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# A WAVELET-LOGARITHMIC DECREMENT TO IDENTIFY DAMPING IN MDOF SYSTEMS : THEORY AND APPLICATIONS <sup>†</sup>

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## ABSTRACT

A wavelet-based formula similar to the logarithmic decrement formula is introduced to estimate damping in multi-degree-of-freedom systems from time-domain responses. The new formula is then validated upon both numerical and in-situ experiments.

## INTRODUCTION

mercredi 7-fvr.-2001 In the frame of applications, the study of continuous systems modeled by partial differential equations (PDE) with boundary and initial conditions for both displacements and stresses is performed using a discretization of PDE, via a Rayleigh-Ritz procedure for instance. So, systems with  $N$  degree-of-freedom may be obtained, governed by equations

$$\mathbf{M} \ddot{\mathbf{X}} + \mathbf{C} \dot{\mathbf{X}} + \mathbf{K} \mathbf{X} = \mathbf{f}(t) \quad (1)$$

where  $\mathbf{M}$ ,  $\mathbf{C}$ ,  $\mathbf{K}$  respectively denote a mass, a damping and a stiffness matrix;  $\mathbf{f}(t)$  stands for the external forcing: let us assume that it does not depend on displacement vector  $\mathbf{X}$  and velocity vector  $\dot{\mathbf{X}}$ . The term  $\mathbf{C}\dot{\mathbf{X}}$  corresponds to a mathematical expression of viscous damping proportional to velocity. In the modal basis, one is lead to study the dynamic response of  $N$  uncoupled

single-degree-of-freedom oscillators governed by equations:

$$\ddot{q}_j + 2\xi_j \Omega_j \dot{q}_j + \Omega_j^2 q_j = f_j(t) \quad 1 \leq j \leq N \quad (2)$$

As the experimental setup provides signals recorded in the physical basis written as a linear combination of eigenmodes  $(q_j)_{1 \leq j \leq N}$  and given by

$$X_i(t) = \sum_{j=1}^N A_{ij} \cdot q_j(t) \quad (3)$$

the main stake is also to analyze damped vibrations such as

$$f(t) = f_{\omega}(t) + \sum_{i=1}^{N-1} f_{\omega_i}(t) \quad (4)$$

with

$$\begin{cases} f_{\omega}(t) = A e^{-ct} \cos(\omega t + \varphi) \cdot H_y(t) \\ f_{\omega_i}(t) = A_i e^{-c_i t} \cos(\omega_i t + \varphi_i) \cdot H_y(t) \end{cases} \quad (5)$$

where  $A$  and  $A_i = \alpha_i A$  denote the amplitudes,  $c$  and  $c_i = \delta_i c$  denote the viscous dampings,  $\omega$  and  $\omega_i = \omega + \varepsilon_i$  denote a sorted

<sup>†</sup>Some of the material in this paper was presented in references (Lamarque *et al.*, 2000; Hans *et al.*, 2000).

sequence of natural frequencies,  $\varphi$  and  $\varphi_i$  stand for the phases of the fundamental mode  $f_\omega$  and the  $i^{\text{th}}$  harmonic response  $f_{\omega_i}$ ,  $H_y$  referring to the Heaviside function.

Last decade, numerous applications in mechanics have been extensively using wavelets. According to damping identification, Ruzzene and al (Ruzzene *et al.*, 1997) and Staszewski (Staszewski, 1997) performed a continuous wavelet analysis of the free response of dynamic systems to extract modal features: they separated modal contributions thanks to the rapidly decreasing properties of the progressive Morlet wavelet. In (Staszewski, 1997), Staszewski also introduced a wavelet reconstruction formula to approximate the impulse response of MDOF systems. The main purpose of this paper is to analyze a damped signal recorded in the physical basis with wavelet patterns optimally localized both in time and in frequency domains and to extract 'local' information to estimate the damping ratio. As a result, a wavelet logarithmic decrement formula is introduced.

The paper is organized as follows: In section 1, we investigate the determination of viscous damping by using a continuous wavelet transform. In section 2, a wavelet logarithmic decrement formula is introduced in the frame of a superabundant multiresolution analysis well adapted to analyze sampled signals. In section 3, numerical experiments are then proceeded to test reliability and noise resistance of the wavelet based formula using the dynamic response of exact single-degree-of-freedom systems perturbed with a Gaussian noise. These simulations are achieved by comparison with a pre-filtered logarithmic decrement procedure. As practical applications involve multiple degree-of-freedom systems, the wavelet logarithmic formula is finally used in section 4 in the study of a civil engineering building to extract damping estimates of the first and second modes from in-situ shock test responses. Finally, a few conclusions and future prospective works are outlined.

## 1 CONTINUOUS WAVELET TRANSFORM – DAMPING ESTIMATE

Let's consider the continuous wavelet transform of a signal  $f$  defined by

$$W_f^\phi(b, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(t) \overline{\phi\left(\frac{t-b}{a}\right)} dt \quad \forall (b, a) \in \mathbb{R} \times \mathbb{R}^+ \quad (6)$$

bound to the scaling function  $\phi \in L^2(\mathbb{R})$  whose support is completely included in  $\mathbb{R}$  and checking some oscillation properties (Daubechies, 1988, p. 909-910)

$$C_\phi = \int_{-\infty}^{+\infty} \frac{|\hat{\phi}(\omega)|^2}{|\omega|} d\omega < +\infty \quad (7)$$

Integrating by part the wavelet transform of signal  $f$  defined in (4), a damping estimate may be exhibited when equating

$$c = \frac{1}{(m-n)T} \ln \left| \frac{W_f^\phi(nT, a)}{W_f^\phi(mT, a)} \right| + \sum_{i=1}^{N-1} \alpha_i \left| \frac{e^{(1-\delta_i)mT} c - e^{(1-\delta_i)nT} c}{(m-n)T} \right| O\left(\frac{1}{\inf_{1 \leq i < N} (\epsilon_i a)}\right) \quad (8)$$

where  $T = \frac{2\pi}{\omega}$  is the fundamental pseudo-period of signal  $f$ . This damping estimate may be simplified as soon as the analyzing scale  $a$  accurately separates the natural frequencies, given by the following admissibility conditions :

$$\begin{cases} a\omega = O(1) & \rightarrow \text{Focus on mode } \omega \\ \inf_{1 \leq i < N} (\epsilon_i a) \gg 1 & \rightarrow \text{Filtering of modes } \omega_i \end{cases} \quad (9)$$

Assuming the analyzing wavelet  $\phi$  is well localized both in time and frequency domains, a relationship between the dilation parameter  $a$  and the corresponding frequency  $f$  which is focused by the wavelet transform may be given by  $a = \frac{f_\phi}{f}$ ,  $f_\phi$  denoting the frequency that matches the maximum value of the Fourier transform of  $\phi$ .

The procedure can straightforwardly be generalized to extract the damping  $c_i$  attached to the  $i^{\text{th}}$  mode of frequency  $\omega_i$  as soon as it is applied to successive filtered components of signal  $f$  cancelling the fundamental and the first  $(i-1)$  harmonic modes (low-pass filtering):

$$c_i \simeq \frac{1}{(m-n)T_i} \ln \left| \frac{W_{f-f_\omega-\sum_{j=1}^{i-1} f_{\omega_j}}^\phi(nT_i, a)}{W_{f-f_\omega-\sum_{j=1}^{i-1} f_{\omega_j}}^\phi(mT_i, a)} \right| \quad (10)$$

where  $T_i = \frac{2\pi}{\omega_i}$  is the pseudo-period of the  $i^{\text{th}}$  frequency mode.

## 2 WAVELET FORMULA IN THE FRAME OF A SUPER-ABUNDANT ANALYSIS

For numerical purposes, a discrete wavelet decrement formula is derived from its continuous counterpart (8) in the frame of a superabundant analysis (Mallat, 1989; Perrier, 1991). Considering a periodic Spline multiresolution analysis of  $L^2(\frac{\mathbb{R}}{2^J})$  (Chui, 1992; Lemarié, 1988) bound to scaling function  $\phi$ , we define an intermediate scaling function by :

$$\Phi_{\alpha k}(x) = \sqrt{\alpha} \sum_{r \in \mathbb{Z}} \phi(\alpha(x+r - \frac{k}{2^J})), \quad k \in [0, 2^J - 1] \quad (11)$$

where  $\alpha = \frac{2^{J-1}}{m}$  with  $1 \leq m \leq 2^{J-1}$ ,  $J$  standing for the finer resolution scale of the multiresolution analysis.

As reliability of formula (8) strongly depends on an efficient detection of successive coefficient extrema, it is common to use an 'oblique' analysis which associates  $2^J$  coefficients to  $2^J$  sampling points for each scale  $1 \leq \alpha \leq 2^{J-1}$ .

Introducing intermediate scaling coefficients  $O_{\alpha,k}(f)$  given by the inner products

$$O_{\alpha k}(f) = \langle f, \Phi_{\alpha,k} \rangle \quad (12)$$

signal  $f$  may be approximated by the oblique set of scaling patterns

$$f \simeq f_{\alpha} = \sum_{k=0}^{2^J-1} O_{\alpha,k}(f) \Phi_{\alpha,k} \quad (13)$$

An extension of formula (8) in the superabundant context may be depicted by

$$c \simeq \frac{2^J}{k-l} \ln \left| \frac{O_{\alpha,l}(f)}{O_{\alpha,k}(f)} \right| \quad (14)$$

involving coefficients  $(O_{\alpha,k}(f))_{0 \leq k < 2^J}$  of signal  $f$ . In this context, the dilation parameter  $a = \alpha$  is set to both capture the fundamental mode  $f_{\omega}$  and to filter the higher frequency modes  $(f_{\omega_i})_{1 \leq i < N}$  and eventually additional noisy perturbations in the wavelet pyramidal decomposition of  $f$ . Moreover, shifting parameters are chosen so that  $b_k = k/2^J$  and  $b_l = l/2^J$  point towards local extrema of coefficients  $(O_{\alpha,k}(f))_k$ .

### 3 DAMPING IDENTIFICATION OF A SINGLE DEGREE-OF-FREEDOM SYSTEM

The aim being to illustrate the efficiency and the robustness of the wavelet decrement formula, numerical simulations are proceeded to identify damping of a class of single degree-of-freedom systems whose response is perturbed by a random noise given by :

$$x(t) = X e^{-ct} \cos(\omega t) + B(t) \quad (15)$$

with amplitude and frequency set to  $X = 1$  and  $\omega = 10\pi \text{ rad.s}^{-1}$ ,  $B(t)$  standing for a Gaussian noise of null mean and of standard deviation  $B$ . The experimental set-up consists in first perturbing damped oscillations characterized by a given damping ratio  $c$  ranging between 0.1% and 100% with a noise of standard deviation  $B \in [0, 50]$  (%). Super-abundant scaling coefficients

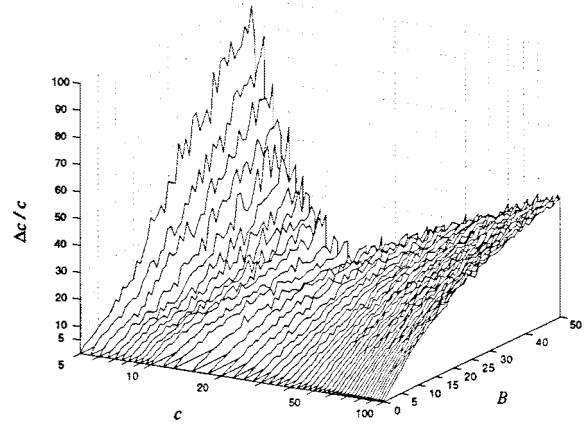


Figure 1. Standard logarithmic decrement with wavelet pre-filtering : Relative error distribution  $\frac{\Delta c}{c}$  (%) in terms of the damping ratio  $c$  (%) and the noise level  $B$  (%).

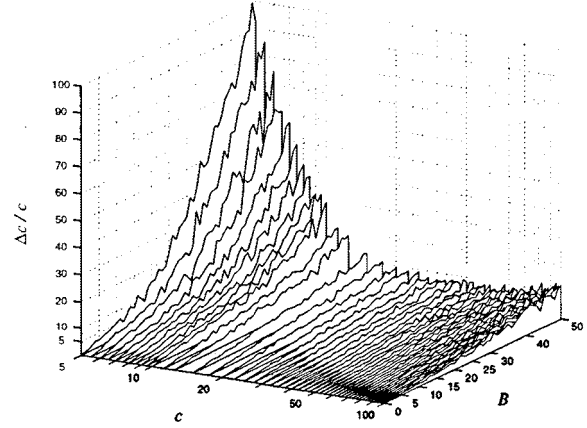


Figure 2. Wavelet-based logarithmic decrement (14): Relative error distribution  $\frac{\Delta c}{c}$  (%) in terms of the damping ratio  $c$  (%) and the noise level  $B$  (%).

$(O_{\alpha,k}(x))_k$  of signal  $x(t)$  are then computed for a convenient scale  $\alpha$ . After having automatically recognized local coefficients extrema, the wavelet decrement formula (14) is used to provide estimates of the damping ratio  $c$ . This scheme is repeated according to several random processes and for each set of data.

Finally, a relative error map is displayed on Figure 2 and compared with the one computed with a standard logarithmic decrement technique associated with a wavelet pre-filtering (see

Figure 1). Generally speaking, Figure 2 reveals a good behavior of the wavelet decrement formula : the mean relative error is indeed very often below the threshold of 10% according to  $c \in [10\%, 100\%]$ , with a linear growth in terms of the noise level. Wavelet-based formula's performance highlights an excellent noise resistance of the so-called identification procedure, sensitively improving estimates for  $c$  ranging in large damping area ( $c \in [20\%, 100\%]$ ) as well as in medium damping area ( $c \in [5\%, 20\%]$ ).

As physical systems are often characterized by damping of weak and even very weak amplitude, a similar study was lead in the range of  $c \in [0.1\%, 5\%]$ . Few damping estimates are gathered in Table 1 and attest the wavelet analysis is not only a wavelet filtering of the damped signal. As for the estimation of very small damping ( $c \simeq 0.1\%$ ), Table 1 shows formula (14) is greatly improving the forecasts compared to the ones computed with its competitive methods whose estimations literally explode. Generally speaking, the experimenter may consider the wavelet logarithmic decrement is bringing a sensitive improvement with forecasts' errors roughly less than 20% off the 'ultimate' damping value, while at the same time its classical counterpart give unrealistic values easily reaching levels of 150%.

| $c$ (%) | $b$ (%) | Logarithmic Decrement<br>(Wavelet pre-filtering) |        | Wavelet Decrement<br>(Superabundant analysis) |        |
|---------|---------|--|--------|---|--------|
|         |         | $\frac{\Delta c}{c}$ (%)                         | $c$    | $\frac{\Delta c}{c}$ (%)                      | $c$    |
| 0.1     | 0.0     | 151  | 0.0025 | 0   | 0.0010 |
|         | 0.1     | 160  | 0.0026 | 3.5   | 0.0010 |
|         | 0.5     | 191  | 0.0029 | 22  | 0.0012 |
| 1       | 0       | 0  | 0.010  | 0   | 0.010  |
|         | 1       | 6  | 0.010  | 2.5   | 0.0010 |
|         | 2       | 11   | 0.010  | 4   | 0.010  |
|         | 5       | 73   | 0.017  | 19.5  | 0.011  |
| 100     | 0       | 0.1  | 1.001  | 0   | 0.999  |
|         | 10      | 23   | 1.229  | 1.5   | 1.001  |
|         | 20      | 35   | 1.344  | 2.7   | 1.005  |
|         | 50      | 19.5   | 1.084  | 10.6  | 1.093  |

Table 1. Comparison of performances of the wavelet decrement formula versus the standard logarithmic decrement with wavelet pre-filtering.

#### 4 DAMPING IDENTIFICATION OF A CIVIL ENGINEERING BUILDING

The topic is here to test the reliability of the wavelet decrement formula used to identify non-dimensional damping ratios of



Figure 3. The tested eight-levels building (1970). Height: 22 m. Length: 30 m. Width: 14 m.

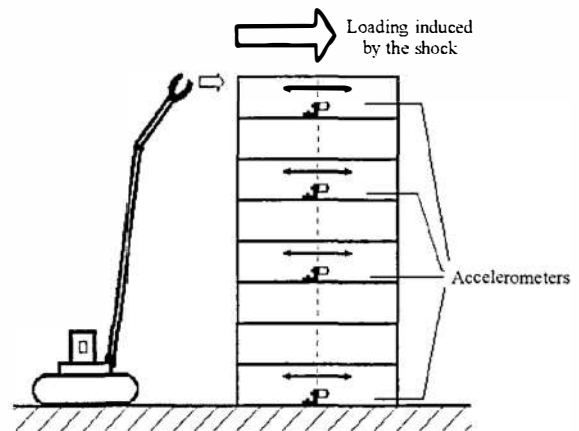


Figure 4. Experimental set up of the shock test.

modal responses of a regular eight-story civil engineering building made in 1970 and depicted in Figure 3.

The wavelet-based procedure is directly applied on in-situ shock test responses of the building recorded in the transverse direction. The testing procedure consists in using the pliers of a mechanical shovel to create impacts at the top level of the structure in the longitudinal direction. Vibrations are simultaneously recorded by four accelerometers located in the middle of the 1<sup>st</sup>, 4<sup>th</sup>, 6<sup>th</sup> and 8<sup>th</sup> levels of the building as illustrated in Figure 4. Typical time and frequency responses of the 8<sup>th</sup> floor displacement in the transverse direction are displayed in Figure 5 ac-

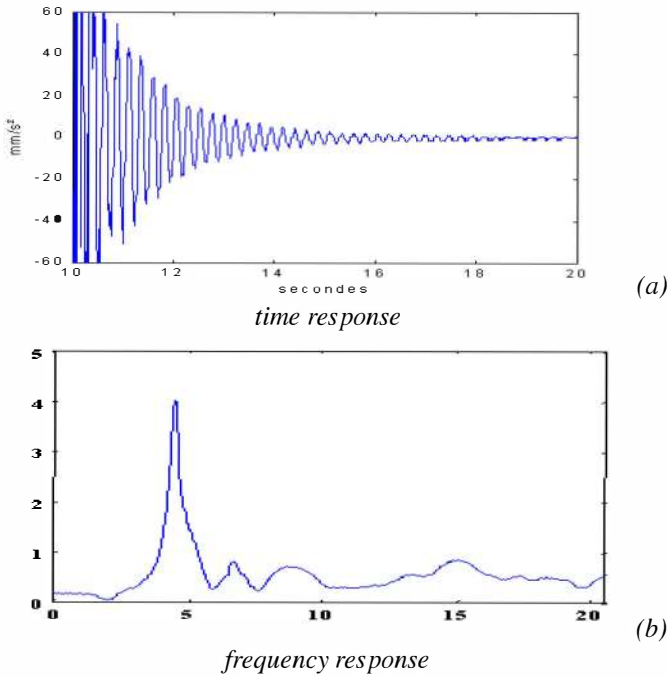


Figure 5. Shock test response of the top level in the transverse direction.

cording to a maximum acceleration amplitude of  $10^{-2}$  g. Spectral response highlights a linear behavior of the structure exhibiting two bending modes excited at respectively  $f_1 = 4.5$  Hz and  $f_2 = 15.7$  Hz. Hence, the dynamic response recorded in the physical basis is mainly related to a superposition of these two eigenfrequency modes eventually including additional noisy components due to the experimental setup.

Updating the wavelet decrement formula (14) to extract the non-dimensional damping  $\xi_i$  related to the  $i^{\text{th}}$  component response  $q_i$  in the modal basis, one obtain the following formula :

$$\xi_i = \frac{\delta_i}{\sqrt{4\pi^2 + \delta_i^2}} \text{ with } \delta_i \simeq \ln \left| \frac{O_{\alpha,1}(f-f_{\omega} - \sum_{j=1}^{i-1} f_{\omega_j})}{O_{\alpha,k}(f-f_{\omega} - \sum_{j=1}^{i-1} f_{\omega_j})} \right| \quad (16)$$

As the key point is to choose an accurate analyzing scale  $\alpha$  that will permit to isolate eigenfrequency modes in the signal response, a superabundant wavelet decomposition map displayed in Figure 6 is used to identify which detail channels are bound to the different eigenfrequency modes. If one is interested in identifying the damping using a standard superabundant analysis, formula (16) is applied with an accurate analyzing scale  $\alpha$  to identify damping related to the first mode. The procedure is successively repeated with the higher modes by cutting-off channels

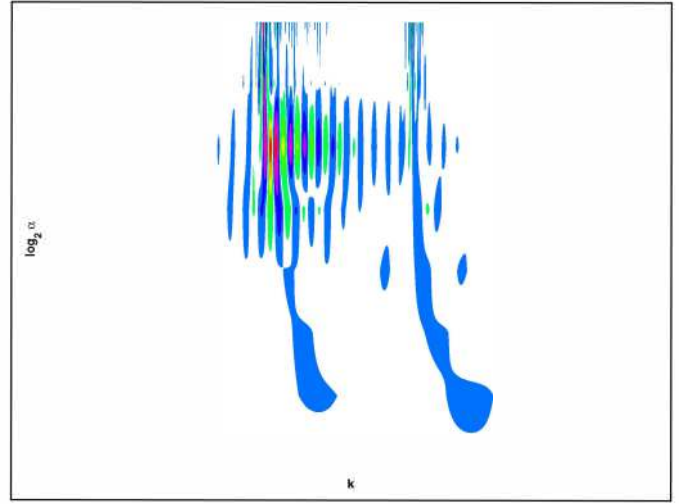


Figure 6. Superabundant wavelet decomposition of the 8<sup>th</sup> level shock time-response in the transverse direction.

related to the previous modes. Here it is noticeable on Figure 6 that the first mode  $f_1 = 4.5$  Hz is related to channels  $2^8 \leq \alpha \leq 2^9$  whereas the first harmonic  $f_2 = 15.7$  Hz is bound to channels  $\alpha \simeq 2^{10}$ . A filtered counterpart may be implemented selecting specific channels to re-build modal contributions as depicted in Figure 7. These graphics are also useful to set the time-window parameters required by the wavelet decrement formula to locate the coefficients extrema.

Damping estimates of the first and second eigenfrequency modes computed using either the wavelet formula or a logarithmic decrement associated with a wavelet pre-filtering are gathered in Table 2. Damping estimates produced from experimental data are in good agreement with magnitude orders that remain stable. Generally speaking, simulations demonstrate damping estimates sensitively depends on the acceleration level.

## CONCLUSION

This study allowed us to build a new procedure to identify the damping ratio of dynamic systems leaning on a wavelet analysis of the time-response. The corresponding formula which is based upon rigorous mathematical developments has been validated first analytically in the context of a general continuous wavelet transform and then numerically by comparison with exact reference solutions in the frame of a superabundant multiresolution analysis. Simulations using in-situ time-responses of a civil engineering building loaded under shock excitations demonstrate the reliability of the proposed method. Using standard identification techniques, damping estimates are often perturbed by important errors related to multi-modal contributions,



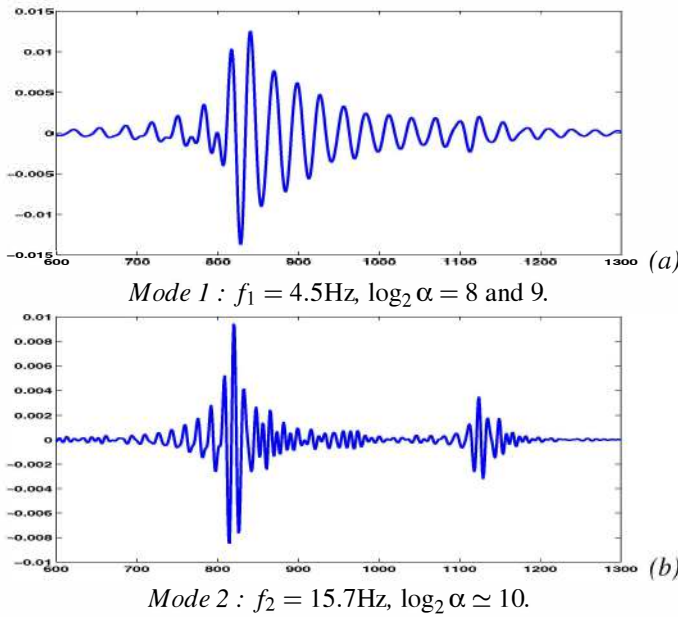


Figure 7. Filtered modal responses to shock test in the transverse direction.

| $\xi$                          |        | Mode 1 : $f_1 = 4.5\text{Hz}$  |        |        |                                     |  |
|--------------------------------|--------|--------------------------------|--------|--------|-------------------------------------|--|
| Method                         | Lev. 8 | Lev. 6                         | Lev. 4 | Lev. 1 | Mean $\pm$ dev.                     |  |
| Filtered Logarithmic Decrement | 0.033  | 0.036                          | 0.037  | 0.039  | <b>0.036 <math>\pm</math> 0.001</b> |  |
| Wavelet Decrement              | 0.033  | 0.035                          | 0.036  | 0.038  | <b>0.036 <math>\pm</math> 0.001</b> |  |
|                                |        | Mode 2 : $f_2 = 15.7\text{Hz}$ |        |        |                                     |  |
| Filtered Logarithmic Decrement | 0.063  | 0.055                          | 0.046  | 0.057  | <b>0.057 <math>\pm</math> 0.003</b> |  |
| Wavelet Decrement              | 0.062  | 0.035                          | 0.039  | 0.033  | <b>0.038 <math>\pm</math> 0.006</b> |  |

Table 2. Estimates of the non-dimensional damping ratio of eigenfrequency modes 1 and 2.

nonlinear effects or excitation defaults for instances: here, damping estimates computed with the Wavelet-Logarithmic decrement are accurate with magnitude orders that remain stable. Future prospects will now focus on the wavelet analysis of dynamic systems using a frequency domain approach.

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