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STUDY OF AN ELASTOPLASTIC MODEL WITH AN INFINITE NUMBER OF DEGREES OF FREEDOM

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ABSTRACT

We deal with the dynamical behavior of continuous elastoplastic model of Masing, consisting of infinite number of springs and dry-friction elements. Using theory of differential inclusions we provide existence and uniqueness result. Moreover, we prove that continuous model is the limit of the discrete Masing model when the number of degrees of freedom tends to infinity. Starting from known results of numerical analysis, we build an implicit Euler-like numerical scheme of order one.

NOMENCLATURE

- A differential inclusion
- B elastoplastic model
- C numerical scheme
- D hysteretic cycle

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INTRODUCTION

In this paper, we study an elastoplastic model with infinite number of degrees of freedom. The Prandtl model, also called model of Masing, is known as mechanical model (see (Fougères Sidoroff 1989)). In a numerical point of view, there are some schemes for elastoplastic model (Crandall Evans 1975; Glowinski 1976 a; Glowinski 1976 b) but these scheme can not work for Prandtl model in dynamics or do not provide estimates of order of convergence.

The Prandtl model, consisting of one material point connected in parallel with one spring and a finite number of associations in series of one spring and Saint-Venant elements (see figure 1) has been presented in (Bastien Schatzman Lamarque 2000; Bastien 2000). We provided differential inclusions that govern this model. We also gave a numerical scheme which permitted us to make numerical experiments exhibiting hysteretic cycles characterizing elastoplastic models. In (Bastien Schatzman 1999; Bastien Schatzman 2000; Bastien 2000), we have proved that order of the numerical scheme is one. Order one has been founded again via numerical experiments: numerical error

is clearly independent on the number of degrees of freedom of the Prandtl model. In (Bastien Schatzman 1999; Bastien 2000), we observed that the cycle of hysteresis of our rheological model possesses a limit when the number of freedom tends to infinity.

Here, we discuss that all the properties are still valid if the number of degrees of freedom of the Prandtl model becomes infinite. So, we present a continuous Prandtl model (see figure 4). This model is also known as Masing model ((Fougères Sidoroff 1989)). In section 1, this model is described: we replace discrete equations that govern the discrete Prandtl model by the corresponding continuous equations. Indeed a continuous differential inclusion is obtained. In section 2, we explain how that this differential inclusion possesses a unique solution thanks to the classical Brezis's work (Brezis 1973). We give also smoothness results. This solution is the limit of the discrete solution with P degrees of freedom when P tends to infinity. Analysis of this limit process is made in section 3. In section 4, we build a numerical scheme which discretizes the continuous Prandtl model and we discuss on its order one of convergence. In section 5, we study cycles of hysteresis of the continuous model. A few numerical examples are given in section 6.

1 DESCRIPTION OF THE CONTINUOUS MODEL

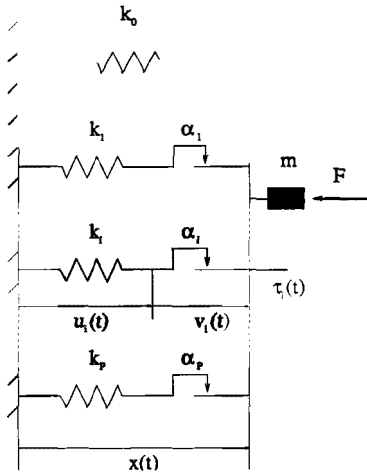


Figure 1. The discrete Prandtl model

In (Bastien Schatzman Lamarque 2000; Bastien 2000), we presented the Prandtl model with a finite number of degrees of freedom (see figure 1): let x be the abscissa of the material point, let u_i be the displacement of the i -th spring (with stiffness k_i) and let v_i be the displacement of the i -th Saint-Venant element (with threshold α_i). The material point of mass m is submitted to an

external force F . Denote τ_i the force exerted by the i -th spring. The constitutive law of the i -th spring is

$$\tau_i = -k_i u_i. \quad (1)$$

We write the constitutive law of the i -th Saint-Venant under the form

$$\tau_i \in -\alpha_i \sigma(\dot{v}_i), \quad (2)$$

where the graph σ (see figure 2) is defined on \mathbb{R}^2 by

$$\sigma(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0. \end{cases}$$

The graph σ is maximal monotone (for details, see (Brezis

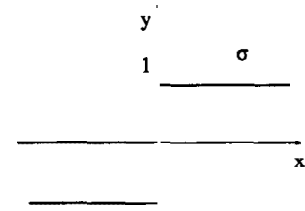


Figure 2. The graph σ

1973)). We proved that the discrete Prandtl model is governed by the following differential system: almost everywhere on $[0, T]$

$$\dot{x}(t) = y(t), \quad (3a)$$

$$y(t) = \frac{1}{m} \left(F(t) - k_0 x(t) - \sum_{i=1}^P k_i u_i(t) \right), \quad (3b)$$

$$\forall i \in \{1, \dots, P\}, \quad \dot{u}_i(t) + \partial \Psi_{[-\eta_i, \eta_i]}(u_i(t)) \ni y(t), \quad (3c)$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \\ \forall i \in \{1, \dots, P\}, \quad u_i(0) = u_{0,i} \in [-\eta_i, \eta_i]. \quad (3d)$$

Here, we set

$$\forall i \in \{1, \dots, P\}, \quad \eta_i = \alpha_i/k_i. \quad (4)$$

The function $\psi_{[-1,1]}$ is the indicatrix function of $[-1, 1]$ and it is defined by

$$\psi_{[-1,1]}(x) = \begin{cases} 0 & \text{if } x \in [-1, 1], \\ +\infty & \text{if } x \notin [-1, 1]. \end{cases}$$

This function is convex, proper and lower semi-continuous on \mathbb{R} ; so its sub-differential $\partial\psi_{[-1,1]}$ is a maximal monotone operator on \mathbb{R} , defined by

$$\partial\psi_{[-1,1]}(x) = \begin{cases} \emptyset & \text{if } x \in]-\infty, -1[\cup]1, +\infty[, \\ \{0\} & \text{if } x \in]-1, 1[, \\ \mathbb{R}_- & \text{if } x = -1, \\ \mathbb{R}_+ & \text{if } x = 1. \end{cases}$$

The graph $\partial\psi_{[-1,1]}$ is presented in figure 3 and it is inverse of

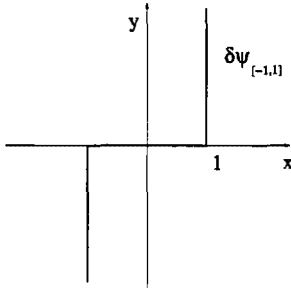


Figure 3. The graph $\partial\psi_{[-1,1]}$

graph σ , that is to say

$$\forall x, y \in \mathbb{R}, \quad x \in \sigma(y) \iff y \in \partial\psi_{[-1,1]}(x).$$

Using uniqueness and existence results of Brezis (see. (Brezis 1973)), we proved in (Bastien Schatzman Lamarque 2000; Bastien 2000) that system (3) possesses a unique solution.

We replace formally the parallel association of P associations in series of one spring with stiffness k_i and of one Saint-Venant element with threshold α_i by the parallel association of an infinite number of associations in series of one spring with

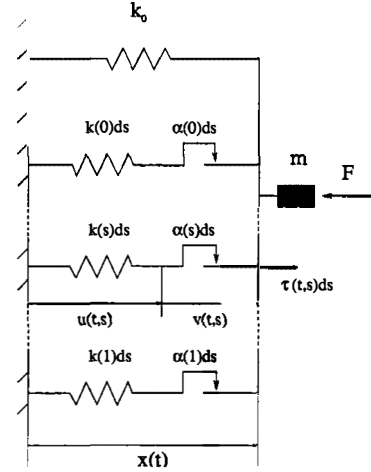


Figure 4. The continuous Prandtl model

infinitesimal stiffness $k(s)ds$ and one Saint-Venant element with infinitesimal threshold $\alpha(s)ds$ for $s \in [0, 1]$ (see figure 4).

We use notations similar to the notations of the discrete model. The material point of abscissa $x(t)$ and of mass m is submitted to an external force F . Let us denote $\tau(t,s)ds$ the force exerted by the spring with stiffness $k(s)ds$. Like (1), the constitutive law of the spring with stiffness $k(s)ds$ is

$$\tau(t,s) = -k(s)u(t,s). \quad (5)$$

Similar to (2), the constitutive law of the Saint-Venant element with threshold $\alpha(s)ds$ is

$$\tau(t,s) \in -\alpha(s)\sigma\left(\frac{\partial v}{\partial t}(t,s)\right). \quad (6)$$

Formally, the limit case of (3) gives the continuous behavior of the continuous Prandtl model: we obtain the following differential system: almost everywhere on $[0, T]$,

$$\dot{x}(t) = y(t), \quad (7a)$$

$$\dot{y}(t) = \frac{1}{m} \left(F(t) - k_0 x(t) - \int_0^1 k(s)u(t,s)(s) \right), \quad (7b)$$

almost everywhere on $[0, T] \times [0, 1]$

$$\frac{\partial u}{\partial t}(t,s) + \partial\psi_{[-\eta(s), \eta(s)]}(u(t,s)) \ni y(t), \quad (7c)$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \\ \text{and a.e. on } [0, 1] \quad u(0, s) = u_0(s) \in [-\eta(s), \eta(s)], \quad (7d)$$

where we set with respect to (4),

$$\forall s \in [0, 1], \quad \eta(s) = \frac{\alpha(s)}{k(s)}. \quad (8)$$

In section 3, we explain how to prove rigorously equations (7) like a limit case of (3).

2 EXISTENCE, UNIQUENESS AND REGULARITY RESULTS

The system (7) is a differential inclusion in the Hilbert space $\mathcal{H} = \mathbb{R} \times \mathbb{R} \times L^2(0, 1)$; so we can apply the theory of Brezis (Brezis 1973). Let $x_0, y_0 \in \mathbb{R}$, $m \in \mathbb{R}_+^*$ and $k_0 \in \mathbb{R}_+$. Define a function f from $[0, T] \times \mathcal{H}$ to \mathcal{H} by

$$f(t, (x, y, z)) = \begin{pmatrix} y \\ (F(t) - k_0 x - (k, z)) / m \\ y \otimes \mathbf{1} \end{pmatrix}, \quad (9)$$

With these notations, (7) is equivalent to

$$\dot{X} + \partial\psi(X) \ni f(\cdot, X), \quad (10)$$

$$X(0) = X_0, \quad (11)$$

where we set

$$X_0 = \begin{pmatrix} x_0 \\ y_0 \\ u_0 \end{pmatrix} \text{ and } X(t) = \begin{pmatrix} x(t) \\ y(t) \\ u(t, \cdot) \end{pmatrix}. \quad (12)$$

Observing that $\partial\psi$ is the sub-differential of the function convex ψ of \mathcal{H} defined by forall $x, y \in \mathbb{R}$, for all $z \in L^2(0, 1)$, for all $s \in [0, 1]$,

$$\psi(x, y, z(s)) = \{0\} \times \{0\} \times \Psi_{[-1, 1]} \left(\frac{z(s)}{\eta(s)} \right), \quad (13)$$

we can use the Brezis's theory on the differential inclusions (Brezis 1973): the system (10) and (11) possesses a unique solution. So, we obtain the

Proposition 2.1. *Let F essentially bounded and let u_0, η and k square integrable. We assume that, almost everywhere on $[0, 1]$,*

$$|u_0(s)| \leq \eta(s), \quad k(s) > 0, \quad \eta(s) > 0. \quad (14)$$

Then there exists a unique solution (x, y, u) of differential system (7). Moreover, the derivatives of x, y and u are square integrable.

According to the proposition 2.1, the derivative of the function u is square integrable. Moreover, with hypothesis of smoothness of the functions u_0 and η , we can give a regularity result on u vs its second variable.

Proposition 2.2. *Under the hypothesis of proposition 2.1, assume moreover that the derivatives of u_0 and η are essentially bounded and there exists $\gamma > 0$ such that*

$$\eta(s) \geq \gamma \text{ a.e. on } [0, 1]. \quad (15)$$

Let (x, y, u) be the unique solution of (7). Then the derivatives of u is essentially bounded.

The proof is based on the comparison between the two solutions u and \tilde{u} of the two following differential systems

$$\frac{\partial u}{\partial t}(t, s) + \partial\Psi_{[-1, 1]} \left(\frac{u(t, s)}{\eta(s)} \right) \ni y(t), \text{ a.e. on }]0, T[\times]0, 1[, \quad (16)$$

$$\frac{\partial \tilde{u}}{\partial t}(t, s) + \partial\Psi_{[-1, 1]} \left(\frac{\tilde{u}(t, s)}{\tilde{\eta}(s)} \right) \ni \tilde{y}(t), \text{ a.e. on }]0, T[\times]0, 1[, \quad (17)$$

and estimate of $u - \tilde{u}$ is based on a Gronwall's lemma (Bastien Lamarque schatzman 2001), which gives the following result: there exists C such that, for all $t \in [0, T]$, for all $h \in]0, 1]$,

$$\forall s \in [0, 1 - h], \quad |u(t, s) - u(t, s + h)| \leq Mh.$$

3 CONVERGENCE OF THE DISCRETE MODEL TO THE CONTINUOUS MODEL

Let $P \in \mathbb{N}^*$ and $\delta s = 1/P$. The discrete system (3) is equivalent to the continuous system (7) where the functions $u_{0, \delta s}$, $k_{\delta s}$ and $\eta_{\delta s}$ of $L^2(0, 1)$ are defined by

$$\forall i \in \{1, \dots, P\}, \quad \forall s \in [s_{i-1}, s_i], \quad u_{0, \delta s}(s) = u_0(s_i), \quad (18a)$$

$$k_{\delta s}(s) = k(s_i), \quad (18b)$$

$$\eta_{\delta s}(s) = \eta(s_i). \quad (18c)$$

Denote the function $u_{\delta s}$ from $[0, T] \times [0, 1]$ to \mathbb{R} defined by

$$\forall t \in [0, T], \quad \forall i \in \{1, \dots, P\}, \quad \forall s \in]s_{i-1}, s_i[, \\ u_{\delta s}(t, s) = u_{i, \delta s}(t), \quad (19)$$

we can remark that $(x_{\delta s}, y_{\delta s}, u_{\delta s})$ is the solution of the system (7) expressed with the functions $u_{0, \delta s}$, $k_{\delta s}$ and $\eta_{\delta s}$.

Using an estimate between the solutions of (16) and (17), we can prove

Proposition 3.1. *Let $\delta s = 1/P$, F essentially bounded on $[0, T]$ and let u_0 , η and k with derivatives essentially bounded which verifies (14) and (15). Then there exists a constant C such that, for all $P \in \mathbb{N}^*$,*

$$\max_{t \in [0, T]} |x(t) - x_{\delta s}(t)| \leq C \delta s, \quad (20)$$

$$\max_{t \in [0, T]} |y(t) - y_{\delta s}(t)| \leq C \delta s, \quad (21)$$

$$\max_{(t, s) \in [0, T] \times [0, 1]} |u(t, s) - u_{\delta s}(t, s)| \leq C \delta s. \quad (22)$$

The proof is based on comparison between solutions of two differential inclusions and a Gronwall's lemma. Thus, the solution $(x_{\delta s}, y_{\delta s}, u_{\delta s})$ of discrete Masing model is a good approximate of solution (x, y, u) of continuous Masing model when the number of degrees of freedom P of the discrete Masing model tends to infinity.

4 NUMERICAL SCHEME

From the property 3.1 we have an estimate of the difference between the solution (x, y, z) of the continuous Prandtl model and the solution $(x_{\delta s}, y_{\delta s}, u_{\delta s})$, of the discrete continuous Prandtl model. This system can be rewritten under the form

$$\dot{u} + \partial\phi(u) \ni f(t, u), \quad (23)$$

where u is a function from $[0, T]$ to \mathbb{R}^P , ϕ is a convex function from \mathbb{R}^P to \mathbb{R} and its sub-differential $\partial\phi$ is a multivalued maximal monotone graph from \mathbb{R}^P to \mathbb{R}^P (see (Brezis 1973)). In (Bastien 2000; Bastien Schatzman 1999; Bastien Schatzman 2000), we have proved that the Euler implicit numerical scheme defined by

$$\frac{U^{p+1} - U^p}{h} + \partial\phi(U^p) \ni f(ph, U^p), \quad (24)$$

(where $h = T/N$) converges to the solution of (23), with an error in $O(h)$, that is to say

$$\max_{0 \leq p \leq N} \|U^p - u(ph)\| \leq Ch. \quad (25)$$

We can approximate the solution $(x_{\delta s}, y_{\delta s}, u_{\delta s})$ of the discrete continuous Prandtl model by using numerical scheme (24). We have seen in proposition 3.1 that $(x_{\delta s}, y_{\delta s}, u_{\delta s})$ is an approximation of continuous solution (x, y, u) of differential system (7). Let $N, P \in \mathbb{N}^*$. We set $\delta s = 1/P$, $\delta t = T/N$ and we define the values $(x_{\delta s}^p)_{0 \leq p \leq P}$, $(y_{\delta s}^p)_{0 \leq p \leq P}$, $(u_{\delta s, i}^p)_{0 \leq p \leq P, 0 \leq i \leq N}$ by, for all $p \in \{0, \dots, N-1\}$,

$$x_{\delta s}^{p+1} = \delta t y_{\delta s}^p + x_{\delta s}^p, \quad (26a)$$

$$y_{\delta s}^{p+1} = \frac{\delta t}{m} \left(F(t_p) - k_0 x_{\delta s}^p - \sum_{i=1}^P \delta s k_i u_{\delta s, i}^p \right) + y_{\delta s}^p, \quad (26b)$$

$$\forall i \in \{1, \dots, P\}, \quad u_{\delta s, i}^{p+1} = \text{proj}_{[-\eta_i, \eta_i]} \left(\delta t y_{\delta s}^p + u_{\delta s, i}^p \right), \quad (26c)$$

and

$$x_{\delta s}^0 = x_0, \quad (26d)$$

$$y_{\delta s}^0 = y_0, \quad (26e)$$

$$\forall i \in \{1, \dots, P\}, \quad u_{\delta s, i}^0 = u_{0, i}. \quad (26f)$$

Here, $\text{proj}_{[-\eta_i, \eta_i]}$ is the projection on the interval $[-\eta_i, \eta_i]$. Let $x_{\delta s, \delta t}$, $y_{\delta s, \delta t}$ and $u_{\delta s, \delta t}$ be the functions of $C^0([0, T], \mathbb{R})$, $C^0([0, T], \mathbb{R})$ and $C^0([0, T], L^2(0, T))$, based on the linear interpolation of the values $(x_{\delta s}^p)_{0 \leq p \leq N}$, $(y_{\delta s}^p)_{0 \leq p \leq N}$, $(u_{\delta s, i}^p)_{0 \leq p \leq P, 0 \leq i \leq N}$. We can observe that numerical scheme (26) is the numerical scheme (24) applied to differential system (3) (with function $\eta_{\delta s}$, $k_{\delta s}$ and $u_{0, \delta s}$) written under the form (23). According of estimate (25), we can prove that the numerical scheme has order one with respect to δs and δt :

Proposition 4.1. *Under the hypothesis of proposition 3.1, then there exists a constant C such that, for all $\delta s, \delta t \in \mathbb{R}_+^*$,*

$$\max_{t \in [0, T]} |x(t) - x_{\delta s, \delta t}(t)| \leq C(\delta s + \delta t),$$

$$\max_{t \in [0, T]} |y(t) - y_{\delta s, \delta t}(t)| \leq C(\delta s + \delta t),$$

$$\max_{(t, s) \in [0, T] \times [0, 1]} |u(t, s) - u_{\delta s, \delta t}(t, s)| \leq C(\delta s + \delta t).$$

5 STUDY OF THE HYSTERESIS CYCLES

In (Bastien Schatzman Lamarque 2000; Bastien 2000), we studied the hysteresis cycles of the Prandtl model with a finite number of degrees of freedom. If we assume x increasing on $[0, T]$, we have proved that the data of the physical parameters of the Prandtl model permit us to find partially the hysteresis cycle, that is to say the curve $\{x(t), F(t) - m\ddot{x}(t)\}_{t \in [0, T]}$; this curve is convex, polygonal composed of $n + 1$ segment lines. Reciprocally, the data of one half hysteresis cycle permits us to identify the physical parameters of Prandtl model.

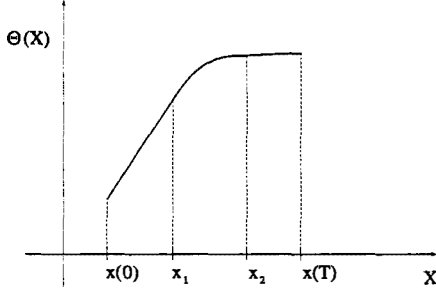


Figure 5. The half continuous hysteresis cycle.

We study again the hysteresis cycle for the continuous Prandtl model. If x is increasing on $[0, T]$, we can prove that the functions η and k permit us to find the cycle; this curve is the union of two segment lines and of a strictly concave part (see figure 5). Reciprocally, the data of one half hysteresis cycle permits us to find the physical parameters of Prandtl model. Nevertheless, this identification is not one-to-one: we prove that a family of functions (η, k) possesses the same hysteresis cycle. If the function η is arbitrarily fixed, then the shape of hysteresis cycle permits us to find the function k thanks to a differential equation between experimental curve Θ and the function k : for all $s \in [0, 1]$

$$2(\eta(1) - \eta(0))\Theta''(2(\eta(1) - \eta(0))s + 2\eta(0)) = -k(s). \quad (27)$$

This parameter identification permits us to identify an elastoplastic model with experimental data: we suppose that a curve $(x, f - m\ddot{x})$ is known, via an approximation with a finite set of values. With a small step of interpolation, we can obtain a continuous curve which interpolates this finite set of values and we can determine the function k with the differential equation (27) by choosing arbitrarily the function η .

6 NUMERICAL SIMULATION

The functions η , k and u_0 are defined by

$$\forall s \in [0, 1], \quad \eta(s) = s + 0.1, \quad (28)$$

$$k(s) = 1, \quad (29)$$

$$u_0(s) = 0. \quad (30)$$

The values of $(\eta_i)_i$, $(k_i)_i$ and $(u_{i,0})_i$ are defined by (18). We have chosen

$$T = 500, \quad k_0 = x_0 = \dot{x}_0 = 0, \quad F(t) = 0.45 \cos(0.5t). \quad (31)$$

We have plotted some hysteresis cycles $\{x(t), F(t) - m\ddot{x}(t)\}_{400 \leq t \leq 500}$ in the figures 6, 7, 8, 9 and 10 in appendix A. For this simulation, we have chosen $N = 10^6$; for the figures 6 and 7, we have chosen $P = 3$ and $P = 10$; in this figures, we can see polygonal cycles similar to those of the discrete Prandtl model (see figures 20 through 22 of (Bastien Schatzman Lamarque 2000)). For the figures 8, 9 and 10, where $P \in \{100, 700, 1500\}$, the obtained cycles seem to be the cycles of a continuous model. The polygonal shape disappears: we obtain a continuous smooth curve with two segments and a concave part.

CONCLUSION

By using the maximal monotone frame of (Bastien Schatzman Lamarque 2000; Bastien 2000), we can prove existence and uniqueness results for the continuous dynamical Masing model. Moreover, due to results of numerical analysis presented in (Bastien Schatzman 1999; Bastien Schatzman 2000; Bastien 2000), we provide an implicit Euler numerical scheme with order 1. By using this numerical scheme, we give cycle of hysteresis. Then we explain how to identify (partially) Masing model.

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Appendix A: Curves of numerical simulations

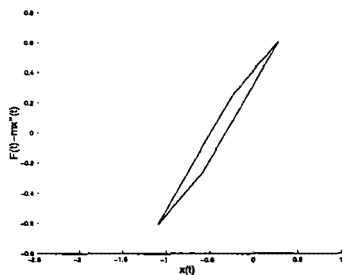


Figure 6. Curve $\{x(t), F(t) - m\ddot{x}(t)\}_{400 \leq t \leq 500}$ for $P = 3$

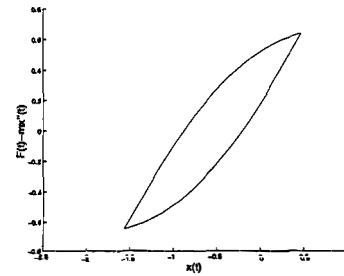


Figure 7. Curve $\{x(t), F(t) - m\ddot{x}(t)\}_{400 \leq t \leq 500}$ for $P = 10$

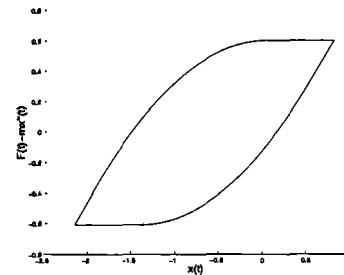


Figure 8. Curve $\{x(t), F(t) - m\ddot{x}(t)\}_{400 \leq t \leq 500}$ for $P = 100$

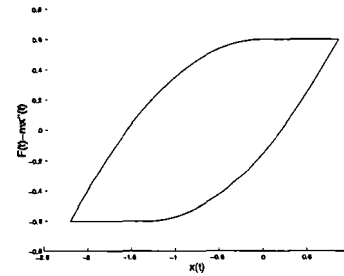


Figure 9. Curve $\{x(t), F(t) - m\ddot{x}(t)\}_{400 \leq t \leq 500}$ for $P = 700$

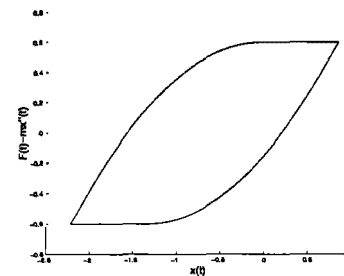


Figure 10. Curve $\{x(t), F(t) - m\ddot{x}(t)\}_{400 \leq t \leq 500}$ for $P = 1500$.