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# First Return Time near grazing linear modes of discrete vibro-impact systems

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## Abstract

The paper deals with the dynamics of conservative  $N$ -degree-of-freedom vibro-impact systems involving one unilateral contact condition and a linear free flow. Among all possible trajectories, grazing linear modes (GLM) exhibit a contact occurrence with a vanishing incoming velocity which generates mathematical difficulties. We show that the First Return Time (FRT) to the chosen Poincaré section of the orbits which are sufficiently close to a GLM is always near a multiple of the fundamental period of the mode. This yields an infinite partition of the neighborhood of the GLMs on the Poincaré section along with square root singularities of the FRT depending on each set of the partition.

**Keywords:** Unilateral contact, Grazing Linear Mode, First Return Time, square root singularity

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## 1 Introduction

In the area of discrete unilateral contact dynamics, it is well known since the works by Nordmark [7, 11, 10] that singular behaviors occur near grazing orbits. The stability analysis of nonsmooth modes of vibration relies on a fine understand of the dynamics in the vicinity of grazing linear modes (GLM) since the so-called *backbone curves* emerge from GLMs [9, 14]. In the present

work, the dynamics of a conservative mechanical system with  $N$  degrees-of-freedom (dof), one of which being unilaterally constrained, is of interest, see Figure 1. The governing equations read

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r} & (1.1a) \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 & (1.1b) \\ u_N(t) \leq d, \quad R(t) \leq 0, \quad (u_N(t) - d)R(t) = 0 & (1.1c) \\ \dot{\mathbf{u}}^+(t)^\top \mathbf{M}\dot{\mathbf{u}}^+(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) = 2\mathbf{E}(\mathbf{u}(t), \dot{\mathbf{u}}^+(t)) = 2\mathbf{E}(\mathbf{u}(0), \dot{\mathbf{u}}(0)) & (1.1d) \end{cases}$$

with

$$\mathbf{M} = \text{diag}(m_j)_{j=1}^N; \quad \mathbf{K} = (k_{ij})_{i,j=1}^N; \quad \mathbf{u}(t) = (u_j)_{j=1}^N; \quad \mathbf{r}(t) = (\mathbf{0}, R(t))$$

where  $\dot{u}_j$  and  $\ddot{u}_j$  represent the velocity and acceleration of mass  $j$ ,  $j = 1, \dots, N$ , respectively. Equation (1.1a) says that mass  $N$  is constrained on the right side by a rigid obstacle at a distance



Figure 1: A unilaterally constrained  $N$ -degree-of-freedom chain with  $d > 0$ .

$d > 0$  from its equilibrium. The other masses are not assumed to be constrained in any way. The scalar quantity  $R(t)$  is the reaction force induced by the obstacle on mass  $N$  at the time of gap closure. Generally,  $R(t)$  is a measure [2]. However, it is a Lipschitz function which is as regular as  $\ddot{u}_N$  for solutions with sticking phases [13].

Matrices  $\mathbf{M}$  and  $\mathbf{K}$  are symmetric positive definite and define properly the non-negative energy  $\mathbf{E}$ . Hence, there exists a matrix  $\mathbf{P}$  of  $\mathbf{M}$ -orthogonal eigenmodes which diagonalizes both  $\mathbf{M}$  and  $\mathbf{K}$ , that is  $\mathbf{P}^\top \mathbf{M} \mathbf{P} = \mathbf{I}$  and  $\mathbf{P}^\top \mathbf{K} \mathbf{P} = \mathbf{\Omega}^2 = \text{diag}(\omega_j^2)_{j=1, \dots, N}$  where  $\mathbf{I}$  is the  $N \times N$  identity matrix;  $\omega_j^2$  are the eigenfrequencies and  $T_j$ , the linear periods with  $\omega_j T_j = 2\pi$ ,  $j = 1, \dots, N$ .

We denote  $\mathbf{U} = (u_1, \dots, u_N)$  and  $\mathbf{V} = \dot{\mathbf{U}}$ , the vectors of displacements and velocities, respectively, of the  $N$ -mass system.

The present paper is devoted to the investigation of the dynamics of such systems near GLM. For this purpose, the First Return Time (FRT) to the Poincaré section is studied. Under generic assumptions on the system, a stability result on the FRT and the instability of the GLMs is proved.

## 2 Main results on the First Return Time

The main results of the paper are now outlined. First, the considered assumptions and notations are stated. The definitions of the chosen Poincaré section  $\mathcal{H}_\varphi$  along with the corresponding FRT are explained, from which the number of loops of orbits near GLM can be mathematically defined. This is the first main result of the paper.

Then, any sufficiently small neighborhood  $\mathcal{V}^\varepsilon \in \mathcal{H}_\varphi$  of the contact point of the GLM is partitioned into non-overlapping subsets  $\mathcal{V}_n^\varepsilon$  corresponding to the orbits with exactly  $n$  loops. Such sets  $\mathcal{V}_n^\varepsilon$  are carefully investigated. In particular,  $\mathcal{V}_0^\varepsilon$  is always empty for  $\mathcal{V}^\varepsilon$  small enough. In contrast,  $\mathcal{V}_1^\varepsilon$  is never empty and more generally infinitely many  $\mathcal{V}_n^\varepsilon$  are never empty. For a specified GLM, the FRT on each  $\mathcal{V}_n^\varepsilon$  enjoys a common square root singularity near the multiple of the fundamental linear period.

**Assumptions** Throughout the paper, the following framework is considered.

1. There is no internal resonance in the system.

2. GLM always exist, that is  $P_{Nk} \neq 0, \forall k$ .

Mathematically, the first assumption means that the frequencies  $\omega_1, \dots, \omega_N$  are  $\mathbb{Z}$  independent. The second assumption is very general and already discussed in [9]. It always holds for a classical spring-mass chain.

**Physical and modal coordinates** The problem is stated in the physical coordinates  $W = (U, V) = (u_1, \dots, u_N, v_1, \dots, v_N)$  but most proofs in the paper use modal coordinates. For a fixed time, says  $t = 0$  mostly used in the paper, there is a linear change of variables between physical coordinates and modal coordinates. The classical modal coordinates are denoted by  $(\alpha, \beta) = (\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N)$  with the relationship

$$U(t) = \sum_{k=1}^N (\alpha_k \cos \omega_k t + \beta_k \sin \omega_k t) P_{.k}, \quad (2.1)$$

where  $P_{.k}$  is column  $k$  of  $P$ , that is, the eigenvector corresponding to  $\omega_k$ . The modal coordinates restricted to the last mass are  $(a, b) = (a_1, \dots, a_N, b_1, \dots, b_N)$ ,

$$u_N(t) = \sum_{k=1}^N a_k \cos \omega_k t + b_k \sin \omega_k t, \quad a_k = \alpha_k P_{Nk}, \quad b_k = \beta_k P_{Nk}. \quad (2.2)$$

The relation with these two modal coordinates are one-to-one since the last component of the eigenvectors is  $P_{Nk} \neq 0$ . This is a linear diagonal change of variables between the classical modal coordinates and the modal coordinates of the last mass. In the paper, due to the importance of the last mass, the modal coordinates of the last mass are essentially used and simply called the *modal coordinates*.

## 2.1 Non-transverse Poincaré section $\mathcal{H}_\varphi$

In this paper, many proofs are based on a precise analysis of the linear dynamics. Let us define the contact hyperplane  $\mathcal{H} = \{(U, V) \in \mathbb{R}^{2N}, u_N = d\}$ , its “exit” subset  $\mathcal{H}^- = \{(U, V) \in \mathcal{H}, v_N < 0\}$  and its “entrance” subset  $\mathcal{H}^+ = \{(U, V) \in \mathcal{H}, v_N > 0\}$ . A natural choice for the Poincaré section is at least  $\mathcal{H}^-$  since we want to focus our attention on orbits instantaneously leaving  $\mathcal{H}$  and coming back to  $\mathcal{H}$ . The set  $\mathcal{H}^+$  corresponds to orbits reaching  $u_N = d$  and going to  $\mathcal{H}^-$  through the Newton impact law  $v_N^- = -v_N^+$ .<sup>1</sup>

The state corresponding to a grazing contact with vanishing in-coming velocity  $v_N^- = 0$  is challenging. A proposition in [12] states that  $\mathcal{H}^0 = \{(U, V) \in \mathcal{H}, v_N = 0\} = \mathcal{H}_1^0 \cup \mathcal{H}_\infty^0$  where  $\mathcal{H}_\infty^0$  is the set of states in  $\mathcal{H}$  generating orbits which come back infinitely many times to  $\mathcal{H}$  and  $\mathcal{H}_1^0$  generating orbits which reach  $\mathcal{H}$  only once for all time in the future and in the past. In other words,  $\mathcal{H}_1^0$  is a set where the FRT cannot be defined. As a consequence,  $\mathcal{H}_1^0$  must be excluded from the Poincaré section, which is thus chosen to be

$$\mathcal{H}_\varphi = \mathcal{H}^- \cup \mathcal{H}_\infty^0. \quad (2.3)$$

The present paper describes the set  $\mathcal{H}_1^0$  more precisely than [12]. It is the convex hull of the points in  $\mathcal{H}^0$  associated to all GLMs of the system which is never empty. This is an important feature not outlined in [12].

---

<sup>1</sup>Notice that the notations are slightly different than in [12]. Here, the notations chosen for  $\mathcal{H}^\pm$  are given by the sign of the velocity of the last mass.

**Proposition 2.1 — Characterization of trajectories with only one grazing contact at all time (set  $\mathcal{H}_1^0$ ).** Assume that at  $t = 0$ ,  $u_N(0) = d$ . Then, the trajectory  $\mathbf{W}(t)$  has one grazing contact for all time if and only if the displacement of mass  $N$  reads

$$u_N(t) = \sum_{k=1}^N a_k \cos \omega_k t, \quad \text{with } a_k \geq 0 \text{ and } \sum_{k=1}^N a_k = d. \quad (2.4)$$

Consequently, the set  $\mathcal{H}_1^0$  is the convex hull of all the contact points in  $\mathcal{H}^0$  corresponding to the  $N$  GLMs, vertices excluded. In particular  $\mathcal{H}_1^0$  is a convex bounded and non-empty set with dimension  $N - 1$ .

This characterisation of solutions with only one contact for all time (one contact at  $t = 0$  and no contact in the future  $t > 0$  and the past  $t < 0$ ) is not indicated in [12]. It appears that there are simply convex combinations of all the GLMs touching the wall at  $t = 0$ . Note that  $\mathcal{H}_1^0$  is expressed in modal coordinates, where the modal coordinates are non-negative and  $b_k = 0$ ,  $k = 1, \dots, N$ . The proof is provided in Appendix A.

The set  $\mathcal{H}_1^0$  plays an important role in the dynamics near GLMs as shown in the remainder. In some sense, this set induces a quite complicated dynamics near GLMs. As shown later,  $\mathcal{V}_\infty^\varepsilon \subset \mathcal{H}_1^0$ .

## 2.2 First Return Time and loops

In order to study the FRT function in the vicinity of a GLM, it is convenient to use the Euclidean distance associated to the total energy of the system

$$E(\mathbf{W}) = E(\mathbf{U}, \mathbf{V}) = \frac{1}{2}(\mathbf{V}^\top \mathbf{M} \mathbf{V} + \mathbf{U}^\top \mathbf{K} \mathbf{U}) = E(\mathbf{W}(0)), \quad (2.5)$$

which is conserved during the free dynamics. The FRT is known to be a discontinuous function [12]. However, the neutral stability of the free linear dynamics yields a kind of continuous dependence of the FRT with respect to the state on  $\mathcal{H}_\varphi$ . For a sufficiently small perturbation  $\varepsilon$  of a GLM, we prove in the present paper, that

$$d(\text{FRT}(\mathbf{W}_0), T_j \mathbb{N}) := \min_{n \in \mathbb{N}} |\text{FRT}(\mathbf{W}_0) - nT_j| = \mathcal{O}(\sqrt{\varepsilon}) \quad (2.6)$$

where  $\varepsilon$  is defined in Theorem 2.2 and where  $\mathbf{W}_0$  is the initial state on the Poincaré section. The square root dependence of the FRT is directly related to the square root singularity [11, 10]. The GLM  $j$  for  $j \in \{1, \dots, N\}$  is fixed and the result is given for that chosen GLM.

**Theorem 2.2 — FRT near multiples of the fundamental period.** Let  $\mathbf{W}_0^G \in \mathcal{H}_\varphi$  be the state corresponding to the  $j^{\text{th}}$  grazing linear mode and  $T_j$ , the corresponding fundamental period. Then, there exists  $\varepsilon_j > 0$  and a positive constant  $C_j$  such that for all  $0 < \varepsilon < \varepsilon_j$ , there is a neighborhood

$$\mathcal{V}^\varepsilon = \{\mathbf{W}_0 \in \mathcal{H}_\varphi \mid E(\mathbf{W}_0 - \mathbf{W}_0^G) < \varepsilon^2\} \quad (2.7)$$

such that for each  $\mathbf{W}_0 \in \mathcal{V}^\varepsilon$ , there is a unique  $n \geq 1$ ,  $n \in \mathbb{N}$

$$|\text{FRT}(\mathbf{W}_0) - nT_j| \leq C_j (E(\mathbf{W}_0 - \mathbf{W}_0^G))^{1/4} < C_j \sqrt{\varepsilon} < T_j/2 \quad (2.8)$$

where  $\text{FRT}(\mathbf{W}_0)$  is the First Return Time to the Poincaré section of the orbit associated to  $\mathbf{W}_0$ .

The upper bound  $T_j/2$  ensures that  $n$ , the number of loops, is uniquely defined, see Definition 2.1. The parameter  $\varepsilon$  represents the size of the perturbation of the GLM. The energy is quadratic in  $\varepsilon$ . The inequality (2.8) used to define the number of loops is also related to the square root singularity since the FRT depends on the square root of the perturbation.

The next step is to define the number of loops in the vicinity of the considered GLM in  $\mathcal{H}_\varphi$ .

**Definition 2.1 — Number of loops.** For  $0 < \varepsilon < \varepsilon_1$  and for all  $\mathbf{W}_0 \in \mathcal{V}^\varepsilon$ , let  $\ell : \mathcal{V}^\varepsilon \rightarrow \mathbb{N}^*$  be a function defined by  $\ell(\mathbf{W}_0) = n$  where  $n$  satisfies

$$|\text{FRT}(\mathbf{W}_0) - nT_j| \leq C_j\sqrt{\varepsilon} < T_j/2. \quad (2.9)$$

Geometrically, for  $\varepsilon$  small enough,  $\ell(\mathbf{W}_0) = n$  is associated to the orbit projected in the plane  $(u_N, v_N)$  of mass  $N$  taking exactly  $n$  rounds around  $(0, 0)$  to come back to  $\mathcal{H}_\varphi$ . In fact, we have

$$\ell(\mathbf{W}_0) = \text{round}(\text{FRT}(\mathbf{W}_0)/T_j) \quad (2.10)$$

where  $\text{round}(x) = n$  is the closest integer to  $x$ ,  $n - 1/2 \leq x < n + 1/2$ . The quantity  $\varepsilon_1$  in Equation (2.8) guarantees that the function  $\ell$  is well defined as it assigns a unique  $n$  to each  $\mathbf{W}_0$  in  $\mathcal{V}^\varepsilon$ . Thus, the set  $\mathcal{V}^\varepsilon$  can be partitioned in the set of states generating orbits with exactly  $n$  loops:

$$\mathcal{V}_n^\varepsilon = \{\mathbf{W}_0 \in \mathcal{V}^\varepsilon \mid \ell(\mathbf{W}_0) = n\}. \quad (2.11)$$

A consequence of Theorem 2.2 is that

$$\mathcal{V}^\varepsilon = \bigcup_{n \geq 0} \mathcal{V}_n^\varepsilon. \quad (2.12)$$

It is important to say that the previous study [12] on the FRT was restricted to  $\mathcal{V}_1^\varepsilon$ . One aim of the present work is to extend the investigation to all  $\mathcal{V}_n^\varepsilon$ ,  $n > 1$ . A natural question is concerned with the possible emptiness of the sets  $\mathcal{V}_n^\varepsilon$ . The numerical computations in [9] of nonlinear modes with one impact per period for the subharmonic of  $\omega_j$  shows that there exists  $n > 1$ ,  $\mathcal{V}_n^\varepsilon \neq \emptyset$ . Below, mathematical proofs are given about the existence of orbits with large  $n$ .

It is proven in [12] that states in  $\mathcal{H}_\varphi$  come back to  $\mathcal{H}$  after a finite time, however without any precise information about this return time. We prove now that near GLMs, solutions with large number of loops exist, and more precisely, with a number of loops greater than any fixed number and in any neighborhood of GLMs.

**Theorem 2.3 — Infinitely many non-empty sets  $\mathcal{V}_n^\varepsilon$ .** *There are infinitely many non-empty sets  $\mathcal{V}_n^\varepsilon$ .*

The proof of this theorem uses the perturbation of states in  $\mathcal{H}_1^0$ .

**Theorem 2.4 — Empty set  $\mathcal{V}_0^\varepsilon$ .** *For  $\varepsilon > 0$  small enough,  $\mathcal{V}_0^\varepsilon = \emptyset$ .*

In other words, a micro-contact does not occur for a sufficiently small perturbation of the GLM. A micro-contact is a contact occurrence before an entire loop is completed, that is  $\text{FRT} \leq T_j/2$ .

**Proposition 2.5 — Non-empty  $\mathcal{V}_1^\varepsilon$ .** *Let  $\mathcal{V}_1^\varepsilon$  be the set of orbits with First Return Times close to  $T_1$ , then its interior  $\overset{\circ}{\mathcal{V}}_1^\varepsilon$  is non-empty.*

Indeed  $\mathcal{V}_1^\varepsilon \neq \emptyset$  since there is at least the grazing linear mode with the fundamental period  $T_j$ . However, we exhibit later a subset of  $\mathcal{V}_1^\varepsilon$  with nonempty interior.

**Square root singularity** The study of the FRT on each  $\mathcal{V}_n^\varepsilon$  culminates in the square root singularity of all GLMs. Asymptotic expansions of FRT with square root singularity is proven on each  $\mathcal{V}_n^\varepsilon$ . For  $n = 1$ , such a singular expansion is provided in [12]. The expansion is made more precise for  $n = 1$ , as well as simplified and generalized for  $n > 1$ , in a whole neighborhood of the GLM.

### 3 First Return Time always almost a multiple of a fundamental period

Before stating the proof, Lemma 3.1 is needed.

**Lemma 3.1.** *If  $|1 - \cos \theta| \leq \delta$  with  $0 < \delta < 1$  and  $\theta \geq 0$  then there exists  $n \in \mathbb{N}$  such that  $|\theta - n2\pi| \leq 2\sqrt{\delta}$ .*

This is geometrically clear and moreover, the square root singularity of Nordmak [11, 7] emerges in a simple way.

*Proof.* Consider the cosine function. It decreases on  $[0, \pi]$ . As a consequence, for  $\cos \theta \geq 1 - \delta$ , it follows that  $\theta \leq \tau$  where  $\cos \tau = 1 - \delta$ . Also

$$\arccos(1 - \delta) = \int_{1-\delta}^1 \frac{dy}{\sqrt{1-y^2}} < \int_{1-\delta}^1 \frac{dy}{\sqrt{1-y}} \quad (3.1)$$

since  $0 < y < 1$  and hence  $\arccos(1 - \delta) < 2\sqrt{\delta}$  or  $\tau < 2\sqrt{\delta}$ . This implies that  $\theta < 2\sqrt{\delta}$ . Since  $\cos$  is an even function, this property also extends to  $[-\pi, 0]$  and  $|\theta| \leq 2\sqrt{2\delta}$  in  $[-\pi, \pi]$ . Expanding this result to the whole non-negative domain of the cosine function, we conclude that  $\theta \in [n2\pi - 2\sqrt{\delta}, n2\pi + 2\sqrt{\delta}]$  for some  $n \in \mathbb{N}$  which concludes the proof of the lemma. Also, note that a Taylor expansion of cosine shows that  $\tau \sim \sqrt{2\delta} < 2\sqrt{\delta}$  when  $\tau \rightarrow 0$ .  $\square$

Theorem 2.2 can now be proven.

*Proof.* Before starting the proof, one has to take a sufficiently small  $\varepsilon_0$  to avoid orbits with sticking contact [13]. It is already known that there is no such orbit near a GLM [12]. However, it is incidentally proven again in the sequel of the present paper in a different fashion. With such a  $\varepsilon_0 > 0$ , we are considering  $0 < \varepsilon_1 \leq \varepsilon_0$ .

The core ingredient of the proof is the neutral stability of the linear dynamics which guarantees that any orbit starting near  $\mathbf{W}_0^G$  stays close to the grazing linear mode during the free dynamics. As a consequence, the corresponding FRT also lies near a multiple of  $T_j$ .

Let us denote the Euclidean norm related to the mechanical energy by  $\|\mathbf{W}\|_E^2 = \mathbf{E}(\mathbf{W})$ . The dynamics away from the unilateral interaction with the obstacle is linear and a grazing linear mode  $\mathbf{W}_G(t)$  is a linear solution. Assume that  $\mathbf{W}(t)$  is the solution associated with the state  $\mathbf{W}_0$  on  $\mathcal{H}_\varphi$  and  $T(\mathbf{W}_0)$  is the corresponding FRT. For all  $0 \leq t \leq T(\mathbf{W}_0)$ , it is known that  $\mathbf{W}(t) - \mathbf{W}_G(t)$  is also a solution of the linear equation  $M\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0}$ . Hence, for every  $\mathbf{W}_0$  in  $\mathcal{V}^\varepsilon$ ,

$$\|\Delta\mathbf{W}(t)\|_E := \|\mathbf{W}(t) - \mathbf{W}_G(t)\|_E = \|\mathbf{W}_0 - \mathbf{W}_0^G\|_E \leq \varepsilon. \quad (3.2)$$

Let  $\|\mathbf{W}\|^2 = \mathbf{W}^T \mathbf{W}$  be the square of the classical Euclidean norm in  $\mathbb{R}^{2N}$ , which is different from the norm  $\|\cdot\|_E$  associated to the energy  $E$ . Consider the Rayleigh quotient  $\|\Delta\mathbf{W}(t)\|_E^2 / \|\Delta\mathbf{W}(t)\|^2$ . It is well known that

$$\lambda_{\min} \leq \frac{\|\Delta\mathbf{W}(t)\|_E^2}{\|\Delta\mathbf{W}(t)\|^2} \leq \lambda_{\max}, \quad (3.3)$$

which implies  $\|\Delta \mathbf{W}(t)\| \leq c \|\Delta \mathbf{W}(t)\|_E$  where  $c = 1/\sqrt{\lambda_{\min}} > 0$ . Denoting  $u_{\text{GLM},N}(t)$  the last component of the displacement of the GLM and combining Equations (3.2) and (3.3) along with the basic inequality  $|u_N(t) - u_{\text{GLM},N}(t)| \leq \|\Delta \mathbf{W}(t)\|$  yields

$$|u_N(t) - u_{\text{GLM},N}(t)| \leq c\varepsilon. \quad (3.4)$$

At  $T = \text{FRT}(\mathbf{W}_0)$ , the displacement of mass  $N$  for the orbit starting at  $\mathbf{W}_0$  is  $u_N(T) = d$  and it is  $u_{\text{GLM},N}(T) = d \cos \omega_j T$  for the corresponding grazing linear mode. Accordingly, Equation (3.4) implies  $|d - d \cos \omega_j T| \leq c\varepsilon$ . Since  $\varepsilon < \varepsilon_1 = d/c$ , it follows that  $c\varepsilon < d$ . Hence, by Lemma 3.1, there exists  $n \in \mathbb{Z}$  such that  $|\omega_j T - n2\pi| \leq 2\sqrt{c\varepsilon/d}$  that is

$$|T - nT_j| \leq 2\sqrt{\frac{c\varepsilon}{d\omega_j}} \quad (3.5)$$

where  $T_j = 2\pi/\omega_j$ . Denoting  $C_j = 2\sqrt{c/d\omega_j}$ , we have proven that, for all  $\mathbf{W}_0 \in \mathcal{V}^\varepsilon$ , there exists  $n \in \mathbb{N}$  such that  $|\text{FRT}(\mathbf{W}_0) - nT_j| \leq C_j\sqrt{\varepsilon}$ .

We now show that  $n$  is unique for each initial state  $\mathbf{W}_0$ . Since  $\varepsilon < \varepsilon_1$  and with the chosen value of  $\varepsilon_1$ , we have  $C_j\sqrt{\varepsilon} < T_j/2$  and it follows that  $|\text{FRT}(\mathbf{W}_0) - kT_j| < T_j/2$ , for all  $j = 1, \dots, N$  which implies that  $I_j^k \cap I_j^m = \emptyset$  for any  $k \neq m$ , where  $I_j^m = (mT_j - T_j/2; mT_j + T_j/2)$ . In other words, there is no overlapping between any intervals  $I_j^k$ , for all  $k = 1, 2, \dots$ . Therefore, the neighborhood  $\mathcal{V}^\varepsilon$  of  $\mathbf{W}_0^G$  is the disjoint union of the sets  $\mathcal{V}_n^\varepsilon$  for  $n \geq 0$ :  $\mathcal{V}^\varepsilon = \bigcup_{n \geq 0} \mathcal{V}_n^\varepsilon$ . Consequently,  $\text{FRT}(\mathbf{W}_0)$  belongs to the interval defining  $n$  loops, only.  $\square$

## 4 Return to Poincaré section after $n$ loops

The set  $\mathcal{V}_n^\varepsilon$  represents the subset of  $\mathcal{V}^\varepsilon$  where the emerging orbit takes exactly  $n$  loops to come back to  $\mathcal{H}_\varphi$ . Its topology and the fact that it is empty or not has to be clarified. Unless it is empty, it is neither closed nor open. Nevertheless, it is proven in this section that

1.  $\forall n \geq 0, \Omega_n^\varepsilon = \bigcup_{m \geq n} \mathcal{V}_m^\varepsilon \neq \emptyset$  is an open set.
2.  $\mathcal{V}_0^\varepsilon = \emptyset$  for  $\varepsilon$  small enough.
3.  $\mathcal{V}_1^\varepsilon \neq \emptyset$  with nonempty interior  $\mathcal{V}_1^\circ \neq \emptyset$ .

### 4.1 Large number of loops

To prove the existence of orbits emanating from  $\mathcal{H}_\varphi$  and featuring a large number of loops before they return to  $\mathcal{H}_\varphi$ , the set  $\mathcal{H}_1^0$  is used. Only two degrees-of-freedom, denoted  $j$  and  $k$ , of the  $N$ -dof system are needed to prove this result. We consider an orbit with one and only one contact for all time, which corresponds to a state on  $\mathcal{H}_1^0$ , not included in  $\mathcal{H}_\varphi$ . The idea is to use some properties of the set  $\mathcal{H}_1^0$  lying outside  $\mathcal{H}_\varphi$  in order to exhibit interesting features of  $\mathcal{H}_\varphi$ . From Proposition 2.1, the function  $\psi(t)$  is the displacement of the last mass, that is

$$\psi(t) = d((1 - \alpha) \cos \omega_k t + \alpha \cos \omega_j t) \quad (4.1)$$

with  $\omega_k < \omega_j$  and  $0 < \alpha < 1$  where  $\alpha$  quantifies the perturbation of the GLM  $j$ . Such a function reaches its maximum  $d$  once and only once at  $t = 0$ . However, in the future, for some specific times, it will approach  $d$  closer and closer, that is [6]

$$\forall t > 0, \quad \psi(t) < d \quad \text{but} \quad \forall \tau > 0, \quad \lim_{T \rightarrow +\infty} \max_{[\tau, T]} \psi(t) = d. \quad (4.2)$$

In order to be in  $\mathcal{H}_\varphi$ , it is enough to slightly disturb  $\psi$ . Let  $\gamma > 0$  be small and let us define

$$\psi_\gamma(t) = \psi(t) - d\gamma \sin \omega_j t. \quad (4.3)$$



It should first be checked that  $\psi$  really corresponds to  $\dot{u}_N(0^+) < 0$ . This is true because  $\gamma$  is strictly positive. We have  $\psi_\gamma(t) < \psi(t)$  for all  $t \in (0, T_j/2)$  since  $\sin \omega_j t > 0$  and  $\gamma$  is small. Let us show the existence of a solution with  $n$  loops before it returns to  $\mathcal{H}$ , where  $n \geq 1$ . Let  $I = [T_j/2, nT_j + T_j/2]$ . By the continuity of  $\psi$  on the compact set  $I$ , there exists  $h > 0$  such that  $\psi(t) < d - h$  for all  $t \in I$ . If  $0 < \gamma < h/d$ , then  $\psi_\gamma(t) < \psi(t) + \gamma d < d$ , for all  $t$  in  $I$ . However

$$\sup \psi_\gamma = d(1 - \alpha + \sqrt{\alpha^2 + \gamma^2}) > d \quad (4.4)$$

and there must be a time  $t > nT_j + T_j/2$  for which  $\psi(t) > d$  [6]. In other words, there exists a solution with a FRT greater than  $nT_j$ .

**Remark 4.1.** *The number of loops  $\ell$  depends on the parameter  $\gamma$  and  $\lim_{\gamma \rightarrow 0^+} \ell(\gamma) = \infty$ . This explains why  $\mathcal{H}_1^0$  in the neighbourhood of the considered GLM is equal to  $\mathcal{V}_\infty^\varepsilon$ . When  $\gamma = 0$ ,  $\psi_0$  corresponds to a state in  $\mathcal{H}_1^0$ . On one hand, when  $\gamma > 0$ ,  $\psi_\gamma$  corresponds to a state in  $\mathcal{V}_{\ell(\gamma)}^\varepsilon$ . Thus, passing to the limit, we have the equality between  $\mathcal{H}_1^0$  and  $\mathcal{V}_\infty^\varepsilon$  in the neighborhood of the GLM. On the other hand, Equation (4.2) can be interpreted as  $\psi_0$  reaching  $d$  at infinity, which corresponds to an infinite number of loops.*

The previous result can be written with the following formalism. Let  $\Omega_n^\varepsilon$  be the set of initial states generating orbits for which mass  $N$  does not return to  $\mathcal{H}$  after  $n$  loops, that is

$$\max_{[T_j/2, nT_j + T_j/2]} u_N(t) < d \quad (4.5)$$

Note that  $\Omega_n^\varepsilon$  is defined only by a strict inequality on a continuous function and so is an *open set*. In fact the inequality is also valid on  $]0, T_j/2[$  and will be discussed below as a consequence of  $\mathcal{V}_0^\varepsilon = \emptyset$  for  $\varepsilon$  small enough. The sequence of such sets is non increasing, that is  $\Omega_{n+1}^\varepsilon \subset \Omega_n^\varepsilon$  and  $\Omega_0^\varepsilon = \mathcal{V}^\varepsilon$ . As a consequence, it can be expressed as a union of disjoint sets,  $\mathcal{V}_n^\varepsilon = \Omega_{n-1}^\varepsilon - \Omega_n^\varepsilon$ , explaining why  $\mathcal{V}_n^\varepsilon$  is neither an open set nor a closed one, if it is not empty. Then, Theorem 2.3 can be simply rewritten in the following manner:

$$\forall n \geq 0, \quad \Omega_n^\varepsilon = \bigcup_{m \geq n} \mathcal{V}_m^\varepsilon \neq \emptyset. \quad (4.6)$$

In other words, for any  $n$ , there always exists a solution near the GLMs which comes back to  $\mathcal{H}$  in more than  $n$  loops.

**Modal characterization of the Poincaré section near a GLM** Consider the GLM number  $j$  with  $j = 1$  and an attendant perturbation—other perturbations can be enforced by relabelling the frequencies. We use a modal representation of  $u_N = \phi$  on the time interval  $(0, \text{FRT})$ :

$$\phi(t) = (d + \delta_1) \cos \omega_1 t + \sum_{i=2}^N a_i \cos \omega_i t + \sum_{i=1}^N b_i \sin \omega_i t. \quad (4.7)$$

Why do we use two notations for  $u_N$ , that is  $u_N$  and  $\phi$ ? The interest is that the function  $\phi$  can be studied for all time without imposing the unilateral constraint. Indeed, when there is a contact, the modal coordinates of  $u_N$  change due to the impact laws. In contrast,  $\phi$  can be used for all time using the classical tools of almost periodic functions [6]. In particular, it is possible that  $\sup_{t>0} \phi(t) > d$ . Accordingly,  $\phi$  does not corresponds to  $u_N$  for all time. This technique was also used in [9, 12] which allows classical smooth analysis tools on  $\phi$ .

Due to the special role of the first mode, the modal coefficient  $a_1$  is split in two parts:  $d$ , the modal coefficient of the first GLM and  $\delta_1$ , a perturbation of the first GLM, with  $a_1 = d + \delta_1$ . The modal coordinates are not all independent since  $\phi(0) = d$  which entails

$$0 = \delta_1 + \sum_{i=2}^N a_i. \quad (4.8)$$

Up to this constraint, these coordinates are in bijection with  $\mathcal{H}_\rho$  through a constant linear transformation. Computing the relative size of  $\mathcal{V}_1^\varepsilon$  in the modal coordinates  $(a_i, b_i)$  is of the same order as in the physical coordinates. It means that

$$\lim_{\varepsilon \rightarrow 0^+} |\mathcal{V}_1^\varepsilon|/|\mathcal{V}^\varepsilon| \quad (4.9)$$

where  $|\cdot|$  stands for the volume of the set, is the same when it is computed in physical coordinates or in modal coordinates.

The set of *exiting* states at time  $t = 0^+$  is

$$\mathcal{H}_{\text{exit}} = \{(u_1(0^+), \dots, u_N(0^+), v_1(0^+), \dots, v_N(0^+)) | \phi(0) = d, \exists \tau > 0, \forall t \in (0, \tau), \phi(t) < d\}.$$

It should be noted that  $\mathcal{H}_{\text{exit}} = \mathcal{H}_\rho \cup \mathcal{H}_1^0$ . The set  $\mathcal{H}_\rho$  can also be described with conditions on the successive derivatives to ensure that the solution of the free dynamics with such initial state on  $\mathcal{H}_\rho$  leaves the wall. Accordingly, the following conditions must be satisfied:

- (0):  $d = \phi(0)$ ,
- (1):  $0 \geq \phi'(0)$ ,
- (2): if  $0 = \phi'(0)$  then  $0 \geq \phi''(0)$ ,
- (3): if  $0 = \phi'(0) = \phi''(0)$  then  $0 \geq \phi^{(3)}(0)$ ,

...

$$(2N - 1): \text{ if } 0 = \phi'(0) = \dots = \phi^{(2N-2)}(0) \text{ then } 0 \geq \phi^{(2N-1)}(0).$$

There is no additional condition because the dimension of  $\mathcal{H}_\rho$  is  $2N - 1$ . The case when all the first  $2N - 1$  derivatives are zero corresponds to a unique special orbit not near GLM 1 (at a fixed positive distance). This solution is the *flattest* grazing solution which exhibits the maximal power-root singularity with the power  $1/(2N - 1)$  under generic assumptions [12].

In the coordinates  $(\mathbf{a}, \mathbf{b})$ ,  $\mathcal{H}_\rho$  is characterized as follows:

- (0):  $0 = \delta_1 + \sum_{i=2}^N a_i$ ,
- (1):  $0 \geq \sum_{i=1}^N b_i \omega_i$ ,
- (2): if  $0 = \sum_{i=1}^N b_i \omega_i$  then  $d\omega_1^2 \geq -\delta_1 \omega_1^2 - \sum_{i=2}^N a_i \omega_i^2$ ,
- (3): if  $0 = \sum_{i=1}^N b_i \omega_i$  and  $d\omega_1^2 = -\delta_1 \omega_1^2 - \sum_{i=2}^N a_i \omega_i^2$  then  $0 \geq -\sum_{i=1}^N b_i \omega_i^3$ ,

...

$$(2N - 1): \text{ if } 0 = \sum_{i=1}^N b_i \omega_i, \dots, d\omega_1^{2N-2} = (-1)^{N-1} (\delta_1 \omega_1^{2N-2} + \sum_{i=2}^N a_i \omega_i^{2N-2}) \text{ then } 0 \geq (-1)^{N-1} \sum_{i=1}^N b_i \omega_i^{2N-1}.$$

It is clear here that condition  $(2N - 1)$  is the last one. Otherwise, for the condition  $(2N)$ , due to the invertibility of the Vandermonde matrix, the modal coordinates  $(\mathbf{a}, \mathbf{b})$  are uniquely determined and correspond to the *flattest* grazing orbit associated to GLM 1. This aspect is detailed in [12].

Near GLM 1, consider that  $\phi$ , an  $\varepsilon$ -perturbation of  $t \mapsto \phi_0(t) = d \cos \omega_1 t$  corresponding to the GLM. Again,  $\varepsilon^2$  represents the mechanical energy of the perturbation. Note that  $\phi_0(0) = d$ ,  $\phi_0'(0) = 0$ ,  $\phi_0''(0) = -d\omega_1^2 < 0$ . Thus, locally, for a sufficiently small positive  $\varepsilon$

$$\phi(t) = d + (\mathbf{a} \cdot \mathbf{1}) + (\mathbf{b} \cdot \boldsymbol{\omega})t - (d\omega_1^2 + \mathbf{a} \cdot \boldsymbol{\omega}^2 + \mathcal{O}(\varepsilon)t + \mathcal{O}(d\omega_1^4 + \varepsilon)t^2) \frac{t^2}{2} \quad (4.10)$$

with  $\mathbf{a} = (\delta_1, a_2, \dots, a_N) = \mathcal{O}(\varepsilon)$ ,  $\mathbf{b} = (b_1, \dots, b_N) = \mathcal{O}(\varepsilon)$ ,  $\mathbf{1} = (1, \dots, 1)$ ,  $\boldsymbol{\omega}^k = (\omega_1^k, \dots, \omega_N^k)$ . The term  $\mathcal{O}(\varepsilon)t$ , i.e.  $\mathcal{O}(\varepsilon)t^3/2$  in the developed expansion, corresponds to the remainder  $(\mathbf{b} \cdot \boldsymbol{\omega}^3)t^3/6$  of the sine functions while the term  $\mathcal{O}(d\omega_1^4 + \varepsilon)t^2$ , i.e. in  $t^4$ , corresponds to the remainder  $(d\omega_1^4 + \mathbf{a} \cdot \boldsymbol{\omega}^4)t^4/24$  of the functions cosine functions.

To guarantee  $\phi(t) < d$  for a small positive  $t$ , it is collectively required that  $0 = \mathbf{a} \cdot \mathbf{1}$ ,  $0 \geq \mathbf{b} \cdot \boldsymbol{\omega}$ , and  $0 < d\omega_1^2 + \mathbf{a} \cdot \boldsymbol{\omega}^2$ . The last inequality is not so strict since  $\mathbf{a} = \mathcal{O}(\varepsilon)$ . However, the last strict inequality is mandatory to absorb the remainder of order  $\mathcal{O}(\varepsilon)t^3 + \mathcal{O}(1)t^4$ .

## 4.2 No micro-contacts

This subsection shows that there is no micro-contact for  $\varepsilon$  small enough, or, in other words, that  $\mathcal{V}_0^\varepsilon$  is an empty set. Clearly, for a larger  $\varepsilon$ , it is possible to have a return to  $\mathcal{H}_\varphi$  with a FRT  $< T_1/2$ , see Figure 2. First  $\mathcal{V}_0^\varepsilon = \emptyset$  is proven, because it is a step in the proof for  $\mathcal{V}_1^\varepsilon \neq \emptyset$ .

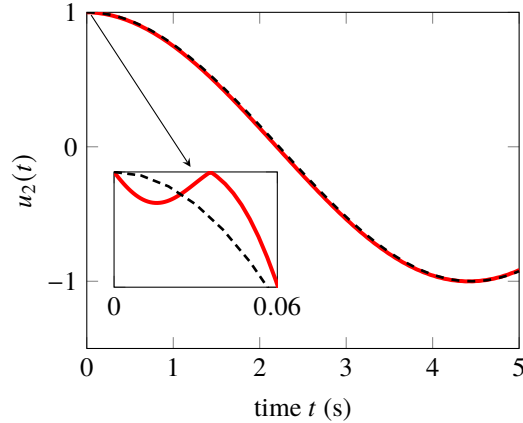


Figure 2: A solution with a micro-contact [solid red] for  $N = 2$  with  $m_1 = m_2 = 1$ ;  $k_1 = 1$ ,  $k_2 = 3000$ ,  $\varepsilon = 0.05$ . The contact appears just at the beginning for a time FRT  $\approx 0.04$  s. The black dashed line corresponds to the first GLM.

*Proof of Theorem 2.4.* To prove that  $\mathcal{V}_0^\varepsilon = \emptyset$ , it suffices to check that FRT  $> T_1/2$  for  $\varepsilon > 0$  small enough, that is to say  $\phi(t) < d, \forall t \in (0, T_1/2)$ . The proof has two steps. The first step is to reduce the interval  $(0, T_1/2)$  where  $\phi < d$  to a small interval  $(0, \tau(\varepsilon))$  where  $\tau(\varepsilon) \rightarrow 0$  when  $\varepsilon \rightarrow 0$ . The second step uses a Taylor expansion to show that  $\phi < d$  on  $(0, \tau)$ . This step needs careful derivations because  $\phi(0) = d$  so the smaller  $t$  is, the closer  $\phi$  is to  $d$ .

**Step 1: Reduce the proof on  $(0, \tau)$  with  $\tau = O(\sqrt{\varepsilon})$**  Note that  $\phi$  is an  $O(\varepsilon)$  perturbation of the mode given by  $u_N(t) = d \cos \omega_1 t$ . Let  $C_1$  be the positive constant such that  $|O(\varepsilon)| < C_1 \varepsilon$ , that is  $\phi(t) < d \cos \omega_1 t + C_1 \varepsilon$ . The function  $t \mapsto d \cos \omega_1 t$  is decreasing on  $(0, T_1/2)$ , so that, for  $\varepsilon$  small enough,  $C_1 \varepsilon < 2d$  and there exists a unique  $\tau \in (0, T_1/2)$  such that  $\cos \omega_1 \tau = 1 - (C_1/d)\varepsilon > -1$ . Such  $\tau$  is an implicit function of  $\varepsilon$  and since  $\cos'(0) = 0$  and  $\cos''(0) < 0$ , then  $\tau = \tau(\varepsilon) = O(\sqrt{\varepsilon}) \leq C_2 \sqrt{\varepsilon}$ . Thus, on  $[\tau, T_1/2]$ ,  $\phi(t) < d \cos \omega_1 t + C_1 \varepsilon \leq d$  which concludes the first step.

**Step 2: Taylor expansions on  $(0, \tau)$**  It should be shown that  $\phi(t) < d$  on  $(0, \tau)$ . This is clear since in the last outer parenthesis of Equation (4.10), only the term  $d\omega_1^2$  is a real positive constant whereas the other terms have an order of at least  $\varepsilon$ .

Accordingly, the inequality is true on  $(0, \tau)$  and the proof is complete.  $\square$

## 4.3 Single loop

This section deals with the non-emptiness of  $\mathcal{V}_1^\varepsilon$ , the set with returns in exactly one loop, see Proposition 2.5. We provide two proofs: first, a simple one only using two dof and then, a more general one trying to identify a large subset of  $\mathcal{V}_1^\varepsilon$  using  $N$  dof. The simple proof gives a continuum of solutions as in the proof of the existence of solutions with a large number of loops, see Theorem 2.3.

*Proof.* Let us assume that  $d = 1$ . Recall that  $V_1 = \{W_0 \in \mathcal{H}_\varphi : |T(W_0) - T_1| < C\sqrt{\varepsilon}\}$  where  $\varepsilon > 0$  is small enough. Consider the solution where the displacement of mass  $N$  coincides with the function  $\varphi(t) = (1 + \varepsilon) \cos \omega_1 t - \varepsilon \cos \omega_2 t$ . At  $t = T_1$ ,  $\varphi(T_1) = 1 + \varepsilon - \varepsilon \cos \omega_2 T_1 > 1$  since

$\cos \omega_2 T_1 < 1$  (by the  $\mathbb{Z}$ -independence of  $\omega_1$  and  $\omega_2$ , i.e. the non internal resonance assumption). Also,  $\varphi'(T_1) = \varepsilon \omega_2 \sin \omega_2 T_1 \neq 0$ . Thus, there exists  $t < T_1$  such that  $\varphi(t) = 1$ , which means the mass  $N$  reaches the obstacle at time  $t$ .  $\square$

A subset of  $\mathcal{V}_1^\varepsilon$  written in the modal coordinates as in Equation (4.7) can be clearly identified, for  $\varepsilon$  small enough. It proves Proposition 2.5.

**Lemma 4.1 — An explicit subset of  $\mathcal{V}_1^\varepsilon$ .** Let  $\mathcal{V}_1^{\varepsilon++}$  be the set of  $(\delta_1, a_2, \dots, b_N)$  such that

$$\delta_1^2 + \sum_{i=2}^N a_i^2 + \sum_{i=1}^N b_i^2 \leq O(\varepsilon^2), \quad (4.11a)$$

$$\delta_1 + \sum_{i=2}^N a_i = 0, \quad \sum_{i=1}^N b_i \omega_i < 0, \quad (4.11b)$$

$$\delta_1 + \sum_{i \geq 2}^N a_i \cos \omega_i T_1 + b_i \sin \omega_i T_1 > 0. \quad (4.11c)$$

then, for  $\varepsilon$  small enough,  $\emptyset \neq \mathcal{V}_1^{++} = \mathcal{V}_1^{\varepsilon++} \subset \mathcal{V}_1^\varepsilon \subset \mathcal{H}\varphi$ .

Equation (4.11a) means that the mechanical energy is smaller than  $\varepsilon^2$ . Equation (4.11b) insures that the mass  $N$  leaves  $\mathcal{H}$  with a negative velocity at time  $t = 0$ . Equation (4.11c) yields  $\text{FRT} < T_1$ .

*Proof.* Let  $\varepsilon > 0$  be small enough such that  $\mathcal{V}_0^\varepsilon = \emptyset$ . Such a function  $\phi$  with the chosen modal coordinates corresponds to a mass  $N$  leaving the wall with a number of loops  $\ell \geq 1$ . However, Equation (4.11c) forces  $\phi(t) = d$  for  $0 < t < T_1$ . Accordingly,  $\ell = 1$  which concludes the proof.

The set  $\mathcal{V}_1^{\varepsilon++}$  clearly has a nonempty interior. The three inequalities define a convex bounded subdomain of  $\mathcal{H}\varphi$ . This set is delimited by two hyperplanes given by the normal vectors

$$\mathbf{V}_1 = (0, \dots, 0, \omega_1, \dots, \omega_n), \quad (4.12a)$$

$$\mathbf{V}_2 = (1, \cos \omega_2 T_1, \dots, \cos \omega_n T_1, 0, \sin \omega_2 T_1, \dots, \sin \omega_n T_1). \quad (4.12b)$$

These two normals of  $\mathbb{R}^{2N}$  define two different hyperplanes in  $\mathbb{R}^{2N}$  and also two different hyperplanes in  $\mathcal{H}$  due to the assumption of non internal resonance, hence  $\mathcal{V}_1^{++} \neq \emptyset$ .  $\square$

## 5 Square root singularity near the grazing linear modes

### 5.1 Square root expansion of the First Return Time

In this section, an asymptotic expansion of the FRT is provided in a whole neighborhood of  $W_H^G$ , the state on  $\mathcal{H}\varphi$  corresponding to the first GLM,  $j = 1$  to fix the notation. Except on  $\mathcal{V}_\infty^\varepsilon$ , the asymptotic expansion of the FRT is considered on  $\cup_{1 \leq n < \infty} \mathcal{V}_n^\varepsilon$ . The result is expressed in terms of the modal coordinates  $(\delta_1, a_2, \dots, a_N, b_1, \dots, b_N)$  as in Equation (4.7) which is repeated for convenience:

$$u_N(t) = (d + \delta_1) \cos \omega_1 t + \sum_{k=2}^N a_k \cos \omega_k t + \sum_{k=1}^N b_k \sin \omega_k t \quad \text{where } a_1 = d + \delta_1. \quad (5.1)$$

These modal coordinates satisfy the following conditions for a small perturbation of  $W_H^G$ :

$$\delta_1 + \sum_{k=2}^N a_k = 0, \quad \sum_k b_k \omega_k \leq 0, \quad \|(\delta_1, a_2, \dots, b_N)\|_E \leq \varepsilon. \quad (5.2)$$

Recall that the acceleration is negative for  $\varepsilon$  small enough, so the previous conditions are necessary and sufficient to leave  $\mathcal{H}$  at  $t = 0^+$ .

To take into account Equation (5.2),  $\delta_1$  is replaced by  $(a_2, \dots, a_N)$  and Equation (5.1) reads

$$u_N(t) = d \cos \omega_1 t + \sum_{k=2}^N a_k (\cos \omega_k t - \cos \omega_1 t) + \sum_{k=1}^N b_k \sin \omega_k t. \quad (5.3)$$

**Theorem 5.1 — FRT: asymptotic expansion for  $n$  loops.** *There exists a positive sufficiently small  $\varepsilon$  such that for all  $n$ ,  $1 \leq n < +\infty$ , there exists  $g_n \in C^\infty(\mathcal{V}_n^\varepsilon, ]T_1/2, +\infty[)$  such that the FRT to  $\mathcal{H}$  on  $\mathcal{V}_n^\varepsilon$  reads*

$$\begin{aligned} \text{FRT}(a_2, \dots, b_N) &= T_n^*(a_2, \dots, b_{N-1}) + g_n(-\sqrt{|b_N - b_N^*|}, a_2, \dots, b_{N-1}), \\ &= nT_1 - \gamma \sqrt{|b_N - b_N^*|} + \mathcal{O}(\|(a_2, \dots, b_N)\|) \\ &\leq T_n^*(a_2, \dots, b_{N-1}) \end{aligned} \quad (5.4)$$

$$T_n^*(a_2, \dots, b_{N-1}) = nT_1 + t_n^*(a_2, \dots, b_{N-1}) \quad (5.5)$$

$$|b_N - b_N^*| = \text{sign}(\sin(\omega_N n T_1))(b_N - b_N^*(a_2, \dots, b_{N-1})) =: \mathcal{B}_n(a_2, \dots, b_N) \quad (5.6)$$

where  $\gamma > 0$  is independent of  $n$ ,  $g_n$  is a smooth analytic function,  $t_n^*$  is a key function to localize the square root singularity. The function  $t_n^*$  is defined near 0 by the implicit equation

$$\dot{u}_N(nT_1 + t_n^*) = 0, \quad (5.7)$$

depending on  $(a_2, \dots, b_{N-1})$  and  $b_N^*$  is a function depending on  $t_n^*$  and is the modal component  $b_N$  such that  $u_N(nT_1 + t_n^*) = d$ .

This result needs the following comments:

1.  $\text{FRT}(0, \dots, 0) = nT_1$  means that the FRT is discontinuous on  $\mathcal{H}_\varphi$ . In fact, when  $(a_2, \dots, b_N) = (0, \dots, 0)$ , the orbit corresponds to the GLM for which  $\text{FRT} = T_1$ . However, on  $\mathcal{V}_n^\varepsilon$ , the corresponding orbits are near the trajectory of the GLM with exactly  $n$  loops and the FRT is in the vicinity of  $nT_1$ :

$$\lim_{\mathcal{V}_n^\varepsilon \ni (a_2, \dots, b_N) \rightarrow (0, \dots, 0)} \text{FRT}(a_2, \dots, b_N) = nT_1. \quad (5.8)$$

This discontinuity of the FRT at  $(a_2, \dots, b_N) = (0, \dots, 0)$  is due to the considered choice of the Poincaré section, tangent to the grazing orbit and the existence of orbits with many loops before they come back to  $\mathcal{H}_\varphi$ .

2. The implicit theorem is only needed once to obtain this expansion, precisely to define the critical time  $T_n^* = nT_1 + t_n^*$  when the return time occurs with a grazing contact. In fact, the linearity with respect to the modal coordinates allows us to essentially use the inverse function theorem with only one variable. The other variables become parameters.
3. The important inequality

$$\text{FRT}(a_2, \dots, b_N) \leq T_n^*(a_2, \dots, b_{N-1}) = \text{FRT}(a_2, \dots, b_{N-1}, b_N^*) \quad (5.9)$$

means that the FRT for the orbit with the initial modal coordinates  $(a_2, \dots, b_N)$  is always smaller than or equal to the FRT with a grazing contact and the initial modal coordinates  $(a_2, \dots, b_{N-1}, b_N^*)$ . In fact, the acceleration of  $u_N$  is negative, so the velocity decreases before contact. However, the velocity has to be non-negative before contact. Thus, contact on  $\mathcal{H}_\varphi$  occurs before the velocity vanishes, or exactly at the same time for a grazing return.

4.  $\text{FRT} \leq nT_1$  if the square root term dominates the remainder or if  $t_n^* \leq 0$  from the previous discussion. The reduction of the FRT with respect to the linear time was already emphasized in [9].

*Proof.* The proof involves many steps, all collected in this section. The proof of the last theorem of Section 2.2 in [12] for  $n = 1$  is extended to all  $n$  and simplified with a new formula.

**FRT not a closed-form expression of  $(\delta_1, \dots, b_N)$**  The FRT is a positive time  $T = nT_1 + t$ ,  $|t| < T_1/2$ , solution of the equation  $u_N = d$ . Since  $\delta_1$  is functionally related to the other modal coordinates through  $\delta_1 + \sum_{k=2}^N a_k = 0$ , we choose to remove this parameter in order to work with a set of free parameters:

$$\begin{aligned} d &= F(T, a_2, \dots, b_N) = u_N(T, a_2, \dots, b_N) \\ &= d \cos \omega_1 T + \sum_{k=2}^N a_k (\cos \omega_k T - \cos \omega_1 T) + \sum_{k=1}^N b_k \sin \omega_k T. \end{aligned} \quad (5.10)$$

Note that there are usually many  $T$  solutions to this equation. In particular, there are solutions  $T$  such that the condition  $u_N < d$  is violated on  $(0, T)$ . These  $T$  correspond to ghost *non-admissible* solutions [9]. Nevertheless, to perform the asymptotic expansion of the FRT, it is simple first to perform the asymptotic expansion of  $T$  with respect to the other parameters using smooth analysis and then obtain the asymptotic expansion of FRT by only considering parameters in  $\mathcal{V}_n^\varepsilon$  and  $T$  near  $nT_1$ . At this point, it is useful to ignore the admissibility conditions for the function  $u_N$  and to study all the roots of the function  $F$ . We already know that  $T$  is near  $nT_1$  by Theorem 2.2. The number of loops  $n$  is now fixed and we focus on the FRT  $= T$  near  $nT_1$ , or in other words

$$T = nT_1 + t \quad (5.11)$$

with small time  $t$ . As already noticed in [11, 10, 12], grazing contact implies that the implicit function theorem cannot be used in Equation (5.10). This is due to the condition  $\partial_T F(nT_1, 0, \dots, 0) = \dot{u}_N(nT_1) = 0$ , at  $T = nT_1$ . The strategy is to choose another variable, for instance  $b_N$  since  $\partial_{b_N} F(nT_1, 0, \dots, 0) = \cos \omega_2 nT_1 \neq 0$  due to the non internal resonance assumption. Note that  $b_1$  is the only forbidden choice. Accordingly,  $b_N$  can be seen as a function of  $t$  and the  $2N - 2$  other variables  $a_2, \dots, a_N, b_1, \dots, b_{N-1}$ . The grazing and the non-zero acceleration  $\partial_t^2 F(nT_1, 0, \dots, 0) = \ddot{u}_N(nT_1, 0, \dots, 0) = -d\omega_1^2 < 0$  induce a parabolic flatness. Then, expressing  $t$ , which corresponds to the FRT including the non-admissible orbits, as a function of  $b_N$  leads the square root singularity in the function  $g_n$ .

**Flatness near  $(nT_1; 0, \dots, 0)$**  When the modal coordinates vanish, except  $a_1 = d$ ,  $F(T; 0, \dots, 0)$  in Equation (5.10) corresponds to  $u_N(T)$  for the GLM, i.e.  $0 = \dot{u}_N(nT_1) = \partial_T F(nT_1; 0, \dots, 0)$ . Thus, we cannot express the FRT as a function of the modal coordinates with the inverse function theorem or the implicit function theorem. The flatness is only quadratic since  $0 > -d\omega_1^2 = \ddot{u}_N(nT_1) = \partial_T^2 F(nT_1; 0, \dots, 0)$ . The strategy consists in choosing a modal coordinate as a function of  $T$ , to then exhibit the parabolic flatness of this function and to finally invert this function to obtain a square-root singularity for  $T$  as a function of this modal coordinate. The chosen modal coordinate is  $b_N$ .

**$b_N$  as a function  $B_N(T, a_2, \dots, b_{N-1})$**  Equation (5.10) yields

$$\begin{aligned} b_N &= B_N(t = T - nT_1; a_2, \dots, b_{N-1}) \\ &= \frac{d \cos \omega_1 T - \sum_{k=2}^N a_k (\cos \omega_k T - \cos \omega_1 T) - \sum_{k=1}^{N-1} b_k \sin \omega_k T}{\sin \omega_N T}. \end{aligned} \quad (5.12)$$

The function  $B_N$  is well defined if  $\sin \omega_N T \neq 0$ . This is true for  $T$  near  $nT_1$  since  $\sin \omega_N nT_1 \neq 0$ . The flatness of the grazing linear mode  $\dot{u}_N(nT_1; 0, \dots, 0)$  yields a flatness of  $F$  and then of  $B_N$  with respect to  $T$ .

**Location of the flatness of  $B_N$ : critical time  $T_n^* = nT_1 + t_n^*$  and critical  $b_N, b_N^*$**  An important point is that the flatness of  $B_N$  is not only located at  $t = 0$ , i.e.  $T = nT_1$  when the parameter  $(a_2, \dots, b_{N-1}) \neq (0, \dots, 0)$ . The flatness location moves at a time  $T_n^*$  near  $nT_1$ . Consequently,  $b_N = B_N(t; a_2, \dots, b_{N-1})$  has to be explored near  $t_n^* = T_n^* - nT_1$ . The time  $t_n^*$  or equivalently  $T_n^* = nT_1 + t_n^*$  has to satisfy

$$0 = \dot{B}_N(nT_1 + t_n^*), \quad b_N^* = B_N(t_n^*; a_2, \dots, b_{N-1}). \quad (5.13)$$

Consequently,  $b_N^*$  which is  $B_N$  evaluated at time  $t_n^*$ , is a function of  $t_n^*$  and the other free parameters.

Note that Equation (5.13) is related to the grazing contact condition  $0 = \dot{u}_N(nT_1 + t_n^*; a_2, \dots, b_{N-1}, b_N^*)$ . The quantities  $t_n^*$  and  $b_N^*$  are defined with respect to the free modal coordinates  $(a_2, \dots, b_{N-1})$ . From the differentiation of Equation (5.10) with respect to  $t$ , the first and second time derivatives of  $B_N$  satisfy

$$0 = -d\omega_1 \sin \omega_1 T - \sum_{k=2}^N a_k (\omega_k \sin \omega_k T - \omega_1 \sin \omega_1 T) + \sum_{k=1}^N b_k \omega_k \cos \omega_k T + \dot{B}_N \sin \omega_N T \quad (5.14)$$

$$0 = d\omega_1^2 \cos \omega_1 T + \sum_{k=2}^N a_k (\omega_k^2 \cos \omega_k T - \omega_1^2 \cos \omega_1 T) + \sum_{k=1}^N b_k \omega_k^2 \sin \omega_k T - 2\omega_N \dot{B}_N \cos \omega_N T - \ddot{B}_N \sin \omega_N T. \quad (5.15)$$

Equation (5.14) defines  $T_n^*$  by saying that  $\dot{B}_N = 0$ . Equation (5.15) is then used to check that  $\ddot{B}_N \neq 0$  near  $T_n^*$  to get the parabolic flatness of  $B_N$ . Accordingly,  $T_n^*$  is defined by the implicit equation

$$0 = -d\omega_1 \sin \omega_1 T_n^* - \sum_{k=2}^N a_k (\omega_k \sin \omega_k T_n^* - \omega_1 \sin \omega_1 T_n^*) + \sum_{k=1}^N b_k \omega_k \cos \omega_k T_n^*, \quad (5.16)$$

where the last modal coordinates  $b_N$  has to be replaced by the function  $B_N(T_n^*; a_2, \dots, b_{N-1})$ . As a consequence,  $T_n^* = T_n^*(a_2, \dots, b_{N-1})$  is solution to the longer equation

$$d\omega_1 \sin \omega_1 T_n^* - B_N(T_n^*; a_2, \dots, b_{N-1}) \cos \omega_N T_n^* = - \sum_{k=2}^N a_k (\omega_k \sin \omega_k T_n^* - \omega_1 \sin \omega_1 T_n^*) + \sum_{k=1}^{N-1} b_k \omega_k \cos \omega_k T_n^*. \quad (5.17)$$

Notice that the dependence on  $n$  only arrives when we look for a  $T^*$  near  $nT_1$ . In order to see that Equation (5.16) admits a unique solution  $T_n^*$  near  $nT_1$ , we evaluate Equations (5.14) and (5.15) when  $T = nT_1$  and  $\dot{B}_N = 0$ :

$$0 = - \sum_{k=2}^N a_k \omega_k \sin \omega_k nT_1 + \sum_{k=1}^N b_k \omega_k \cos \omega_k nT_1, \quad (5.18)$$

$$0 = d\omega_1^2 + \sum_{k=2}^N a_k (\omega_k^2 \cos \omega_k T - \omega_1^2) + \sum_{k=1}^N b_k \omega_k^2 \sin \omega_k nT_1 - \ddot{B}_N \sin \omega_N nT_1. \quad (5.19)$$

The functions  $b_N^*$  and  $\ddot{B}_N$  are explicit at  $T = nT_1$  via Equations (5.18) and (5.19):

$$(\omega_N \cos \omega_N nT_1) b_N^* = \sum_{k=2}^N a_k \omega_k \sin \omega_k nT_1 - \sum_{k=1}^{N-1} b_k \omega_k \cos \omega_k nT_1 \quad (5.20)$$

$$(\sin \omega_N nT_1) \ddot{B}_N = d\omega_1^2 + \sum_{k=2}^N a_k (\omega_k^2 \cos \omega_k nT_1 - \omega_1^2) + \sum_{k=1}^N b_k \omega_k^2 \sin \omega_k nT_1 = d\omega_1^2 + \mathcal{O}(\varepsilon). \quad (5.21)$$

The critical value  $b_N^*$  of  $b_N$  is a linear function of  $(a_2, \dots, b_{N-1})$  but a nonlinear function of  $T_n^*$ . The sign of the second time derivative of  $B_N$  is the same as the sign of  $\sigma_n = \sin \omega_N nT_1$ . The expansion has to be achieved near  $T^* = nT_1 + t^*$ , i.e.,

$$T = nT_1 + t = nT_1 + t_n^* + \Delta t = T_n^* + \Delta t, \quad \Delta t = t - t^* = T - T_n^*. \quad (5.22)$$

Hence, the Taylor expansion of  $B_N$  with respect to  $t$  near  $t_n^* = T_n^* - nT_1$  yields a quadratic remainder expressed in the integral form with non-negative quantities:

$$\begin{aligned} 0 &\leq \sigma_N (B_N(T^* + \Delta t; a_2, \dots, b_{N-1}) - b_N^*(a_2, \dots, b_{N-1})) \\ &= |b_N - b_N^*(a_2, \dots, b_{N-1})| = (\Delta t)^2 R_n(\Delta t; a_2, \dots, b_{N-1}) \end{aligned} \quad (5.23)$$

with

$$R_n(\Delta t; a_2, \dots, b_{N-1}) = \int_0^1 (1-s) |\ddot{B}_N|(T^* + s\Delta t; a_2, \dots, b_{N-1}) ds > 0. \quad (5.24)$$

where  $R_n(0; 0, \dots, 0) = d\omega_1^2/2 > 0$ , since  $T^*(0, \dots, 0) = nT_1$ . The positive sign of  $R_n$  is preserved for  $t$  and  $\varepsilon$  small enough.

**FRT as a function of modal coordinates** The quantity  $\text{FRT} - nT_1 = t$  is expressed as a function of  $(a_2, \dots, b_N)$ . As in [12], the time  $t$  is now expressed as a function of the modal coordinates with a special role for  $b_N - b_N^*$  which displays a parabolic flatness, see Equation (5.23). To this end, the function

$$\psi_n(\Delta t; a_2, \dots, b_{N-1}) = \Delta t \sqrt{R_n(\Delta t; a_2, \dots, b_{N-1})} \quad (5.25)$$

is used where  $(a_2, \dots, b_{N-1})$  only plays the role of a vector parameter. The information from Equation (5.23) is only in terms of the square of the time in the form  $|b_N - b_N^*| = (\Delta t)^2 R_n$ . The sign of  $\Delta t$  is given by the admissibility condition that the velocity of the last mass is non-negative at the first return time:

$$0 \leq \dot{u}_N(T^* + \Delta t - 0) = \dot{u}_N(T^* - 0) + \ddot{u}_N(T^* - 0)\Delta t + \mathcal{O}(\Delta t)^2 = 0 - d\omega_1^2 \Delta t + \mathcal{O}(\Delta t)^2. \quad (5.26)$$

That means that, locally,  $\Delta t \leq 0$ . This information is important and rewritten as

$$\Delta t \leq 0 \quad \Rightarrow \quad \text{FRT} \leq T^* = nT_1 + t^*. \quad (5.27)$$

The function  $\psi_n$  is a diffeomorphism in  $\Delta t$ , eventually reducing the time interval, as shown later. From Equation (5.23),  $|b_N - b_N^*| = (\psi_n(\Delta t))^2$ , thus,

$$\Delta t = \text{FRT}(a_2, \dots, b_N) - T^* = g_n \left( -\sqrt{|b_N - b_N^*(a_2, \dots, b_{N-1})|}; a_2, \dots, b_{N-1} \right), \quad (5.28)$$

where  $g_n(A_1; a_2, \dots, b_{N-1}) = \psi_n^{(-1)}(A_1; a_2, \dots, b_{N-1})$  on  $\mathcal{V}_n^\varepsilon$ , the inverse of  $\psi_n$  being taken with respect to the first argument  $A_1$  only.

Let us show that  $\psi_n$  is a diffeomorphism with respect to its first argument and uniformly with respect to the other arguments  $(a_2, \dots, b_{N-1})$ . With Lemma 5.2, the proof is complete.  $\square$



**Lemma 5.2 — Diffeomorphism  $\psi_n$  with respect to the time.** *Let  $n$  be fixed. There exist  $\varepsilon_3 > 0$  and  $\tau > 0$  such that for all  $\varepsilon < \varepsilon_3$  and for  $|t| \leq \tau$ ,  $\psi_n$  is a time diffeomorphism with the property*

$$0 < \inf_{\mathcal{V}_n^\varepsilon} |\dot{\psi}_n|. \quad (5.29)$$

*Proof.* The function  $\psi_n$  is smooth as soon as  $R_n > 0$  since  $R_n$  is smooth. The positive sign of  $R_n$  comes from (1)  $R_n(0; 0, \dots, 0) = d\omega_1^2/2 > 0$  and (2) the estimate on  $\dot{\varphi}$ , eventually reducing  $\varepsilon$  and the size of the time intervals. Also,  $\psi_n$  is a diffeomorphism with respect to  $t$  as soon as

$$\dot{\psi}_n > 0 \quad \text{with} \quad \dot{\psi}_n = \sqrt{R_n} + t \frac{\dot{R}_n}{2\sqrt{R_n}}. \quad (5.30)$$

First, with  $\varepsilon$  small enough well chosen, we have  $\inf R_n(t; a_2, \dots, b_n) > \omega_1^2 d/4 > 0$ . For  $\tau > 0$  small enough can be chosen such that the positive sign of  $\dot{\psi}_n$  still holds and for instance for all  $|t| < \tau$ ,  $\inf \dot{\psi}_n(t; a_2, \dots, b_n) > \omega_1 \sqrt{d/8} > 0$  which concludes the proof.  $\square$

## 5.2 Square root singularity for the First Return Map

The singularity of the FRM near GLMs is now discussed. This singularity plays a role in the expected instability of GLMs [10]. The investigation of solution stability for smooth dynamical systems is well known [3, 8]. However, in the present work it involves a square root singularity in the Poincaré map [2]. A square root singularity for nonsmooth systems is intuitively worse than a positive Lyapunov exponent for smooth systems. The instability near grazing orbits was already conjectured in the literature [2, 10, 11] but never mathematically completely proven. The present paper extends to  $\mathcal{V}^\varepsilon$ , a whole neighborhood of the GLMs in the contact hyperplane  $\mathcal{H}$ , the square root expansion restricted in  $\mathcal{V}_1^\varepsilon$  in [12]. The expansions are exactly the same as in [12], only the remainder depends on the number of loops, that is on  $\mathcal{V}_n^\varepsilon$ . Let us briefly recall the main points from [12, Section 6.2]:

- The singularity in the FRM only comes from the singularity in FRT. The flow of the free dynamics is denoted by  $S(t, W_0)$  which is simply a matrix with almost periodic coefficients. Furthermore, all time derivatives of its coefficients are bounded for all time. In other words,  $S(t, W_0)$  is the solution of the free linear dynamics at time  $t$  with initial data  $W_0$ . The FRM can be expressed as suggested in [12] although with  $\mathcal{H}_\varphi$  on the side of  $\mathcal{H}$  where  $v_N \leq 0$

$$\mathbf{FRM}(W_0) = \mathbf{Sym} \cdot S(\mathbf{FRT}(W_0), W_0) \quad (5.31)$$

where  $W_0 \in \mathcal{H}_\varphi$  and  $\mathbf{Sym}$  is the constant matrix of the symmetry representing the action of the impact law which changes only one physical components  $v_N(t+0) = -v_N(t-0)$ .

- The FRT has a different expansion in each  $\mathcal{V}_n^\varepsilon$  for all  $0 < n < \infty$ . Note that, unlike FRT,  $\mathbf{FRM}$  is a continuous function and can be expressed in modal coordinates as

$$\mathbf{FRM}(a_2, \dots, b_N) = W_0^G - \gamma \sqrt{|b_N - b_N^*|} \mathbf{Sym} \cdot \mathbf{C} + \mathcal{O}(\|(a_2, \dots, b_N)\|) \quad (5.32)$$

where  $W_0^G$  stands for the GLM in modal coordinates; the vector  $\mathbf{C}$  is constituted of the so-called square root instability coefficients [12, Section 6.2]. It should be noted that the size of the remainder depends on  $\mathcal{V}_n^\varepsilon$ . Our study does not control the remainder uniformly with respect to  $n$ .

- The vector  $\mathbf{C}$  is never the zero vector for a GLM, see [12, last Remark of Section 6], and the square root singularity of the FRM thus really occurs.

In order to thoroughly explore the expected instability of a GLM, a strategy is to understand what occurs on  $\mathcal{H}_\varphi$  because during the free flight, the dynamics known to be neutrally stable.

The main term of the asymptotic expansion of the FRM is the same in all  $\mathcal{V}_n^\varepsilon$  but the remainder depends on  $\mathcal{V}_n^\varepsilon$  and is not uniformly bounded. In [12], the instability of an orbit always staying in  $\mathcal{V}_1^\varepsilon$  (as defined in the present article) is established. However, it is not clear whether such an orbit always stays in  $\mathcal{V}_1^\varepsilon$ . This means that we cannot directly use the framework of the square root instability mathematically justified in [12]. A complete mathematical proof of the instability of the GLMs is still an open problem but the present study brings more arguments towards the instability of GLMs after [2, 10, 11, 12].

## 6 Conclusion

This paper expands on the general study of the First Return Time (FRT) to the Poincaré section [12] for the local study in the vicinity of a Grazing Linear Mode (GLM). The proofs exposed in [12] are simplified and new results listed below are obtained.

1. This study describes the local behavior of the FRT in the whole neighborhood  $\mathcal{V}^\varepsilon$  of a grazing linear mode. In [12], the study was conducted only for a perturbation of the GLM with a perturbed linear period which corresponds, in the present work, to  $\mathcal{V}_1^\varepsilon$  only.
2. The FRT lies in the vicinity of a multiple of the fundamental period of a GLM, which implies  $\mathcal{V}^\varepsilon = \bigcup_{0 < n < +\infty} \mathcal{V}_n^\varepsilon$ .
3. Explicit and simple characterization in modal coordinates of the set  $\mathcal{V}_\infty^\varepsilon \neq \emptyset$  corresponding to orbits with only one contact for all time is given. In particular, this set is never empty which is not proven in [12]. This is a useful tool to prove that infinitely many  $\mathcal{V}_n^\varepsilon$  are not empty.
4. The fact that  $\mathcal{V}_0^\varepsilon = \emptyset$  that means there is no micro-contact for sufficiently small perturbations. This is an unusual result for a dissipative contact law with chattering. It emerges here as a property of the conservative impact law [1].
5. The asymptotic expansion of the FRT in all  $\mathcal{V}_n^\varepsilon$  is provided in a simplified way compared to [12] (which covers only  $n = 1$ ) and the same for the First Return Map.

## A Solutions with only one contact for all time

The solutions with one and only one contact for all time play an important role in this study. We recall that there is no solution with  $k$  contacts for all time with  $1 < k < +\infty$  [12]. It is quite a strange feature that the First Return Time and the First Return Map are not defined only for  $k = 1$  since for all other orbits that touch the contact hyperplane once, they touch it infinitely many times [12].

We provide additional information on  $\mathcal{H}_1^0$  complementing [12]. In particular, in [12], it is not clear that  $\mathcal{H}_1^0$  is not empty. As shown below, it is actually never empty. The set is the convex combination of the states on  $\mathcal{H}$  corresponding to GLMs without the GLMs themselves.

*Proof.* The proof of Proposition 2.1 is derived using the results in [12] and the well-known properties of almost periodic functions [6]. Fixing the sole contact time at  $t = 0$ , the  $a_k$  modal coordinates of such a  $u_N$  are necessarily non-negative. All  $b_k = 0$ ,  $k = 1, \dots, N$  and  $u_N$  reads

$$u_N(t) = \sum_{k=1}^N a_k \cos \omega_k t, \quad a_k \geq 0, \quad k = 1, \dots, N, \quad \sum_{k=1}^N a_k = d. \quad (\text{A.1})$$

Moreover, at least two  $a_k$  are strictly positive. If only one  $a_k$  is positive and then equal to  $d$ , we recognize a GLM that periodically features a contact. This is the reason why the intricate nature

of the dynamics starts with  $N = 2$ . This is also a consequence of the  $\mathbb{Z}$ -independence of the frequencies  $\omega_k$ ,  $k = 1, \dots, N$  [6] and

$$\sup_{\mathbb{R}} u_N = \sum_k a_k. \quad (\text{A.2})$$

Moreover, at  $t = 0$ ,  $u_N(0) = \sum_k a_k = d$ . From the  $\mathbb{Z}$ -independence of the frequencies, the functions  $\cos \omega_k t$  reach their maximum together at time  $t = 0$  and never come back at the same time to their maximum 1 so  $u_N(t) < d$  for all  $t \neq 0$ . We have then shown that such  $u_N$  in Equation (A.1) corresponds to solutions having their maximum  $d$  only once at  $t = 0$ .

Let us recall why  $a_k \geq 0$  and  $b_k = 0$  for all  $k$ . A general solution of the linear differential system has its last component of the form

$$u_N(t) = \sum_{i=1}^N a_k \cos \omega_k t + b_k \sin \omega_k t \quad (\text{A.3})$$

which induces

$$u_N(0) = \sum_{i=1}^N a_k = d. \quad (\text{A.4})$$

Moreover, the supremum on  $\mathbb{R}$  has to be  $d$  [6], that is

$$\sup_{\mathbb{R}} u_N = \sum_k \sqrt{a_k^2 + b_k^2} = d. \quad (\text{A.5})$$

The only solution is thus  $b_k = 0$  and  $a_k = |a_k| \geq 0$ ,  $k = 1, \dots, N$  and the proof of Proposition 2.1 is complete.  $\square$

## B Comparisons with Nordmark's seminal works

In his seminal works on the square root instability, Nordmark considered a general smooth flow with a periodic grazing orbit [11, 10, 7] (see also [5, 4, 2]). He suggested the use of a Poincaré section, hereafter denoted NPS, transverse to a fixed periodic orbit. The NPS is located anywhere on this periodic orbit except at the contact hyperplane. On one “half” of the NPS, the corresponding First Return Map (denoted NFRM below) features a square-root singularity, while it does not on the second half.

In the present work, the Poincaré section is exactly the contact hyperplane. It seems to be the worst possible choice from the classical smooth analysis point-of-view. However, it is a natural choice to observe the effects of the nonsmooth contact conditions on the dynamics. It is also the simplest Poincaré section to compute the FRT. Moreover, we consider a linear free flow while Nordmark considered a general smooth nonlinear free flow. In [12] the square root singularity is exhibited on the contact hyperplane which corresponds exactly to the first “half” part of NPS with the singularity. In the present paper, we describe what happens in the other “half” part of NPS, near GLMs. It is partitioned in infinitely many subsets  $\mathcal{V}_n^\varepsilon$ , where, on each of them, the corresponding orbits do exactly  $n$  loops (for the last mass) before returning to the contact hyperplane and the square-root singularity emerges exactly after  $n$  loops. This means that the  $n$  iterated NFRM function is singular on this subset. An important feature in the present paper is the simple geometric description and proof of existence of the orbits that touch the contact hyperplane only once in the past and in the future, see Appendix A. This is obviously made possible because the free dynamics is linear.

The set  $\mathcal{V}_\infty$  is key to prove the existence of orbits featuring very large numbers of loops. During the free dynamics, the orbits with large numbers of loops introduced in the present paper stay close to the solutions starting from  $\mathcal{V}_\infty$ . A natural question is the following: for any  $n > 1$ , are there orbits exactly featuring  $n$  loops? It is an open problem. On the other side, the existence of solutions with  $n$  as large as desired shows that the instability problem is challenging. The free dynamics is neutrally stable, and no instability can be generated before the orbit touches the contact hyperplane. Since the duration of the free dynamics can be as big as we want, no clear instability with a precise rate can be exhibited.

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