Lévy-Itô decomposition. Along the paper we give some numerical simulation of stochastic models that match our results namely some stable driven Ornstein-Uhlenbeck and Cox–Ingersoll–Ross processes.

AMS Subject Classification: 60G17, 60G52, 60H10, 60H20, 65C30, 65C20, 60E07.

Key words and phrases: Stochastic differential equation; Non-Lipschitz condition, Strong approximation, Stable process.

1. INTRODUCTION

Models such as Stochastic Differential Equations (SDEs) are natural choice to model the time evolution of dynamic systems and have a variety of applications in many disciplines such as population dynamics, epidemiology, finance and biology. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions. Moreover, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is sufficiently rich to contain other processes considered in this paper that aims to study the following stochastic differential equation (SDE) driven by α -stable processes, $\alpha \in (1, 2)$:

$$\begin{cases} dX_t = F(t, X_t)dt + G(t, X_{t-})dZ_t, & t \in [0, T] \\ X_0 = x \in \mathbb{R}, \end{cases} \quad (1)$$

where x_0 is a starting point and $T > 0$ a time horizon, under some conditions on functions F and G . In the theory of stochastic processes and its applications, it is important to find existence conditions for solutions of SDE. There are several notions of existence and uniqueness. A weak solution is a solution that satisfy the SDE (1). A strong solution is a predictable process (X_t, \mathcal{F}_t^Z) that satisfy the SDE (1). By its definition, a weak solution which is predictable is a strong solution. The pathwise uniqueness holds if two any solutions are indistinguishable. Some results are available on the existence under various conditions on the coefficients of the above SDE. The existence and uniqueness of solutions to the above SDE under Lipschitz or linear growth conditions are well know results in stochastic calculus as well as for general Lévy processes Z , see e.g. Rong (2006); Fu and Li (2010); Protter (1991). For example, assuming that the drift and diffusion coefficients are continuous and have at most linear growth; then there is weak existence in L^p for any $p \in (0, \alpha)$ of SDE (1), see Proposition 2 in Fournier (2013) for $\alpha \in (0, 2)$ and $\alpha \neq 1$. This weak existence is contained in Rong (2006).

For non-Lipschitz coefficients and considering α -Stable driven SDE, some results have been studied before. In Bass et al. (2004); Komatsu (1982), when $\alpha \in (1, 2)$ and Z is symmetric with $\alpha \in (1, 2)$, it was shown that if $F = 0$ and $G(t, x) = G(x)$ has modulus of continuity ρ satisfying $\int_{0+} \frac{1}{\rho(x)^\alpha} = \infty$, there is a strong solution and the solution is pathwise unique. In particular, if the diffusion term $G(t, x) = G(x)$ is Hölder continuous of order α^{-1} then, the strong existence and pathwise uniqueness holds. This condition is the exact analogue of the Yamada–Watanabe condition for SDE driven by a Brownian motion. When the drift term $F(t, x) = F(x)$ is Hölder continuous of order $\beta > 2 - \alpha$ with $\beta \in (0, 1)$ and $G(t, x) = G(x)$ is Lipschitz continuous and bounded then, the existence and uniqueness of a strong solution is proved in Mikulevičius and Xu (2018). In Wu and Hao (2023), if the drift term is a continuous β -Hölder function of order $\beta \leq 1 - \frac{\alpha}{2}$ with $\alpha \in (1, 2)$ then, the existence and uniqueness of a strong solution is proved, thanks to the deep connection between the Kolmogorov equations and SDE. Also in Priola (2012), the existence and uniqueness of strong solutions is shown as well as pathwise uniqueness for SDE driven by non-degenerate symmetric stable processes Z with values in the higher dimensional space \mathbb{R}^d , whenever the diffusion term $G(t, x) = G(x)$ is bounded and the drift term is β -Hölder continuous under the condition $\beta > 1 - \frac{\alpha}{2}$ and $\alpha \in [1, 2)$. The proof is based on the associated integro-differential operators of Kolmogorov type. In Pamen and Taguchi (2017), when $G(t, x) = G(x)$ is bounded and the drift function $F(t, x)$ is bounded β -Hölder continuous in space and η -Hölder in time with $\eta \in [1/2, 1]$ and $\beta \in (0, 1)$, the pathwise uniqueness was proved by applying Gronwall’s lemma and using the elliptic version of the Kolmogorov equation and regularity of its solution. Recently, in Athreya et al. (2020), they established strong existence and uniqueness whenever $G = 1$ and the

drift function $F(t, x) = F(x)$ belongs to the Besov Hölder space C^β for $\beta > \frac{1}{2} - \frac{\alpha}{2}$ and Z symmetric with $\alpha \in (1, 2)$. For time-dependent Sobolev drifts $F(t, x)$, the pathwise uniqueness in \mathbb{R}^d is proved in Zhang (2013).

Assume that $G(t, x) = G(x)$ is non-decreasing and Hölder continuous with index $1 - \frac{1}{\alpha}$ and $F(t, x) = F(x)$ is the sum of a Lipschitz-continuous function and a non-increasing function. It is shown in Li and Mytnik (2011); Fu and Li (2010) that the strong existence holds if $\alpha \in (1, 2)$ and Z has only positive jumps, as well as pathwise uniqueness. This latter result was extended in Fournier (2013) as follows. For stable process with not necessarily positive jumps, but under some conditions on the positive and negative jump activity; the weak existence and pathwise uniqueness is established assuming for example that, the drift function $F(t, x) = F(x)$ is a sum of Lipschitz-continuous function and a non-increasing function; the diffusion coefficient $G(t, x) = G(x)$ is a sum of a Lipschitz-continuous function and Hölder continuous with index lying in $[1 - \alpha^{-1}, \alpha^{-1}]$ for the case $\alpha \in (1, 2)$. For stable process Z with only positive jumps, they also show that strong existence holds when the diffusion coefficient is a sum of a Lipschitz-continuous and Hölder function with index $1 - \frac{1}{\alpha}$ which is similar to Li and Mytnik (2011). For symmetric stable process Z , strong existence holds also in Fournier (2013) whenever the diffusion term is Hölder continuous with index $\beta = \frac{1}{\alpha}$, as in Komatsu (1982) or Bass et al. (2004). The technique in Fournier (2013) is based on Ito formula for jump processes and some technical integrals computing (see Lemma 7 and 9 in Fournier (2013)) and an approximating sequence iteration method.

It is possible to appreciate the basics of how to simulate SDEs numerically with just a background knowledge of Euler-Maruyama' scheme which is a well-known method to approximate any solution. Some results are now available concerning the approximate solutions of SDEs driven by stable process as well as the speed of convergence of the approximation. The Itô-Tanaka technique has traditionally been used to obtain convergence rates for Euler schemes which has important applications in stochastic financial theory. In Pamen and Taguchi (2017), when $G = 1$ and the drift function $F(t, x)$ is a bounded β -Hölder continuous in space with $\beta \in (0, 1)$ and, η -Hölder in time with $\eta \in [1/2, 1]$, a strong approximate solution is proved and they provide the rate of convergence for the Euler-Maruyama approximation. More precisely, using the the Itô-Tanaka trick method and assuming that a strong solution exists with $\alpha + \beta > 2$, the rate of strong convergence was derived in the form n^{-1} if $p \geq \frac{2}{\beta}$ and on the form $n^{-p\beta/2}$ for d -dimensional truncated symmetric α -stable process when $d \geq 2$ for $\alpha \in (1, 2)$ and $1 \leq p\beta < 2$ or $p \in [1, 2]$. In the same way, Mikulevičius and Xu (2018) provide a strong approximate solution and a rate of convergence of order $n^{-p\beta/\alpha}$ when $p \in (0, \alpha)$ and Z is a symmetric stable process of index

$\alpha \in (1, 2)$ whenever $F(t, x) = F(x)$ is β -Hölder continuous with $\beta > 1 - \frac{\alpha}{2}$ and G being bounded Lipschitz continuous function. See also Wu and Hao (2023) for similar conditions.

In this paper, we consider the following SDE:

$$X_t = x_0 + \int_0^t F(s, X_s)ds + \int_0^t g(s)\phi(X_{s-})dZ_s, \quad t \in [0, T], \quad (2)$$

where Z is an α -stable process with $\alpha \in (1, 2)$ not necessarily symmetric and g is a bounded function on $[0, T]$ under the following assumptions :

(A1) The function ϕ is Hölder continuous with exponent $\beta \in (0, \frac{\alpha}{2})$ that is

$$|\phi(y) - \phi(z)| \leq c_\phi |y - z|^\beta \quad c_\phi > 0 \quad y, z \in \mathbb{R}.$$

(A2) $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous L -Lipchitz function that is :

$$|F(t, y) - F(t, z)| \leq L|y - z| \quad L > 0 \quad y, z \in \mathbb{R}.$$

We investigate the existence of consistency strong Euler-Maruyama approximate solution defined as

$$\tilde{X}_t^n = x_0 + \int_0^t F(\pi_s^n, \tilde{X}_{\pi_s^n}^n)ds + \int_0^t g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n)dZ_s, \quad (3)$$

where

$$\pi_t^n = \frac{k}{n} \quad \text{if } t \in \left[\frac{k}{n}, \frac{(k+1)}{n} \right], \quad k = 0, \dots, n-1.$$

While this process is pleasant to handle from a theoretical point of view, it is not easy for simulation purpose except at points $t_k = \frac{k}{n}$. From a practical point of view, we therefore prefer its linearly interpolated version between instants t_k and t_{k+1} .

We assume in this paper that $T = 1$. Compared to the above mentioned previous works, we established a new result on strong approximate solution as well as the rate of convergence rate for the new SDE (2). Moreover our methodology of proof is new and not based on Itô-Tanaka trick method but, based on Lévy-Itô decomposition together with a truncation method for stable integrals developed in Manou-Abi (2015). We have not found a direct proof like our technique. Our result extend the previous mentioned work on strong approximate solution to the case of Lipschitz drift function and Hölder continuous diffusion coefficients. This paper is organized as follows. In section 2, primary analytic tools are discussed, notation and preliminaries are introduced. In section 3, we first give some preliminaries moments bound results of any solution of the SDE (2). After that, we present our main theorem, which extends the result of Mikulevičius and Xu (2018); Pamen and Taguchi (2017); Wu and Hao (2023) for Lipchitz drift function F , Hölder condition diffusion function $\phi \neq 1$ or ϕ bounded if Z is an α -stable process with index $\alpha \in (1, 2)$. Namely if ϕ is β -Hölder continuous

with index $\beta \in [\frac{1}{2}, \frac{\alpha}{2})$, there exists a strong consistent approximate solution of (2) and the rate of convergence is of order $n^{-\beta^2}$. In Section 4, we consider stochastic models as well as numerical simulation of their solutions, that match our results. The last section is dedicated to the proofs of the main results.

2. PRELIMINARIES

In this part, we recall the definition of Lévy process and introduce the α -stable process as a member of the class of Lévy process. We also introduce some notations that will be useful along the paper. We follow the presentation in Ken-Iti (1999); Applebaum (2009). We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and satisfying the usual conditions as above.

Definition 2.1. *A real stochastic process $(Z_t)_{t \in [0, T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is said to be a Lévy process if $Z_0 = 0$ almost surely (a.s.) with the following properties:*

1. *Z is stochastically continuous i.e., for any $\epsilon > 0$ and $t, s \geq 0$,*

$$\lim_{s \rightarrow 0} \mathbb{P}(|Z_{t+s} - Z_t| > \epsilon) = 0.$$

2. *Z has stationary increments, i.e., for $0 \leq s \leq t$, the distribution of $Z_t - Z_s$ does not depend on s , namely has the same distribution as Z_{t-s} .*
3. *Z has independent increments, i.e., for $0 \leq s \leq t$, $Z_t - Z_s$ is independent of \mathcal{F}_s .*
4. *The paths of Z are \mathbb{P} -a.s. right-continuous in $t \geq 0$ and has left limits in $t > 0$ (càdlàg).*

Note that the last item above can be dropped, as one can prove that there always exists a càdlàg modification of the stable process Z . In this way, the Brownian motion is an example of Lévy process without jumps. Set $\Delta Z_t = Z_t - Z_{t-}$. The jumps part of the stochastic process Z can be described by its Poisson random measure (jump measure of Z on interval $[0, t]$) defined as

$$\mu(t, A) = \sum_{0 \leq s \leq t} \mathbb{I}_A(\Delta Z_s), \quad A \in \mathcal{B}(\mathbb{R}^*),$$

the number of jumps of Z on the interval $[0, t]$ whose size lies in the set A bounded below. For such A , the process $\mu(\cdot, A)$ is a Poisson process and the Lévy measure ν is defined by $\nu(A) := \mathbb{E}(\mu(1, A))$. In the sequel, we denote the compensated random martingale measure by

$$\tilde{\mu}(t, A) = \mu(t, A) - t\nu(A).$$

A particular class of Lévy processes that can contain infinitely many jumps (unbounded jumps) is the following class of α -stable process. Now, we follow the presentation in Cattiaux and S. Manou-Abi (2014); Manou-Abi (2015).

Definition 2.2. A real stochastic process $(Z(t))_{t \in [0, T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is said to be an α -stable process with characteristics (b, c_+, c_-) , if it is a Lévy process with the following characteristic function :

$$\varphi_{Z_t}(u) = \exp t \left(iub + \int_{-\infty}^{+\infty} (e^{iuy} - 1 - iuy \mathbb{1}_{|y| \leq 1}) \nu(dy) \right), \quad t \in [0, T],$$

where b stands for the drift parameter of Z and ν the Lévy measure defined on $\mathbb{R} \setminus \{0\}$ by

$$\nu(dx) := \frac{dx}{|x|^{\alpha+1}} (c_+ 1_{\{x>0\}} + c_- 1_{\{x<0\}}).$$

The parameters c_+, c_- above are non-negative with furthermore $c_+ + c_- > 0$ and $c_+ = c_-$ when $\alpha = 1$. The process is said to be symmetric if $c_+ = c_- := c$. It is said to be strictly α -stable if $b = 0$. In the case $\alpha \in (1, 2)$, the drift parameter is given by

$$b := - \int_{|y|>1} y \nu(dy) = - \frac{(c_+ - c_-)}{\alpha - 1} \quad \text{when } \alpha \neq 1.$$

An α -stable process is closely related to the notion of self-similar process. The process Z is said to be strictly α -stable if we have the self-similarity property

$$k^{-1/\alpha} (Z(kt))_{t \in [0, T]} \stackrel{d}{=} (Z(t))_{t \in [0, T]},$$

where $k > 0$ and the equality $\stackrel{d}{=}$ is understood in the sense of finite dimensional distributions. Note that α -stable processes are interesting due to the self-similarity property and the fact that the Lévy measure and the Lévy-Itô decomposition are almost totally explicit for the one dimensional case. In order to control the jump size of Z and the moment behavior of a stable stochastic integral, let us introduce the following truncation method developed in our past paper Manou-Abi (2015) and also in Joulin (2007). As a Lévy process, Z is a semimartingale with respect to $\mathcal{F}_t^Z := \sigma(Z_s : s \in [0, t])$, $t \in [0, T]$ assuming it satisfies the usual hypothesis, that is completeness and right-continuity. The Lévy-Itô decomposition is given by

$$Z_t = b_R t + \int_0^t \int_{|x| \leq R} x (\mu - \sigma)(ds, dx) + \int_0^t \int_{|x| > R} x \mu(ds, dx), \quad t \in [0, T],$$

where R is some arbitrary positive truncation level (classically chosen to be 1) and μ is a Poisson random measure on $[0, T] \times \mathbb{R}$ with intensity $\sigma(dt, dx) = dt \otimes \nu(dx)$. Here b_R is the drift parameter given by

$$b_R := b + \int_{1 < |x| \leq R} x \nu(dx).$$

Denote respectively by

$$Z^{R-} = \int_0^t \int_{|x| \leq R} x (\mu - \sigma)(ds, dx) \quad \text{and} \quad Z^{R+} = \int_0^t \int_{|x| > R} x \mu(ds, dx).$$

The first one has a compactly supported Lévy measure and is a square-integrable martingale with infinitely many jumps bounded by R on each compact time interval, whereas the second one is an integrable compound Poisson process with jumps larger than R . Let K be a $(\mathcal{F}_t^Z)_{t \in [0, T]}$ -predictable process belonging to $L^2(\Omega \times [0, T])$ that is

$$\int_0^T \mathbb{E}[K_t^2] dt < +\infty.$$

Then the stable integral $X_T := \int_0^T K_s dZ_s$ is well-defined as a stochastic integral with respect to the Lévy-Itô decomposition of Z , that is $X = A^R + X^{R-} + X^{R+}$ where for $t \in [0, T]$:

$$A_t^R := b_R \int_0^t K_s ds, \quad X_t^{R-} := \int_0^t K_s dZ_s^{R-} \quad \text{and} \quad X_t^{R+} := \int_0^t K_s dZ_s^{R+}.$$

The second aforementioned process is a square-integrable martingale whereas the two other integrals are constructed in the Lebesgue-Stieltjes sense.

It is also well known in Rosinski and Woyczynski (1986), that if Z is a symmetric stable process and K a \mathcal{F}^Z -predictable process belonging to $L^\alpha(\Omega \times [0, T])$, i.e.,

$$\int_0^T |K_t|^\alpha dt < \infty \quad a.s.$$

then the stable integral $X_T := \int_0^T K_s dZ_s$ is also well defined. The following elementary and classical Gronwall inequality will be useful in order to establish some estimates in this paper.

Lemma 2.1. *Let $C_1, C_2 > 0$ and $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function such that for all $t \in [0, T]$,*

$$H(t) \leq C_1 + C_2 \int_0^t H(r) dr.$$

Then

$$H(t) \leq C_1 e^{C_2 t}, \quad t \in [0, T].$$

A simpler proof can be handle in the following way.

Proof. From the assumption (2.1), we deduce that

$$\frac{d}{dt} \left(e^{-K_2 t} \int_0^t \phi(r) dr \right) \leq K_1 e^{at} e^{-K_2 t}.$$

Integrating this inequality yields to

$$e^{-K_2 t} \int_0^t \phi(r) dr \leq \frac{K_1 e^{at}}{K_2} (1 - e^{-K_2 t}),$$

so that

$$\phi(t) \leq K_1 e^{at} + K_2 \int_0^t \phi(r) dr \leq K_1 e^{at} e^{K_2 t} \leq K_1 e^{(a+K_2)T}, \quad \forall t \in [0, T].$$

□

In the sequel, we shall be concerned with the following convention on constants. Throughout the paper, C and C_* denotes a positive boundary constant whose value may change from line to line. Constants denoted by $C_{\beta,L}$, $C_{\beta,L,T}$, $C_{\beta,\alpha,g}$, $C_{\phi,\beta,\alpha}$, C_ϕ , $C_{\beta,\alpha,g,\phi,b}$ are all positive and their precise values are not important. The dependence of constants on parameters if needed will be mentioned inside brackets, for example, we will frequently make use of the following constants :

$$\eta(g) := \left(\sup_{t \in [0,1]} |g(t)|^2 \right)^{1/2}, \quad C_1(\alpha) = \frac{c_+ + c_-}{\alpha}, \quad C_2(\alpha) = \frac{c_+ + c_-}{2 - \alpha},$$

$$C(\beta, L) = \max \left(2^{p-1}L, 2^{p-1} \sup_{t \in [0,1]} |F(t, 0)|^{2\beta} \right), \quad C(\phi) = \max \left(2\phi(0)^2, 2c_\phi^2 \right)$$

$$C(\alpha, b) = 8b^2 + 8C_1(\alpha)C_2(\alpha), \quad \kappa(\alpha, b, \phi) = C(\alpha, b) \|\phi\|_\infty^2 + 4\|\phi\|_\infty^2 C_2(\alpha) + C_1(\alpha).$$

3. MAIN RESULTS

In this section, we aims to state the main theorems of this paper. Firstly, we give some preliminaries results about the p -th moment and the stochastic continuity of any given solution X_t of the SDE (2). Secondly, we prove the consistency of the strong approximate solution (3) as well as the underlined rate of convergence.

3.1. Boundedness and stochastic continuity of any solution. We start by giving an L^p -boundedness property of the stochastic integral process $\int_0^t g(s)\phi(X_s)dZ_s$.

Lemma 3.1. *Let Z be an α -stable process with index $\alpha \in (1, 2)$ and $(X_t)_{t \in [0,1]}$ any solution of the SDE (2) starting at point x_0 . If ϕ is bounded then for all $p < \alpha$, we have :*

$$\sup_{t \in [0,1]} \mathbb{E} \left| \int_0^t g(s)\phi(X_s)dZ_s \right|^p \leq \eta(g)^p \left(1 + \frac{p}{\alpha - p} \kappa(\alpha, b, \phi) \right). \quad (4)$$

Next, we give the following boundedness property of any solution of the SDE (2) available for continuous bounded function ϕ or function ϕ satisfying assumption **(A1)**.

Lemma 3.2. *Let Z be an α -stable process with index $\alpha \in (0, 2)$ and $(X_t)_{t \in [0,T]}$ any solution of the SDE (2) starting at point x_0 . Under assumptions **(A1)** and **(A2)**, if $\beta \in [\frac{1}{2}, \frac{\alpha}{2})$, there exists a positive constant C depending on $L, \beta, \alpha, g, c_\phi, b, c_+, c_-$ such that*

$$\sup_{t \in [0,1]} \mathbb{E} |X_t|^{2\beta} \leq C(|x_0|^{2\beta} + 1). \quad (5)$$

Remark 3.1. *This result is applicable if we replace assumption **(A2)** by F is a sum of a Lipschitz function and a negative function, according to the methodology of the proof. As concerned the moment estimation in Lemma*

3.2; note that under linear growth conditions, it was stated in Fournier (2013), Proposition 2. But we give here a different proof, which is a new methodology we shall describe in the following lines.

For the sake of simplicity, we only give the proof of Lemma 3.2.

Proof. We have for all $\beta \geq \frac{1}{2}$,

$$\begin{aligned} \mathbb{E}|X_t|^{2\beta} &\leq 3^{2\beta-1}|x_0|^{2\beta} + 3^{2\beta-1}\mathbb{E}\left|\int_0^t F(s, X_s)ds\right|^{2\beta} \\ &\quad + 3^{2\beta-1}\mathbb{E}\left|\int_0^t g(s)\phi(X_s)dZ_s\right|^{2\beta} \\ &\leq 3^{2\beta-1}|x|^{2\beta} + 3^{2\beta-1}\mathbb{E}\left(\int_0^t |F(s, X(s))|ds\right)^{2\beta} \\ &\quad + 3^{2\beta-1}\mathbb{E}\left|\int_0^t g(s)\phi(X_s)dZ_s\right|^{2\beta} \\ &= 3^{2\beta-1}\left(|x_0|^{2\beta} + J_1(t, \beta) + J_2(t, \beta)\right), \end{aligned}$$

where

$$J_1(t, \beta) = \mathbb{E}\left(\int_0^t |F(s, X_s)|ds\right)^{2\beta} \quad \text{and} \quad J_2(t, \beta) = \mathbb{E}\left|\int_0^t g(s)\phi(X_s)dZ_s\right|^{2\beta}.$$

Note that from the globally Lipschitz condition **(A2)**, we can deduce the following linear growth condition:

$$|F(t, x)|^{2\beta} \leq C_{\beta, L}(1 + |x|^{2\beta}). \quad (6)$$

Using Hölder inequality we have:

$$J_1(t, \beta) \leq t \int_0^t \mathbb{E}|F(s, X_s)|^{2\beta} ds \leq C_{\beta, L, T} \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds. \quad (7)$$

Note that

$$J_2(t, \beta) = \mathbb{E}\left|\int_0^t g(s)\phi(X_s)dZ_s\right|^{2\beta} = 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P}\left(\left|\int_0^t g(s)\phi(X_s)dZ_s\right| \geq x\right) dx.$$

Now let us first established a tail bound for the stochastic integral $\int_0^t g(s)\phi(X_s)dZ_s$ appearing in $J_2(t, \beta)$ when ϕ satisfy assumption **(A1)**. We have

$$\begin{aligned}
\mathbb{P} \left(\left| \int_0^t g(s) \phi(X_s) dZ_s \right| \geq x \right) &\leq \mathbb{P} \left(\int_0^t |b_R g(s) \phi(X_s)| ds \geq \frac{x}{2} \right) \\
&\quad + \mathbb{P} \left(\int_0^t \int_{|y| \leq R} |y g(s) \phi(X_s)| \tilde{\mu}(ds, dy) \geq \frac{x}{2} \right) \\
&\quad + \mathbb{P} \left(\int_0^t \int_{|y| \leq R} |y g(s) \phi(X_s)| \mu(ds, dy) > 0 \right) \\
&:= R_1(x) + R_2(x) + R_3(x).
\end{aligned}$$

By Chebychev's inequality and Cauchy-Schwarz inequality, we have:

$$R_1(x) \leq \frac{4b_R^2}{x^2} \mathbb{E} \left(\int_0^t |g(s)| |\phi(X_s)| ds \right)^2 \leq \eta^2(g) \frac{4b_R^2}{x^2} t \int_0^t \mathbb{E}(\phi(X_s)^2) ds. \quad (8)$$

But

$$\begin{aligned}
b_R^2 &\leq 2b^2 + 2\nu\{y \in \mathbb{R} : 1 < |y| \leq R\} \int_{1 < |y| \leq R} y^2 \nu(dy) \\
&\leq 2b^2 + 2\nu\{y \in \mathbb{R} : |y| > 1\} \int_{|y| \leq R} y^2 \nu(dy) \\
&\leq 2b^2 + 2C_1(\alpha)R^2 \int_{|y| \leq R} \nu(dy) \\
&\leq 2b^2 + 2C_1(\alpha)C_2(\alpha)R^{2-\alpha},
\end{aligned}$$

so that

$$R_1(x) \leq t \eta^2(g) \left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_0^t \mathbb{E} \phi^2(X_s) ds.$$

From Hölder condition, we have the following linear growth condition:

$$|\phi(x)|^2 \leq C(\phi) (1 + |x|^{2\beta}). \quad (9)$$

Therefore,

$$R_1(x) \leq tC \eta^2(g) \left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds. \quad (10)$$

From Chebychev's inequality and isometry formula for Poisson stochastic integrals and Hölder condition, we have

$$\begin{aligned}
R_2(x) &\leq \frac{4}{x^2} \mathbb{E} \left(\int_0^t \int_{|y| \leq R} |y g(s) \phi(X_s)| \tilde{\mu}(ds, dy) \right)^2 \\
&\leq \frac{4}{x^2} \int_{|y| \leq R} y^2 \nu(dy) \int_0^t g^2(s) \mathbb{E} \phi(X_s)^2 ds \\
&\leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) \int_0^t \mathbb{E} \phi(X_s)^2 ds
\end{aligned}$$

and,

$$R_2(x) \leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) C(\phi) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}). \quad (11)$$

Now, let us deal with $R_3(x)$. Note that

$$N_t = \int_0^t \int_{|y| > R} y g(s) \phi(X_s) \mu(ds, dy)$$

is a compound Poisson stochastic integral. The first jump time T_1^R of the Poisson process $\mu(\{y \in \mathbb{R} : |y| > R\} \times [0, t])$ on the set $\{y \in \mathbb{R} : |y| > R\}$ is exponentially distributed with parameter $\nu(\{y \in \mathbb{R} : |y| > R\})$, see e.g. Ken-Iti (1999), Theorem 21.3. We have

$$\begin{aligned}
\mathbb{P}(N_t > 0) &= 1 - \mathbb{P}(N_t = 0) \\
&\leq 1 - \mathbb{P}(T_1^R > t) \\
&= 1 - \exp -t\nu(\{y \in \mathbb{R} : |y| > R\})
\end{aligned}$$

and,

$$R_3(x) \leq t\nu(\{y \in \mathbb{R} : |y| > R\}) \leq tC_1(\alpha) R^{-\alpha}. \quad (12)$$

By combining the above three estimates in (10), (11) and (12), we obtain

$$\begin{aligned}
\mathbb{P} \left(\left| \int_0^t g(s) \phi(X_s) dZ_s \right| \geq x \right) &\leq tC \eta^2(g) \left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \\
&\quad + \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) C \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds + tC_1(\alpha) R^{-\alpha}.
\end{aligned}$$

Set

$$R = \frac{x}{\eta(g)} \geq 1.$$

Thus for all $x \geq \eta(g)$, we have

$$\begin{aligned}
\mathbb{P} \left(\left| \int_0^t g(s) \phi(X_s) dZ_s \right| \geq x \right) &\leq tC_1(\alpha) x^{-\alpha} \eta(g)^\alpha + 4C_2(\alpha) x^{-\alpha} \eta(g)^\alpha C(\phi) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \\
&\quad + x^{-\alpha} \eta(g)^\alpha t C(\phi) C(\alpha, b) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds.
\end{aligned}$$

Since

$$\eta(g)^\alpha \int_{\eta(g)}^{+\infty} x^{2\beta-1-\alpha} dx = \frac{\eta(g)^{2\beta}}{\alpha - 2\beta},$$

we have

$$\begin{aligned} \mathbb{E} \left(\int_0^t |g(s)\phi(X_s)| dZ_s \right)^{2\beta} &= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(s)\phi(X_s)| dZ_s \geq x \right) dx, \\ &\leq \eta(g)^{2\beta} + 2\beta \int_{\eta(g)}^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(s)\phi(X_s)| dZ_s \geq x \right) dx \\ &\leq \eta(g)^{2\beta} + tC_1(\alpha) \frac{\eta(g)^{2\beta}}{\alpha - 2\beta} \\ &\quad + \frac{\eta(g)^{2\beta}}{\alpha - 2\beta} C(\phi) \left(tC(\alpha, b) + 4C_2(\alpha) \right) \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds, \end{aligned}$$

so that

$$J_2(t, \beta) \leq 3^{2\beta-1} \left(C_{\beta, \alpha, g} + C_{\beta, \alpha, g, \phi, b} \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \right). \quad (13)$$

From (7) and (13), we have

$$\mathbb{E}|X_t|^{2\beta} \leq 3^{2\beta-1} \left(|x_0|^{2\beta} + C_{\beta, \alpha, g} + C_{\beta, \alpha, g, \phi, b, L} \int_0^t (1 + \mathbb{E}|X_s|^{2\beta}) ds \right)$$

and, using the Gronwall's lemma, we obtain the desired result. \square

We can deduce Lemma 3.1 using the same methodology. Now, let us establish the following result.

Theorem 3.3. *Let Z be an α -stable process with index $\alpha \in (0, 2)$ and $(X_t)_{t \in [0, 1]}$ any solution of the SDE (2) starting at point x_0 . Under assumptions **(A1)** and **(A2)**, if $\beta \in [\frac{1}{2}, \frac{\alpha}{2})$, there exists a positive constant C_* depending on $L, \beta, \alpha, g, c_\phi, b, c_+,$ and c_- such that*

$$\sup_{t \in [0, 1]} \mathbb{E} |X_{t+h} - X_t|^{2\beta} \leq C \left(h^{2\beta} + h^{2\beta^2+1} + h^{2\beta^2-2\beta+1} + h^{2\beta^2-2\beta+2} \right) \quad (14)$$

This result ensures that, the paths of any solution for the SDE (2) are stochastically continuous when h is small enough and under the conditions of Theorem 3.3. Hence according to (Peszat and Zabczyk (2007), Prop. 3.21), X has a predictable modification.

Remark 3.2. *The above result remains true if we remove assumption **(A1)** and assuming that ϕ is a continuous bounded function. The relation (14) is new. For $h \rightarrow 0$, and $\forall \beta \in (\frac{1}{2}, \alpha/2)$ with $\alpha \in (1, 2)$, we have*

$$\sup_{t \in [0, 1]} \mathbb{E} |X_{t+h} - X_t|^{2\beta} \leq Ch^{\beta^2},$$

since $2\beta^2 - 2\beta + 1, 2\beta^2 - 2\beta + 2, 2\beta^2 + 1$ and 2β are all greater than β^2 .

3.2. Error and rate of the approximation. We aims here to study the consistency of the above mentionned continuous time version of the Euler-Maruyama approximation of SDE (2) given by

$$\tilde{X}_t^n = x_0 + \int_0^t F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) ds + \int_0^t g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) dZ_s.$$

We established the following result.

Theorem 3.4. *Let Z be an α -stable process with index $\alpha \in (1, 2)$ and $(X_t)_{t \in [0, T]}$ any solution of the SDE (2) starting at point x_0 . Under assumptions **(A1)** and **(A2)**, if $\beta \in [\frac{1}{2}, \frac{\alpha}{2})$, there exists a positive constant C_* depending on $L, \beta, \alpha, g, c_\phi, b, c_+, c_-$ such that*

$$\sup_{t \in [0, T]} \mathbb{E} |\tilde{X}_t^n - X_t|^{2\beta} \leq C_* n^{-\beta^2}. \quad (15)$$

Remark 3.3. *Note that, this result is again applicable if we remove assumption **(A1)** and assuming that ϕ is a continuous bounded function. It extend the work of Mikulevičius and Xu (2018); Wu and Hao (2023) in the case of Lipschitz drift function and Hölder continuous diffusion coefficient. Moreover, we are not limited to symmetric stable process and positive stable processes.*

Comment 3.1. *Let Z be an α -stable process with index $\alpha \in (1, 2)$. Under assumptions **(A1)** and **(A2)**, if g is a constant, there exists a weak solution of SDE (2) since from (6) and (9), function F and ϕ are continuous and have at most linear growth. The proof of this weak existence can also be summarized by using Aldous' criterion (Jacod and Shiryaev (2013)) and by imitating Fournier (2013) since the sequence $(\tilde{X}_t^n)_{n \geq 1}$ defined in (3) is tight in the Skorokhod space and by continuity we have*

$$\left(\tilde{X}_t^n, \int_0^t F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) ds, \int_0^t g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) dZ_s \right)$$

converges in law to

$$\left(X_t, \int_0^t F(s, X_s) ds, \int_0^t g(s) \phi(X_s) dZ_s \right).$$

Passing to the limit in equation (3), we deduce that there exists a weak solution of SDE (2). Now, from Theorem 3.3, one can state that this weak solution of the SDE (2) has a predictable version. This ensure the existence of a strong solution of the SDE (2) and extend by the way, the work of Fournier (2013). This extend also the work of Bass et al. (2004) for not necessarily symmetric stable process where strong existence is proved for $\beta = \frac{1}{\alpha}$. More precisely, we cover the case $\beta = \frac{1}{\alpha}$ with $\alpha \in (1, \sqrt{2}]$ and for not necessarily symmetric stable process. Our mains Theorems extends also the work of Li and Mytnik (2011) for not necessarily positive stable process with $\alpha \in (1, \sqrt{2}]$.

4. EXAMPLES

In this section, we illustrate usefulness of the strong approximate solution results of the SDE (2) we have established in the preceding section.

Let us first recall the random walk approximation method for α -stable process simulation, due to the self-similarity property and the property of stationary independent increments. Of course others methods are available like the series approximation of Lévy process, see Janicki et al. (1997) for more details. The following proposition based on the algorithm of Chambers et al. (1976) for α -stable distributions, allow to simulate α -stable processes.

Proposition 4.1 (Discretized trajectory for a strictly α -stable process). *Let $Z = (Z_t)_{t \in [0, T]}$ be a strictly- α stable process.*

Step 1 *Simulate n independent, uniformly distributed random variables Φ on $[-\pi/2, \pi/2]$ and n independent and identically distributed random variable W as an exponentially with parameter 1.*

Step 2 *Compute ΔZ_i for $i = 1, \dots, n$ as follows.*

(1) *If $\alpha \neq 1$*

$$\Delta Z_i = \sigma \left(\frac{T}{n} \right)^{1/\alpha} \frac{\sin(\alpha(\Phi - \phi_0))}{\cos(\Phi)^{1/\alpha}} \left(\frac{\cos(\Phi - \alpha(\Phi - \phi_0))}{W} \right)^{\frac{1-\alpha}{\alpha}}.$$

(2) *If $\alpha = 1$*

$$\Delta Z_i = \sigma \left(\frac{T}{n} \right)^{1/\alpha} \frac{2}{\pi} \left(\left(\frac{\pi}{2} + \beta\Phi \right) \tan(\Phi) - \beta \log \left(\frac{\frac{1}{2}\pi W \cos(\Phi)}{\frac{1}{2}\pi + \beta\Phi} \right) \right)$$

where

$$\phi_0 = -\frac{\beta\pi}{2} \frac{1 - |1 - \alpha|}{\alpha}.$$

Step 3 *The discretized trajectory of Z is given by*

$$Z(t_i) = \sum_{k=1}^i \Delta Z_k.$$

From a practical point of view, we consider the linear interpolated version between instants t_i and t_{i+1} for graphical representations. Figure 1 show typical sample paths for some strictly and symmetric α -processes. When α is closed to 2 and β to 0 the sample paths resemble to that of Brownian motions. For $\alpha = 1$ and $\beta = 0$, the sample paths corresponds to the that of Cauchy process. We can see the existence of jumps in the simulation, for instance when α is not close to 2.

Now, we consider the following SDE models.

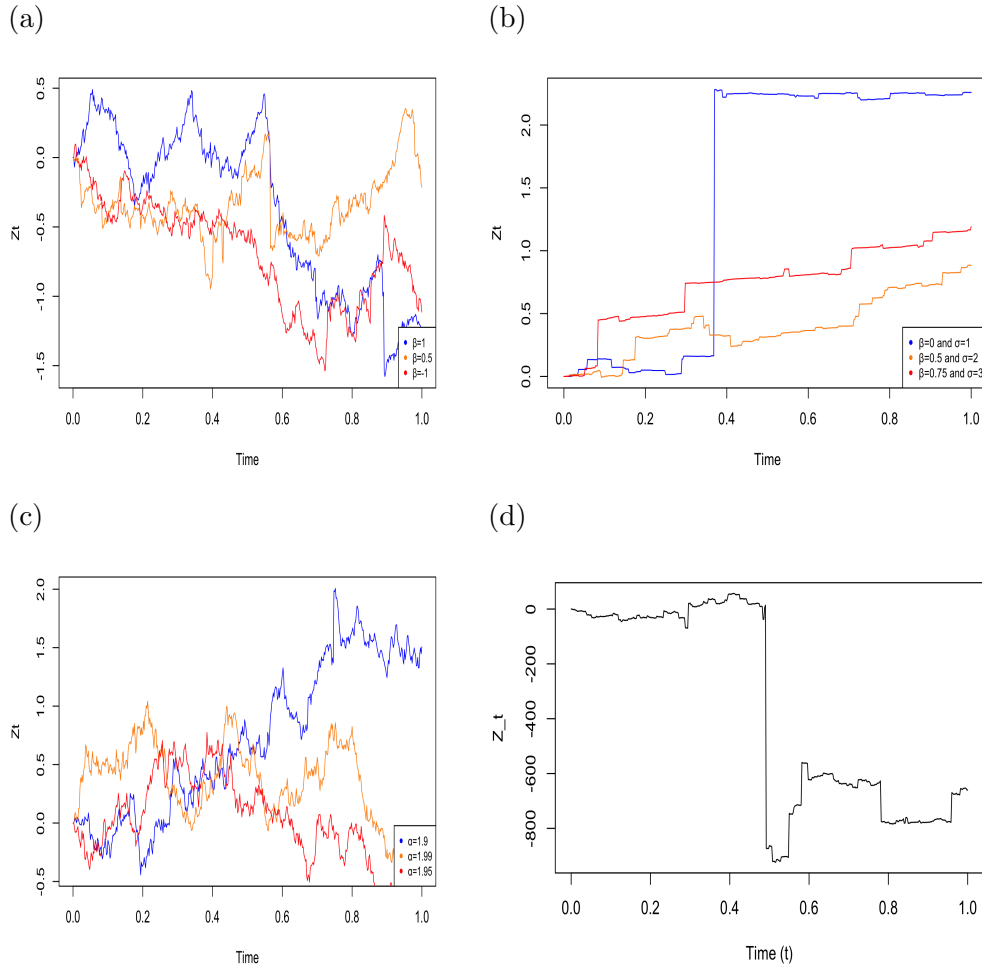


FIGURE 1. Paths of a standard strictly 1.7-stable process with different skewness (a), 0.7-stable process with different scale and skewness (b), the symmetric and standard α -stable process with different index of stability (c) and the case of standard 1-stable process (d).

Example 4.1. (*Stable driven OU process*) Consider the following Ornstein-Uhlenbeck (OU) process driven by a stable process $(Z_t)_{t \geq 0}$ with parameter θ and $\rho > 0$ is defined by

$$dX_t = \theta X_t dt + \rho dZ_t, \quad X_0 = x_0 \in \mathbb{R}. \tag{16}$$

and has the integral representation

$$X_t(\omega) = e^{-at} \left(x_0 + \rho \int_0^t e^{as} dZ_s(\omega) \right), \quad \text{for } t \in [0, T]. \tag{17}$$

Clearly since $F(t, x) = \theta x$, $g(t) = 1$ and $\phi(x) = \rho$ satisfy the Lipschitz and boundedness condition, there exists a strong approximate solution. We give in Figure 2 (a) a path approximate solution for $\theta = 1.2$, $\alpha = 1.7$, $\beta = 0$, $\rho = 1$ with $n = 1000$ sample size. Another classical example is given by the following Stable driven Cox–Ingersoll–Ross (CIR) models. Such models are popular in stochastic modelling for description of interest rates in finance and population dynamics.

Example 4.2. (*Stable driven Cox–Ingersoll–Ross process, Yang (2017)*)

A Stable driven Cox–Ingersoll–Ross process can be defined as follows

$$dX_t = (\lambda - \theta X_t)dt + \rho|X_t|^q dZ_t, \quad X_0 \geq 0,$$

where $q \in [0, 1)$ λ, ρ, θ are real constants.

When $q = \alpha^{-1}$ it is the so called SCIR model given in Bayraktar and Clément (2023) and there is a weak existence solution when Z is a positive stable process. When $q = 1 - \alpha^{-1}$, the existence of a strong approximate result is contained in Mikulevičius and Xu (2018) for positive stable process Z and for $q \in (1 - \alpha^{-1}, \alpha^{-1})$, a weak existence solution holds in Fournier (2013) for $\alpha \in (1, 2)$. Note that, $F(t, x) = \lambda - \theta x$ is a Lipschitz function, $g(t) = \rho$ and if the Hölder $\phi(x) = |x|^q$ satisfy $q \in [\frac{1}{2}, \frac{\alpha}{2})$, then we can say that, there exists a strong approximate solution. In particular one can choose $q = \alpha - 1$ and $\alpha \in [1.5, 2)$ to obtain in Figure 2 (b) a strong approximate path solution for $\lambda = 0.2$, $\theta = 1.2$ $\alpha = 1.7$, $\beta = 0$, $\rho = 1$ and $q = \alpha - 1$ with $n = 1000$ sample size. We give in Figure 2 (c) an approximate path solution for $\lambda = 0.2$, $\theta = 1.2$ $\alpha = 1.7$, $\beta = 0$, $\rho = 1$ and $q = 1 - \alpha^{-1}$ with $n = 1000$ sample size. We introduce the following Lotka and Volterra models, famous in population dynamics (in the case where the solution is positive):

Example 4.3. (*Stable driven Lotka–Volterra process*)

A Stable driven Lotka–Volterra process can be defined as follows

$$dX_t = X_t(\lambda - \theta X_t)dt + \rho|X_t|^q dZ_t, \quad X_0 \geq 0,$$

where $q \in [\frac{1}{2}, \frac{\alpha}{2})$ λ, ρ, θ are real constants.

When $q = 1$, it is proposed in Zhang et al. (2017) when Z is a positive stable process. Note that, $F(t, x) = x(\lambda - \theta x) = \lambda x - \theta x^2$ is the sum of a Lipschitz function and a negative function if $\theta > 0$. We consider $g(t) = \rho$ and $\phi(x) = |x|^q$ Hölder continuous with $q = 1/2$. In particular there exists a strong approximate solution. We give in Figure 2 (d) an approximate path solution for $\lambda = 0.2$, $\theta = 1.2$ $\alpha = 1.7$, $\beta = 0$, $\rho = 1$ and $q = 1/2$ with $n = 1000$ sample size.

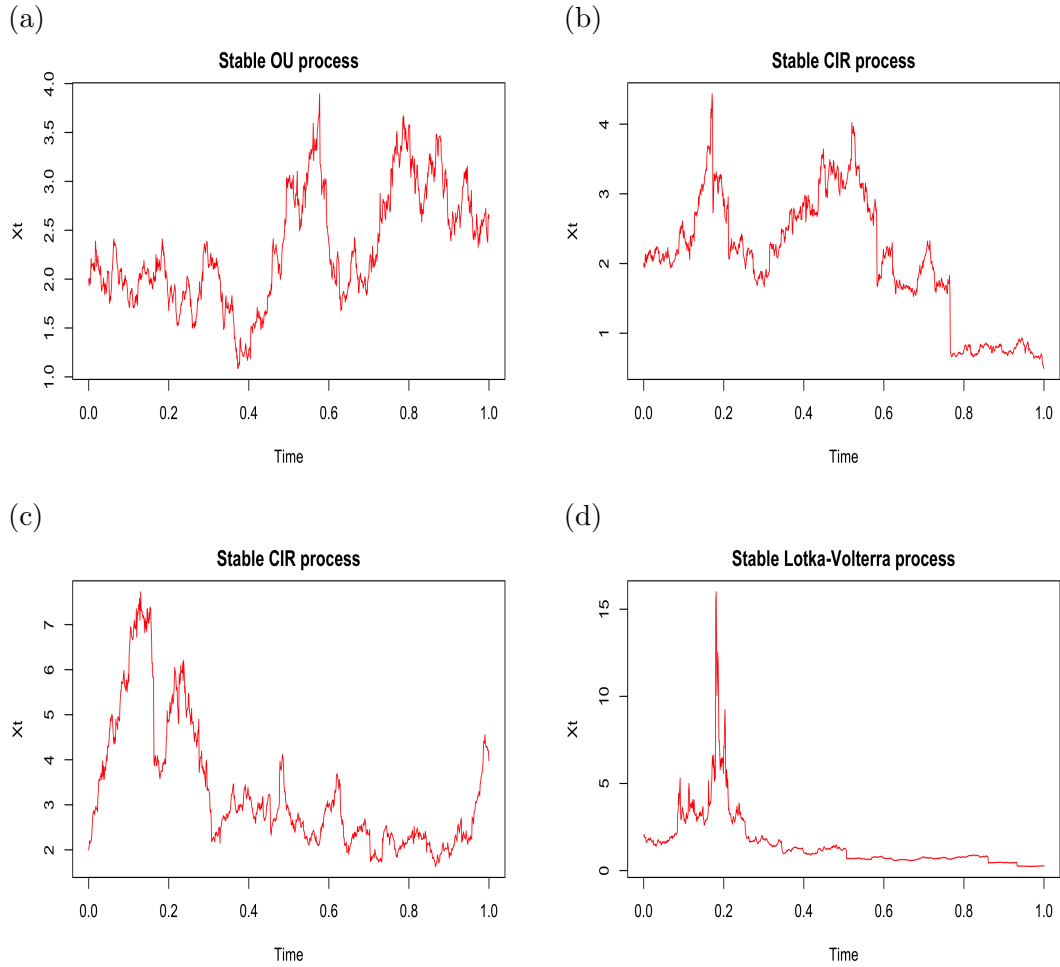


FIGURE 2. An approximate path of the solution of a standard strictly stable driven OU process with sample size $n = 1000$, $\rho = 1$, $\theta = 1.2$, $\alpha = 1.7$ and $\beta = 0$ (a); of a standard strictly stable driven CIR process with sample size $n = 1000$, $\rho = 1$, $\theta = 1.2$, $\alpha = 1.7$, $\beta = 0$, $\lambda = 0.2$ and $q = \alpha^{-1}$ (b); of a standard strictly stable driven CIR process with sample size $n = 1000$, $\rho = 1$, $\theta = 1.2$, $\alpha = 1.7$, $\beta = 0$, $\lambda = 0.2$ and $q = 1 - \alpha^{-1}$ (c) and the case of a standard strictly stable driven Lotka–Volterra process with $\lambda = 0.2$, $\theta = 1.2$, $\alpha = 1.7$, $\beta = 0$, $\rho = 1$, $q = 1/2$ and $n = 1000$ (d).

5. PROOFS OF THE MAIN THEOREMS

5.1. Proof of Theorem 3.3.

Proof. We have

$$\begin{aligned} X_{t+h} - X_t &= \int_0^{t+h} F(s, X_s) ds - \int_0^t F(s, X_s) ds \\ &\quad + \int_0^{t+h} g(s) \phi(X_s) dZ_s - \int_0^t g(s) \phi(X_s) dZ_s. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}|X_{t+h} - X_t|^{2\beta} &\leq 2^{2\beta} \mathbb{E} \left| \int_t^{t+h} F(s, X_s) ds \right|^{2\beta} \\ &\quad + 2^{2\beta} \mathbb{E} \left| \int_t^{t+h} g(s) \phi(X_s) dZ_s \right|^{2\beta} \\ &:= I_1(t, \beta) + I_2(t, \beta) \end{aligned}$$

Using Hölder inequality and assumption **(A2)**, we have the following estimate

$$\begin{aligned} I_2(t, \beta) &\leq 2^{2\beta-1} h^{2\beta-1} \int_t^{t+h} \mathbb{E}|F(s, X_s)|^{2\beta} ds \\ &\leq 2^p h^{2\beta-1} C \int_t^{t+h} (1 + \mathbb{E}|X_s|^{2\beta}) ds \\ &\leq C_\beta h^{2\beta}, \end{aligned}$$

since $1 + \mathbb{E}|X_s|^{2\beta}$ is finite, according to Lemma 3.2.

Now, let us deal with the estimation of $I_2(t, \beta)$. Imitating the previous argument we have

$$\begin{aligned} \mathbb{P} \left(\left| \int_t^{t+h} g(s) \phi(X_s) dZ_s \right| \geq x \right) &\leq \mathbb{P} \left(\int_t^{t+h} |b_R g(s) \phi(X_s)| ds \geq \frac{x}{2} \right) \\ &\quad + \mathbb{P} \left(\int_t^{t+h} \int_{|y| \leq R} |y g(s) \phi(X_s)| \tilde{\mu}(ds, dy) \geq \frac{x}{2} \right) \\ &\quad + \mathbb{P} \left(\int_t^{t+h} \int_{|y| \leq R} |y g(s) \phi(X_s)| \mu(ds, dy) > 0 \right) \\ &:= R_1(x, h) + R_2(x, h) + R_3(x, h). \end{aligned}$$

By Chebychev's inequality and Cauchy-Schwarz inequality, we have :

$$R_1(x, h) \leq \frac{4b_R^2}{x^2} \mathbb{E} \left(\int_t^{t+h} |g(s)| |\phi(X_s)| ds \right)^2 \leq \eta^2(g) \frac{4b_R^2}{x^2} h \int_t^{t+h} \mathbb{E}(\phi(X_s))^2 ds. \quad (18)$$

$$R_1(x, h) \leq hC \eta^2(g) \left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right) \int_t^{t+h} (1 + \mathbb{E}|X_s|^{2\beta}) ds \quad (19)$$

$$\leq h^2C \eta^2(g) \left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right), \quad (20)$$

since $1 + \mathbb{E}|X_s|^{2\beta}$ is bounded. Next, we use Chebychev's inequality and isometry formula for Poisson stochastic integrals and Hölder condition to have

$$\begin{aligned} R_2(x, h) &\leq \frac{4}{x^2} \mathbb{E} \left(\int_t^{t+h} \int_{|y| \leq R} |y g(s) \phi(X_s)| \tilde{\mu}(ds, dy) \right)^2 \\ &\leq \frac{4}{x^2} \int_{|y| \leq R} y^2 \nu(dy) \int_t^{t+h} g^2(s) \mathbb{E} \phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) \int_t^{t+h} \mathbb{E} \phi(X_s)^2 ds \\ &\leq \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) C \int_t^{t+h} (1 + \mathbb{E}|X_s|^{2\beta}) ds \\ &\leq hC \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g). \end{aligned}$$

Now, let us deal with the compound Poisson stochastic integral appearing in $R_3(x, h)$. Note that

$$N_{t,h} = \int_t^{t+h} \int_{|y| > R} y g(s) \phi(X_s) \mu(ds, dy)$$

have the same law as

$$N_h = \int_0^h \int_{|y| > R} y g(s) \phi(X_s) \mu(ds, dy),$$

so that we have

$$\mathbb{P}(N_h > 0) \leq hC_1(\alpha) R^{-\alpha}.$$

By combining the three estimates, we obtain

$$\begin{aligned} \mathbb{P} \left(\left| \int_t^{t+h} g(s) \phi(X_s) dZ_s \right| \geq x \right) &\leq hC_1(\alpha) R^{-\alpha} + hC \frac{4C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) \\ &\quad + h^2C \eta^2(g) \left(\frac{8b^2}{x^2} + \frac{8C_1(\alpha)C_2(\alpha)R^{2-\alpha}}{x^2} \right). \end{aligned}$$

Therefore choosing the truncation level

$$R = \frac{x}{h^\beta \eta(g)} \geq 1$$

and rearranging the terms, we obtain for all $x \geq h^\beta \eta(g)$:

$$\begin{aligned} \mathbb{P} \left(\left| \int_t^{t+h} g(s) \phi(X_s) dZ_s \right| \geq x \right) &\leq C_1(\alpha) h^{\alpha\beta+1} x^{-\alpha} \eta(g)^\alpha + 4C h^{\beta(\alpha-2)+1} \eta^\alpha(g) C_2(\alpha) x^{-\alpha} \\ &\quad + 8b^2 C h^{2+\alpha\beta-2\beta} x^{-\alpha} \eta(g)^\alpha + x^{-\alpha} \eta(g)^\alpha C h^{2+\alpha\beta-2\beta} 8C_1(\alpha) C_2(\alpha). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E} \left(\int_t^{t+h} |g(s) \phi(X_s)| dZ_s \right)^{2\beta} &= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_t^{t+h} |g(s) \phi(X_s)| dZ_s \geq x \right) dx, \\ &\leq \eta(g)^{2\beta} h^{2\beta^2} \\ &\quad + 2\beta \int_{h^\beta \eta(g)}^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_t^{t+h} |g(s) \phi(X_s)| dZ_s \geq x \right) dx. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E} \left(\int_t^{t+h} |g(s) \phi(X_s)| dZ_s \right)^{2\beta} &\leq \eta(g)^{2\beta} h^{2\beta^2} \\ &\quad + \frac{2\beta \eta(g)^{2\beta}}{\alpha - 2\beta} \left(C_1(\alpha) h^{2\beta^2+1} + 4C h^{2\beta^2-2\beta+1} C_2(\alpha) + 8b^2 C h^{2\beta^2-2\beta+2} + 8C_1(\alpha) C_2(\alpha) h^{2\beta^2-2\beta+2} \right), \end{aligned}$$

because

$$\eta(g)^\alpha \int_{h^\beta \eta(g)}^{+\infty} x^{2\beta-1-\alpha} dx = \frac{\eta(g)^{2\beta}}{\alpha - 2\beta} h^{\beta(2\beta-\alpha)}.$$

Finally, we have

$$\sup_{t \in [0, T]} \mathbb{E} \left(|X_{t+h} - X_t|^{2\beta} \right) \leq C \left(h^{2\beta} + h^{2\beta^2+1} + h^{2\beta^2-2\beta+1} + h^{2\beta^2-2\beta+2} \right).$$

But

$$2\beta^2-2\beta+1 \geq \beta^2, \quad 2\beta^2-2\beta+2 \geq \beta^2, \quad 2\beta^2+1 \geq \beta^2, \quad 2\beta \geq \beta^2 \quad \forall \beta \in \left(\frac{1}{2}, \alpha/2 \right) \quad \alpha \in (1, 2),$$

so that

$$\sup_{t \in [0, T]} \mathbb{E} \left(|X_{t+h} - X_t|^{2\beta} \right) \leq C h^{\beta^2} \quad \text{when } h \rightarrow 0.$$

□

5.2. Proof of Theorem 3.4.

Proof. Note that

$$\sup_{t \in [0, T]} \mathbb{E} |\tilde{X}_t^n - X_t|^{2\beta} \leq \sup_{k=0, \dots, n-1} \sup_{t \in [t_k, t_{k+1}]} \mathbb{E} |\tilde{X}_t^k - X_t|^{2\beta}.$$

To establish the convergence of \tilde{X}^n to X , let's write the explicit expression for $Z_t = \tilde{X}_t^n - X_t$.

$$\tilde{X}_t^n - X_t = \int_0^t \left[F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) - F(s, X_s) \right] ds + \int_0^t \left[g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(s) \phi(X_s) \right] dZ_s.$$

We have

$$\begin{aligned} \mathbb{E} |Z_t|^{2\beta} &\leq (2t)^{2\beta-1} \int_0^t \mathbb{E} \left| F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) - F(s, X(s)) \right|^{2\beta} ds \\ &\quad + 2^{2\beta-1} \mathbb{E} \left| \int_0^t (g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(s) \phi(X_s)) dZ_s \right|^{2\beta}. \\ &:= P_1(t, \beta) + 2^{2\beta-1} P_2(t, \beta) \end{aligned}$$

But

$$\begin{aligned} \mathbb{E} |F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) - F(s, X_s)|^{2\beta} &\leq 3^{2\beta-1} \mathbb{E} |F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) - F(\pi_s^n, X_{\pi_s^n})|^{2\beta} \\ &\quad + 3^{2\beta-1} \mathbb{E} |F(\pi_s^n, X_{\pi_s^n}) - F(s, X_{\pi_s^n})|^{2\beta} \\ &\quad + 3^{2\beta-1} |F(s, X_{\pi_s^n}) - F(s, X_s)|^{2\beta} \\ &\leq 3^{2\beta-1} \mathbb{E} |F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) - F(\pi_s^n, X_{\pi_s^n})|^{2\beta} \\ &\quad + 3^{2\beta-1} \mathbb{E} |F(s, X_{\pi_s^n}) - F(s, X_s)|^{2\beta} \end{aligned}$$

and,

$$\begin{aligned} P_1(t, \beta) &\leq (6t)^{2\beta-1} \mathbb{E} \left(\int_0^t |F(\pi_s^n, \tilde{X}_{\pi_s^n}^n) - F(\pi_s^n, X_{\pi_s^n})|^{2\beta} ds \right) \\ &\quad + (6t)^{2\beta-1} \mathbb{E} \left(\int_0^t |F(s, X_{\pi_s^n}) - F(s, X_s)|^{2\beta} ds \right) \\ &\leq C_{\beta, L, T} \left(\mathbb{E} \left(\int_0^t |\tilde{X}_{\pi_s^n}^n - X_{\pi_s^n}|^{2\beta} ds \right) + \mathbb{E} \left(\int_0^t |X_{\pi_s^n} - X_s|^{2\beta} ds \right) \right) \\ &\leq C_{\beta, L, T} \int_0^t \mathbb{E} (Z_s^{2\beta}) ds + C_{\beta, L, T} \int_0^t \mathbb{E} |X_{\pi_s^n} - X_s|^{2\beta} ds \\ &\leq C_{\beta, L, T} \int_0^t \mathbb{E} |Z_s|^{2\beta} ds + C n^{-\beta^2}, \end{aligned}$$

where in the last inequality, we apply Theorem 3.3 for $h = n^{-1}$ together with Remark 3.2. In the same way, we have

$$\begin{aligned} |g(\phi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(s) \phi(X_s)| &\leq |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| \\ &\quad + |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(s) \phi(X_{\pi_s^n})| + |g(s) \phi(X_{\pi_s^n}) - g(s) \phi(X_s)| \\ &\leq |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| + |g(s) \phi(X_{\pi_s^n}) - g(s) \phi(X_s)|, \end{aligned}$$

so that

$$\begin{aligned} \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(s) \phi(X_s)| dZ_s \geq x \right) &\leq \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\phi_s^n) \phi(X_{\pi_s^n})| dZ_s \geq x/2 \right) \\ &\quad + \mathbb{P} \left(\int_0^t |g(s) \phi(X_{\pi_s^n}) - g(s) \phi(X_s)| dZ_s \geq x/2 \right). \end{aligned}$$

Thus,

$$\begin{aligned} P_2(t, \beta) &= \mathbb{E} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(s) \phi(X_s)| dZ_s \right)^{2\beta} \\ &= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(s) \phi(X_s)| dZ_s \geq x \right) dx, \\ &\leq 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| dZ_s \geq x/2 \right) dx, \\ &\quad + 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(s) \phi(X_{\pi_s^n}) - g(s) \phi(X_s)| dZ_s \geq x/2 \right) dx \\ &\leq 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| dZ_s \geq x \right) dx \\ &\quad + 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(s) \phi(X_{\pi_s^n}) - g(s) \phi(X_s)| dZ_s \geq x \right) dx \\ &:= P_{2,1}(t, \beta) + P_{2,2}(t, \beta). \end{aligned}$$

Firstly, we have

$$\mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| dZ_s \geq x \right) \leq R_1^*(x) + R_2^*(x) + R_3^*(x),$$

where

$$R_1^*(x, h) \leq \left(\frac{8b^2 t \eta^2(g)}{x^2} + t \frac{8C_1(\alpha) C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) \right) c_\phi^2 \int_0^t \mathbb{E} |Z_s|^{2\beta} ds,$$

$$R_2^*(x, h) \leq \frac{4C_2(\nu) R^{2-\alpha}}{x^2} \eta^2(g) c_\phi^2 \int_0^t \mathbb{E} |Z_s|^{2\beta} ds$$

and,

$$R_3^*(x, h) \leq t C_1(\alpha) R^{-\alpha}.$$

Therefore choosing again the same truncation level

$$R = \frac{x}{h^\beta \eta(g)} \geq 1$$

and rearranging the terms, we obtain for all $x \geq h^\beta \eta(g)$:

$$\begin{aligned} \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| dZ_s \geq x \right) &\leq C_1(\alpha) h^{\alpha\beta+1} x^{-\alpha} \eta(g)^\alpha \\ &\quad + 4c_\phi^2 h^{\beta(\alpha-2)+1} \eta^\alpha(g) C_2(\alpha) x^{-\alpha} \int_0^t \mathbb{E}|Z_s|^{2\beta} ds \\ &\quad + 8b^2 c_\phi^2 h^{2+\alpha\beta-2\beta} x^{-\alpha} \eta(g)^\alpha \int_0^t \mathbb{E}|Z_s|^{2\beta} ds \\ &\quad + x^{-\alpha} \eta(g)^\alpha c_\phi^2 h^{2+\alpha\beta-2\beta} 8C_1(\alpha) C_2(\alpha) \int_0^t \mathbb{E}|Z_s|^{2\beta} ds. \end{aligned}$$

We have

$$\begin{aligned} P_{2,1}(t, \beta) &= 2\beta \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| dZ_s \geq x \right) dx \\ &\leq \eta(g)^{2\beta} h^{2\beta^2} + 2\beta \frac{\beta \eta(g)^{2\beta}}{\alpha - 2\beta} C_1(\alpha) h^{2\beta^2+1} \\ &\quad + \left(4c_\phi^2 h^{2\beta^2-2\beta+1} C_2(\alpha) + 8b^2 c_\phi^2 h^{2\beta^2-2\beta+2} + 8C_1(\alpha) C_2(\alpha) h^{2\beta^2-2\beta+2} c_\phi^2 \right) \int_0^t \mathbb{E}|Z_s|^{2\beta} ds, \end{aligned}$$

because

$$\eta(g)^\alpha \int_{h^\beta \eta(g)}^{+\infty} x^{2\beta-1-\alpha} dx = \frac{\eta(g)^{2\beta}}{\alpha - 2\beta} h^{\beta(2\beta-\alpha)}.$$

Secondly, we have

$$\mathbb{P} \left(\int_0^t |g(s) \phi(X_{\pi_s^n}) - g(s) \phi(X_s)| dZ_s \geq x \right) \leq R_1^{**}(x, h) + R_2^{**}(x, h) + R_3^{**}(x, h),$$

where

$$R_1^{**}(x, h) \leq \left(\frac{8b^2 t \eta^2(g)}{x^2} + t \frac{8C_1(\alpha) C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) \right) c_\phi^2 \int_0^t \mathbb{E}|X_{\pi_s^n} - X_s|^{2\beta} ds,$$

$$R_2^{**}(x, h) \leq \frac{4C_2(\nu) R^{2-\alpha}}{x^2} \eta^2(g) c_\phi^2 \int_0^t \mathbb{E}|X_{\phi_s^n} - X_s|^{2\beta} ds \leq \frac{4C_2(\nu) R^{2-\alpha}}{x^2} \eta^2(g) c_\phi^2 t C h^{\beta^2}$$

and,

$$R_3^{**}(x, h) \leq t C_1(\alpha) R^{-\alpha}.$$

Since $|\pi_s^n - s| \leq h$ for $t \in [t_k, t_{k+1})$, we apply Theorem 3.3 together with Remark 3.2, to obtain

$$R_1^{**}(x, h) \leq \left(\frac{8b^2 t \eta^2(g)}{x^2} + t \frac{8C_1(\alpha) C_2(\alpha) R^{2-\alpha}}{x^2} \eta^2(g) \right) c_\phi^2 t C h^{\beta^2}.$$

Set

$$R = xh^{-\beta}\eta(g)^{-1} \geq 1, \quad x \geq h^\beta\eta(g).$$

Therefore choosing again the same truncation level

$$R = \frac{x}{h^\beta\eta(g)} \geq 1$$

and rearranging the terms, we obtain for all $x \geq h^\beta\eta(g)$:

$$\begin{aligned} \mathbb{P} \left(\int_0^t |g(s) \phi(X_{\phi_s^n}) - g(s) \phi(X_s)| dZ_s \geq x \right) &\leq C_1(\alpha) h^{\alpha\beta+1} x^{-\alpha} \eta(g)^\alpha \\ &\quad + 4c_\phi^2 h^{\beta(\alpha-2)+1} \eta^\alpha(g) C_2(\alpha) x^{-\alpha} t h^{\beta^2} \\ &\quad + 8b^2 c_\phi^2 h^{2+\alpha\beta-2\beta} x^{-\alpha} \eta(g)^\alpha t h^{\beta^2} \\ &\quad + x^{-\alpha} \eta(g)^\alpha c_\phi^2 h^{2+\alpha\beta-2\beta} 8C_1(\alpha) C_2(\alpha) t h^{\beta^2}. \end{aligned}$$

We have

$$\begin{aligned} P_{2,2}(t, \beta) &= \int_0^{+\infty} x^{2\beta-1} \mathbb{P} \left(\int_0^t |g(\pi_s^n) \phi(\tilde{X}_{\pi_s^n}^n) - g(\pi_s^n) \phi(X_{\pi_s^n})| dZ_s \geq x \right) dx \\ &\leq \eta(g)^{2\beta} h^{2\beta^2} + \frac{2\beta\eta(g)^{2\beta}}{\alpha-2\beta} C_1(\alpha) h^{2\beta^2+1} \\ &\quad + \left(4c_\phi^2 h^{2\beta^2-2\beta+1} C_2(\alpha) + 8b^2 c_\phi^2 h^{2\beta^2-2\beta+2} + 8C_1(\alpha) C_2(\alpha) h^{2\beta^2-2\beta+2} c_\phi^2 \right) h^{\beta^2}. \end{aligned}$$

Finally, there exists positive constant C (that depend on all the the model parameters) and, $C_{\beta,L}$ such that:

$$\mathbb{E}(|Z_t|^{2\beta}) \leq (C_{\beta,L} + C) \int_0^t \mathbb{E}(|Z_s|^{2\beta}) ds + Ch^{\beta^2}$$

and, we use the Gronwall Lemma's, to obtain the desired result. \square

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DECLARATION STATEMENT

The author have seen and agree with the contents of the manuscript. The author certify that the submission is an original work and is not under review at any other publication. No potential conflict of interest was reported by the author.

DATA AVAILABILITY STATEMENTS

Data sharing not applicable to this article as no datasets were generated or analysed during the current study. The R codes for the simulation study are available from the author on reasonable request.

REFERENCES

- Applebaum, D. (2009). *Lévy processes and stochastic calculus*. Cambridge university press.
- Athreya, S., O. Butkovsky, and L. Mytnik (2020). Strong existence and uniqueness for stable stochastic differential equations with distributional drift.
- Bass, R. F., K. Burdzy, and Z.-Q. Chen (2004). Stochastic differential equations driven by stable processes for which pathwise uniqueness fails. *Stochastic processes and their applications* 111(1), 1–15.
- Bayraktar, E. and E. Clément (2023). Estimation of a pure-jump stable cox-ingersoll-ross process. *arXiv preprint arXiv:2304.02386*.
- Cattiaux, P. and M. S. Manou-Abi (2014). Limit theorems for some functionals with heavy tails of a discrete time markov chain. *ESAIM: Probability and Statistics* 18, 468–482.
- Chambers, J. M., C. L. Mallows, and B. Stuck (1976). A method for simulating stable random variables. *Journal of the american statistical association* 71(354), 340–344.
- Fournier, N. (2013). On pathwise uniqueness for stochastic differential equations driven by stable lévy processes. In *Annales de l’IHP Probabilités et statistiques*, Volume 49, pp. 138–159.
- Fu, Z. and Z. Li (2010). Stochastic equations of non-negative processes with jumps. *Stochastic Processes and their Applications* 120(3), 306–330.
- Jacod, J. and A. Shiryaev (2013). *Limit theorems for stochastic processes*, Volume 288. Springer Science & Business Media.
- Janicki, A., Z. Michna, and A. Weron (1997). Approximation of stochastic differential equations driven by stable lévy motion. *Applicationes Mathematicae* 24(2), 149–168.
- Joulin, A. (2007). On maximal inequalities for stable stochastic integrals. *Potential Analysis* 26, 57–78.
- Ken-Iti, S. (1999). *Lévy processes and infinitely divisible distributions*, Volume 68. Cambridge university press.
- Komatsu, T. (1982). On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations of jump type.
- Li, Z. and L. Mytnik (2011). Strong solutions for stochastic differential equations with jumps. In *Annales de l’IHP Probabilités et statistiques*, Volume 47, pp. 1055–1067.
- Manou-Abi, S. (2015). *Théorèmes limites et ordres stochastiques relatifs aux lois et processus stables*. Ph. D. thesis, Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier).

- Mikulevičius, R. and F. Xu (2018). On the rate of convergence of strong euler approximation for sdes driven by levy processes. *Stochastics* 90(4), 569–604.
- Pamen, O. M. and D. Taguchi (2017). Strong rate of convergence for the euler–maruyama approximation of sdes with hölder continuous drift coefficient. *Stochastic Processes and their Applications* 127(8), 2542–2559.
- Peszat, S. and J. Zabczyk (2007). *Stochastic partial differential equations with Lévy noise: An evolution equation approach*, Volume 113. Cambridge University Press.
- Priola, E. (2012). Pathwise uniqueness for singular sdes driven by stable processes.
- Protter, P. (1991). Stochastic integration and differential equations—a new approach. *Bull. Amer. Math. Soc* 25, 170–180.
- Rong, S. (2006). *Theory of stochastic differential equations with jumps and applications: mathematical and analytical techniques with applications to engineering*. Springer Science & Business Media.
- Rosinski, J. and W. Woyczynski (1986). On itô stochastic integration with respect to p-stable motion: inner clock, integrability of sample paths, double and multiple integrals. *The Annals of Probability*, 271–286.
- Wu, M. and Z. Hao (2023). Sde driven by cylindrical stable process with distributional drift and applicationwe. *arXiv preprint arXiv:2305.18139*.
- Yang, X. (2017). Maximum likelihood type estimation for discretely observed cir model with small α -stable noises. *Statistics & Probability Letters* 120, 18–27.
- Zhang, X. (2013). Stochastic differential equations with sobolev drifts and driven by stable processes. In *Annales de l’IHP Probabilités et statistiques*, Volume 49, pp. 1057–1079.
- Zhang, Z., X. Zhang, and J. Tong (2017). Exponential ergodicity for population dynamics driven by α -stable processes. *Statistics & Probability Letters* 125, 149–159.