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# On dually-CPT and strong-CPT posets

Liliana Alc3n<sup>a,b</sup>, Martin Charles Golumbic<sup>c</sup>, Noem3 Gudi3o<sup>a,b</sup>, Marisa Gutierrez<sup>a,b</sup>, and Vincent Limouzy<sup>d,e</sup>

<sup>a</sup>*Centro de Matem3tica CMaLP, FCE, Universidad Nacional de La Plata. Argentina*

<sup>b</sup>*CONICET, Buenos Aires, Argentina*

<sup>c</sup>*Department of Computer Science, University of Haifa, Mt. Carmel, Haifa, Israel*

<sup>d</sup>*Universit3 Clermont Auvergne, Clermont Auvergne INP, CNRS, Mines Saint-Etienne, LIMOS, F-63000 Clermont-Ferrand, France*

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## Abstract

A poset is a containment of paths in a tree (CPT) if it admits a representation by containment where each element of the poset is represented by a path in a tree and two elements are comparable in the poset if and only if the corresponding paths are related by the inclusion relation. Recently Alc3n, Gudi3o and Gutierrez [1] introduced proper subclasses of CPT posets, namely dually-CPT, and strongly-CPT. A poset  $\mathbf{P}$  is dually-CPT, if and only if  $\mathbf{P}$  and its dual  $\mathbf{P}^d$  both admit a CPT representation. A poset  $\mathbf{P}$  is strongly-CPT, if and only if  $\mathbf{P}$  and all the posets that share the same underlying comparability graph admit a CPT representation. Where as the inclusion between Dually-CPT and CPT was known to be strict. It was raised as an open question by Alc3n, Gudi3o and Gutierrez [1] whether strongly-CPT was a strict subclass of dually-CPT. We provide a proof that both classes actually coincide.

## 1 Introduction

A poset is called a containment order of paths in a tree (CPT for short) if it admits a representation by containment where each element of the poset corresponds to a path in a tree and for two elements  $x$  and  $y$ , we have  $x < y$  in the poset if and only if the path corresponding to  $x$  is properly contained in the path corresponding to  $y$ .

Several classes of posets are known to admit specific containment models, for example, containment orders of circular arcs on a circle [14, 15], containment orders of axis-parallel boxes in  $\mathbb{R}^d$  [12], or containment orders of disks in the plane [3, 5, 6] to cite just a few. All the aforementioned classes, as well as CPT, generalize the class CI of containment

orders of intervals on a line [4]. It is well known that this class coincides with the class of 2-dimensional posets and are also equivalent to the transitive orientations of permutation graphs [9].

In 1984, Corneil and Golubic observed that a graph  $G$  may be the comparability graph of a CPT poset, yet a different transitive orientation of  $G$  may not necessarily have a CPT representation, (see Golubic [10]). This stands in contrast to poset dimension, interval orders, unit interval orders, box containment orders, tolerance orders and others which are comparability invariant. Golubic and Scheinerman [12] called such classes *strong containment poset classes*.

Recently, interest in CPT posets has been revived and several groups of researchers have considered various aspects of this class [1, 2, 11, 13]. Since the CPT posets are not a strong containment class, Alc3n, Gudi3no and Gutierrez [1] introduced the study of the subclasses dually-CPT and strongly-CPT posets. A poset  $\mathbf{P}$  is called *dually-CPT* if  $\mathbf{P}$  and its dual  $\mathbf{P}^d$  admit a CPT representation. A poset  $\mathbf{P}$  is called *strongly-CPT* if  $\mathbf{P}$  and all the posets that share the same underlying comparability graph admit CPT representations. From the definition it is clear that the class of strongly-CPT posets is included in the class of dually-CPT posets. Many families of separating examples are now known between the class of dually-CPT and general CPT posets, however, concerning the strongly and dually-CPT, it was left as an open problem for many years to determine whether the inclusion is strict or if the two classes coincide.

We present in this paper a solution to this question with the following main theorem.

**Theorem 1.** *A poset  $\mathbf{P}$  is strongly-CPT if and only if it is dually-CPT.*

To prove our main result we rely on the link between modular decomposition of the underlying comparability graph and its transitive orientations. Our strategy consists of considering a dually-CPT poset and proving that any poset with the same comparability graph also admits a CPT representation. At first we consider the representation and perform some modifications to obtain a representation with particular properties. Once this is done, we rely on the specific structure of modules in dually-CPT posets, and we provide a method to obtain the representation of any poset with the same comparability graph.

The paper is organized as follows: In Section 2, we present the definitions related to posets, CPT and modular decomposition and recall some fundamental results that we will use throughout the paper. In Section 3, we prove that for dually-CPT posets it is possible to obtain a representation where no element of a strong module is represented by a trivial path. Then, in Section 4, we show how to modify a CPT representation of a dually-CPT poset so that either the paths of a strong module do not end on a trivial path or the considered module admits very specific properties. Finally, in Section 5, we show how to use an operation called substitution to prove our main result.

## 2 Definitions and notations

A *partially ordered set* or *poset* is a pair  $\mathbf{P} = (X, P)$  where  $X$  is a finite non-empty set and  $P$  is a reflexive, antisymmetric and transitive binary relation on  $X$ . The elements of  $X$  are also called *vertices* of the poset. As usual, we write  $x \leq y$  in  $\mathbf{P}$  for  $(x, y) \in P$ ; and  $x < y$  in  $\mathbf{P}$  when  $(x, y) \in P$  and  $x \neq y$ . If  $x < y$  or  $y < x$ , we say that  $x$  and  $y$  are *comparable* in  $\mathbf{P}$  and write  $x \perp y$ . When there is no relationship between  $x$  and  $y$  we say that they are *incomparable* and write  $x \parallel y$ . An element  $x$  is *covered* by  $y$  in  $\mathbf{P}$ , denoted by  $x <: y$  in  $\mathbf{P}$ , when  $x < y$  and there is no element  $z \in X$  for which  $x < z$  and  $z < y$ . The *down-set*  $\{x \in X : x < z\}$  and the *up-set*  $\{x \in X : z < x\}$  of an element  $z$  are denoted by  $D(z)$  and  $U(z)$ , respectively. We let  $D[z] = D(z) \cup \{z\}$  and  $U[z] = U(z) \cup \{z\}$ . The *dual* of  $\mathbf{P} = (X, P)$  is the poset  $\mathbf{P}^d = (X, P^d)$  where  $x \leq y$  in  $\mathbf{P}^d$  if and only if  $y \leq x$  in  $\mathbf{P}$ .

A *containment representation*  $R_{\mathbf{P}}$  or *model* of a poset  $\mathbf{P} = (X, P)$  maps each element  $x$  of  $X$  into a set  $W_x$  in such a way that  $x < y$  in  $\mathbf{P}$  if and only if  $W_x$  is a proper subset of  $W_y$ . We identify the containment representation  $R_{\mathbf{P}}$  with the set family  $\{W_x\}_{x \in X}$ .

A poset  $\mathbf{P} = (X, P)$  is a *containment order of paths in a tree*, or *CPT* poset for brevity, if it admits a containment representation  $R_{\mathbf{P}} = \{W_x\}$  where every  $W_x$  is a path of a tree  $T$ , which is called the *host tree* of the model. When  $T$  is a path,  $\mathbf{P}$  is said to be a *containment order of intervals* or *CI* poset for short. (We generally consider a path as the set of vertices that induces it.)

The comparability graph  $G_{\mathbf{P}}$  of a poset  $\mathbf{P} = (X, P)$  is the simple graph with vertex set  $V(G_{\mathbf{P}}) = X$  and edge set  $E(G_{\mathbf{P}}) = \{xy : x \perp y\}$ . In what follows, a poset  $\mathbf{P}$ , such that  $G_{\mathbf{P}}$  is complete (resp. without edges), is called a *total order* (resp. an *empty order*). We say that two posets are *associated* if their comparability graphs are isomorphic. A graph  $G$  is a *comparability graph* if there exists some poset  $\mathbf{P}$  such that  $G = G_{\mathbf{P}}$ .

A *transitive orientation*  $\vec{E}$  of a graph  $G$  is an assignment of one of the two possible directions,  $\vec{xy}$  or  $\vec{yx}$ , to each edge  $xy \in E(G)$  in such a way that if  $\vec{xy} \in \vec{E}$  and  $\vec{yz} \in \vec{E}$  then  $\vec{xz} \in \vec{E}$ . The graphs whose edges can be transitively oriented are exactly the comparability graphs [7, 8, 9]. Furthermore, given a transitive orientation  $\vec{E}$  of a graph  $G$ , we let  $\mathbf{P}_{\vec{E}}$  denote the poset  $(V(G), P_{\vec{E}})$  where  $u < v$  in  $\mathbf{P}_{\vec{E}}$  if and only if  $\vec{uv} \in \vec{E}$ . The comparability graph of  $\mathbf{P}_{\vec{E}}$  is  $G$ . Thereby, the transitive orientations of  $G$  are put in one-to-one correspondence with the posets whose comparability graphs are  $G$ .

Let  $\mathbf{P} = (X, P)$  be a poset. A set  $M \subseteq X$  is a *module* (*homogeneous set* [7]) if for every  $y \in X - M$ , either  $y \perp x$  for all  $x \in M$ , or  $y \parallel x$  for all  $x \in M$ . The whole set  $X$  and the singleton sets  $\{x\}$ , for any  $x \in X$ , are modules of  $\mathbf{P}$ . These modules are called *trivial modules*. A poset  $\mathbf{P}$  is *prime* or *indecomposable* if all its modules are trivial. Otherwise  $\mathbf{P}$  is *decomposable* or *degenerate*. A module  $M$  is *strong* if for all modules  $M'$  either  $M \cap M' = \emptyset$  or  $M \subseteq M'$  or  $M' \subseteq M$ .

A module (respectively, strong module)  $M \neq X$  is called *maximal* if there exists no module (respectively, strong module)  $Y$  such that  $M \subset Y \subset X$ .

**Theorem 2.** (*Modular decomposition theorem*) [7] *Let  $\mathbf{P} = (X, P)$  be a poset with at least two vertices. Then exactly one of the following three conditions is satisfied:*

- (i)  $G_{\mathbf{P}}$  is not connected and the maximal strong modules of  $\mathbf{P}$  are the connected components of  $G_{\mathbf{P}}$ .
- (ii)  $\overline{G_{\mathbf{P}}}$  is not connected and the maximal strong modules of  $\mathbf{P}$  are the connected components of  $\overline{G_{\mathbf{P}}}$ .
- (iii)  $G_{\mathbf{P}}$  and  $\overline{G_{\mathbf{P}}}$  are connected. There is some  $Y \subseteq X$  and a unique partition  $\mathcal{S}$  of  $X$  such that
  - (a)  $|Y| \geq 4$ ,
  - (b)  $\mathbf{P}[Y]$  is the biggest prime subposet of  $\mathbf{P}$  (in the sense that it is not included in any other prime subposet),
  - (c) for every part  $S$  of the partition  $\mathcal{S}$ ,  $S$  is a module of  $\mathbf{P}$  and  $|S \cap Y| = 1$ .

The previous theorem defines a partition  $\mathcal{M}(\mathbf{P}) = \{M_1, \dots, M_k\}$  of  $X$ , which is called the *canonical partition* or *maximal modular partition* of  $\mathbf{P}$ . In the first case,  $G_{\mathbf{P}}$  is said to be *parallel* or *stable* and the partition is formed by the vertices of the connected components of  $G_{\mathbf{P}}$ . In the second case,  $G_{\mathbf{P}}$  is *series* or *clique* and the partition is formed by the vertices of each connected component of  $\overline{G_{\mathbf{P}}}$ . And, in the last case,  $G_{\mathbf{P}}$  is *neighborhood* or *prime*, and the partition is  $\mathcal{S}$ .

The *quotient poset* of  $\mathbf{P}$ , denoted by  $\mathbf{P}/\mathcal{M}(\mathbf{P})$ , has a vertex  $v_i$  for each part  $M_i$  of  $\mathcal{M}(\mathbf{P})$ ; and two vertices  $v_i$  and  $v_j$  of  $\mathbf{P}/\mathcal{M}(\mathbf{P})$  are comparable if and only if for all  $x \in M_i$  and for all  $y \in M_j$ ,  $x \perp y$  in  $\mathbf{P}$ .

The quotient poset is *empty* (iff  $G_{\mathbf{P}}$  is parallel), a *total order* (iff  $G_{\mathbf{P}}$  is series) or *indecomposable* (iff  $G_{\mathbf{P}}$  is neighborhood).

On some occasions, when referring to a module, we will mean the subposet induced by it. For instance, we will say that a module  $M$  of  $\mathbf{P}$  is *CI* or that it is prime, meaning that  $\mathbf{P}(M)$  is. This will be clear from the context and will cause no confusion.

**Theorem 3.** [7] *Given posets  $\mathbf{P}$  and  $\mathbf{P}'$ , if  $G_{\mathbf{P}} = G_{\mathbf{P}'}$  and  $\mathbf{P}$  is indecomposable, then  $\mathbf{P}' = \mathbf{P}$  or  $\mathbf{P}' = \mathbf{P}^d$ .*

**Proposition 4.** [7] *Given posets  $\mathbf{P}$  and  $\mathbf{P}'$ , if  $G_{\mathbf{P}} = G_{\mathbf{P}'}$ , then  $\mathbf{P}$  and  $\mathbf{P}'$  have the same strong modules and, consequently,  $\mathcal{M}(\mathbf{P}) = \mathcal{M}(\mathbf{P}')$ .*

Given a vertex  $v$  of a poset  $\mathbf{P} = (X, P)$  and a poset  $\mathbf{H} = (X_1, H)$ , *substituting or replacing  $v$  by  $\mathbf{H}$  in  $\mathbf{P}$*  results in the poset  $\mathbf{P}_{\mathbf{H} \rightarrow v} = (X - \{v\} \cup X_1, P_{\mathbf{H} \rightarrow v})$  such that  $P_{\mathbf{H} \rightarrow v} = P - \{(x, y) : x = v \vee y = v\} \cup H \cup \{(x, y) : x \in X_1 \wedge y \in U(v)\} \cup \{(x, y) : y \in X_1 \wedge x \in D(v)\}$ .

**Theorem 5.** *Let  $\mathcal{M}(\mathbf{P}) = \{M_1, \dots, M_k\}$  be the maximal modular partition of a connected poset  $\mathbf{P} = (X, P)$  whose quotient is prime, and call  $\mathbf{H}$  the quotient poset  $\mathbf{P}/\mathcal{M}(\mathbf{P})$ . A poset  $\mathbf{Q}$  is associated to  $\mathbf{P}$  if and only if there exist posets  $\mathbf{Q}_i$  for  $1 \leq i \leq k$  such that  $\mathbf{Q}_i$  is associated to  $\mathbf{P}_i = \mathbf{P}(M_i)$  for each  $i$ , and  $\mathbf{Q}$  is obtained by replacing each vertex  $v_i$  of  $\mathbf{H}$  by the poset  $\mathbf{Q}_i$  or replacing each vertex  $v_i$  of  $\mathbf{H}^d$  by the poset  $\mathbf{Q}_i$ .*

**Theorem 6.** [7] *A poset  $\mathbf{P}$  is CI if and only if the quotient poset and all the maximal strong modules of  $\mathbf{P}$  are CI.*

**Lemma 7.** [1] *If  $z$  is a vertex of a CPT poset  $\mathbf{P}$  then the subposet induced by the closed down-set of  $z$  is CI. In particular, if  $\mathbf{P}$  is dually-CPT, then also the subposet induced by the closed up-set of  $z$  is CI.*

**Remark 8.** [7] *Let  $\mathbf{P}$  and  $\mathbf{P}'$  be associated posets. Then,  $\mathbf{P}$  is a CI poset if and only if  $\mathbf{P}'$  is a CI poset. In particular,  $\mathbf{P}$  is a CI poset if and only if  $\mathbf{P}^d$  is a CI poset.*

**Theorem 9.** *Let  $\mathbf{P} = (X, P)$  be a connected dually-CPT poset. Then the quotient poset of  $\mathbf{P}$  is dually-CPT and every maximal strong module of  $\mathbf{P}$  is CI. In particular, if the quotient poset is CI, then  $\mathbf{P}$  is CI.*

*Proof.* Let  $\mathcal{M}(\mathbf{P}) = \{M_1, \dots, M_k\}$  be the maximal modular partition of  $\mathbf{P}$ . The quotient poset  $\mathbf{H} = \mathbf{P}/\mathcal{M}(\mathbf{P})$  is a subposet of  $\mathbf{P}$ , so  $\mathbf{H}$  is dually-CPT. We can assume that  $\mathbf{P}$  is not empty, and since  $\mathbf{P}$  is connected we have that  $\mathbf{H}$  is connected, and so every vertex  $v_i$  of  $\mathbf{H}$  is in the down-set or in the up-set of some other vertex. Which implies that in  $\mathbf{P}$  the whole module  $M_i$  is in the up-set or in the down-set of some other vertex. It follows from Lemma 7 that each  $\mathbf{P}_i = \mathbf{P}(M_i)$  is CI. Therefore, by Theorem 6, if  $\mathbf{H}$  is CI, then  $\mathbf{P}$  is CI.  $\square$

The converse of Theorem 9 is not true in general. For instance, if in the quotient poset  $\mathbf{H}$  there exists a vertex  $v_i$  such that in any CPT representation of  $\mathbf{H}$  the corresponding path  $W_{v_i}$  is reduced to a vertex, then for  $\mathbf{P}$  to be CPT the module  $M_i$  has to be a singleton.

In a representation  $R_{\mathbf{P}}$  of a CPT poset  $\mathbf{P}$ , a subset  $X$  of paths of  $R_{\mathbf{P}}$  is called *one-sided* if all the paths that represent  $X$  arrive at a vertex  $a$  of the host tree and all paths of  $X$ , except possibly one trivial path, pass through a vertex  $b$  of  $T$  neighbor of  $a$ . If all the paths of  $X$  arrive at a vertex  $a$  and  $X$  is not one-sided, then it is called *two-sided*.

Addressing that issue in the proof of the main theorem will requires the following lemmas and properties.

**Property 10.** [4, 10] *Every CI poset admits a CI representation where the intersection of all the intervals used in the representation is a non-trivial interval.*

### 3 Trivial paths into modules

The goal of this section is to prove that for any dually-CPT poset  $\mathbf{P}$ , there exists a representation  $R_{\mathbf{P}}$  where all the elements contained in strong modules are represented by non-trivial paths.

At first we prove that if an element of module is represented by a trivial path, it does mean that the module (all its elements) are not greater than any other element not in the module.

**Lemma 11.** *Let  $\mathbf{P}$  be a poset and let  $M$  be a strong module of  $\mathbf{P}$ . If there exists a representation  $R_{\mathbf{P}}$  where an element  $x$  of  $M$  is represented by a trivial path, then all the elements of  $M$  are not greater than any element of  $\mathbf{P}$  not in  $M$ .*

*Proof.* Let us proceed by contradiction and let us assume that there exists an element  $z \notin M$  such that  $z < x$ . Then in any representation  $R_{\mathbf{P}}$  we have  $W_z \subset W_x$  but since  $W_x$  is already a trivial path it cannot properly contain some other object.  $\square$

Hence from the previous lemma, if in a representation  $R_{\mathbf{P}}$  an element of a module is represented by a trivial path, the module is a minimal subset of  $\mathbf{P}$ .

**Lemma 12.** *Let  $M$  be a strong module of a CPT poset  $\mathbf{P}$ , if in a representation  $R_{\mathbf{P}}$  one of its elements is represented as a trivial path, then there exists an element  $x$  not in  $M$  such that the path  $W_x$  contains all the paths representing the elements of  $M$ .*

*Proof.* Since the poset is connected, and by the previous lemma, we know that the module cannot contain any other element, to ensure the connection outside the module, there might be at least one element  $x$  that is greater than every element of  $M$ .  $\square$

**Lemma 13.** *Let  $M$  be a strong module of a CPT poset  $\mathbf{P}$ . If in a representation  $R_{\mathbf{P}}$  one of its elements  $z$  is represented as a trivial path, then this path is hosted on some vertex  $a$  of  $T$ . If for an element  $x$  not in  $M$  its path  $W_x$  passes through  $a$ , then  $W_x$  has to contain all the paths corresponding to the elements of  $M$ .*

*Proof.* From the definition of a module, every element not in the module is either completely disconnected from  $M$  or completely connected to  $M$ . In that case, if for an element  $x$ , in a representation  $R_{\mathbf{P}}$  its path  $W_x$  passes through  $a$ , then it is connected to the element  $z$ . Hence it has to be connected to every element of  $M$ . In addition, in a transitive orientation of a graph, the containment relation between  $x$  and the elements of  $M$  is the same for every element of  $M$ . Thus if  $W_x$  contains  $W_z$  it contains all the paths of the elements of  $M$ .  $\square$

**Lemma 14.** *Let  $M$  be a strong module of a CPT poset  $\mathbf{P}$ . If in a representation  $R_{\mathbf{P}}$  one of its elements is represented as a trivial path and  $M$  is a clique or prime module, then there exists at least one element of  $M$  represented as a non-trivial path.*

In the case of dually-CPT posets, the next three lemmas consider the presence of trivial paths in a representation of strong modules and show how to obtain an equivalent representation where all the elements of the module are non-trivial paths. For these lemmas, we consider each strong module to be a CI poset and the element of the module represented by a trivial path is denoted by  $z$ .

**Lemma 15.** *Let  $M$  be a strong CI clique module of a dually-CPT poset  $\mathbf{P}$ . If an element  $z$  of  $M$  is represented by a trivial path in a representation  $R_{\mathbf{P}}$ , then there exists a representation  $R'_{\mathbf{P}}$  where  $z$  is represented as non-trivial path.*

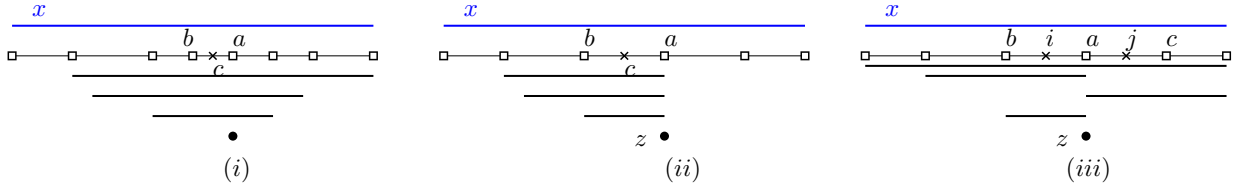


Figure 1: Representation of cliques modules with trivial paths.

*Proof.* By Lemma 12 we know that there exists an element  $x$  such that all the paths of  $M$  are contained in  $W_x$  in all CPT representations. Let us consider three cases.

(1) Suppose the trivial path of  $z$  is not an extremity of any path that represents the elements of  $M$ . Let  $a$  be the vertex of  $T$  that hosts the trivial path of  $z$ . Since  $W_z$  is not an extremity of any path of  $M$ ,  $a$  admits at least one neighbor  $b$  in  $T$  such that all the paths of  $M$  (except for  $z$ ) pass through  $b$  (see Figure 1(i)). Let us subdivide the edge  $a, b$  by adding a vertex  $c$ . Then it suffices to replace the trivial path of  $z$  by a non-trivial path that goes from  $c$  to  $a$  in  $T$ . The containment relations among  $M$  are preserved and no new containment relation is added nor deleted with respect to the elements not in  $M$ .

(2) Suppose now, the trivial path of  $z$  is a common extremity for all the elements of  $M$  and  $M$  is one-sided (see Figure 1(ii)). We proceed as in the previous case; we consider a vertex  $b$  of  $T$  that is a neighbor of  $a$  and such that all the paths of  $M$  except for  $z$  pass through  $b$ . Since  $M$  is a clique, it only admits at most one element represented by a trivial path, such a vertex  $b$  exists, then we subdivide the edge by adding a vertex  $c$  and the path of  $z$  goes from  $a$  to  $b$ . Note that the technique still works if some paths of  $M$  continue after  $a$ .

(3) Suppose now, the trivial path of  $z$  is the common extremity for some paths of the module in a 2-sided manner (see Figure 1(iii)). Let  $b$  and  $c$  be two vertices of  $T$  that are neighbors of  $a$ , such that  $b$  and  $c$  lie on the path of  $x$ ,  $x$  being an element not in  $M$  that contains all elements of  $M$ . We can partition the elements of  $M$  into three sets:  $B$  the elements for which paths arrive at  $a$  and pass through  $b$ ,  $C$  defined in a similar way but *w.r.t.*  $c$  instead of  $b$ , and  $A$ , the paths of  $M$  that go through  $b$  and  $c$ . This time we need to subdivide the edges  $a, b$  and  $a, c$  of  $T$ . We add a vertex  $i$  between  $a$  and  $b$  and a vertex  $j$  between  $a$  and  $c$ . Then it suffices to extend the paths of  $B$  until  $j$  and the paths of  $C$  until  $i$ . The path of  $z$  now goes from  $i$  to  $j$ . By subdividing several times the edges  $a, b$  and  $a, c$ , we can make sure that all the extremities are distinct.  $\square$

**Lemma 16.** *Let  $M$  be a strong CI stable module of a dually-CPT poset  $\mathbf{P}$ . If an element  $z$  of  $M$  is represented by a trivial path in a representation  $R_{\mathbf{P}}$ , then there exists a representation  $R'_{\mathbf{P}}$  where  $z$  is represented as non-trivial path.*



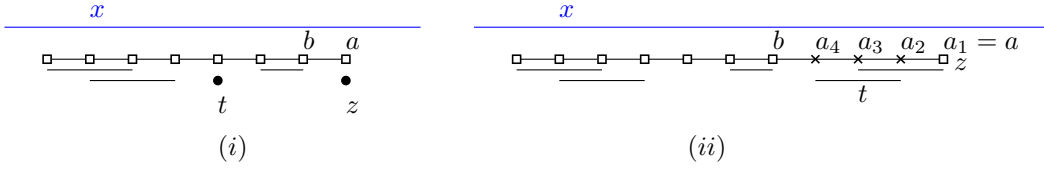


Figure 2: (i) Representation of a stable module with elements represented by trivial paths; (ii) transformation to eliminate trivial paths from the representation.

*Proof.* Let us first remark that in a strong stable module, several elements can be represented as trivial paths in a representation  $R_{\mathbf{P}}$ . In addition, if an element of  $M$  is represented by a trivial path, the trivial path is disjoint from all the other paths representing the elements of  $M$ . Let  $z$  be such an element. We will transform  $R_{\mathbf{P}}$  such that all the elements of  $M$  represented by trivial paths in  $R_{\mathbf{P}}$  will be represented by non-trivial paths. Let  $a$  be the vertex of  $T$  that hosts the path of  $z$ . Thanks to Lemma 12, we know that there exists an element  $x$  of  $\mathbf{P}$  such that in  $R_{\mathbf{P}}$  the paths of the elements of  $M$  are contained in the path of  $x$ . Since  $M$  is a non-trivial module it contains at least two elements, hence in  $R_{\mathbf{P}}$  there exists a vertex  $b$  of  $T$  that is adjacent to  $a$ , and  $b$  is contained in all the paths of the elements not in  $M$  that contain  $M$ , since such a path has to contain  $W_z$  and all the other elements of  $M$ .

Let us denote by  $U = \{u_1, u_2, \dots, u_k\}$  the elements of  $M$  that are represented by trivial paths in  $R_{\mathbf{P}}$ . To obtain an equivalent representation  $R'_{\mathbf{P}}$ , we subdivide  $2k - 1$  times the edge  $a, b$ . We then rename  $a$  as  $a_1$ , and we number the newly created vertices  $a_2, a_3, \dots, a_{2k}$  (the transformation is presented in Figure 2). In this new representation each element  $u_i$  of  $U$  is replaced by a path that goes from  $a_i$  to  $a_{k+i}$  in  $T$ .

It remains to prove that this representation is equivalent. First observe that for any element  $x$  connected to  $M$ , its path in  $R_{\mathbf{P}}$  contains all the elements of  $M$ . By the choice of vertex  $b$  to perform the transformation, we can guarantee that any path of such an element  $x$  will pass through  $a, b$  in  $R_{\mathbf{P}}$ . Since we subdivided this edge to obtain  $R'_{\mathbf{P}}$ , this path will still pass through  $a$  and  $b$  and all the vertices introduced by the transformation.

Now for any element  $y$  not connected to  $M$ , we know by Lemma 13 that no path of such an element will pass through  $a$ .  $\square$

**Lemma 17.** *Let  $M$  be a strong CI prime module of a dually-CPT poset  $\mathbf{P}$ . If an element  $z$  of  $M$  is represented by a trivial path in a representation  $R_{\mathbf{P}}$ , then there exists a representation  $R'_{\mathbf{P}}$  where  $z$  is represented as non-trivial path.*

*Proof.* For this proof, we consider three cases: (1) either  $W_z$  the trivial path of  $z$  is properly contained (*i.e.*  $W_z$  is not an extremity of any path of the element of  $M$ ) in all the paths of the elements of the module  $M$ , or (2) there exists at least two elements  $q$  and  $r$  of  $M$  such that  $W_z$  is the right bound of  $W_q$  and the left bound of  $W_r$ , or (3) the path  $W_z$  is the right (respectively left) bound for some paths representing elements of  $M$ , and is not the

left (respectively right) bound of any elements of  $M$ . These three cases are illustrated in Figure 3(i) – (iii).

(1) Let  $a$  be the vertex of  $T$  that hosts  $W_z$ , the trivial path representing  $z$ . By hypothesis, all the paths that represent the elements of  $M$  properly contain  $W_z$  and thus pass through vertex  $a$ . Since it is a proper containment, no path of elements of  $M$  (other than  $z$ ) starts or finishes at  $a$ . Thus  $a$  admits at least one neighbor  $b$  in  $T$  such that all the paths that represent elements of  $M$ , except for  $z$ , pass through  $b$ . To obtain a new representation  $R'_{\mathbf{P}}$  we subdivide the edge  $a, b$  by adding a vertex  $d$ . Then  $W_z$  in  $R'_{\mathbf{P}}$  is replaced by the path  $a, d$ . (See Figure 3(iv)). Since the representation of  $W_z$  is the only modification of the representation, by the previous discussion all the paths that represent the elements of  $M$  pass through  $a$  and  $b$  and as a consequence pass through  $a$  and  $d$  since  $d$  is in between  $a$  and  $b$ . By Lemma 13 we know that all the paths of the elements not in  $M$  that pass through  $a$  will also contain all the paths of  $M$ . Hence the modification of  $W_z$  preserves the containment relation of  $R_{\mathbf{P}}$ .

(2) Let us now consider that there exist at least two elements  $q$  and  $r$  of  $M$  such that in  $R_{\mathbf{P}}$ , the vertex  $a$  is the right bound of the path  $W_q$  and the left bound of the path  $W_r$  (see Figure 3(ii)). Let us denote by  $L$  the set of elements of  $M$  for which  $a$  is the right bound in the representation  $R_{\mathbf{P}}$  and similarly let us denote by  $R$  the set of elements of  $M$  for which  $a$  is the left bound in  $R_{\mathbf{P}}$ . Let us remark that  $L \cap R = \emptyset$  and some elements of  $M \setminus (L \cup R)$  might not be empty. Let  $b$  be the neighbor of  $a$  in  $T$  such that the paths of the elements of  $L$  pass through  $b$ . And similarly let  $c$  be the neighbor of  $a$  in  $T$  such that the paths of the elements of  $R$  pass through  $c$ . To obtain a new representation  $R'(P)$  we subdivide the edge  $a, b$   $|R| + 1$  times, and the edge  $a, c$   $|L| + 1$  times. The added vertices are called  $ab_i$  for the vertices between  $a$  and  $b$  and  $ac_j$  for the vertices between  $a$  and  $c$ . Let  $ab_1$  and  $ac_1$  be the neighbors of  $a$  in  $R'_{\mathbf{P}}$ . The path  $W_z$  now goes from  $ab_1$  to  $ac_1$ . The left bound of the paths of the elements of  $R$  are moved on the  $ab_i$  vertices. The coordinates are chosen to preserve the containment relation. We proceed symmetrically for the paths of the elements in  $L$ . It remains to prove that the obtained representation still corresponds to  $\mathbf{P}$ . Again we know by Lemma 13 that no path of an element not connected to  $M$  passes through  $a$ , by construction it remains valid for  $a$  and for all the newly introduced vertices. For any other path their relation to  $W_z$  and the paths of the elements of  $L$  and  $R$  are unchanged. If the path  $W_s$  of an element  $s$  was containing the path  $W_l$  of an element  $l$  of  $L$  in  $R_{\mathbf{P}}$ , it is still the case in  $R'_{\mathbf{P}}$ . In that case the left bound of  $W_l$  is contained in  $W_s$  and the right bound of  $W_s$  will be at the right of  $c$  in  $R_{\mathbf{P}}$ . This property will be preserved in  $R'_{\mathbf{P}}$ . Similarly if both paths  $W_s$  and  $W_l$  were overlapping in  $R_{\mathbf{P}}$ , they are still overlapping in  $R'_{\mathbf{P}}$ .

(3) Since  $W_z$  is the right (*resp.* left) bound of some paths representing some elements of  $M$ , and is not the left (*resp.* right) bound of any other elements of  $M$ , there exists a vertex  $b$  in  $T$  adjacent to  $a$  and such that all the paths representing elements of  $M$  that end at  $a$  pass through  $b$ . To obtain the new representation  $R'_{\mathbf{P}}$  it suffices to subdivide this edge one time. Let  $c$  the newly introduced vertex. Then the trivial path  $W_z$  in  $R_{\mathbf{P}}$  is replaced by a

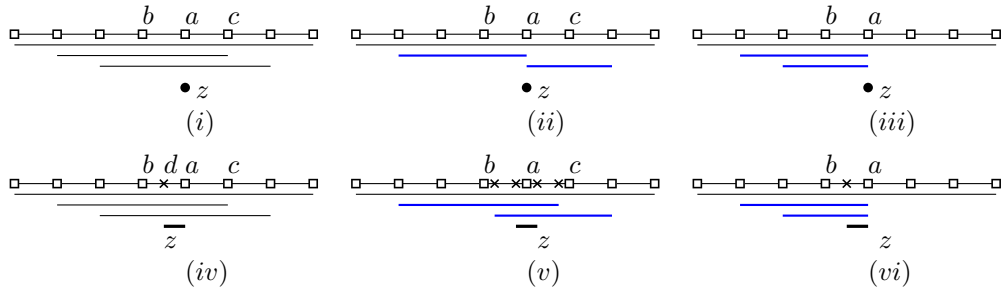


Figure 3: Representation of prime modules with the element  $z$  represented as trivial path.

path going from  $a$  to  $c$ . By the transformation, we can observe that all the paths that were containing  $W_z$  in  $R_{\mathbf{P}}$  still contain  $W_z$  in  $R'_{\mathbf{P}}$ . Let  $s$  be an element of  $\mathbf{P}$  such that  $W_z \subset W_s$  in  $R_{\mathbf{P}}$ . If  $W_s$  was containing  $W_z$  it had to pass through  $a$  and  $b$ , thus by subdividing  $a, b$  we can also conclude that this paths will pass through  $a$  and  $c$ , the added vertex, in  $R'_{\mathbf{P}}$ .  $\square$

**Theorem 18.** *If  $\mathbf{P}$  is dually-CPT and in a representation  $R_{\mathbf{P}}$  some elements of strong modules are represented by trivial paths, then there exists an equivalent representation  $R'_{\mathbf{P}}$  where all the paths representing elements of strong modules are non-trivial paths.*

*Proof.* It is a direct consequence of Lemmas 15, 16 and 17 and the fact that each time a trivial path is replaced by a non-trivial one, no trivial path is created in  $R'_{\mathbf{P}}$ .  $\square$

From the preceding theorem, we know how to obtain a representation a dually-CPT poset where all the elements contained in non-trivial strong modules are represented by non-trivial paths. Hence, in this representation some elements that do not belong to strong modules might be represented by trivial paths.

## 4 Ending of modules onto trivial paths

In the previous section we proved that for a dually-CPT poset, one can always obtain a representation where no element of a strong module is represented by a trivial path. It therefore remains to consider how the paths that represent a strong module  $M$  can connect to an element  $z$ , not contained in a strong module, where  $z$  is represented by a trivial path in the representation  $R_{\mathbf{P}}$ . Since we need to reconfigure the containment relation inside the module, this operation could be prevented or constrained if the trivial path is misplaced. In the case where the trivial path is in the middle of the paths of the module, it will be easy to reconfigure the containment relation. In the opposite case, if all the paths representing elements of a module arrive at a trivial path, we cannot perform the intended operation as planned. In this section, we will identify the problematic situations, and we will show how

to overcome these problems. As in the previous section, we will perform local changes to the representation to suppress problematic cases.

When the paths that represent elements of a module are connected to a trivial path in a representation, several configurations could arise. The most favorable one, is when the trivial path is properly contained in the paths of the module (*i.e.* the trivial path does not lie on any extremity of the path of the module). Actually this is a configuration we aim at obtaining. The other two configurations is when all the path have their extremities that end at a trivial path, or just some of them end at this trivial path. In most cases we will be able to reconfigure our representation to obtain a representation that is favorable to our purpose.

## 4.1 Complete ending of a module on a trivial path

Let us assume that all paths in  $R_{\mathbf{P}}$  corresponding to elements of  $M$  have all their extremities end at a vertex  $a$  of the host tree  $T$ . In that case, there are several possibilities: either all the paths that represent  $M$  will arrive at  $a$  by passing by a vertex  $b$  of  $T$  and such that  $a, b$  is an edge of  $T$ , or there is another vertex  $c$  that is a neighbor of  $a$  in  $T$  different from  $b$  and such that some paths of the module pass through  $c$ .

In this section, even if it is not explicitly stated, the representation of the module  $M$  will contain the trivial path of  $z$  located at the vertex  $a$  in  $T$ .

**Remark 19.** *If a strong module of a dually-CPT poset  $\mathbf{P}$  is two-sided in a representation  $R_{\mathbf{P}}$ , then the induced graph is not connected. Hence the strong module is a stable module.*

**Lemma 20.** *Let  $M$  be a strong module of a dually-CPT poset  $\mathbf{P}$ . If  $M$  is one-sided in a representation  $R_{\mathbf{P}}$  and the poset induced by  $M$  is connected, then  $M$  is a clique module.*

*Proof.* If the graph induced by  $M$  is connected,  $M$  is either a clique module or a prime module. If  $M$  is a clique module, then there is nothing to prove. If  $M$  is a prime module, then the graph induced by  $M$  necessarily contains an induced  $P_4$ . Let us show that it is not possible to represent a  $P_4$  as a CPT representation where all the paths end up at a same vertex  $a$  of the host tree  $T$ . Consider the representation of the  $P_4$  as presented in Figure 4(i) with the containment relation represented in Figure 4(ii).

For a contradiction, let us assume that such a representation exists. Since the paths of 2 and 4 have to contain the path of 3 and all these paths have to arrive at vertex  $a$  of the host tree, we have a configuration similar to the one depicted in Figure 4(iii) and a part of the host tree is depicted in Figure 4(iv). Since 2 and 4 are not connected, their paths have to diverge in  $T$ . Call  $x$  the vertex of  $T$  where these paths diverge. It remains to represent the path of 1. Since 1 is connected to 2 but not to 4, call  $y$  the vertex of  $T$  where the path of 1 begins. The vertex  $y$  has to lie in the proper part of the path of 2 (see Figure 4(iv)), and this path, by hypothesis, has to go all the way to  $a$ . But in that case it has to contain the path of 3, hence there is a contradiction.  $\square$

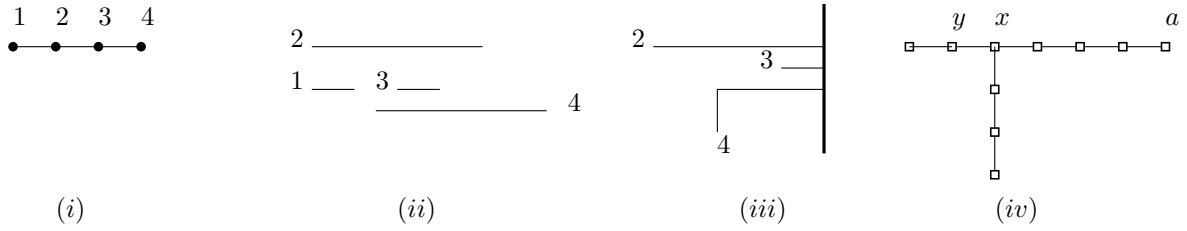


Figure 4: (i) a  $P_4$ , (ii) a CI representation of  $P_4$ , (iii) tentative representation with all the paths arriving at a vertex, (iv) the host tree of the tentative representation.

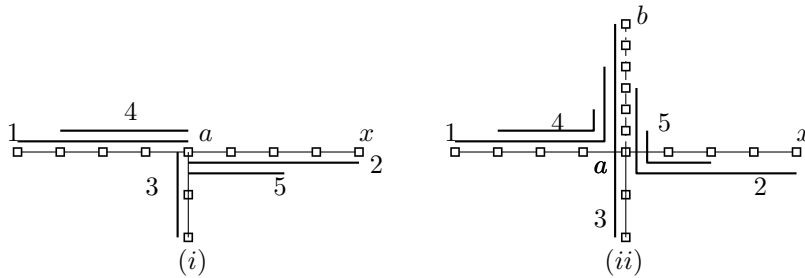


Figure 5: Example of modification on a representation of a poset  $\mathbf{P}$

We have proven that if in the representation of a strong module all its paths arrive at a same vertex of the host tree, then the module is either a clique or a stable module. We now consider in which cases can we obtain an alternative representation where all the paths do not arrive at a same vertex of the host tree. When the modification is possible, we will show how by starting from  $R_{\mathbf{P}}$  one can obtain an equivalent representation  $R'_{\mathbf{P}}$ , that is, a containment representation that still corresponds to  $\mathbf{P}$ .

**Lemma 21.** *Let  $M$  be a strong module of a dually-CPT poset  $\mathbf{P}$  and  $R_{\mathbf{P}}$  a representation of  $\mathbf{P}$  where all the paths of  $M$  arrive at a same vertex. If there is no element of  $\mathbf{P}$  that contains the elements of  $M$ , then there exists an alternative representation  $R'_{\mathbf{P}}$  of  $\mathbf{P}$  where all the paths of  $M$  will have different endpoints.*

*Proof.* Let us assume that all the paths of a strong module  $M$  arrive at a vertex  $a$  in the representation  $R_{\mathbf{P}}$ . If there is no element of  $\mathbf{P} \setminus M$  that contains all the paths of  $M$ , then we can add a new branch to the host tree starting at  $a$  and ending at  $b$  (see Figure 5). Let us denote by  $k$  the cardinality of  $M$ . In order to guarantee that all paths end on a dedicated vertex, the new branch needs to have at least  $k$  new vertices. It is easy to make sure that the containment relation inside the module is not altered in this new representation. It is simple to notice that the previous containments of  $\mathbf{P}$  are preserved by this modification and no new containment is added since the branch only contains paths of  $M$ .  $\square$



Figure 6: Modification of the representation of a two-sided stable module.

We now consider the case when there is at least one element  $x$  not in  $M$  that is greater than all the elements of  $M$ . In that case  $M$  is either one-sided or two-sided. Let us start with this second case.

**Lemma 22.** *Let  $M$  be a strong stable module of a dually-CPT poset  $\mathbf{P}$  and let  $x$  be an element of  $\mathbf{P} \setminus M$  that contains all the elements of  $M$ . Let us assume that in a representation  $R_{\mathbf{P}}$  all the paths of  $M$  arrive at a vertex  $a$  of  $T$ . Then there exists an equivalent representation  $R'_{\mathbf{P}}$  where all the endpoints of the paths of  $M$  near  $a$  are distinct.*

*Proof.* By hypothesis, since the elements of  $M$  are all contained in an element  $x$  of  $\mathbf{P}$ , it means that in any representation  $R_{\mathbf{P}}$  of  $\mathbf{P}$  the union of the paths of  $M$  is a path. If in a representation  $R_{\mathbf{P}}$  of  $\mathbf{P}$  all the paths of  $M$  arrive at  $a$ , let  $b$  and  $c$  be the immediate neighbors of  $a$  on  $T$  along the path that hosts all the paths of  $M$ . Since the strong module considered is stable and in the representation every element lies under the path of  $x$ , the module is two-sided at  $a$ . Since  $M$  is two-sided in the representation, its elements can be partitioned into two sets  $B$  and  $C$  as follows: An element  $r$  is in  $B$  if its path in  $R_{\mathbf{P}}$  passes by the vertex  $b$ . Similarly, an element  $s$  is in  $C$  if its path in  $R_{\mathbf{P}}$  passes by  $c$  (see Figure 6). To obtain  $R'_{\mathbf{P}}$  it suffices to subdivide the edges  $a, b$  and  $a, c$  of  $T$ . All the paths of the elements of  $B$  that previously ended at  $a$  will now end between  $a$  and  $c$ . Hence it is necessary to add  $|B|$  new vertices between  $a$  and  $c$ . In a symmetric manner, the paths of the elements of  $C$  will be elongated to end on a new vertex between  $a$  and  $b$ ; thus it is necessary to add  $|C|$  new vertices between  $a$  and  $b$ . It is simple to see that the introduced modification does not alter the containment relationship. Any path that contained all the elements of  $M$  will still contain all the elements of  $M$ . And any path that crossed the section of tree spanned by the elements of  $M$  but did not contain them, will still not contain them.  $\square$

**Lemma 23.** *Let  $M$  be a strong clique module of a dually-CPT poset  $\mathbf{P}$  and let  $x$  be an element of  $\mathbf{P} \setminus M$  that contains all the elements of  $M$ . If in a representation  $R_{\mathbf{P}}$  all the paths of  $M$  arrive at a vertex  $a$ , then  $M$  does not contain any other strong module.*

*Proof.* Because of the element  $x$ , the union of all the paths of the elements of  $M$  in  $R_{\mathbf{P}}$  is included in the path of  $x$  and hence itself forms a path. Since all these paths are bounded at  $a$ , then for any pair of elements  $p$  and  $q$  of  $M$  either the path of  $p$  is contained in the path of  $q$  or the converse. There is no pair of non-adjacent vertices. As a consequence it does not contain any other module.  $\square$

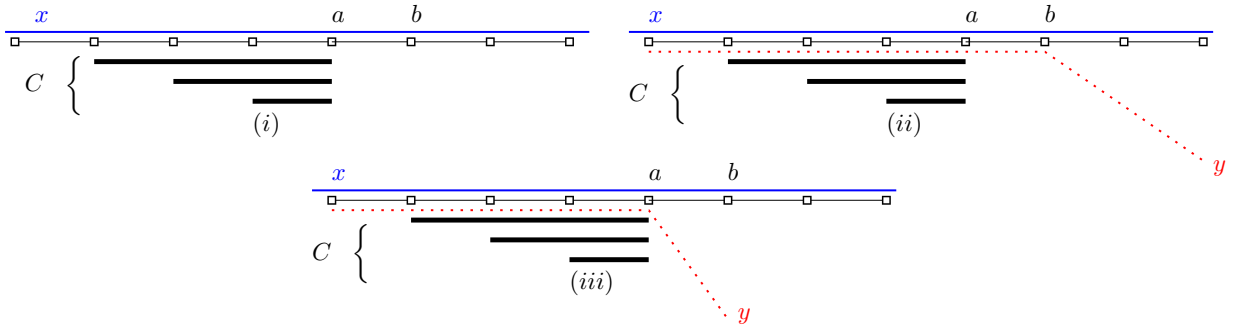


Figure 7: Configuration of a clique path. The set of paths  $C$  represents the paths of a strong clique module  $M$

Let  $M$  be a strong clique module with representation  $R_{\mathbf{P}}$  where all the paths of the elements of  $M$  stop at a vertex  $a$ . We say that  $M$  is *free* in  $R_{\mathbf{P}}$  if there is at least one vertex  $b$  of  $T$  such that  $a, b$  is an edge of  $T$ , no path of  $M$  passes through  $b$  and all the paths that contain the paths of  $M$  pass through  $b$ . (See Figure 7 (i) and (ii).)

**Lemma 24.** *Let  $M$  be a strong clique module of a dually-CPT poset  $\mathbf{P}$ , and let  $x$  be an element of  $\mathbf{P} \setminus M$  that contains all the elements of  $M$ . If  $M$  is free in a representation  $R_{\mathbf{P}}$  where all the paths of  $M$  arrive at a vertex  $a$ , then we can find a representation  $R'_{\mathbf{P}}$  where all the endpoints of  $M$  arrive on different vertices of  $T$ .*

*Proof.* Since  $M$  is free in  $R_{\mathbf{P}}$  we can re-use the technique used in Lemma 22 by subdividing the edge  $a, b$  of  $T$ .  $\square$

Thanks to Lemmas 21, 22, and 24, we know how modify a representation in almost all the cases. However, one case is not covered, namely, when the module is a clique and it is blocked. We say that a strong clique module bounded at a vertex  $a$  in a representation  $R_{\mathbf{P}}$  is *blocked* if it is not free. There are two reasons why  $M$  may be blocked: (1) It might be because a path that contains the elements of  $M$  also stops at  $a$ , or (2) because there are two elements  $x$  and  $y$  that contain all the elements of  $M$  and in  $R_{\mathbf{P}}$  their corresponding paths diverge at  $a$ . (See Figure 7(iii).)

**Remark 25.** *Let  $M$  be a strong clique blocked module of a dually-CPT. From Lemma 23, we know that it does not contain any other strong module. Hence, a reconfiguration of this subposet is just a matter of relabelling the elements.*

## 4.2 Partial ending of a module on a trivial path

In the previous section, we proved that whenever a module is connected to an element  $z$  of  $\mathbf{P}$  represented by a trivial path in a representation  $R_{\mathbf{P}}$  and all paths that represent the element of  $M$  end at this path, we can either alter the representation to ensure that all

the paths do not end on that trivial path or, the module is a clique and does not contain any other modules. Hence it is possible to alter the containment relation.

If, in the completely opposite direction, a module  $M$  is connected to an element  $z$  represented by a trivial path, but no path that represents an element of  $M$  ends at this trivial path, it does not create any problem to change the containment relation of the module.

The last case to consider is when  $M$  is connected to a trivial path, but only some paths of  $M$  (not all) end at this trivial path. We will prove that in that case an equivalent representation, where no path of  $M$  ends at this trivial path, can be obtained.

**Lemma 26.** *Let  $M$  be a strong module of a dually-CPT poset  $\mathbf{P}$  connected to an element  $z$  ( $z \notin M$ ). If in a representation  $R_{\mathbf{P}}$  the element  $z$  is represented by a trivial path  $W_z$  and the paths of some elements of  $M$  end at the path of  $W_z$  and some other paths of elements of  $M$  properly contain  $W_z$ , then there exists an equivalent representation  $R'_{\mathbf{P}}$  where no element of  $M$  ends at a trivial path.*

*Proof.* Let  $I$  denote the set of elements not in  $M$  such that the paths of the elements of  $I$  are contained in the paths of the elements of  $M$ . In the representation  $R_{\mathbf{P}}$  all the paths that represent the element of  $I$  are all contained in  $\bigcap_{m \in M} P_m$ .

Since by hypothesis not all the paths of  $M$  end at a trivial path, if there are some elements of  $M$  that end paths represented by trivial paths, there are at most two trivial paths in that situation. Call these trivial paths  $y$  and  $z$ .

Let us assume that the part common to all the paths of  $M$  in  $R_{\mathbf{P}}$  is on a horizontal line, and that *w.l.o.g.* that  $W_y$  is the leftmost and  $W_z$  is the rightmost of this common part. We assume further, in the representation  $R_{\mathbf{P}}$ , that  $W_z$  lies on vertex  $a$  of  $T$  and  $W_y$  lies on vertex  $b$  of  $T$ .

We denote by  $L$  (resp.  $R$ ) the set of all elements of  $M$  whose paths in  $R_{\mathbf{P}}$  end at  $b$  (resp. at  $a$ .) Note that there is at most one element of  $M$  that belongs to both  $L$  and  $R$ , since the containment relation is proper.

There are two cases to consider: (1) either there is no element  $x$  such that all the paths of  $M$  are contained in the path of  $x$ , or (2) such an element  $x$  exists.

(1) For the first case, let us assume that such an element does not exist. Hence there is no path in  $R_{\mathbf{P}}$  that contains any path of the elements of  $M$ . In that case, to obtain an equivalent representation, in  $T$  we can add one path with  $|M|$  new vertices connected to  $a$  and another path with  $|M|$  new vertices connected to  $b$ . Since the poset induced by  $M$  is CI, it suffices to represent this module as a containment of intervals using these new branches for the endpoints. The transformation process is presented in Figure 8.

The containment relation between elements of  $R$  (resp.  $L$ ) and  $I$  remain unchanged. Moreover, for any element  $q$  not connected to  $M$ , since the endpoints of the paths of the elements of  $M$  have been relocated in the two new branches, there is no containment relation between  $W_q$  and the paths of the elements of  $M$ , since  $W_q$  does not contain any of the new branches in  $R'_{\mathbf{P}}$ .



(2) Let us now consider the case when there is an element  $x$  not in  $M$  such that in  $R_{\mathbf{P}}$  the path of  $x$  contains all the paths of the elements of  $M$ . In the host tree  $T$  we denote by  $c$  the neighbor of  $a$  such that no path of  $R$  passes through  $c$  but some paths of elements of  $M$  do (by our initial hypothesis). Let  $d$  be the neighbor of  $b$  in  $T$  such that paths of some elements of  $M$  pass through but no element of  $L$  does.

To obtain an alternative representation  $R'_{\mathbf{P}}$  we subdivide the edge  $a, c$   $|R|$  times and subdivide the edge  $b, d$   $|L|$  times. (This transformation is presented in Figure 9). Then it is just a matter of extending the paths of the elements in  $R$  such that they end on a vertex located between  $a$  and  $c$ . For each element of  $R$ , its new ending vertex is determined according to the containment relation in  $R$ . For the elements of  $L$ , we proceed in a similar manner.

It remains to prove that the new representation still represents the poset  $\mathbf{P}$ . The only paths that are transformed are the paths that correspond to elements of  $R$  and  $L$ . Without loss of generality, let  $l$  be an element of  $R$  and let  $W_l$  be its path in  $R'_{\mathbf{P}}$ . Since  $W_l$  has been extended, it is clear that all the paths in  $R_{\mathbf{P}}$  that were contained in  $W_l$  remain contained in  $R'_{\mathbf{P}}$ . In addition, since the extension occurred between  $a, c$  or  $b, d$ . Equivalently let  $k$  be an element of  $\mathbf{P}$ . If  $W_l \subset W_k$  in  $R_{\mathbf{P}}$  then  $W_l \subset W_k$  in  $R'_{\mathbf{P}}$ . If  $k$  is an element of  $R$ , by the transformation we ensure that the containment relation is preserved. If  $k$  is not an element of  $R$ , then in  $R_{\mathbf{P}}$ , the path  $W_k$  passed by vertex  $c$  of  $T$ , hence by extending  $W_l$ , it will not reach  $c$ , then it is still contained in  $W_k$  in  $R'_{\mathbf{P}}$ .

Let us now consider an element  $q$  such that  $W_q$  intersects  $W_l$  but there is no containment relation in  $R_{\mathbf{P}}$ . If  $W_l \cup W_q$  is not a path in  $R_{\mathbf{P}}$  then it contains a claw pattern and this pattern will be preserved in  $R'_{\mathbf{P}}$ . Let us now consider the case when  $W_l \cup W_q$  forms a path in  $R_{\mathbf{P}}$ . If  $W_q$  passes through  $a$  in  $R_{\mathbf{P}}$  it has one endpoint contained between the endpoint of  $W_l$ . Thus the first endpoint of  $W_q$  is at the left of  $a$  in  $R_{\mathbf{P}}$  and the endpoint at the right of  $c$  (possibly  $c$ ). Since  $W_l$  does not reach  $c$  in  $R'_{\mathbf{P}}$ , the overlap relation is preserved in the new representation. If both paths were disjoint, they remain disjoint in  $R'_{\mathbf{P}}$ . □

From Lemmas 21, 22, 24 and Remark 25, we can summarize the results of this section with the following theorem:

**Theorem 27.** *Let  $\mathbf{P}$  be a dually-CPT poset. Either for each strong module  $M$  of  $\mathbf{P}$  there exists a representation  $R_{\mathbf{P}}$  such that all the paths of  $M$  do not end on a trivial path, or  $M$  is a clique blocked module.*

We call a representation that fulfills the condition of the previous theorem a *normalized representation*.

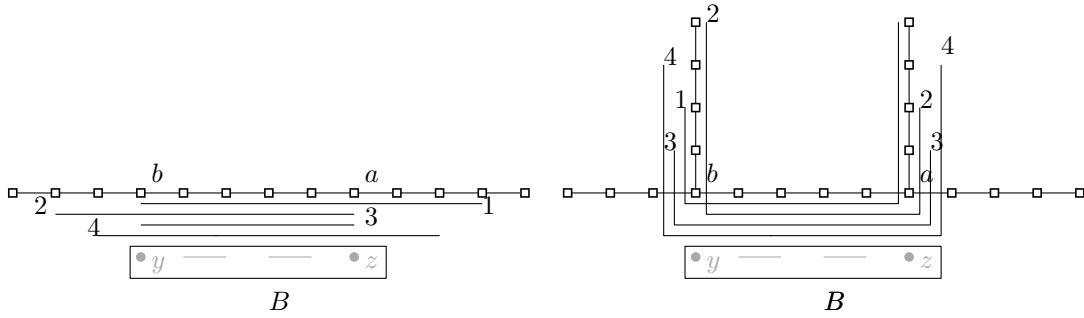


Figure 8: Illustration of case (1) of Lemma 26. Elements 1, 2, 3 and 4 are parts of the modules. The module is connected to the elements represented by the paths in the box  $B$ . Elements 1 and 3 belong to  $L$  and elements 2 and 3 belong to  $R$ .

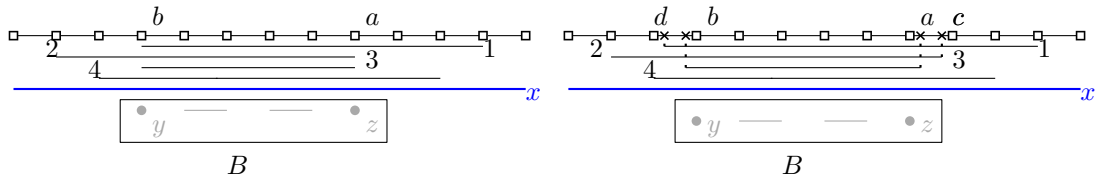


Figure 9: The same example as in Figure 8, but this time there is an element  $x$  not in  $M$  that contains all the elements of  $M$  and that prevents performing the modification of case (1).

## 5 Substitution

The last step to obtain our main result is to prove that for any dually-CPT poset  $\mathbf{P}$  all the posets  $\mathcal{Q} = \{\mathbf{Q}_1, \dots, \mathbf{Q}_l\}$  that are associated to  $\mathbf{P}$  admit a CPT representation. Let us consider one particular poset  $\mathbf{Q}$  of this set. If  $\mathbf{Q}$  is associated to  $\mathbf{P}$  it means by definition that their underlying comparability graphs are identical. We assume that  $\mathbf{P}$  is not CI, otherwise the results already stand from Theorem 6. Thus we deduce that the quotient poset of  $\mathbf{P}$  is not CI, by Theorem 9, and thus is prime. Since  $\mathbf{P}$  and  $\mathbf{Q}$  are associated, by Property 4 they admit the same set of strong modules. The quotient  $\mathbf{H}$  of  $\mathbf{P}$  is obtained by keeping one element of each strong maximal module and the quotient  $\mathbf{K}$  of  $\mathbf{Q}$  is either equal to  $\mathbf{H}$  or to its dual  $\mathbf{H}^d$ . Let us consider that  $\mathbf{H}$  is equal to  $\mathbf{K}$ .

To obtain a representation for  $\mathbf{Q}$ , we will use the normalized representation  $R_{\mathbf{P}}$  obtained for  $\mathbf{P}$ . From  $R_{\mathbf{P}}$  it is immediate to obtain a representation  $R_{\mathbf{H}}$  for  $\mathbf{H}$  as it suffices to keep one path for each strong module of  $\mathbf{P}$ . In addition, since it is obtained by removing paths from a normalized representation, we can consider that all the paths that correspond to strong modules which are not clique blocked modules, do not end on trivial paths of other elements. Then we will show that for such elements, we can replace this path by an arbitrary CI poset. Finally, to obtain a CPT representation for  $\mathbf{Q}$  it suffices to replace

each path that is a representative of a strong module, by the corresponding CI poset in  $\mathbf{Q}$ . For the clique blocked modules, as they do not contain other strong modules, they correspond to total orders, hence the representation can be preserved, but the labelling has to be changed to suit the total order in  $\mathbf{Q}$ .

Let  $v_0$  be an element of  $\mathbf{H}$  that is a representative of some maximal strong module of  $\mathbf{P}$  that is not a clique blocked module. Let  $W_{v_0} = (x_1, x_2, \dots, x_k)$  be its path in  $R_{\mathbf{H}}$ . We will assume that  $k$  is at least 4. We will show how to replace  $W_{v_0}$  by a CI poset  $\mathbf{N}$ . Let  $R_{\mathbf{N}} = \{I_i\}_{1 \leq i \leq n}$  be a CI representation of a poset  $\mathbf{N}$  whose vertices are  $u_1, u_2, \dots, u_m$ .

Assume that the intervals  $I_i$  (subpaths of a path  $I$ ) are non-trivial, no two of them share an end vertex and there is an edge  $cd$  of  $I$  contained in the total intersection of the intervals  $I_i$  – this assumption is guaranteed by Proposition 10. Name  $a$  and  $b$  the end vertices of the interval union of the intervals  $I_i$ . Clearly  $[c, d] \subset [a, b]$ . We also assume that  $a, b, c$  and  $d$  are distinct, and that neither  $c$  nor  $d$  are end vertices of an interval  $I_i$ .

**Replacement process.** *The process of replacing in the representation  $R_{\mathbf{H}}$  the path  $W_{v_0}$  by the intervals  $\{I_i\}_{1 \leq i \leq n}$  of the representation  $R_{\mathbf{N}}$  consists of:*

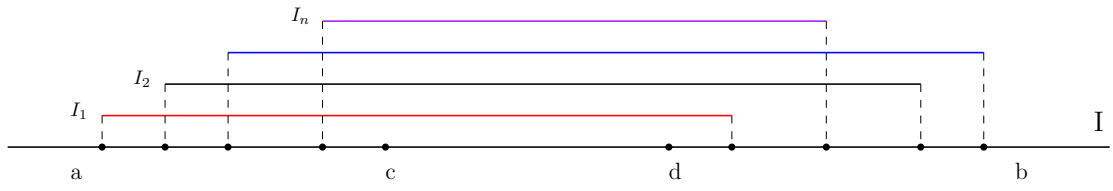
- (i) *subdividing the edges  $x_1x_2$  and  $x_{k-1}x_k$  of  $T$  by adding in each one  $n - 1$  vertices.*
- (ii) *subdividing the edge  $cd$  of  $I$  by adding as many vertices as there are in  $T$  between  $x_2$  and  $x_{k-1}$ .*
- (iii) *removing from  $R_{\mathbf{H}}$  the path  $W_{v_0}$  and embedding in its place the intervals of  $S$  in such a way that the vertices  $a, c, d, b$  and all others between them match with the vertices  $x_1, x_2, x_{k-1}, x_k$  and all others between them, respectively, as it is shown in Figure 10.*

**Lemma 28.** *If in  $R_{\mathbf{H}}$  the path  $W_{v_0}$  that represents a module of a dually-CPT poset  $\mathbf{P}$  does not end on trivial paths, then we can obtain the representation  $R_{\mathbf{H}\mathbf{N} \rightarrow v_0}$  by replacing  $W_{v_0}$  by the intervals  $\{I_i\}_{1 \leq i \leq n}$  of the representation  $R_{\mathbf{N}}$  in  $R_{\mathbf{H}\mathbf{N} \rightarrow v_0}$ . If any of the paths  $I_i$  contains (resp. is contained in) a path  $W_v$ , then all the paths  $I_i$  contain (resp. are contained in)  $W_v$ .*

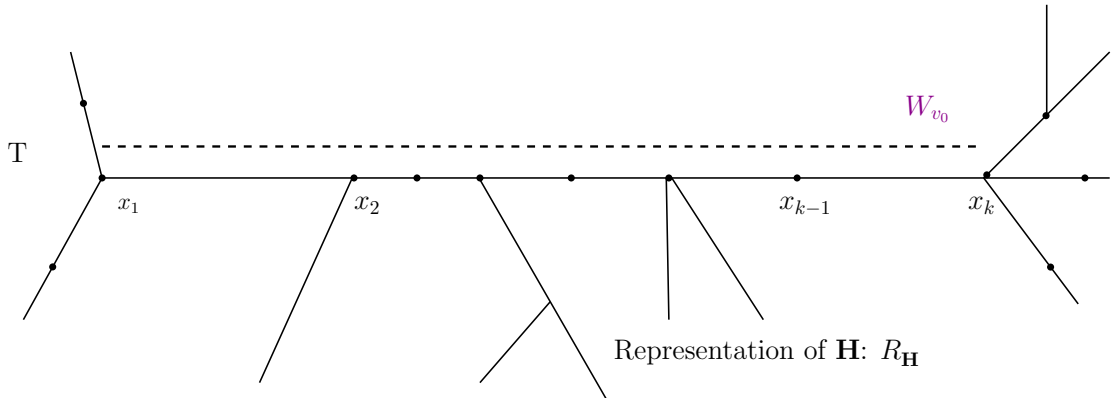
*Moreover, a path  $W_v$  of  $R_{\mathbf{H}}$  contains (is contained in)  $W_{v_0}$  if and only if  $W_v$  contains (is contained in) every one of the intervals  $I_i$  in  $R_{\mathbf{H}\mathbf{N} \rightarrow v_0}$ .*

*Proof.* This result is a direct consequence of two facts: first, that in  $R_{\mathbf{N}}$  no interval  $W_v$  of  $R_{\mathbf{H}}$  has an end-vertex between  $x_1$  and  $x_2$ , nor between  $x_{k-1}$  and  $x_{k+1}$ , and second, that in  $R_{\mathbf{N}}$ , all the intervals  $I_i$  contain the interval  $x_2x_{k-1}$ . See Figure 10.  $\square$

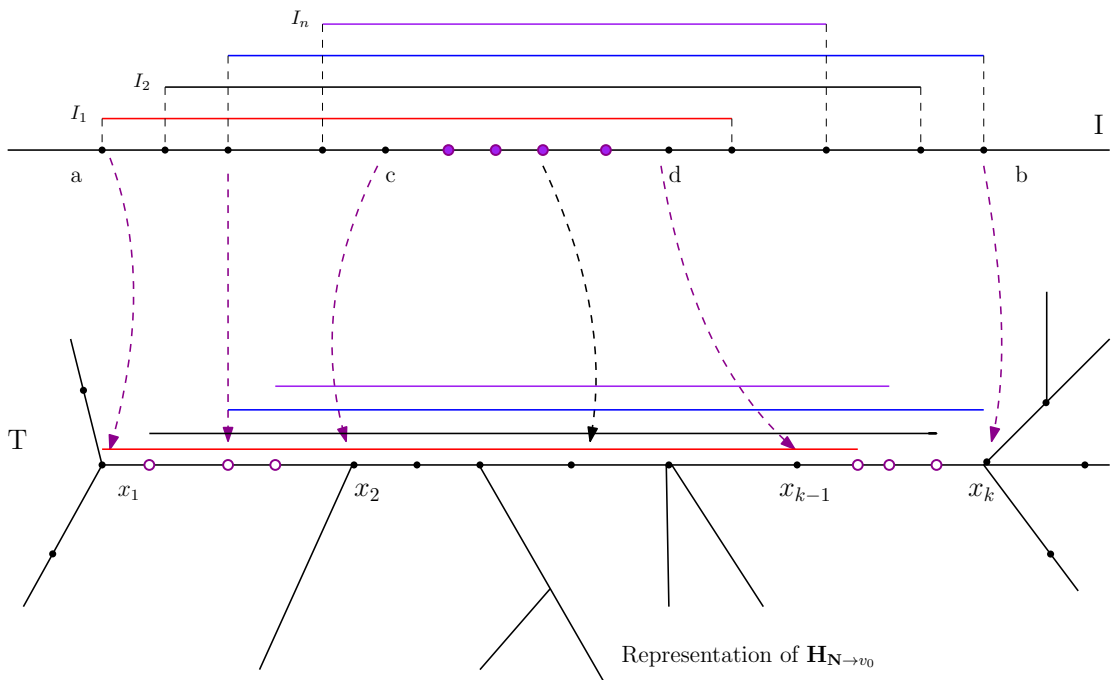
**Lemma 29.** *If in  $R_{\mathbf{H}}$  the path  $W_{v_0}$ , that represents a blocked clique module of a dually-CPT poset  $\mathbf{P}$ , ends on a trivial path, then we can obtain the representation  $R_{\mathbf{H}\mathbf{N} \rightarrow v_0}$  by replacing  $W_{v_0}$  by the a collection of paths that represent a clique.*



Representation of CI poset  $\mathbf{N}$ :  $R_{\mathbf{N}}$



Representation of  $\mathbf{H}$ :  $R_{\mathbf{H}}$



Representation of  $\mathbf{H}_{\mathbf{N} \rightarrow v_0}$

- these are the vertices added in the first step of the Replacement process
- these are the vertices added in the second step of the Replacement process

Figure 10: Description of the Replacement process

*Proof.* Let us assume that  $W_z$  is the trivial path that  $W_{v_0}$  ends on in  $R_{\mathbf{H}}$ . Let us denote by  $a$  the vertex of the host tree that hosts  $W_z$ . Since the containment relation is proper, we can assume that  $W_{v_0}$  passes through at least two vertices of the host tree. One of the extremities of  $W_{v_0}$  is  $a$ . Let us call the other extremity  $b$ . Since the length of  $W_{v_0}$  is at least two, we know there exists in the host tree a vertex  $c$  that is the immediate neighbor of  $b$  on the path going to  $a$ . The vertex  $c$  is possibly equal to  $a$ . By subdividing an appropriate number of times the edge  $bc$  of the host tree, we can add as many paths as we need to place a clique module. From the transformation, it is easy to see that the containment relation is preserved with respect to the module.  $\square$

We restate here our main theorem:

**Theorem 30.** *A poset  $\mathbf{P}$  is strongly-CPT if and only if it is dually-CPT.*

*Proof.* Let  $\mathbf{H} = \mathbf{P}/\mathcal{M}(\mathbf{P})$  be the quotient poset, where  $\mathcal{M}(\mathbf{P}) = \{M_1, \dots, M_k\}$  is the maximal modular partition of  $\mathbf{P}$ .

Since  $\mathbf{P}$  is a dually-CPT poset and  $\mathbf{H}$  is a subposet of  $\mathbf{P}$ , then  $\mathbf{H}$  and  $\mathbf{H}^d$  admit a normalized CPT-representation. If  $\mathbf{H}$  is CI, by Remark 8 and Theorem 9,  $\mathbf{P}$  is CI and so strongly-CPT. Thus let us assume that  $\mathbf{H}$  is a prime dually-CPT poset.

Let  $\mathbf{Q}$  be an associated poset of  $\mathbf{P}$  and let  $\mathbf{K}$  be its quotient poset. Since  $\mathbf{P}$  and  $\mathbf{Q}$  are associated, an immediate consequence is that  $\mathbf{H}$  and  $\mathbf{K}$  are associated; in addition by hypothesis they are both prime, hence by Theorem 3,  $\mathbf{K}$  is either equal to  $\mathbf{H}$  or to  $\mathbf{H}^d$ . Let us assume, *w.l.o.g.*, that  $\mathbf{H} = \mathbf{K}$ .

We will prove that  $\mathbf{Q}$  admits a CPT representation. By Theorem 5 and *w.l.o.g.*, we assume that  $\mathbf{Q}$  is obtained by replacing in  $\mathbf{H}$  each vertex  $v_i$  of  $\mathbf{H}$  for  $\mathbf{Q}_i = \mathbf{Q}(M_i)$ . By Proposition 4,  $\mathbf{P}$  and  $\mathbf{Q}$  possess the same strong modules and by Theorem 9 since  $\mathbf{P}$  is dually-CPT, all the strong modules of  $\mathbf{P}$  and  $\mathbf{Q}$  are CI. For each  $\mathbf{Q}_i$  we have a CI representation.

Let  $R_{\mathbf{H}}$  be a CPT representation of  $\mathbf{H}$ , obtained from a normalized representation of  $\mathbf{P}$ . The representation is obtained by only keeping one path for each strong module of  $\mathbf{P}$ .

For each path  $W_{v_i}$  that corresponds to a module  $M_i$  of  $\mathbf{Q}$ , if  $W_{v_i}$  does not end on a trivial path of  $R_{\mathbf{H}}$  then it corresponds to a module which is not a blocked clique module, hence by Lemma 28, we can replace  $W_{v_i}$  by a CI representation of  $\mathbf{Q}_i$ .

The only remaining case is if  $W_{v_i}$  ends on a trivial path in  $R_{\mathbf{H}}$ . In that case, it means that it corresponds to a blocked clique module of  $\mathbf{P}$  in the representation  $R_{\mathbf{P}}$ . Hence by Lemma 29, we can replace  $W_{v_i}$  by a CI representation of the maximal strong clique module  $\mathbf{Q}_i$ .

By proceeding in that way for each maximal strong module, we are able to obtain a CPT representation  $R_{\mathbf{Q}}$  of  $\mathbf{Q}$ .  $\square$

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