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# RESOLVING A CLEARING MEMBER'S DEFAULT

## A RADNER EQUILIBRIUM APPROACH

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**ABSTRACT.** For vanilla derivatives that constitute the bulk of investment banks' hedging portfolios, central clearing through central counterparties (CCPs) has become hegemonic. A key mandate of a CCP is to provide an efficient and proper clearing member default resolution procedure. When a clearing member defaults, the CCP can hedge and auction or liquidate its positions. The counterparty credit risk cost of auctioning has been analyzed in terms of XVA metrics in Bastide, Crépey, Drapeau, and Tadese (2023). In this work we assess the costs of hedging or liquidating. This is done by comparing pre- and post-default market equilibria, using a Radner equilibrium approach for portfolio allocation and price discovery in each case. We show that the Radner equilibria uniquely exist and we provide both analytical and numerical solutions for the latter in elliptically distributed markets. Using such tools, a CCP could decide rationally on which market to hedge and auction or liquidate defaulted portfolios.

**Keywords:** financial markets, exchanges, central counterparties (CCPs), default resolution, Radner equilibrium, price impact, entropic risk measure, expected shortfall, hedging, auctioning, liquidation, market risk, credit risk, cross-valuation adjustments (XVA).

**2020 Mathematics Subject Classification:** 91B50, 91B05, 91B26, 91G20, 91G40, 91G45.

### 1. INTRODUCTION

Financial actors can discover counterparties for their transactions by participating in an exchange. In the case of derivatives, an exchange is backed by a central counterparty (CCP). As surveyed in Gregory (2014, Sections 13.3.5, 13.3.6 and 13.4) and Menkveld and Vuillemeijer (2021), a CCP transforms counterparty credit risk into liquidity risk by netting and managing multiple flows of collateral, including a default fund that is pooled among the clearing members. A CCP is also responsible for rewiring or liquidating the CCP portfolios

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Python notebooks reproducing the results of this paper are available on <https://github.com/mekonnenta/CCP-Radner.git>

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of a defaulted clearing member within a few days of the default. As emphasized in Table 1, an ordered resolution of clearing members' defaults might actually be the most tangible

	<b>Pros</b>	<b>Cons</b>
<b>Netting</b>	Multilateral netting benefit	Loss of bilateral netting across asset classes
<b>Counterparty credit risk and liquidity risk</b>	Reduced default risk of the CCP itself and reduced "domino effects" between members	Concentration risk if a major CCP were to default. Joint membership and feedback liquidity issues. - about 30 major CCPs today and only a few prominent ones (CME, LCH, Eurex, ICE,...)
<b>Information</b>	Better information of the regulator via access to the repositories of the CCP	Opacity of the default fund for the clearing members, which are not in a position of estimating their risks and the corresponding costs with accuracy
<b>Costs</b>	Default resolution cheaper: - Bilateral trading means a completely arbitrary transaction network. - <i>An orderly default procedure cannot be done manually. It requires an IT network, whether it is CCPs, block-chain technology,...</i>	High cost of raising funding initial margins (at least, if funded by unsecured borrowing)

TABLE 1. Pros and cons of CCPs (Gregory, 2014; Menkveld and Vuillemeij, 2021; Albanese, Armenti, and Crépey, 2020).

benefit of CCPs. However, a CCP is faced with several possibilities for settling the default of a clearing member. This paper provides quantitative tools to assess and compare the costs of alternative default resolution strategies.

So far, the optimal settlement of the portfolio of a clearing member has been mostly addressed from the financial engineering viewpoint of optimal liquidation schedules accounting for exogenous liquidity constraints (Avellaneda and Cont, 2013; Vicente, Cerezetti, Faria, Iwashita, and Pereira, 2015). In this paper, we look at the default resolution procedure from a less dynamic but more endogenous, economic equilibrium perspective, with offer and demand determined by clearing member strategies, positions, and preferences. Our default resolution market costs arise from the perturbation of the market equilibrium triggered by the default resolution procedure. To estimate these costs, one has to know the values of the traded assets as well as the positions of all market participants, in both the pre- and post-default markets. A classical method for valuing a derivative is to compute the expectation of its discounted payoffs under an equivalent martingale measure. But in a realistic, incomplete market, this strategy may not perform properly, because of the non-uniqueness of a pricing measure. An equilibrium approach may then be more appropriate. As surveyed in Magill and Shafer (1991) or Radner (1982), the concept of competitive or Walras equilibrium in economics was first stated in deterministic and static

settings. Debreu (1959) extends it to a stochastic framework based on a sequence of market future conditions, but with trading taking place only at time zero. Radner (1968, 1972) revisit the approach in a multi-period and stochastic setting, well suited to the pricing of financial assets, financial derivatives in particular, with inherently uncertain future cash flows. We use a Radner equilibrium approach to discover prices and reallocate a defaulted derivative portfolio among trading participants, under two (main) possible specifications: either the CCP hedges the portfolio by trading on an exchange, or it liquidates the portfolio on an exchange. The ensuing costs are determined by comparing pre- and post-default Radner equilibria. This is done in a one-period specification of the discrete-time Radner equilibrium approach of Cheridito, Horst, Kupper, and Pirvu (2015). Though the application of Radner equilibrium in static and dynamic problems is not new, using it to estimate CCP close-out costs appears to be a novel approach.

On top of hedging and liquidation, there is a third vertex to the triangle of the possible close-out procedures, namely auctions. Hedging and liquidation are driven by equilibria between the agents, whereby the market decides endogenously the optimal prices and asset allocation. Auctioning, instead, is of a game theoretical nature. Actual default resolution procedures are in fact a combination of the three, with implications not only in terms of price impact liquidity risk as emphasized in the above, but also of counterparty credit risk, the way addressed in the last section of the paper.

**Outline.** Section 2 provides our Radner equilibrium market model. Section 3 introduces the related comparative statics approach for the analysis of the market costs of hedging or liquidating a defaulted clearing member portfolio, either on the exchange of the CCP of the defaulter, or on an external exchange. Sections 4 and 5 detail these costs in the case of entropic and expected shortfall risk measures. Section 6 analyzes the additional impact of counterparty credit risk, based on XVA specifications detailed in Section A.

**Standing notation.** Given vectors  $x \in \mathbb{R}^m$  and  $y \in \mathbb{R}^n$  (understood as column matrices),  $x^\top$  is the transpose of  $x$  and  $(x, y)$  is the vector of  $\mathbb{R}^{m+n}$  formed by stacking  $x$  above  $y$ . We denote by  $\mathcal{N}_n(\mu, \Gamma)$ , the  $n$ -variate Gaussian distribution with mean  $\mu$  and covariance matrix  $\Gamma$ , and by  $\phi$  and  $\Phi$ , the standard univariate Gaussian probability density and cumulative density functions; by  $\mathcal{E}_n(\mu, \Gamma, \psi)$ , the  $n$ -variate elliptical distribution with mean  $\mu$ , covariance matrix  $\Gamma$ , and characteristic generator function  $\psi$ , by  $\mathcal{T}_n(\mu, \Gamma, \nu)$ , the  $n$ -variate Student  $t$ -distribution of degree of freedom  $\nu$  with mean  $\mu$  and covariance matrix  $\Gamma$ , and by  $t_\nu$  and  $T_\nu$ , the standard univariate Student  $t$  probability density and cumulative density functions of degree  $\nu$ . Throughout the paper,  $(\Omega, \mathcal{A}, \mathbb{P})$  denotes a fixed probability space, with expectation, variance and covariance operators  $\mathbb{E}$ ,  $\text{Var}$  and  $\text{Cov}$ ;  $\mathcal{L}^0$  and  $\mathcal{L}^1$  respectively denote the space of all the measurable and integrable random variables (identified in the  $\mathbb{P}$  almost sure sense),  $\mathcal{X}$  is a linear subspace of  $\mathcal{L}^1$  containing the constants. Capital letters denote random vectors with components in  $\mathcal{X}$ . Bold letters refer to the solution of a Radner equilibrium.

For a function  $f: \mathbb{R}^m \rightarrow \mathbb{R}$ , its directional derivative at  $x$  in the direction of  $y$  is defined as

$$\mathcal{D}_y f(x) = \lim_{\epsilon \searrow 0} \frac{f(x + \epsilon y) - f(x)}{\epsilon}; \quad (1.1)$$

for  $f$  convex, a point  $y \in \mathbb{R}^m$  is said to be a subgradient of  $f$  at  $x$ , denoted  $y \in \partial f(x)$ , if

$$f(z) \geq f(x) + y^\top(z - x), \quad z \in \mathbb{R}^m; \quad (1.2)$$

the convex conjugate  $f^*$  of  $f$  is defined as

$$f^*(y) = \sup \left\{ y^\top x - f(x); x \in \mathbb{R}^m \right\}, \quad y \in \mathbb{R}^m. \quad (1.3)$$

Let  $f_i: \mathbb{R}^m \rightarrow \mathbb{R}$  be convex functions, for  $i$  in a finite set  $E$ . The inf-convolution  $f$  of the  $f_i$  is defined as

$$f(x) = \inf \left\{ \sum_{i \in E} f_i(x_i); \sum_{i \in E} x_i = x \right\}, \quad x \in \mathbb{R}^m. \quad (1.4)$$

## 2. RADNER EQUILIBRIUM MARKET MODEL

We consider a one-period model of a financial exchange, with trading participants indexed by a finite set  $E$ . A unitary position in each of the assets traded on the exchange pays a vector of random payoffs  $P \in \mathbb{R}^m$  (with  $m \geq 1$ ) at the terminal time  $T$ . We assume that each participant  $i$  in  $E$  is endowed with a real valued random receivable  $R_i$  (i.e.  $R_i \geq 0$  means a cash flow promised to the participant  $i$ ). For each  $i \in E$ , the participant  $i$  hedges<sup>1</sup>  $R_i$  by entering a portfolio  $\mathbf{q}_i \in \mathbb{R}^m$  of traded assets. The corresponding market loss of member  $i$  is

$$-R_i + \mathbf{q}_i^\top (\mathbf{p} - P), \quad (2.1)$$

where  $\mathbf{p} \in \mathbb{R}^m$  is the vector of prices of the traded assets at initial time: see Table 2, where

$$\Gamma_i = \begin{bmatrix} \text{Var}(R_i) & \text{cov}_i \\ \text{cov}_i & \Gamma \end{bmatrix}. \quad (2.2)$$

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$\mathbf{p}$	equilibrium prices of the traded (hedging) assets
$\mathbf{q}_i$	equilibrium positions of participant $i$ in the traded (hedging) assets
$\mu$	the vector $\mathbb{E}[P]$
$\Gamma$	the matrix $\text{Cov}(P)$
$R_i$	receivable to be hedged by the participant $i \in E$
$\mu_i$	the vector $\mathbb{E}[(R_i, P)]$
$\text{cov}_i$	the vector $\text{Cov}(R_i, P) = (\text{Cov}(R_i, P_1), \dots, \text{Cov}(R_i, P_m))$
$\text{cov}$	$\sum_{i \in E} \text{cov}_i$
$\Gamma_i$	the covariance matrix $\text{Cov}((R_i, P))$ , i.e. of the vector $(R_i, P)$
$\rho_i$	risk measure of the participant $i$
$\varrho_i$	risk-aversion parameter of an entropic risk measure of the participant $i$
$\varrho$	the number $(\sum_{i \in E} (1/\varrho_i))^{-1}$
$\alpha_i$	confidence level for an expected shortfall risk measure of the participant $i$

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TABLE 2. Main notation relative to an exchange  $E$ .

For the monetary valuation of the risk of the participant  $i$ , we consider a law invariant risk measure  $\rho_i: \mathcal{X} \rightarrow \mathbb{R}$  monotonous, convex<sup>2</sup>, translation equivariant<sup>3</sup> in the sense that<sup>4</sup>, and normalized ( $\rho_i(0) = 0$ ). We want to determine the portfolios  $\mathbf{q}_i$  and the prices  $\mathbf{p}$  endogenously as a Radner equilibrium driven by the offer and demand of all the participants to the exchange:

<sup>1</sup>unhedged market risk can generate significant regulatory capital requirements, rendering the contractual commitments non viable for the trading participant (Bank For International Settlements, 2019, Section MAR23, pp. 64 and 93).

<sup>2</sup>in the sense of McNeil, Frey, and Embrechts (2015, Definition 2.24, p. 74).

<sup>3</sup>also known as translation invariance.

<sup>4</sup>see Pflug and Römisch (2007, Definition 2.2 (i), page 29)  $\rho_i(L + m) = \rho_i(L) + m$  holds for any  $L$  in  $\mathcal{X}$  and real constant  $m$ .

**Definition 2.1.** A matrix of positions  $(\mathbf{q}_i)_{i \in E}$  and a price vector  $\mathbf{p} \in \mathbb{R}^m$  form a Radner equilibrium on  $E$  if

- (optimality condition relative to each market participant  $i \in E$ )

$$\rho_i(-R_i + \mathbf{q}_i^\top (\mathbf{p} - P)) \leq \rho_i(-R_i + q_i^\top (\mathbf{p} - P)), \quad q_i \in \mathbb{R}^m, \quad (2.3)$$

- (zero clearing condition)

$$\sum_{i \in E} \mathbf{q}_i = 0. \blacksquare \quad (2.4)$$

**Remark 2.2.** Since  $R_i$  is assumed to be exogenously given, the price of the corresponding receivable to the participant  $i$  is not part of the equilibrium. We say nothing on this price (assumed exogenously given and in fact implicitly part of  $R_i$  itself in our setup) in the paper, nor on the way it could be impacted (in our setup it is simply not) by the instant default of a participant to the exchange. ■

**2.1. Generic Results.** For each trading participant  $i$  in  $E$ , we consider the convex function  $r_i: \mathbb{R}^m \rightarrow \mathbb{R}$  defined as

$$r_i(q_i) = \rho_i(-R_i - q_i^\top P), \quad q_i \in \mathbb{R}^m.$$

By translation equivariance of  $\rho_i$ , the member  $i$  optimality condition (2.4) can be rewritten as

$$r_i(q_i) \geq r_i(\mathbf{q}_i) + (-\mathbf{p})^\top (q_i - \mathbf{q}_i), \quad q_i \in \mathbb{R}^m,$$

i.e., by (1.2),

$$-\mathbf{p} \in \partial r_i(\mathbf{q}_i). \quad (2.5)$$

By Rockafellar (1970, Theorem 23.5), this is in turn equivalent to

$$r_i(\mathbf{q}_i) = -\mathbf{q}_i^\top \mathbf{p} - r_i^*(-\mathbf{p}), \quad (2.6)$$

where  $r_i^*$  is the convex conjugate (1.3) of  $r_i$ . Note that

$$\begin{aligned} -q_i^\top p - r_i^*(-p) &\leq r_i(q_i), \quad q_i, p \in \mathbb{R}^m, \\ &\text{with equality if and only if } -p \in \partial r_i(q_i), \end{aligned} \quad (2.7)$$

by Rockafellar (1970, Theorem 23.5).

Lemma 2.1 and Theorems 2.2-2.3 below are variants, for a single period model but with unbounded  $(R_i, P)$  (as we want to endorse elliptical factor models later in the paper), of Cheridito, Horst, Kupper, and Pirvu (2015, Theorems 1 and 2). Radner equilibria admit the following dual characterization in terms of the inf-convolution (1.4)  $r$  of the  $r_i$ .

**Lemma 2.1.** A matrix of positions  $(\mathbf{q}_i)_{i \in E}$  and a price vector  $\mathbf{p} \in \mathbb{R}^m$  form a Radner equilibrium on  $E$  if and only if

- (i)  $-\mathbf{p} \in \partial r(0)$ ,
- (ii)  $r(0) = \sum_{i \in E} r_i(\mathbf{q}_i)$ , and
- (iii)  $\sum_{i \in E} \mathbf{q}_i = 0$ .

**Proof.** Let  $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$  be a Radner equilibrium as per Definition 2.1. The zero clearing condition (2.4) yields (iii). By Rockafellar (1970, Theorem 16.4, page 145), the convex conjugate of the inf-convolution of proper convex functions is the sum of the corresponding conjugates, i.e.

$$r^*(-\mathbf{p}) = \sum_{i \in E} r_i^*(-\mathbf{p}).$$

Summing the expression (2.6) across all  $r_i$  and using (iii) gives

$$\sum_{i \in E} r_i(\mathbf{q}_i) = 0 - r^*(-\mathbf{p}) \leq r(0),$$

where the inequality holds by definition (1.3) of the convex conjugate of  $r$ . By definition of  $r(0)$ , the above inequality becomes equality, i.e.

$$r(0) = -r^*(-\mathbf{p}) = \sum_{i \in E} r_i(\mathbf{q}_i).$$

Hence (ii) holds and so does also (i), in view of the equivalence between (2.5) and (2.6), here applied to  $r$  (instead of  $r_i$  there).

Conversely, suppose that  $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$  satisfies (i)–(iii). (iii) is the zero clearing condition (2.4), whereas (i) implies via (2.6) applied to  $r$  that

$$r(0) = -r^*(-\mathbf{p}) = \sum_{i \in E} \left( -\mathbf{q}_i^\top \mathbf{p} - r_i^*(-\mathbf{p}) \right). \quad (2.8)$$

By (2.7) and (2.8),  $-\mathbf{p} \notin \partial r_i(\mathbf{q}_i)$  for some  $i \in E$  would imply that  $r(0) < \sum_{i \in E} r_i(\mathbf{q}_i)$ , contradicting (ii). Hence (2.5), which is equivalent to the member  $i$  optimality condition (2.4), holds for each  $i \in E$ . ■

Since the subgradient of a real valued convex function is non-empty, Lemma 2.1 implies that a Radner equilibrium exists if and only if the inf-convolution  $r$  is attained at 0. It also implies that, whenever a Radner equilibrium exists, the optimal price is unique if and only if  $r$  is differentiable at 0.

**Theorem 2.2.** *If  $\rho_i$  is sensitive to large losses, i.e. <sup>5</sup>  $\lim_{\lambda \rightarrow \infty} \rho_i(\lambda L) = \infty$  for all  $L \in \mathcal{X}$  such that  $\mathbb{P}[L > 0] > 0$ ,  $i$  in  $E$ , then there exists a Radner equilibrium on  $E$ .*

**Proof.** Let  $Q$  be the set of vector of positions  $(q_i)_{i \in E}$  satisfying the zero clearing condition, i.e.

$$Q = \left\{ q \in \mathbb{R}^{m|E|} : q^\top b^k = 0, k = 1, \dots, m \right\},$$

where  $b^k$  is a vector in  $\mathbb{R}^{m|E|}$  such that, for all  $j = 0, \dots, |E| - 1$ , the  $k + jm$  entries of  $b^k$  equal 1 and all the other entries of  $b^k$  are 0. Note that  $Q$  is a non-empty closed convex polyhedral subset of  $\mathbb{R}^{m|E|}$ . Let  $\mu = \mathbb{E}[P]$ . By Dalang, Morton, and Willinger (1990, Theorem 3.3),  $q_i^\top (\mu - P) = 0$  almost surely holds or  $\mathbb{P}[q_i^\top (\mu - P) > 0] > 0$  holds, for any  $q_i$  in  $\mathbb{R}^m$ . The closed proper convex function

$$\mathbb{R}^{m|E|} \ni q = (q_1, \dots, q_{|E|}) \mapsto \sum_{i \in E} \rho_i(-R_i + q_i^\top (\mu - P)) \in \mathbb{R} \quad (2.9)$$

is such that

$$\inf_{q \in Q} \beta(q) = \inf_{\sum_{i \in E} q_i = 0} \sum_{i \in E} r_i(q_i).$$

In view of the comment preceding the statement of the theorem, it suffices to show that  $\beta$  attains its minimum on  $Q$ . Let  $B = \{q \in \mathbb{R}^{m|E|} : \beta(q) \leq \beta(0)\}$ , with recession cone  $0^+B = \{y \in \mathbb{R}^{m|E|} : b + \lambda y \in B, \forall \lambda \geq 0, \forall b \in B\}$ . Let  $0^+\beta = \{y \in \mathbb{R}^{m|E|} : \beta(0^+(y)) \leq 0\}$  denote the recession cone of  $\beta$ , where  $\beta(0^+)$  is its recession function<sup>6</sup>. By Rockafellar (1970, Theorem 8.7, page 70),  $0^+B = 0^+\beta$ . Since  $B$  is a closed convex set containing the origin, Rockafellar (1970, Corollary 8.3.2, page 64) yields

$$0^+B = \{y \in \mathbb{R}^{m|E|} : \lambda y \in B \quad \forall \lambda > 0\}. \quad (2.10)$$

Let  $y = (y_1, \dots, y_{|E|}) \in \mathbb{R}^{m|E|} \setminus \{0\}$  (where each  $y_i$  is in  $\mathbb{R}^m$ ). (i) If  $\mathbb{P}[y_i^\top (\mu - P) > 0] > 0$  holds for some  $i \in E$ , then, by the sensitivity to large losses condition on  $\rho_i$ ,  $\beta(\lambda y)$  goes to infinity as  $\lambda$  goes to infinity, which implies that  $y \notin 0^+B$ . (ii) If, instead,  $y_i^\top (\mu - P) = 0$

<sup>5</sup>see Cheridito et al. (2015, Section 2.3).

<sup>6</sup>the recession function  $\beta(0^+)$  is the map defined on  $\mathbb{R}^{m|E|}$  as  $\beta(0^+(y)) = \inf \{m \in \mathbb{R} : (y, m) \in 0^+\text{epi } \beta\}$ , where  $\text{epi } \beta = \{(q, n) \in \mathbb{R}^{m|E|} \times \mathbb{R} : n = g(q)\}$ .

holds for all  $i \in E$ , then  $\pm y \in 0^+B$  hold by definitions (2.9) of  $\beta$ ,  $B$  and (2.10) of  $0^+B$ . In particular, for any  $y \in \mathbb{R}^{m|E|} \setminus \{0\}$  such that  $y \in 0^+\beta$ , we have  $-y \in 0^+\beta$ . Hence, by Rockafellar (1970, Corollary 8.6.1, page 69), every direction of recession is a direction in which  $\beta$  is constant. Following Rockafellar (1970, Theorem 27.3, page 267), in either case (i) or (ii),  $\beta$  attains its minimum over  $Q$ . ■

**Remark 2.3.** Let  $\mathcal{X}$  be given as the Orlicz heart corresponding to the Young function  $\theta: [0, \infty) \rightarrow [0, \infty)$  given by  $\theta(t) = \exp(t-1) - \exp(-1)$ , i.e.

$$\mathcal{X} = \{L \in \mathcal{L}^0: \mathbb{E}[\theta(c|L|)] < \infty, c > 0\} \subseteq \mathcal{L}^1,$$

where  $\mathcal{L}^0$  is the space of all real valued measurable random variables. An entropic risk measure of the form, for some  $\varrho_i > 0$ ,

$$\rho_i(L) = \frac{1}{\varrho_i} \ln(\mathbb{E}[\exp(\varrho_i L)]), \quad L \in \mathcal{X}, \quad (2.11)$$

is sensitive to large losses (Cheridito et al., 2015, Section 2.3). ■

Regarding the uniqueness of an optimal solution:

**Theorem 2.3.** Let  $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$  be a Radner equilibrium on  $E$ .

(i) If  $r_i$  is differentiable at  $\mathbf{q}_i$  for some  $i$  in  $E$ , then  $\mathbf{p}$  is unique.

(ii) For any  $i \in E$ , if  $r_i$  is differentiable and strictly convex on  $\mathbb{R}^m$  then  $\mathbf{q}_i$  is unique.

**Proof.** Let  $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$  be optimal. If  $r_i$  is differentiable at  $\mathbf{q}_i$ , then Lemma 2.1(i) together with Rockafellar (1970, Theorems 23.2, p. 216 and 25.2, page 244) yield

$$\mathcal{D}_x r_i(\mathbf{q}_i) = x^\top(-\mathbf{p}) \leq \mathcal{D}_x r(0), \quad (2.12)$$

where  $\mathcal{D}_x r_i(\mathbf{q}_i)$  is the directional derivative (1.1) of  $r_i$  at  $\mathbf{q}_i$  along  $x$ . Take  $\hat{q}_i = \mathbf{q}_i + \epsilon x$  and  $\hat{q}_j = \mathbf{q}_j$  for all  $j \neq i$ . By definition (1.4) of the inf-convolution,

$$r(\epsilon x) \leq \sum_{j \in E} r_j(\hat{q}_j) = \sum_{j \neq i} r_j(\mathbf{q}_j) + r_i(\mathbf{q}_i + \epsilon x).$$

This together with Lemma 2.1(ii) yields

$$\mathcal{D}_x r(0) = \lim_{\epsilon \searrow 0} \frac{r(\epsilon x) - r(0)}{\epsilon} \leq \lim_{\epsilon \searrow 0} \frac{\sum_{j \in E} r_j(\hat{q}_j) - r(0)}{\epsilon} = \mathcal{D}_x r_i(\mathbf{q}_i) \leq \mathcal{D}_x r(0),$$

where the second inequality is due to (2.12). Hence  $x \mapsto \mathcal{D}_x r(0)$  is linear. Thus, by Rockafellar (1970, Theorem 25.2, page 244),  $r$  is differentiable at 0, i.e.  $\partial r(0)$  is a singleton, which, in view of Lemma (2.1) (ii) implies (i). As for (ii), if  $r_i$  is a strictly convex and differentiable on  $\mathbb{R}^m$ , then it is closed and proper. Following Rockafellar (1970, Corollary 26.3.1, page 254),  $\partial r_i^*(-\mathbf{p})$  is a singleton and  $\mathbf{q}_i = \nabla r_i^*(-\mathbf{p})$  is unique, by (2.6). ■

**Remark 2.4.** Let  $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$  be optimal. Following Cheridito et al. (2015, Proposition 2), if a risk measure  $\rho_i$  is differentiable<sup>7</sup> at  $-R_i - \mathbf{q}_i^\top P$  for some  $i$  in  $E$ , then  $r_i$  is differentiable at  $\mathbf{q}_i$ , hence the optimal price  $\mathbf{p}$  is unique. ■

<sup>7</sup> $\rho_i$  is differentiable at  $L \in \mathcal{X}$  if there exist a random variable  $W \in \mathcal{X}^*$  (the dual space of  $\mathcal{X}$ ) such that (cf. (1.1))

$$\lim_{\epsilon \searrow 0} \frac{\rho_i(L + \epsilon Y) - \rho_i(L)}{\epsilon} = \mathbb{E}[YW] \quad Y \in \mathcal{X}.$$

In this case, we write  $W = \nabla \rho_i(L)$ .



**2.2. Results Specific to Entropic or Expected Shortfall Risk Measures.** With explicit solutions and regulatory standards in view, from now on,  $\rho_i$  is either an entropic or an expected shortfall risk measure. In elliptical markets, entropic or coherent<sup>8</sup> risk measures lead to analytical expressions for equilibria. We first consider the case of entropic  $\rho_i$  and normally distributed  $(R_i, P)$ .

**Proposition 2.4.** *Let  $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$ ,  $i$  in  $E$ , and  $\Gamma^9$  be invertible. If  $\rho_i(L) = \frac{1}{\varrho_i} \ln(\mathbb{E}[\exp(\varrho_i L)])$  for some  $\varrho_i > 0$ ,  $i$  in  $E$ , then*

$$\mathbf{q}_i = \Gamma^{-1} \left( \frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right), \quad i \in E, \quad \text{and} \quad \mathbf{p} = \mu - \varrho \text{cov}, \quad (2.13)$$

where  $\varrho = \left( \sum_{i \in E} \frac{1}{\varrho_i} \right)^{-1}$  and  $\text{cov} = \sum_{i \in E} \text{cov}_i$ , is a unique Radner equilibrium.

**Proof.** By Theorem 2.2 and Remark 2.3, there exists a Radner equilibrium. In view of (2.2),

$$-R_i - \mathbf{q}_i^\top P \sim \mathcal{N}_1 \left( -\mathbb{E}[R_i] - \mathbf{q}_i^\top \mu, \text{Var}(R_i) + 2\mathbf{q}_i^\top \text{cov}_i + \mathbf{q}_i^\top \Gamma \mathbf{q}_i \right). \quad (2.14)$$

The moment generating function of a standard normal variate  $L$  is  $\mathbb{R} \ni z \mapsto \mathbb{E}[\exp(zL)] = \exp(z\mathbb{E}[L] + \text{Var}(L)z^2/2)$ , hence  $\rho_i(L) = \mathbb{E}[L] + \varrho_i \text{Var}(L)/2$ . This and (2.14) yield

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \frac{\varrho_i \text{Var}(R_i)}{2} + \varrho_i q_i^\top \text{cov}_i + \frac{1}{2} \varrho_i q_i^\top \Gamma q_i \quad (2.15)$$

and

$$\nabla r_i(q_i) = -\mu + \varrho_i \text{cov}_i + \varrho_i \Gamma q_i, \quad i \in E. \quad (2.16)$$

The optimality condition relative to the participant  $i \in E$  yields

$$-\mathbf{p} = \nabla r_i(\mathbf{q}_i) = -\mu + \varrho_i \text{cov}_i + \varrho_i \Gamma \mathbf{q}_i,$$

hence

$$\mathbf{q}_i = \Gamma^{-1} \left( \frac{1}{\varrho_i} (\mu - \mathbf{p}) - \text{cov}_i \right). \quad (2.17)$$

On the other hand, the clearing condition yields

$$\sum_{i \in E} \mathbf{q}_i = 0 = \Gamma^{-1} \left( \frac{1}{\varrho} (\mu - \mathbf{p}) - \text{cov} \right),$$

which is equivalent to

$$\mathbf{p} = \mu - \varrho \text{cov}.$$

(2.17) in turn gives (2.13). ■

We now turn to the case where each  $\rho_i$  is an expected shortfall risk measure (McNeil et al., 2015, page 69)

$$\rho_i(L) = \mathbb{E}\mathbb{S}_{\alpha_i}(L) = \frac{1}{1 - \alpha_i} \int_{\alpha_i}^1 q_u(L) du, \quad L \in \mathcal{X} = \mathcal{L}^1, \quad (2.18)$$

for some  $0 \leq \alpha_i < 1$ , where  $q_u(L)$  is the left  $u$ -quantile of  $L$ .

**Proposition 2.5.** *If  $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$  and  $\rho_i = \mathbb{E}\mathbb{S}_{\alpha_i}$  for some  $0 \leq \alpha_i < 1$ ,  $i \in E$ , then there exists a Radner equilibrium with a unique equilibrium price. If  $\Gamma_i$  is further positive definite,  $i \in E$ , then the Radner equilibrium is unique.*

**Proof.** We divide the proof in three steps.

<sup>8</sup>e.g. expected shortfall.

<sup>9</sup>see Table 2.

• *Existence:* For a univariate normally distributed  $L$ , by McNeil et al. (2015, Example 2.14, page 70),  $\mathbb{E}S_\alpha(L) = \mathbb{E}[L] + \sqrt{\text{Var}(L)} \mathbb{E}S_\alpha(Z)$  with  $Z \sim \mathcal{N}_1(0, 1)$ ,  $\mathbb{E}S_\alpha(Z) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ . This and (2.14) yield

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \mathbb{E}S_{\alpha_i}(Z) \sqrt{\text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma q_i}, \quad q_i \in \mathbb{R}^m. \quad (2.19)$$

By Rockafellar (1970, Corollary 8.5.2), the recession function of  $r_i$  is given by

$$(r_i 0^+)(y) = \lim_{\lambda \searrow 0} \lambda r_i(y/\lambda) = -y^\top \mu + \sqrt{y^\top \Gamma y} \mathbb{E}S_{\alpha_i}(Z).$$

Let  $q_1, \dots, q_{|E|}$  be vectors in  $\mathbb{R}^m$  such that

$$\sum_{i \in E} (r_i 0^+)(q_i) \leq 0 \quad \text{and} \quad \sum_{i \in E} (r_i 0^+)(-q_i) > 0,$$

i.e.

$$-\mu^\top \left( \sum_{i \in E} q_i \right) < \sum_{i \in E} \sqrt{q_i^\top \Gamma q_i} \mathbb{E}S_{\alpha_i}(Z) \leq \mu^\top \left( \sum_{i \in E} q_i \right).$$

Thus  $\sum_{i \in E} q_i \neq 0$ . By Rockafellar (1970, Corollary 9.2.1, page 76), the inf-convolution of real valued convex functions is a real valued convex function. Hence the inf-convolution  $r$  is attained on  $\mathbb{R}^m$  and, by Lemma 2.1, there exists a Radner equilibrium  $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$ .

• *Unique price:* We know that  $-R_i - \mathbf{q}_i^\top P$ ,  $i \in E$ , is a continuous random variable. By Kalkbrenner (2005, Theorem 4.3 and Section 5.2), an expected shortfall is differentiable at continuous random variables. Therefore, by Remark 2.4, the optimal price  $\mathbf{p}$  is unique.

• *Unique portfolio:* If  $\Gamma_i$  is positive definite, then,

$$(1, q_i)^\top \Gamma_i(1, q_i) = \text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma q_i > 0, \quad q_i \in \mathbb{R}^m.$$

This and (2.19) implies that  $r_i$  is differentiable such that

$$\nabla r_i(q_i) = -\mu + \frac{\mathbb{E}S_{\alpha_i}(Z)}{\sqrt{(1, q_i)^\top \Gamma_i(1, q_i)}} (\text{cov}_i + \Gamma q_i), \quad q_i \in \mathbb{R}^m.$$

Following Rockafellar and Wets (1998, Theorem 2.14, page 47), the strict convexity of  $r_i$  is equivalent to

$$r_i(y) > r_i(q_i) + \nabla r_i(q_i)^\top (y - q_i), \quad q_i \neq y.$$

A simple computation reduces this first order condition to

$$\sqrt{[(1, q_i)^\top \Gamma_i(1, q_i)] [(1, y)^\top \Gamma_i(1, y)]} > (1, q_i)^\top \Gamma_i(1, y), \quad q_i \neq y. \quad (2.20)$$

If  $y \neq q_i$ , then  $(1, y)$  is not colinear to  $(1, q_i)$ . Hence, by Johnson and Wichern (2007, Eqn. (2.49), page 79) applied with  $\mathbf{b} = (1, q_i)$ ,  $\mathbf{d} = \Gamma_i(1, y)$ , and  $\mathbf{B} = \Gamma_i$  (hence  $(1, q_i) = \mathbf{b} \neq c\mathbf{B}^{-1}\mathbf{d} = c(1, y)$  for any constant  $c$ ),

$$\left[ (1, q_i)^\top \Gamma_i(1, q_i) \right] \left[ (1, y)^\top \Gamma_i(1, y) \right] > \left[ (1, q_i)^\top \Gamma_i(1, y) \right]^2 \quad \text{holds for any } q_i \neq y.$$

This in turn yields (2.20). Hence by Theorem 2.3 there exists a unique equilibrium. ■

**Remark 2.5.** If  $\text{Var}(R_i) > 0$ ,  $\text{cov}_i = 0$ , and  $\Gamma$  is invertible, then  $z^\top \Gamma_i z = z_1^2 \text{Var}(R_i) + 2z_1 \hat{z}^\top \text{cov}_i + \hat{z}^\top \Gamma \hat{z} > 0$  holds for any  $z = (z_1, \dots, z_{m+1}) \in \mathbb{R}^{m+1} \setminus \{0\}$ , where  $\hat{z} = (z_2, \dots, z_{m+1})$ . Hence  $\Gamma_i$  is positive definite as assumed in the last part of Proposition 2.5.

Instead, the positive definiteness of  $\Gamma_i$  is not guaranteed when  $\text{Var}(R_i) = \text{cov}_i^\top \Gamma^{-1} \text{cov}_i$ , because  $z^\top \Gamma_i z = 0$  for  $z = (-1, \Gamma^{-1} \text{cov}_i)$ . This is for instance the case when  $R_i$  is in the span of  $P$ , i.e.  $R_i = a_i^\top P + b_i$  for some constants  $a_i \in \mathbb{R}^m$  and  $b_i \in \mathbb{R}$ , whence  $\text{cov}_i = \Gamma a_i$  and  $\text{Var}(R_i) = a_i^\top \Gamma a_i = \text{cov}_i^\top \Gamma^{-1} \text{cov}_i$ . ■

**Remark 2.6.** An  $n$ -variate random vector  $L$  has an elliptical distribution written as  $L \sim \mathcal{E}_n(\mu, \Gamma, \psi)$  if its characteristic function is expressed as

$$\mathbb{E}\left[e^{iz^\top L}\right] = \exp(iz^\top \mu) \psi\left(\frac{1}{2}z^\top \Gamma z\right), \quad z \in \mathbb{R}^n,$$

for  $\mu = \mathbb{E}[L]$ ,  $\Gamma = \text{Cov}(L)$ , and a function  $\psi: [0, \infty) \rightarrow \mathbb{R}$  such that  $-\psi'(0) = -1$ . As is well known (Landsman and Valdez, 2003; McNeil, Frey, and Embrechts, 2015), if  $L \sim \mathcal{E}_n(\mu, \Gamma, \psi)$  (or, more specifically<sup>10</sup>,  $\mathcal{T}_n(\mu, \Gamma, \nu)$ ), then  $a^\top L \stackrel{d}{=} a^\top \mu + \sqrt{a^\top \Gamma a} Z$ , where  $Z \sim \mathcal{E}_1(0, 1, \psi)$  (specifically,  $\mathcal{T}_1(0, 1, \nu)$ ). Hence, for any coherent risk measure  $\rho$ ,

$$\rho(a^\top L) = a^\top \mu + \rho(Z) \sqrt{a^\top \Gamma a}. \quad (2.21)$$

Assuming  $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$  (e.g.  $\mathcal{T}_{m+1}(\mu_i, \Gamma_i, \nu)$ ) with  $\Gamma_i$  positive definite,  $i \in E$ , the above implies that

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \rho_i(Z) \sqrt{\text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma_i q_i}. \quad (2.22)$$

The proof of Proposition 2.5 thus works for any law invariant and coherent risk measure  $\rho_i$  differentiable on the linear space spanned by the components of  $(R_i, P)$ ,  $i$  in  $E$ . ■

### 3. THE COMPARATIVE STATICS APPROACH FOR DEFAULT RESOLUTION ANALYSIS

When a clearing member of a CCP defaults, its position is taken over by the CCP. The CCP should then close the defaulter's positions in a way that does not harm the other members or the CCP itself. As outlined in (Bank For International Settlements, 2020), the CCP can settle the defaulter's positions via an auction organized by the CCP between the surviving members (and sometimes invited participants). According to (Bank For International Settlements, 2020, page 7), the chance of a successful auction is increased by hedging the defaulted portfolio's risks prior to the auction:

A CCP should establish a framework for its approach to hedging risks from a defaulted participant's portfolio prior to a default management auction to increase the chance of a successful auction. [...] The goals of a CCP's hedging strategy are generally to minimise the CCP's exposure to the defaulted participant's portfolio and to decrease the overall risk that the portfolio may pose to the CCP and the auction participants. Portfolios with less risk exposure lessen the potential effects of market volatility on the portfolio [...] and time dependency of valuations by auction participants.

A close-out procedure can also involve some liquidation on open markets. As different positions are liquidated separately, hedging prior to liquidation would entail additional costs for liquidating the hedging side of the portfolio. The main default resolution strategies are thus liquidation versus hedging then auctioning. As pointed out in Oleschak (2019),

in cases where the position to be transferred is large in relation to market liquidity or where a central market does not exist, auctions with the surviving agents as bidders is the mechanism of choice.

In any case, the CCP deals with the losses incurred throughout the close-out period by using the collateral of the defaulter, its own resources (skin in the game), and financial resources pooled between the clearing members in the form of a default fund (Gregory, 2014; Biais et al., 2016; Oleschak, 2019). The CCP should assess the adequacy of these financial resources by a careful estimation of the close-out costs of the defaulters' positions, which is the focus of this paper.

<sup>10</sup>A Student  $t$ -distribution is elliptical (Gaunt, 2021).

Let an index  $d$  represent a clearing member of a CCP defaulting instantaneously at time  $0^{11}$ . We want to analyze and compare different close-out procedures, of the ‘hedging or not and liquidation or auctioning’ types, for the CCP portfolio of member  $d$ . These strategies can be implemented on several possible exchanges  $E$ , starting with the one of the CCP itself, denoted hereafter by  $D$ . For each (pre-default) exchange  $E$ , we denote by  $E'$  its advent in the wake of the instant default of  $d$ , depending on the settlement procedure implemented by the CCP. We assume that the different exchanges  $E$  trade the same assets with terminal payoff  $P$ , possibly at different initial prices  $\mathbf{p}$  (interpreted in this setup as “time  $0^-$ ”, pre-default prices), reflecting different market equilibria. Hedging procedures involve the CCP itself in the form of a new trading participant, represented for this purpose by a new index  $c$  (not involved in any exchange  $E$ ). We use similar notation for  $d$  and  $c$  as for participants  $i$  of  $E$  in Section 2 (see Table 2). Although other choices could be used without methodological change in what follows, fault of suitable calibration data in this regard, we assume that any data other than  $\mathbf{p}$  and  $\mathbf{q}_i$  in Table 2 are not affected by the instant default of  $d$ —with the only exception reflected in (3.7) below of  $R_c$ , which represents the post-default receivable<sup>12</sup> of the hedging CCP due to the portfolio of the defaulted member  $d$  taken over by the hedging CCP, whereas the pre-default receivable of the CCP is zero (a CCP should not bear any positions, except for the ones inherited from defaulted market participants during the close-out period of their portfolios).

The pair  $((\mathbf{q}'_i)_{i \in E'}, \mathbf{p}')$  relative to any post-default exchange  $E'$  involved in the settlement of the defaulted portfolio is derived using a Radner equilibrium in  $E'$ . Note that all the receivables and equilibrium portfolios and prices implicitly depend on the corresponding exchange. Regarding prices, we make this dependence explicit hereafter, denoting by  $\mathbf{p}^E$  a pre-default (“time  $0^-$ ”) equilibrium price on  $E$  and by  $\mathbf{p}^{E'}$  a post-default (“time  $0$ ”) equilibrium price on  $E'$ .

**3.1. Price Impact.** We define  $\mathbf{q}_i = 0, i \in E' \setminus E$ , and  $\Delta \mathbf{q}_i = \mathbf{q}'_i - \mathbf{q}_i, i \in E' \cup E$ , hence

$$\sum_{i \in E' \cap E} \mathbf{q}_i + \sum_{i \in E'} \Delta \mathbf{q}_i = \sum_{i \in E'} \mathbf{q}_i + \sum_{E'} \Delta \mathbf{q}_i = \sum_{i \in E'} \mathbf{q}'_i. \quad (3.1)$$

If the CCP chooses to liquidate a portion  $\mathbf{q}_d^l$  of  $\mathbf{q}_d$  and hedge the remaining  $\mathbf{q}_d^h = \mathbf{q}_d - \mathbf{q}_d^l$ , then the incremental positions of the participants to any post-default exchange  $E'$  can be split as  $\Delta \mathbf{q}_i = \Delta \mathbf{q}_i^l + \Delta \mathbf{q}_i^h$ , where  $\Delta \mathbf{q}_i^l$  and  $\Delta \mathbf{q}_i^h$  are the increments implied by the liquidation and hedging legs of the strategy (see e.g. Sections 3.3.7-3.3.8)—with always in particular

$$\Delta \mathbf{q}_c^l = 0, \quad (3.2)$$

as a CCP does not take part as a participant to a liquidation. Since the amount demanded should be equal to the amount supplied on both legs of the strategy, we have

$$\sum_{i \in E'} \Delta \mathbf{q}_i^l = \mathbf{q}_d^l \quad \text{and} \quad \sum_{i \in E'} \Delta \mathbf{q}_i^h = 0, \quad \text{hence} \quad \sum_{i \in E'} \Delta \mathbf{q}_i = \mathbf{q}_d^l \quad (3.3)$$

The first consequence of a default resolution strategy is then a liquidity cost

$$\text{LC} = \sum_E \text{LC}_E, \quad (3.4)$$

<sup>11</sup>Considering several instant defaulters would mainly mean replacing  $\mathbf{q}_d$  by  $\sum_d \mathbf{q}_d$  hereafter, see e.g. Remark 4.1. We refrain from doing so for parsimony of notation.

<sup>12</sup>see (3.14) below.

where

$$\begin{aligned} \text{LC}_E &= \sum_{i \in E' \cap E} \mathbf{q}_i^\top (\mathbf{p}^E - \mathbf{p}'^E) + \sum_{i \in E'} (\Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) \\ &= \sum_{i \in E'} \underbrace{(\mathbf{q}_i + \Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - \mathbf{p}'^E)}_{\text{LC}_i} \end{aligned} \quad (3.5)$$

(as  $\mathbf{q}_i = 0, i \in E' \setminus E$ ) corresponds to margin payments (like in futures markets) by market participants at time 0 in response to the default settlement procedure of  $d$ , i.e. the price they have to pay for the transition from the pre-default to the post-default exchanges.

As reflected in (3.5), it is only the contracts  $\Delta \mathbf{q}_i^l$  involved in the liquidation leg of the strategy, which are old (“time 0–”) contracts with the pre-default prices  $\mathbf{p}^E$ , that deserve margin payments, while the new (“time 0”) contracts  $\Delta \mathbf{q}_i^h$  involved in the hedging leg of the strategy are post-default contracts with the new prices  $\mathbf{p}'^E$ . However, the following reformulation of  $\text{LC}_E$  in terms of the  $\mathbf{q}_i + \Delta \mathbf{q}_i = \mathbf{q}'_i$  (instead of the  $\mathbf{q}_i + \Delta \mathbf{q}_i^l$  natively in (3.5)) is possible:

**Lemma 3.1.** *On each exchange  $E$ ,*

$$\text{LC}_E = \sum_{i \in E' \cap E} \mathbf{q}_i^\top (\mathbf{p}^E - \mathbf{p}'^E) + \sum_{i \in E'} \Delta \mathbf{q}_i^\top (\mathbf{p}^E - \mathbf{p}'^E) = \sum_{i \in E'} (\mathbf{q}'_i)^\top (\mathbf{p}^E - \mathbf{p}'^E). \quad (3.6)$$

*Proof.* By (3.3),  $\sum_{i \in E'} \Delta \mathbf{q}_i^h = 0$ . Hence (3.5) yields

$$\text{LC}_E = \sum_{i \in E'} (\mathbf{q}_i + \Delta \mathbf{q}_i^l + \Delta \mathbf{q}_i^h)^\top (\mathbf{p}^E - \mathbf{p}'^E),$$

where  $\Delta \mathbf{q}_i^l + \Delta \mathbf{q}_i^h = \Delta \mathbf{q}_i$ . ■

**3.2. Funds Transfer Price.** Let

$$\Delta \rho_i = \rho_i(-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P)) - \rho_i(-\mathbf{1}_{i \neq c} R_i + \mathbf{q}_i^\top (\mathbf{p}^E - P)), \quad i \in E'. \quad (3.7)$$

Using (3.1) and  $\Delta \mathbf{q}_i^l + \Delta \mathbf{q}_i^h = \Delta \mathbf{q}_i$ , the post-default market loss of any trading participant  $i \in E'$  is

$$\begin{aligned} &\underbrace{-R_i + \mathbf{q}_i^\top (\mathbf{p}^E - P)}_{\text{pre-default market loss}} + (\Delta \mathbf{q}_i^h)^\top (\mathbf{p}'^E - P) + (\Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - P) \\ &= -R_i + \mathbf{q}_i^\top (\mathbf{p}^E - P) + \Delta \mathbf{q}_i^\top (\mathbf{p}'^E - P) + (\Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) \\ &= -R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P) + \text{LC}_i, \end{aligned} \quad (3.8)$$

for  $\text{LC}_i$  as per (3.5). Hence, by translation equivariance of  $\rho_i$ , the post-default risk of participant  $i$  is

$$\rho_i(-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P)) + \text{LC}_i.$$

The risk incremental of participant  $i$  is therefore  $\text{LC}_i + \Delta \rho_i, i \in E'$ . Accordingly, we assess the market cost (MC) of a default resolution strategy by the funds transfer price

$$\text{MC} = \sum_E \text{MC}_E, \quad \text{where } \text{MC}_E = \text{LC}_E + \sum_{i \in E'} \Delta \rho_i. \quad (3.9)$$

If the CCP of an exchange  $D$  faces the default of a clearing member  $d$ , then this CCP can envision different default resolution procedures, impacting possibly different exchanges  $E'$  (starting with  $D'$  itself), for the CCP portfolio  $\mathbf{q}_d$  of the defaulter (in a pre-default equilibrium on  $D$ ). For each considered default resolution strategy, each of the impacted exchanges  $E'$  (or their corresponding CCPs) would compute its corresponding  $\text{MC}_E$  and communicate it to the CCP of  $D$ . The ensuing MC (3.9) of the strategy is the price that the markets would charge to the CCP of  $D$ , should the latter choose this strategy for resolving  $\mathbf{q}_d$ . The CCP of  $D$  would then choose the most efficient strategy, i.e. the one minimizing

MC. This approach is inspired by a notion of Pareto optimality in financial markets, where numerous exchanges (and trading participants themselves in Section 6) compete with one another.

**3.3. Examples.** The pre-default equilibria  $((\mathbf{q}_i)_{i \in E}, \mathbf{p}^E)$  involved in (3.9) are obtained by direct application of the results of Section 2. We now detail the corresponding post-default Radner equilibria  $((\mathbf{q}'_i)_{i \in E'}, \mathbf{p}^{E'})$  in eight reference cases (without post-default new invited participants other than the CCP itself in the hedging cases, though; extra new invited participants will only be considered later in the paper). The member optimality condition for the post-default market participant  $i \in E'$  is always of the form

$$\rho_i(-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}^{E'} - P)) \leq \rho_i(-R_i + q_i^\top (\mathbf{p}^{E'} - P)), \quad q_i \in \mathbb{R}^m. \quad (3.10)$$

The clearing condition, instead, depends on the considered default resolution strategy.

**3.3.1. The CCP fully liquidates on its own exchange.** As a first default resolution alternative, the CCP may want to liquidate the defaulter's position  $\mathbf{q}_d$  on its own exchange D. Then  $\text{MC}_E = 0, E \neq D$ , and  $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = \mathbf{q}_d$ . As  $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i + \mathbf{q}_d = 0$ , we obtain a post-default equilibrium clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = 0 \quad (3.11)$$

and

$$\text{LC} = \text{LC}_D = 0, \quad \text{MC} = \text{MC}_D = \sum_{D' = D \setminus \{d\}} \Delta \rho_i.$$

**3.3.2. The CCP fully liquidates on another exchange.** If the CCP liquidates  $\mathbf{q}_d$  on some exchange  $E \neq d$  (hence  $E \neq D$ ), then  $\text{MC}_E = 0, E \neq E, D$ , and  $\sum_{i \in E' = E} \Delta \mathbf{q}_i = \mathbf{q}_d$ . As  $\sum_{i \in E} \mathbf{q}_i = 0$ , the ensuing the post-default equilibrium clearing condition on  $E'$  is

$$\sum_{i \in E' = E} \mathbf{q}'_i = \mathbf{q}_d. \quad (3.12)$$

**Remark 3.1.** By change of variables  $\mathbf{z}'_i = \mathbf{q}'_i - k_i \mathbf{q}_d$  and  $R'_i = R_i + k_i \mathbf{q}_d^\top P$ , for reals  $k_i$  such that  $\sum_{i \in E'} k_i = 1$ , the clearing condition (3.12) and the optimality condition (3.10) relative to the post-default equilibrium  $((\mathbf{q}'_i)_{i \in E'}, \mathbf{p}^{E'})$  become  $\sum_{i \in E'} \mathbf{z}'_i = 0$  and

$$\rho_i(-R'_i + (\mathbf{z}'_i)^\top (\mathbf{p}^{E'} - P)) \leq \rho_i(-R'_i + z_i^\top (\mathbf{p}^{E'} - P)), \quad z_i \in \mathbb{R}^m.$$

On  $E'$ , we thus recover a zero clearing condition and member optimality conditions formally similar to Definition 2.1. ■

On the exchange D of the CCP, we have  $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = 0$ , whence the post-default clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = -\mathbf{q}_d. \quad (3.13)$$

Therefore

$$\text{MC} = \text{MC}_E + \text{MC}_D,$$

where

$$\text{MC}_E = \underbrace{\mathbf{q}_d^\top (\mathbf{p}^E - \mathbf{p}^{E'})}_{\text{LC}_E} + \sum_{i \in E' = E} \Delta \rho_i, \quad \text{MC}_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}^{D'})}_{\text{LC}_D} + \sum_{i \in D' = D \setminus \{d\}} \Delta \rho_i.$$

**Remark 3.2.** By change of variable  $\mathbf{z}'_i = \Delta \mathbf{q}_i$  and  $R'_i = R_i + \mathbf{q}_i^\top P, i \in D' = D \setminus \{d\}$ , the clearing condition (3.13) and optimality condition (3.10) relative to the post-default equilibrium  $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}^{D'})$  become  $\sum_{i \in D'} \mathbf{z}'_i = 0$  and

$$\rho_i(-R'_i + (\mathbf{z}'_i)^\top (\mathbf{p}^{D'} - P)) \leq \rho_i(-R'_i + z_i^\top (\mathbf{p}^{D'} - P)), \quad z_i \in \mathbb{R}^m. \quad \blacksquare$$

3.3.3. *The CCP fully hedges on its own exchange.* If  $\mathbf{q}_d$  is not instantaneously liquidated upon the default of member  $d$  at time 0, then the CCP  $c$  of  $d$  endorses at time 0 the receivable

$$R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D), \quad (3.14)$$

which it can hedge by holding on its own exchange  $D$  a portfolio  $\Delta \mathbf{q}_c$  minimizing some risk measure  $\rho_c$ . The corresponding member optimality condition (3.10) for the CCP  $c$ , playing the role of a new post-default trading participant, is

$$\rho_c \left( \mathbf{q}_d^\top (\mathbf{p}^D - P) + (\mathbf{q}'_c)^\top (\mathbf{p}'^D - P) \right) \leq \rho_c \left( \mathbf{q}_d^\top (\mathbf{p}^D - P) + q_c^\top (\mathbf{p}'^D - P) \right), \quad q_c \in \mathbb{R}^d. \quad (3.15)$$

In this case,  $MC_E = 0, E \neq D$ , and  $\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \mathbf{q}_i = 0$  (as, in this hedging case, on the post-default market  $D'$ , the amount demanded must be equal to the amount supplied). Since  $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i = -\mathbf{q}_d$  and  $\Delta \mathbf{q}'_c = \mathbf{q}'_c$ , we obtain a post-default equilibrium clearing condition

$$\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \mathbf{q}'_i = -\mathbf{q}_d \quad (3.16)$$

and

$$MC = MC_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{LC_D} + \sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i.$$

**Remark 3.3.** Much like in Remark 3.2, by change of variable  $\mathbf{z}'_i = \Delta \mathbf{q}_i, i \in (D \setminus \{d\}) \cup \{c\}$ , and  $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}$  and  $R'_c = R_c$ , the clearing condition (3.16) relative to the post-default equilibrium  $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$  can be converted to a zero clearing condition as per Definition 2.1 on  $D'$ . ■

3.3.4. *The CCP fully hedges on another exchange.* The considered CCP of  $d$  can also hedge the portfolio  $\mathbf{q}_d$  that it inherit from member  $d$  (if not liquidated) by trading on an exchange  $E \neq d$ , in which case  $E' = E \cup \{c\}$  and  $R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D)$  (arising from the pre-default Radner equilibrium on the exchange  $D$  of the CCP). In this case,  $MC_E = 0, E \neq E, D$ , and  $\sum_{E' = E \cup \{c\}} \Delta \mathbf{q}_i = 0$  (the amount demanded must be equal to the amount supplied on the post-default exchange  $E'$  where the hedge is implemented). As  $\sum_{i \in E} \mathbf{q}_i = 0$  and  $\Delta \mathbf{q}'_c = \mathbf{q}'_c$ , the ensuing post-default equilibrium clearing condition on  $E'$  is

$$\sum_{i \in E' = E \cup \{c\}} \mathbf{q}'_i = 0. \quad (3.17)$$

The corresponding member optimality condition (3.10) for the CCP  $c$  is

$$\rho_c \left( \mathbf{q}_d^\top (\mathbf{p}^D - P) + (\mathbf{q}'_c)^\top (\mathbf{p}'^E - P) \right) \leq \rho_c \left( \mathbf{q}_d^\top (\mathbf{p}^D - P) + q_c^\top (\mathbf{p}'^E - P) \right), \quad q_c \in \mathbb{R}^d.$$

On the own exchange  $D$  of the CCP, we have  $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = 0$ , whence the post-default clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = -\mathbf{q}_d. \quad (3.18)$$

Therefore  $MC = MC_E + MC_D$ , where

$$MC_E = \underbrace{0}_{LC_E} + \sum_{E' = E \cup \{c\}} \Delta \rho_i, \quad MC_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{LC_D} + \sum_{D' = D \setminus \{d\}} \Delta \rho_i.$$

**Remark 3.4.** As in Remark 3.2 again, by change of variable  $\mathbf{z}'_i = \Delta \mathbf{q}_i$  and  $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}$ , the clearing condition (3.18) relative to the post-default equilibrium  $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$  can be converted to a zero clearing condition as per Definition 2.1 on  $D'$ . ■

3.3.5. *The CCP fully replicates on its own exchange.* By replication, we refer to a default resolution strategy whereby the CCP  $c$  replicates the portfolio  $\mathbf{q}_d$  (if not liquidated) that the CCP inherits from  $d$  by mirroring position  $\mathbf{q}'_c = -\mathbf{q}_d$  on its own exchange  $D$ . In this case  $MC_E = 0, E \neq D$ , and we have  $D' = (D \setminus \{d\}) \cup \{c\}, \Delta \mathbf{q}_c = -\mathbf{q}_d$  (in the replication case, the only admissible trading strategy for  $c$  as a post-default trading participant is  $-\mathbf{q}_d$ ),  $\Delta \rho_c = \rho_c \left( \mathbf{q}_d^\top (\mathbf{p}^D - P) - \mathbf{q}_d^\top (\mathbf{p}'^D - P) \right) = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$ . On the post-default exchange  $D'$  where the hedge is implemented, the amount demanded must be equal to the amount supplied i.e.  $\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \mathbf{q}_i = 0$ , whence the post-default clearing condition

$$\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \mathbf{q}'_i = \underbrace{\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i}_0 + \underbrace{\mathbf{q}'_c}_{-\mathbf{q}_d} = -\mathbf{q}_d. \quad (3.19)$$

Therefore

$$MC = MC_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{LC = LC_D} + \sum_{i \in D \setminus \{d\}} \Delta \rho_i + \underbrace{\Delta \rho_c}_{\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)} = \sum_{i \in D \setminus \{d\}} \Delta \rho_i.$$

The market cost is the same as in the liquidation case of 3.3.1 (note that the embedded post-default Radner equilibria are the same), but its split between LC and  $\sum_{i \in D'} \Delta \rho_i$  is different, see Table 3.

**Remark 3.5.** As in Remark 3.2, by change of variable  $\mathbf{z}'_i = \Delta \mathbf{q}_i, i \in (D \setminus \{d\}) \cup \{c\}$ , and  $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}, R'_c = R_c$ , the clearing condition (3.19) relative to the post-default equilibrium  $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$  can be converted to a zero clearing condition as per Definition 2.1 on  $D'$ . ■

3.3.6. *The CCP fully replicates on another exchange.* Alternatively, the considered CCP of  $d$  can replicate the portfolio  $\mathbf{q}_d$  (if not liquidated) that it inherits from  $d$  by mirroring positions  $\mathbf{q}'_c = -\mathbf{q}_d$  on an external exchange  $E \neq d$ , hence  $E' = E \cup \{c\}$ . As in Section 3.3.5, replication means that the only admissible trading strategy for the post-default trading participant  $c$  is  $-\mathbf{q}_d$ . In this case,  $MC_E = 0, E \neq E, D$ , and  $\sum_{i \in E' = E \cup \{c\}} \Delta \mathbf{q}_i = 0$  (the amount demanded must be equal to the amount supplied on the post-default exchange  $E'$  where the hedge is implemented),  $\sum_{i \in E} \mathbf{q}_i = 0$ . The ensuing post-default clearing condition on  $E'$  is

$$\sum_{i \in E' = E \cup \{c\}} \mathbf{q}'_i = \underbrace{\sum_{i \in E} \mathbf{q}'_i}_{\mathbf{q}_d} + \underbrace{\mathbf{q}'_c}_{-\mathbf{q}_d} = 0.$$

We also have  $\Delta \rho_c = \rho_c \left( \mathbf{q}_d^\top (\mathbf{p}^D - P) - \mathbf{q}_d^\top (\mathbf{p}'^E - P) \right) = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^E)$ .

On the own exchange  $D$  of the CCP, we have  $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = 0$ , whence the post-default clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = -\mathbf{q}_d.$$

Therefore  $MC = MC_E + MC_D$ , where

$$MC_E = \underbrace{0}_{LC_E} + \sum_{i \in E} \Delta \rho_i + \Delta \rho_c \quad \text{and} \quad MC_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{LC_D} + \sum_{i \in D' = D \setminus \{d\}} \Delta \rho_i.$$

**Remark 3.6.** As in Remark 3.2, by change of variable  $\mathbf{z}'_i = \Delta \mathbf{q}_i$  and  $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}$ , we recover member optimally and zero clearing conditions as per Definition 2.1 on  $D'$ . ■



3.3.7. *The CCP partially liquidates and hedges on its own exchange.* The CCP can also liquidate a portion  $\mathbf{q}_d^l$  of the defaulted position  $\mathbf{q}_d$  and hedge the remaining  $\mathbf{q}_d^h = \mathbf{q}_d - \mathbf{q}_d^l$  on its own exchange  $D$ . The amount demanded should be equal to the amount supplied on each leg of the strategy, hence  $\sum_{i \in D \setminus \{d\}} \Delta \mathbf{q}_i^l = \mathbf{q}_d^l$  and  $\sum_{i \in (D \setminus \{d\}) \cup c} \Delta \mathbf{q}_i^h = 0$ , thus  $\sum_{i \in (D \setminus \{d\}) \cup c} \Delta \mathbf{q}_i = \mathbf{q}_d^l$ . As  $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i = -\mathbf{q}_d$ , the ensuing post-default clearing condition on  $D'$  is

$$\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \mathbf{q}_i' = -\mathbf{q}_d^h, \quad (3.20)$$

We assume that both liquidation and hedging happen simultaneously at the same price  $\mathbf{p}^{D'}$ . Hence each trading participant on the post-default market  $D'$  has a single member optimality condition (3.10) (with, in particular,  $R_c = (\mathbf{q}_d^h)^\top (P - \mathbf{p}^{D'})$ ). Then

$$\text{MC} = \text{MC}_D = \underbrace{-(\mathbf{q}_d^h)^\top (\mathbf{p}^D - \mathbf{p}^{D'})}_{\text{LC} = \text{LC}_D} + \sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i.$$

**Remark 3.7.** By change of variables  $\mathbf{z}_i' = \mathbf{q}_i' + k_i \mathbf{q}_d^h$  and  $R_i' = R_i - k_i (\mathbf{q}_d^h)^\top P$ , for reals  $k_i$  such that  $\sum_{i \in D'} k_i = 1$ , the clearing condition (3.20) and the optimality conditions (3.10) relative to the post-default equilibrium  $((\mathbf{q}_i')_{i \in D'}, \mathbf{p}^{D'})$  respectively become  $\sum_{i \in D'} \mathbf{z}_i' = 0$  and

$$\rho_i (-R_i' + (\mathbf{z}_i')^\top (\mathbf{p}^{D'} - P)) \leq \rho_i (-R_i' + z_i'^\top (\mathbf{p}^{D'} - P)), \quad z_i' \in \mathbb{R}^m. \blacksquare$$

3.3.8. *The CCP partially liquidates and hedges on another exchange.* The CCP can also liquidate a portion  $\mathbf{q}_d^l$  of the defaulted position  $\mathbf{q}_d$  and hedge the remaining  $\mathbf{q}_d^h = \mathbf{q}_d - \mathbf{q}_d^l$  on another exchange  $E \neq d$ . In this case,  $\text{MC}_E = 0, E \neq E, D$ . Since the amount demanded should be equal to the amount supplied on each leg of the strategy, we have  $\sum_{i \in E'} \Delta \mathbf{q}_i^l = \mathbf{q}_d^l$  and  $\sum_{i \in E'} \Delta \mathbf{q}_i^h = 0$ , hence  $\sum_{i \in E'} \Delta \mathbf{q}_i = \mathbf{q}_d^l$ . As  $\sum_{i \in E} \mathbf{q}_i = 0$  and  $\mathbf{q}_c = 0$ , the ensuing post-default equilibrium clearing condition on  $E'$  is

$$\sum_{i \in E'} \mathbf{q}_i' = \mathbf{q}_d^l. \quad (3.21)$$

We assume that both liquidation and hedging happen simultaneously on the exchange  $E$  with the same price  $\mathbf{p}^{E'}$ . Hence each trading participant on the post-default market  $E'$  has a single member optimality condition (3.10) (with, in particular,  $R_c = (\mathbf{q}_d^h)^\top (P - \mathbf{p}^{D'})$ ).

Regarding the own exchange of the CCP, the post-default equilibrium clearing condition on  $D'$  is

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}_i' = -\mathbf{q}_d \quad (3.22)$$

Hence  $\text{MC} = \text{MC}_E + \text{MC}_D$ , where

$$\text{MC}_E = \underbrace{(\mathbf{q}_d^l)^\top (\mathbf{p}^{E'} - \mathbf{p}^{E'})}_{\text{LC}_E} + \sum_{E' = E \cup \{c\}} \Delta \rho_i, \quad \text{MC}_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}^{D'})}_{\text{LC}_D} + \sum_{D' = D \setminus \{d\}} \Delta \rho_i.$$

**Remark 3.8.** Similarly to Remarks 3.1 and 3.2, by change of variables, the clearing conditions (3.21) and (3.22) relative to the post-default equilibria  $((\mathbf{q}_i')_{i \in E'}, \mathbf{p}^{E'})$  and  $((\mathbf{q}_i')_{i \in D'}, \mathbf{p}^{D'})$  can be converted to zero clearing conditions on the exchanges  $E'$  and  $D'$ .  $\blacksquare$

In Sections 4-5, we provide explicit or numerical solutions regarding the market cost of default resolutions on  $D$ , hence

$$\text{LC} = \text{LC}_D = \sum_{i \in D'} (\mathbf{q}_i')^\top (\mathbf{p}^D - \mathbf{p}^{D'}), \quad \text{MC} = \text{MC}_D = \text{LC}_D + \sum_{i \in D'} \Delta \rho_i. \quad (3.23)$$

By translation equivariance of the  $\rho_i$ , (3.7) yields

$$\Delta \rho_i = r_i (\mathbf{q}_i') - \mathbf{1}_{i \neq c} r_i (\mathbf{q}_i) + (\mathbf{q}_i')^\top \mathbf{p}^{D'} - \mathbf{q}_i^\top \mathbf{p}^D, \quad i \in D'.$$

	LC	$\sum_E \sum_{i \in E'} \Delta \rho_i$
1.	0	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i$
2.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D) + \mathbf{q}_d^\top (\mathbf{p}^E - \mathbf{p}'^E)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E} \Delta \rho_i$
3.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i$
4.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E \cup \{c\}} \Delta \rho_i$
5.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i$ with $\Delta \rho_c = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$
6.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E \cup \{c\}} \Delta \rho_i$ with $\Delta \rho_c = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^E)$
7.	$-(\mathbf{q}_d^h)^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i$
8.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D) + (\mathbf{q}_d^h)^\top (\mathbf{p}^E - \mathbf{p}'^E)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E \cup \{c\}} \Delta \rho_i$

TABLE 3. Decomposition of the market costs in the eight examples of Section 3.3.

A further computation based on (3.23) then yields

$$\text{MC} = \text{MC}_D = \sum_{i \in D'} (\Delta \mathbf{q}_i)^\top \mathbf{p}^D + \sum_{i \in D'} (r_i(\mathbf{q}'_i) - \mathbf{1}_{i \neq c} r_i(\mathbf{q}_i)). \quad (3.24)$$

#### 4. MARKET COST: THE CASE OF ENTROPIC RISK MEASURES

Throughout this section, we assume that the risk preference of each trading participant  $i$  in  $D \cup D'$  is an entropic risk measure of the form

$$\rho_i(L) = \frac{1}{\varrho_i} \ln(\mathbb{E}[\exp(\varrho_i L)]), \text{ for some } \varrho_i > 0. \quad (4.1)$$

We also assume that each  $(R_i, P)$  is jointly normal, so that  $r_i(q_i)$  is given by (2.15).

##### 4.1. Liquidation on D.

**Proposition 4.1.** *Let  $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$ ,  $i \in D \cup D'$ , and  $\Gamma$  be invertible. If the CCP liquidates the defaulter position  $\mathbf{q}_d$  on its own exchange D, then*

$$\begin{aligned} \text{LC}_D &= 0 \quad \text{and} \\ \text{MC}_D &= \frac{1}{2} \varrho' \mathbf{q}_d^\top \Gamma \mathbf{q}_d - \varrho' \left( \Gamma^{-1} \sum_{j \in D' \setminus D} \frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right)^\top \left( \text{cov}' + \frac{1}{2} \Gamma \mathbf{q}_d \right) + \\ &\quad \sum_{i \in D' \setminus D} \left( \Gamma^{-1} (\varrho \text{cov} - \varrho_i \text{cov}_i) \right)^\top \left( \text{cov}_i + \frac{1}{2} \Gamma \mathbf{q}'_i \right), \end{aligned} \quad (4.2)$$

where  $\varrho' = \left( \sum_{i \in D'} \frac{1}{\varrho_i} \right)^{-1}$ .

**Proof.** By the clearing condition  $\sum_{i \in D'} \mathbf{q}'_i = 0$  (established like (3.11)), (3.23) yields  $\text{LC}_D = \sum_{i \in D'} (\mathbf{q}'_i)^\top (\mathbf{p}^D - \mathbf{p}'^D) = 0$ . Letting D and  $D'$  successively play the role of  $E$  in Proposition 2.4, the pre-default and the post-default equilibrium are uniquely given by

$$\mathbf{q}_i = \Gamma^{-1} \left( \frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right), \quad i \in D; \quad \mathbf{p}^D = \mu - \varrho \text{cov}, \quad (4.3)$$

and

$$\mathbf{q}'_i = \Gamma^{-1} \left( \frac{\varrho'}{\varrho_i} \text{cov}' - \text{cov}_i \right), \quad i \in D'; \quad \mathbf{p}'^D = \mu - \varrho' \text{cov}', \quad (4.4)$$

where  $\varrho = \left( \sum_{i \in D} \frac{1}{\varrho_i} \right)^{-1}$ ,  $\text{cov} = \sum_{i \in D} \text{cov}_i$ ,  $\varrho' = \left( \sum_{i \in D'} \frac{1}{\varrho_i} \right)^{-1}$ ,  $\text{cov}' = \sum_{i \in D'} \text{cov}_i$ . From (2.15), we obtain

$$r_i(\mathbf{q}'_i) - r_i(\mathbf{q}_i) = -\Delta \mathbf{q}_i^\top \mu + \varrho_i \Delta \mathbf{q}_i^\top \left[ \text{cov}_i + \Gamma \left( \mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i \right) \right], \quad i \in D'. \quad (4.5)$$

As also  $\sum_{i \in D'} \Delta \mathbf{q}_i = \mathbf{q}_d$  holds in a liquidation setup and since  $\mathbf{q}_i = 0$  for each trading participant  $i \in D' \setminus D$ , (3.24) yields

$$\begin{aligned} \text{MC}_D &= \mathbf{q}_d^\top (\mathbf{p}^D - \mu) + \sum_{i \in D'} \varrho_i (\Delta \mathbf{q}_i)^\top \left[ \text{cov}_i + \Gamma \left( \mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i \right) \right] \\ &= -\varrho \mathbf{q}_d^\top \text{cov} + \sum_{i \in D'} \varrho_i (\Delta \mathbf{q}_i)^\top \left[ \text{cov}_i + \Gamma \left( \mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i \right) \right], \end{aligned} \quad (4.6)$$

by the second identity in (4.3).

To compute  $\Delta \mathbf{q}_i$  therein, note that

$$\begin{aligned} \text{cov}' &= \text{cov} - \text{cov}_d + \sum_{i \in D' \setminus D} \text{cov}_i = \frac{\varrho}{\varrho_d} \text{cov} + \left( 1 - \frac{\varrho}{\varrho_d} \right) \text{cov} - \text{cov}_d + \sum_{i \in D' \setminus D} \text{cov}_i \\ &= \Gamma \mathbf{q}_d + \frac{\varrho}{\varrho'} \text{cov} - \sum_{i \in D' \setminus D} \left( \frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right), \end{aligned}$$

by the first identity in (4.3) and the fact that  $\frac{1}{\varrho'} = \frac{1}{\varrho} - \frac{1}{\varrho_d} + \sum_{i \in D' \setminus D} \frac{1}{\varrho_i}$ . This implies

$$\varrho' \text{cov}' = \varrho' \Gamma \mathbf{q}_d + \varrho \text{cov} - \varrho' \sum_{i \in D' \setminus D} \left( \frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right). \quad (4.7)$$

The definition of  $\Delta \mathbf{q}_i$ , (4.3), (4.4), and (4.7) yield

$$\Delta \mathbf{q}_i = \begin{cases} \frac{\varrho'}{\varrho_i} \mathbf{q}_d - \frac{\varrho'}{\varrho_i} \Gamma^{-1} \sum_{j \in D' \setminus D} \left( \frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right) & i \in D \setminus \{d\} \\ \frac{\varrho'}{\varrho_i} \mathbf{q}_d - \frac{\varrho'}{\varrho_i} \Gamma^{-1} \sum_{j \in D' \setminus D} \left( \frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right) + \Gamma^{-1} \left( \frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right) & i \in D' \setminus D. \end{cases} \quad (4.8)$$

As  $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i = -\mathbf{q}_d$  and  $\sum_{i \in D'} \Delta \mathbf{q}_i = \mathbf{q}_d$ , substituting (4.8) into (4.6) yields

$$\begin{aligned} \text{MC}_D &= -\varrho \mathbf{q}_d^\top \text{cov} + \left[ \varrho' \mathbf{q}_d - \varrho' \Gamma^{-1} \sum_{j \in D' \setminus D} \left( \frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right) \right]^\top \left( \text{cov}' - \frac{1}{2} \Gamma \mathbf{q}_d \right) + \\ &\quad \sum_{i \in D' \setminus D} \left( \Gamma^{-1} (\varrho \text{cov} - \varrho_i \text{cov}_i) \right)^\top \left( \text{cov}_i + \frac{1}{2} \Gamma \mathbf{q}'_i \right), \end{aligned}$$

whence the expression for  $\text{MC}_D$  in (4.2). ■

**Remark 4.1.** To cope with the case of several instant defaulters  $d$  at time 0, one just needs to replace  $\mathbf{q}_d$ ,  $\text{cov}_d$  and  $\frac{\varrho}{\varrho_d}$  by  $\sum_d \mathbf{q}_d$ ,  $\sum_d \text{cov}_d$  and  $\sum_d \frac{\varrho}{\varrho_d}$  in Proposition 4.1 and its proof. ■

**Remark 4.2.** If the CCP liquidates the defaulter's position among the surviving members, i.e. for  $D' \setminus D = \emptyset$  in the above, then Proposition 4.1 yields

$$\text{MC}_D = \frac{1}{2} \varrho' \mathbf{q}_d^\top \Gamma \mathbf{q}_d \geq 0.$$

Using (4.3), (4.5), and (4.7), we obtain  $\mathbf{p}'^D = \mathbf{p}^D - \varrho' \Gamma \mathbf{q}_d$ . Moreover, (4.8) yields  $\Delta \mathbf{q}_i = \frac{\varrho'}{\varrho_i} \mathbf{q}_d$ ,  $i \in D \setminus \{d\}$ . In the case  $m = 1$  for simplicity, the reason why  $\text{MC} \geq 0$  when the CCP liquidates among the surviving member can thus be explained as follows. If  $\mathbf{q}_d > 0$ ,

then the CCP replaces the defaulter's contract with each surviving member by selling at a "fire sales" price  $\mathbf{p}^{\prime D} < \mathbf{p}^D$ . If  $\mathbf{q}_d < 0$ , then the CCP buys from each surviving member at a "dear" price  $\mathbf{p}^{\prime D} > \mathbf{p}^D$ . In both cases, there is a market cost. ■

**Example 4.3.** Let  $D = \{1, \dots, 15\}$ ,  $d = \{15\}$ ,  $D' \setminus D = \emptyset$ ,  $\varrho_i = 1$ ,  $m = 1$  and  $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$ ,  $i \in D$ . Suppose  $\text{cov}_i = c_i \sigma \sqrt{\text{Var}(R_i)}$ , where  $\sigma^2 = \text{Var}(P)$ ,  $c_i = (-1)^{i+1} 0.8$  (the correlation coefficient between  $R_i$  and  $P$ ), and  $\text{Var}(R_i) = 0.09i^2$ ,  $i \in D$ . Fix  $\sigma = 0.2$ . The corresponding pre- and post-default optimal positions computed from (4.3) and (4.4) are given by Table 4. Note that each  $\mathbf{q}_i$  or  $\mathbf{q}'_i$  is positive (negative)

CM <sub><i>i</i></sub>	1	2	3	4	5	6	7	8
cov <sub><i>i</i></sub>	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
$\mathbf{q}_i$	-0.56	3.04	-2.96	5.44	-5.36	7.84	-7.76	10.24
$\mathbf{q}'_i$	-1.80	1.80	-4.20	4.20	-6.60	6.60	-9.00	9.00
CM <sub><i>i</i></sub>	9	10	11	12	13	14	15	
cov <sub><i>i</i></sub>	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	
$\mathbf{q}_i$	-10.16	12.64	-12.56	15.04	-14.96	17.44	-17.36	
$\mathbf{q}'_i$	-11.40	11.40	-13.80	13.80	-16.20	16.20		

TABLE 4. Pre- and post-default optimal positions of each clearing member  $i$  (CM<sub>*i*</sub>) when the CCP liquidates  $\mathbf{q}_d$  on its own exchange in the entropic case.

provided cov<sub>*i*</sub> is negative (positive) (here and again in Table 7 below), in line with the hedging feature of the exchange. For  $\mu = 2$ , Proposition 4.1 and its proof yield  $\mathbf{p}^D = 1.97$ ,  $\mathbf{p}^{\prime D} = 2.02$ , and  $\text{MC}_D = 0.43$  (with  $\text{LC}_D = 0$  as per the first line in Table 3). ■

As it can be seen from the analytical expressions (4.3)-(4.4) and Table 4, the covariance matrices  $\Gamma_i$ ,  $i \in D \cup D'$ , are the major driving factors for portfolio and price changes. But the number and the risk preferences of the trading participants can also significantly affect these optimal quantities and the market cost:

**Example 4.4.** Let  $D = \{1, \dots, |D| = n + 1\}$ ,  $d = \{n + 1\}$ ,  $\varrho_i = \varrho_1$ , and  $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$ ,  $i \in D$ . Suppose  $\text{cov}_i = 0$ ,  $i \in \{1, \dots, n - 1\}$ , and  $\text{cov}_n = -\text{cov}_{n+1} = \delta$ . We consider two cases.

(i)  $D' \setminus D = \emptyset$ . By (4.3) and (4.4), the pre- and post-default portfolios are then given by

$$\mathbf{q}_i = \begin{cases} 0, & i = 1, \dots, n - 1, \\ -\frac{\delta}{\sigma^2}, & i = n, \\ \frac{\delta}{\sigma^2}, & i = n + 1, \end{cases} \quad \text{and} \quad \mathbf{q}'_i = \begin{cases} \frac{\delta}{n\sigma^2}, & i = 1, \dots, n - 1 \\ -\frac{(n-1)\delta}{n\sigma^2}, & i = n. \end{cases}$$

The pre- and post-default prices are given by

$$\mathbf{p}^D = \mu \quad \text{and} \quad \mathbf{p}^{\prime D} = \mu - \frac{\varrho_1 \delta}{n}.$$

A further computation based on (4.2) yields

$$\text{MC} = \frac{\varrho_1}{2n} \left( \frac{\delta}{\sigma} \right)^2,$$

which decreases to 0 as the number  $n$  of surviving members to  $\infty$ .

(ii)  $D' \setminus D = \{n + 2\}$  with  $\text{cov}_{n+2} = \delta'$ . In this case, the pre-default equilibrium is the

same as in case (i), while the post-default equilibrium is given by

$$\mathbf{q}'_i = \begin{cases} \frac{\delta + \delta'}{(n+1)\sigma^2}, & i = 1, \dots, n-1, \\ \frac{\delta' - n\delta}{(n+1)\sigma^2}, & i = n, \\ \frac{\delta - \delta'n}{(n+1)\sigma^2}, & i = n+2. \end{cases} \quad \text{and} \quad \mathbf{p}^{\text{D}} = \mu - \frac{\varrho_1(\delta + \delta')}{n+1}.$$

A further computation based on (4.2) yields

$$\text{MC} = \frac{2\varrho_1\delta^2 + 2\varrho_1\delta'\delta - \varrho_1(\delta')^2n}{2\sigma^2(n+1)},$$

the sign of which depends on the value of the parameters. ■

**4.2. Hedging on D.** We now turn to the ‘‘hedging on own exchange D’’ case of Section 3.3.3, but with possibly new participants beyond the CCP  $c$  in  $D'$ .

**Proposition 4.2.** *Let  $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$ ,  $i \in D \cup D'$ , with invertible covariance matrix  $\Gamma$  of  $P$ . When the CCP hedges the defaulter position  $\mathbf{q}_d$  on its own exchange D, then*

$$\begin{aligned} \text{LC}_D &= -\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D), \\ \text{MC}_D &= \sum_{i \in D'} \varrho_i (\Delta \mathbf{q}_i)^\top \left[ \text{cov}_i + \Gamma(\mathbf{q}_i + \frac{1}{2}\Delta \mathbf{q}_i) \right] + \left( \frac{\varrho_c \text{Var}(R_c)}{2} - \mathbb{E}[R_c] \right), \end{aligned} \quad (4.9)$$

where  $\varrho' = \left( \sum_{i \in D'} \frac{1}{\varrho_i} \right)^{-1}$ . In the absence of new participants (other than the CCP itself  $c$ ) to  $D'$ ,

$$\text{MC} = \frac{\varrho'^2(\varrho' - \varrho_c)}{2\varrho_c^2} \text{cov}^\top \Gamma^{-1} \text{cov} + \frac{\varrho'}{2} \mathbf{q}_d^\top \Gamma \mathbf{q}_d - \frac{\varrho'}{\varrho_c} \mathbf{q}_d^\top \text{cov}. \quad (4.10)$$

**Proof.** By the clearing condition  $\sum_{i \in D'} \mathbf{q}'_i = -\mathbf{q}_d$  (established like (3.16)), (3.23) yields  $\text{LC}_D = \sum_{i \in D'} (\mathbf{q}'_i)^\top (\mathbf{p}^D - \mathbf{p}'^D) = -\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$ . Applying Proposition 2.4 to  $E = D$ , the pre-default equilibrium is uniquely given by (4.3). As for the post-default equilibrium, if the CCP hedges on D, then, in view of Remark 3.3, introducing the changed variable  $\mathbf{z}'_i = \mathbf{q}'_i - \mathbf{q}_i$  and  $R'_i = R_i + \mathbf{q}'_i^\top P$  yields

$$r'_i(z_i) := \rho_i(-R'_i - z_i^\top P) = \rho_i(-R_i - (z_i + \mathbf{q}_i)^\top P), \quad i \in D'.$$

Following the proof of Proposition 2.4 with  $r'_i$  here in the role of  $r_i$  there, we obtain a unique post-default equilibrium

$$\mathbf{q}'_i = \Gamma^{-1} \left( \frac{\varrho'}{\varrho_i} \text{cov}' - \text{cov}_i \right) - \frac{\varrho'}{\varrho_i} \mathbf{q}_d, \quad i \in D', \quad \text{and} \quad \mathbf{p}^{\text{D}} = \mu - \varrho' \text{cov}' + \varrho' \Gamma \mathbf{q}_d. \quad (4.11)$$

Hence, by (2.15),

$$r_i(\mathbf{q}'_i) - r_i(\mathbf{q}_i) = -\Delta \mathbf{q}_i^\top \mu + \varrho_i (\Delta \mathbf{q}_i)^\top \left[ \text{cov}_i + \Gamma(\mathbf{q}_i + \frac{1}{2}\Delta \mathbf{q}_i) \right], \quad i \in D'.$$

This and  $\sum_{i \in D'} \Delta \mathbf{q}_i = 0$  reduce (3.24) to (4.9).

In the absence of new participants other than the CCP itself to  $D'$ , (4.3) and (4.11) yield

$$\Delta \mathbf{q}_i = \begin{cases} \frac{1}{\varrho_i} \Gamma^{-1} (\varrho' \text{cov}' - \varrho \text{cov}) - \frac{\varrho'}{\varrho_i} \mathbf{q}_d & i \in D \setminus \{d\}, \\ \frac{1}{\varrho_i} \Gamma^{-1} \varrho' \text{cov}' - \mathbf{q}_d - \frac{\varrho'}{\varrho_i} \mathbf{q}_d & i = c. \end{cases}$$

One can check that the value of  $\varrho' \text{cov}'$  given by (4.7) also holds true for the hedging case. Hence, using  $\varrho' \text{cov}'$  given by (4.7) and  $\text{cov}_c = \Gamma \mathbf{q}_d$ , we obtain

$$\Delta \mathbf{q}_i = \begin{cases} \frac{\varrho'}{\varrho_i} \Gamma^{-1} (\Gamma \mathbf{q}_d - \frac{\varrho}{\varrho_c} \text{cov}), & i \in D \setminus \{d\}, \\ \frac{\varrho'}{\varrho_i} \Gamma^{-1} (\Gamma \mathbf{q}_d - \frac{\varrho}{\varrho_c} \text{cov}) + \frac{\varrho}{\varrho_i} \Gamma^{-1} \text{cov} - \mathbf{q}_d, & i = c. \end{cases} \quad (4.12)$$

Since  $R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D)$ , we have  $\text{Var}(R_c) = \mathbf{q}_d^\top \Gamma \mathbf{q}_d$  and  $\mathbb{E}[R_c] = \varrho \mathbf{q}_d^\top \text{cov}$ . Further computations using  $\Delta \mathbf{q}_i$  given by (4.12) reduce (4.9) to (4.10). ■

**Example 4.5.** In the ‘‘CCP hedging on its own exchange’’ case, let  $D = \{1, \dots, 15\}$ ,  $d = \{15\}$ ,  $D' \setminus D = \{c\}$ ,  $\varrho_i = 1 = \varrho_c$ , and  $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$  with  $\text{cov}_i = \sigma c_i \sqrt{\text{Var}(R_i)}$ ,  $\sigma = 0.2$ ,  $c_i = (-1)^{i+1} 0.8$ , and  $\text{Var}(R_i) = 0.09i^2$ ,  $i \in D$ . The corresponding pre- and post-default optimal positions computed from (4.3) and (4.11) are given in Table 5. For

CM <sub><i>i</i></sub>	1	2	3	4	5	6	7	8
cov <sub><i>i</i></sub>	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
$\mathbf{q}_i$	-0.56	3.04	-2.96	5.44	-5.36	7.84	-7.76	10.24
$\mathbf{q}'_i$	-1.76	1.84	-4.16	4.24	-6.56	6.64	-8.96	9.04
CM <sub><i>i</i></sub>	9	10	11	12	13	14	15	c
cov <sub><i>i</i></sub>	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	-0.69
$\mathbf{q}_i$	-10.16	12.64	-12.56	15.04	-14.96	17.44	-17.36	
$\mathbf{q}'_i$	-11.36	11.44	-13.76	13.84	-16.16	16.24		16.80

TABLE 5. Pre- and post-default optimal positions when the CCP hedges on its own exchange  $D$  with  $D' \setminus D = \emptyset$  in the entropic case.

$\mu = 2$ , Proposition 4.2 and its proof yield  $\mathbf{p}^D = 1.97$ ,  $\mathbf{p}^{D'} = 2.02$ , and  $\text{MC}_D = 0.42$  (with, by the third line in Table 3,  $\text{LC}_D = -0.83$ ). ■

Table 6 displays the impacts of the default resolution of  $d$  in terms of  $\text{LC}_i$  and  $\Delta \rho_i$ ,  $i \in D'$ . As can be seen from the table, the impact of the default resolution on the  $\Delta \rho_i$  is almost the same in the liquidation and hedging cases, whereas its impact on the  $\text{LC}_i$  is significantly different in the two cases.

	CM <sub><i>i</i></sub>	1	2	3	4	5	6	7	8
Liquidation	LC <sub><i>i</i></sub>	0.09	-0.09	0.21	-0.21	0.33	-0.33	0.45	-0.45
	$\Delta \rho_i$	-0.06	0.1	-0.18	0.24	-0.30	0.36	-0.42	0.48
Hedging	LC <sub><i>i</i></sub>	0.03	-0.15	0.14	-0.26	0.26	-0.38	0.37	-0.49
	$\Delta \rho_i$	-0.06	0.12	-0.17	0.23	-0.29	0.35	-0.40	0.46
	CM <sub><i>i</i></sub>	9	10	11	12	13	14	c	
Liquidation	LC <sub><i>i</i></sub>	0.56	-0.56	0.68	-0.68	0.80	-0.80		
	$\Delta \rho_i$	-0.53	0.60	-0.65	0.72	-0.77	0.83		
Hedging	LC <sub><i>i</i></sub>	0.49	-0.61	0.60	-0.72	0.72	-0.84	0	
	$\Delta \rho_i$	-0.52	0.58	-0.63	0.69	-0.75	0.81	0.83	

TABLE 6. Impacts of the default resolution on the  $\text{LC}_i$  and  $\Delta \rho_i$  in the liquidation and hedging cases in the entropic risk measure examples 4.3-4.5.

## 5. MARKET COST: THE CASE OF EXPECTED SHORTFALL

Throughout this section, we assume that the risk preferences of each market participant is an expected shortfall  $\mathbb{ES}_{\alpha_i}$  as per (2.18); each vector  $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$  (or,

sometimes, a more specific  $\mathcal{T}_{m+1}(\mu_i, \Gamma_i, \nu_i)$ ; the CCP only operates on its own exchange D. Hence  $\text{MC} = \text{MC}_D$  as in (3.24), with, by (2.21),

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \mathbb{E}\mathcal{S}_{\alpha_i}(Z_i) \sqrt{\text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma q_i} \quad (5.1)$$

where  $Z_i \sim \mathcal{E}_1(0, 1, \psi)$  (or  $Z_i \sim \mathcal{T}_1(0, 1, \nu_i)$ ),  $i \in D \cup D'$ .

**5.1. Liquidation on D.** Let us first consider the liquidation case of Section 3.3.1, but with possibly new participants in  $D'$ . If  $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$  with  $\Gamma_i$  positive definite,  $i \in D \cup D'$ , then, by Proposition 2.6, there exists a unique pre-default Radner equilibrium  $((\mathbf{q}_i)_{i \in D}, \mathbf{p}^D)$ . Following (2.5) and (2.22), the pre-default member  $i \in D$  optimality condition yields

$$\mathbf{p}^D = \mu - \frac{\mathbb{E}\mathcal{S}_{\alpha_i}(Z)}{\sqrt{\text{Var}(R_i) + 2\mathbf{q}_i^\top \text{cov}_i + \mathbf{q}_i^\top \Gamma \mathbf{q}_i}} (\text{cov}_i + \Gamma \mathbf{q}_i), \quad i \in D. \quad (5.2)$$

The zero clearing condition and (5.2) imply

$$\begin{cases} \sum_{i \in D} \mathbf{q}_i = 0 \\ \frac{\mathbb{E}\mathcal{S}_{\alpha_1}(Z)}{\mathbb{E}\mathcal{S}_{\alpha_i}(Z)} \sqrt{\frac{\text{Var}(R_i) + 2\mathbf{q}_i^\top \text{cov}_i + \mathbf{q}_i^\top \Gamma \mathbf{q}_i}{\text{Var}(R_1) + 2\mathbf{q}_1^\top \text{cov}_1 + \mathbf{q}_1^\top \Gamma \mathbf{q}_1}} (\text{cov}_1 + \Gamma \mathbf{q}_1) - \text{cov}_i - \Gamma \mathbf{q}_i = 0, \quad i \in D. \end{cases} \quad (5.3)$$

Therefore, computing the optimal position is equivalent to finding the root of a vector function from  $\mathbb{R}^{m|D|}$  into  $\mathbb{R}^{m|D|}$ . Likewise, the post-default optimal positions solve

$$\begin{cases} \sum_{i \in D'} \mathbf{q}'_i = 0 \\ \frac{\mathbb{E}\mathcal{S}_{\alpha_1}(Z)}{\mathbb{E}\mathcal{S}_{\alpha_i}(Z)} \sqrt{\frac{\text{Var}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}{\text{Var}(R_1) + 2\mathbf{q}'_1{}^\top \text{cov}_1 + \mathbf{q}'_1{}^\top \Gamma \mathbf{q}'_1}} (\text{cov}_1 + \Gamma \mathbf{q}'_1) - \text{cov}_i - \Gamma \mathbf{q}'_i = 0, \quad i \in D' \end{cases} \quad (5.4)$$

and the post-default price is equal to

$$\mathbf{p}^D = \mu - \frac{\mathbb{E}\mathcal{S}_{\alpha_i}(Z_i)}{\sqrt{\text{Var}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}} (\text{cov}_i + \Gamma \mathbf{q}'_i), \quad i \in D'.$$

In the remainder of this subsection, we assume that  $m = 1$ ,  $\alpha_i = 0.975$  for each market participant,  $\mu = 2$ , and  $\sigma = 0.2$ .

**Example 5.1 (Multivariate normal).** Let  $D = \{1, \dots, 15\}$ ,  $d = \{15\}$ ,  $D' \setminus D = \emptyset$ ,  $\varrho_i = 1$ ,  $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$ ,  $\text{cov}_i = c_i \sigma \sqrt{\text{Var}(R_i)}$  with  $c_i = (-1)^{i+1} 0.8$ , and  $\text{Var}(R_i) = 0.09i^2$ ,  $i \in D$ . Solving (5.3)-(5.4) yields the pre- and post-default optimal positions displayed in Table 7. The above results also yield  $\mathbf{p}^D = 1.96$ ,  $\mathbf{p}^D = 2.04$ , and

CM <sub>i</sub>	1	2	3	4	5	6	7	8
cov <sub>i</sub>	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
q <sub>i</sub>	-1.12	2.56	-3.36	5.12	-5.60	7.68	-7.84	10.24
q' <sub>i</sub>	-1.28	2.24	-3.84	4.48	-6.40	6.72	-8.96	8.96
CM <sub>i</sub>	9	10	11	12	13	14	15	
cov <sub>i</sub>	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	
q <sub>i</sub>	-10.08	12.80	-12.32	15.36	-14.56	17.92	-16.80	
q' <sub>i</sub>	-11.52	11.20	-14.08	13.44	-16.64	15.68		

TABLE 7. Pre- and post-default optimal positions when the CCP liquidates on its own exchange D with  $D' \setminus D = \emptyset$  in the expected shortfall case.

$\text{MC}_D = 0.69$  (with  $\text{LC} = 0$  as per the first line in Table 3). ■

**Example 5.2 (Multivariate Student  $t$ ).** Reconsider Example 5.1 but for  $(R_i, P) \sim \mathcal{T}_2(\mu_i, \Gamma_i, \nu_i)$  with  $\nu_i = \nu = 2.5$ ,  $i \in D$ . (5.1) yields

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \mathbb{E}\mathbb{S}_{\alpha_i}(Z) \sqrt{\text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma q_i}$$

with  $Z \sim \mathcal{T}_1(0, 1, \nu)$ ,  $i \in D$ .

By McNeil et al. (2015, Example 2.15, page 71), we obtain  $\mathbb{E}\mathbb{S}_{\alpha_i}(Z) = \sqrt{\frac{\nu-2}{\nu}} \frac{t_\nu(T_\nu^{-1}(\alpha_i)) [\nu + (T_\nu^{-1}(\alpha))^2]}{(1-\alpha)(\nu-1)}$ . By inspection,  $\mathbf{q}$  and  $\mathbf{q}'$  are the same as in Example 5.1, given by Table 7. The above result also yield  $\mathbf{p}^D = 1.94$ ,  $\mathbf{p}'^D = 2.06$ , and  $\text{MC}_D = 1.06$  (with  $\text{LC}_D = 0$  as per the first line in Table 3). ■

**5.2. Hedging on  $D$ .** We now turn to the ‘‘hedging on own exchange  $D$ ’’ case of Section 3.3.3, but with possibly new participants beyond the CCP  $c$  in  $D'$ . Let  $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$  with  $\Gamma_i$  positive definite, for each  $i \in D \cup (D' \setminus \{c\})$ . By Proposition 2.5, there exists a unique pre-default Radner equilibrium  $((\mathbf{q}_i)_{i \in D}, \mathbf{p}^D)$ , which can be computed by (5.2) and (5.3). Since  $R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D)$ , letting  $z = (z_1, z_2, \dots, z_{m+1}) = (z_1, \hat{z})$ , we obtain

$$\mathbb{E} \left[ e^{iz^\top (R_c, P)} \right] = \mathbb{E} \left[ e^{i((z_1 \mathbf{q}_d + \hat{z})^\top P - z_1 \mathbf{q}_d^\top \mathbf{p}^D)} \right] = \exp(i z^\top \mu_c) \psi \left( \frac{1}{2} z^\top \Gamma_c z \right), \quad z \in \mathbb{R}^{m+1},$$

as  $(z_1 \mathbf{q}_d + \hat{z})^\top P - z_1 \mathbf{q}_d^\top \mathbf{p}^D \sim \mathcal{E}_1(z^\top \mu_c, z^\top \Gamma_c z, \psi)$ . Hence  $(R_c, P) \sim \mathcal{E}_{m+1}(\mu_c, \Gamma_c, \psi)$ . However, as  $R_c$  is in the span of  $P$ ,  $\Gamma_c$  is not positive definite (see Remark 2.5). To nevertheless ensure a unique post-default equilibrium (beyond the setup of Proposition 2.5), we assume that  $D' \setminus \{c\} \neq \emptyset$ . By Theorem 2.3, we have a unique post-default price  $\mathbf{p}'^D$  and a unique post-default portfolio  $\mathbf{q}'_i, i \in D' \setminus \{c\}$ . Hence by the post-default clearing condition, we also have a unique post-default position  $\mathbf{q}'_c$ .

Letting  $R'_i = R_i + \mathbf{q}'_i^\top P$  play the role of  $R_i$  in (2.19), we obtain

$$r'_i(z_i) := \mathbb{E}\mathbb{S}_{\alpha_i}(-R'_i - z_i^\top P)$$

$$= -\mathbb{E}[R_i] - (z_i + \mathbf{q}_i)^\top \mu + \mathbb{E}\mathbb{S}_{\alpha_i}(Z) \sqrt{(1, z_i + \mathbf{q}_i)^\top \Gamma_i (1, z_i + \mathbf{q}_i)}, \quad i \in D'$$

with  $Z \sim \mathcal{E}_1(0, 1, \psi)$ . In view of Remark 3.3, by change of variable  $\mathbf{z}'_i = \mathbf{q}'_i - \mathbf{q}_i$ , the proof of Proposition 2.5 shows that, for  $i \in D' \setminus \{c\}$ ,  $r'_i$  is differentiable and strictly convex with

$$-\mathbf{p}'^D = \nabla r'_i(\mathbf{z}'_i) = -\mu + \frac{\mathbb{E}\mathbb{S}_{\alpha_i}(Z)}{\sqrt{(1, \mathbf{z}'_i + \mathbf{q}_i)^\top \Gamma_i (1, \mathbf{z}'_i + \mathbf{q}_i)}} (\text{cov}_i + \Gamma \mathbf{z}'_i + \Gamma \mathbf{q}_i)$$

$$= -\mu + \frac{\mathbb{E}\mathbb{S}_{\alpha_i}(Z)}{\sqrt{\text{Var}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}} (\text{cov}_i + \Gamma \mathbf{q}'_i).$$

For the CCP, the member  $c$  optimality condition gives  $-\mathbf{p}'^D \in \partial r'_c(\mathbf{z}'_c) = \partial r'_c(\mathbf{q}'_c)$ , i.e.  $(r'_c)^*(-\mathbf{p}'^D) = -\mathbf{q}'_c^\top \mathbf{p}'^D - r'_c(\mathbf{q}'_c)$ . Hence computing the optimal post-default position reduces to the root-finding problem

$$\begin{cases} \sum_{i \in D'} \mathbf{q}'_i + \mathbf{q}_d = 0 \\ \frac{\mathbb{E}\mathbb{S}_{\alpha_1}(Z)}{\mathbb{E}\mathbb{S}_{\alpha_i}(Z)} \sqrt{\frac{\text{Var}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}{\text{Var}(R_1) + 2\mathbf{q}'_1^\top \text{cov}_1 + \mathbf{q}'_1^\top \Gamma \mathbf{q}'_1}} (\text{cov}_1 + \Gamma \mathbf{q}'_1) - \text{cov}_i - \Gamma \mathbf{q}'_i = 0, \quad i \in D' \setminus \{c\} \\ r'_c(\mathbf{q}'_c) + \mathbf{q}'_c^\top \mathbf{p}'^D + (r'_c)^*(-\mathbf{p}'^D) = 0. \end{cases} \quad (5.5)$$



**Example 5.3.** In the “hedging on own exchange D” case with  $D' \setminus D = \{c\}$  (as per Section 3.3.3) and expected shortfall risk measures, let  $D = \{1, \dots, 15\}$ ,  $d = \{15\}$ ,  $\alpha_i = 0.975$  and  $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$  (so  $m = 1$ ) with  $\text{cov}_i = \sigma c_i \sqrt{\text{Var}(R_i)}$ ,  $\sigma = 0.2$ ,  $c_i = (-1)^{i+1} 0.8$ , and  $\text{Var}(R_i) = 0.09i^2$ ,  $i \in D$ . The corresponding pre- and post-default optimal positions computed by (5.3) and (5.5) are given in Table 8. For  $\mu = 2$ ,

CM <sub>i</sub>	1	2	3	4	5	6	7	8
cov <sub>i</sub>	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
<b>q<sub>i</sub></b>	-1.12	2.56	-3.36	5.12	-5.60	7.68	-7.84	10.24
<b>q'<sub>i</sub></b>	-1.28	2.24	-3.84	4.48	-6.40	6.72	-8.96	8.96
CM <sub>i</sub>	9	10	11	12	13	14	15	c
cov <sub>i</sub>	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	-0.69
<b>q<sub>i</sub></b>	-10.08	12.80	-12.32	15.36	-14.56	17.92	-16.80	
<b>q'<sub>i</sub></b>	-11.52	11.20	-14.08	13.44	-16.64	15.68		16.80

TABLE 8. Pre- and post-default optimal positions when the CCP hedges on its own exchange D with  $D' \setminus D = \{c\}$  in the expected shortfall case.

we obtain  $\mathbf{p}^D = 1.96$ ,  $\mathbf{p}^{dD} = 2.04$ , and  $\text{MC}_D = 0.69$  (with, by the third line in Table 3,  $\text{LC}_D = -1.39$ ). ■

Table 9 is the expected shortfall analog of Table 6, with qualitatively similar conclusions.

	CM <sub>i</sub>	1	2	3	4	5	6	7	8
Liquidation	LC <sub>i</sub>	0.11	-0.18	0.32	-0.37	0.53	-0.56	0.74	-0.74
	$\Delta\rho_i$	-0.10	0.20	-0.30	0.40	-0.50	0.60	-0.70	0.80
Hedging	LC <sub>i</sub>	0.09	-0.21	0.28	-0.42	0.46	-0.64	0.65	-0.85
	$\Delta\rho_i$	-0.10	0.20	-0.30	0.40	-0.50	0.60	-0.70	0.80
	CM <sub>i</sub>	9	10	11	12	13	14	c	
Liquidation	LC <sub>i</sub>	0.95	-0.93	1.167	-1.11	1.38	-1.30		
	$\Delta\rho_i$	-0.89	0.99	-1.09	1.19	-1.29	1.39		
Hedging	LC <sub>i</sub>	0.84	-1.06	1.02	-1.27	1.21	-1.48	0	
	$\Delta\rho_i$	-0.89	0.99	-1.09	1.19	-1.29	1.39	1.39	

TABLE 9. Impacts of the default resolution on the LC<sub>i</sub> and  $\Delta\rho_i$  in the liquidation and hedging cases in the expected shortfall examples 5.1-5.3.

## 6. CREDIT COST

The MC term (3.9) only addresses the impact of the considered default resolution strategy in terms of mis-hedge of market risk. It remains to address its credit cost (CC), meant to account for counterparty credit risk in a broad sense including the implications of this risk in terms of capital and funding costs. There is in fact empirical evidence that credit risk could actually even dominate market risk (European Bank Authority, 2022, Figure 65 page 67). Accordingly, the overall impact of a clearing member default’s resolution

strategy should not be assessed in terms of MC only, but of the following all-inclusive funds transfer price:

$$\text{FTP} = \text{MC} + \text{CC}, \quad (6.1)$$

which should thus supersede MC in the default resolution approach depicted in the last paragraph of Section 3.2.

**6.1. Structure of the Exchanges.** As depicted in Figure 1, clearing member bank  $a$ 's trades with a CCP are divided into proprietary or house trades  $\mathbf{q}_a$ , which are in effect hedges of the bank's OTC bilateral trading exposures  $R_a = \sum_{o \in O} R_a^o$  (where the non-cleared, end-clients  $o$  are "outside" of the exchange), and back-to-back hedges  $\mathbf{q}_b^a$  of intermediated cleared client trades, through which non-member clients  $b$  (simple participants to the exchange) can access the CCP clearing services.

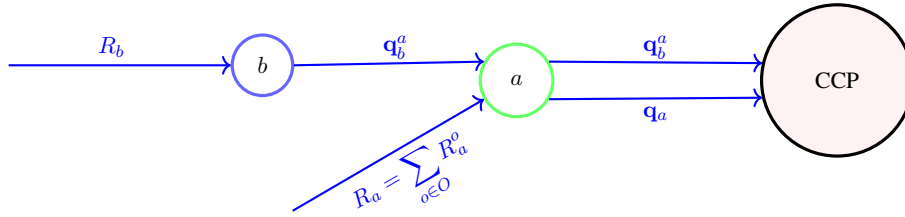


FIGURE 1. Client clearing ( $\mathbf{q}_b^a$  transits from  $b$  to the CCP via  $a$ ) versus bilateral hedging ( $a$  hedges  $R_a = \sum_{o \in O} R_a^o$  by a proprietary trading position  $\mathbf{q}_a$  with the CCP).

Table 10 shows that both categories are really significant and should therefore be encompassed in the analysis. Our next task is to clarify the nature of the Radner equilibrium

	Interest Rates	Credit	Equity	Commodity
ISDA (USA)	90%	83%	26%	
ESMA (Europe)	71%	41%	2%	1%

TABLE 10. Percentage of derivative notional cleared by asset classes. See International Swaps and Derivatives Association (2021, pages 2 and 5) for US as of end of 2020 (no figure for commodity derivatives), European Securities and Markets Authority (2021, page 16) for European interest rates and credit as of end of 2020, and International Swaps and Derivatives Association (2018, page 18) for European equities.

on an exchange  $E$  accounting for this distinction between proprietary and cleared deals. Let  $E = A \cup B$ , with  $A \cap B = \emptyset$ , be the split between the set  $A$  of those participants  $a$  to the exchange that are also clearing members of the CCP and the set  $B$  of simple participants (non clearing members)  $b$ , having recourse to the clearing members for intermediating trades with the CCP. Let  $q_a$  be the proprietary position of member  $a$  and  $q_b^a$  be the position of client (simple participant to the exchange)  $b$  cleared by member  $a$ . As depicted in Figure 1, the position  $q_b^a$  only transits from  $b$  to  $a$  and then passes from  $a$  to the CCP. Hence, even though the total position of  $a$  vis-à-vis the CCP is  $q_a + \sum_{b \in B} q_b^a$ , as  $a$  holds  $-\sum_{b \in B} q_b^a$  vis-à-vis the clients it clears for, the respective positions of  $a$  and  $b$  involved in the Radner equilibrium on  $E$  are  $q_a$  and  $q_b = \sum_{a \in A} q_b^a$ . Once  $((\mathbf{q}_i)_{i \in E=A \cup B}, \mathbf{p}^E)$  has been obtained as the solution of the corresponding Radner equilibrium (2.3)-(2.4), the splits  $\mathbf{q}_b = \sum_{a \in A} \mathbf{q}_b^a$  (for each  $b \in B$ ) should follow in a second stage from pure credit risk considerations.

Similar comments apply “with  $\cdot'$  everywhere” to any post-default Radner equilibrium on  $E' = A' \cup B'$ .

**6.2. Credit Costs XVA Framework.** The settlement of a CCP portfolio of a defaulted clearing member entails a mixture of market and/or counterparty credit risk, which depends on the nature of this portfolio.

Figure 2 and the rows of Table 11 take up the distinction of Section 6.1 between house and intermediation deals. The intermediated client deals of a clearing member are perfectly (back-to-back) hedged by the corresponding intermediating transactions (upper part of Figure 1). If a clearing member  $d$  defaults, its client deals and their static hedge are ported as a package to a surviving clearing member (right panel in Figure 2). As market risk is perfectly hedged throughout, such porting has no market impact, but entails a transfer of counterparty credit risk that can be quantified by XVA costs as per Bastide et al. (2023, Section 7) (second row in Table 11).

The columns of Table 11 distinguish delta-one financial assets, often rolled over time with no upfront payment, such as repo market transactions, equity swaps and futures (Crépey, Bielecki, and Brigo, 2014, Section 4.2.1), from upfront derivative assets traded on a primary market, such as multi-legs swaps and options. Delta-one rolled transactions have a nominal maturity that is infinite, but an effective maturity determined by the next reset date, i.e. typically less than one day, whence a very little counterparty credit risk footprint (second column of Table 11), but possibly significant liquidation or hedging costs. The market costs analysis of the above sections can be enough to deal with a portfolio of delta-one assets. In the case of swaps or derivatives portfolios, however (first column of Table 11), the XVA implications of a default resolution strategy also matter, which is the object of this section.

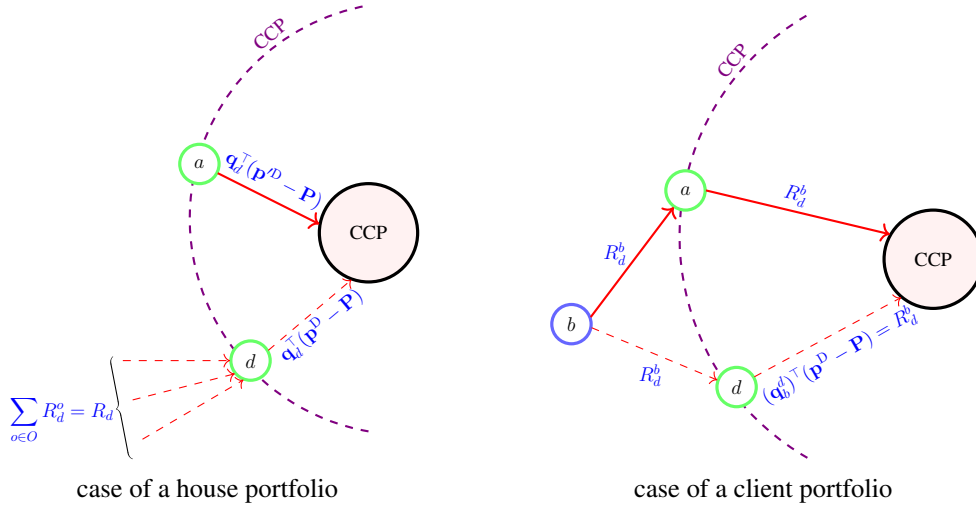


FIGURE 2. Default management of the CCP portfolios of a defaulted clearing member  $d$ . With pre-default positions in red dashed lines and post-default positions in red solid lines: (Left) The receivable  $R_d$  of  $d$  from its OTC bilateral counterparties (which are outside the exchange) is left aside in the default resolution procedure, while the corresponding house (hedging) portfolio  $q_d^h$  is ported to the surviving clearing member  $a$ ; (Right) The client account  $q_b^d$  of the defaulted clearing member  $d$ , as well as the corresponding (mirroring) receivable  $R_d^b$  from its cleared client  $b$ , are ported as a package to the surviving clearing member  $a$ .

Notations as detailed below and in Sections 2-3-A.

Assets		
Cleared	swaps and options	delta-one
centrally via house account	liquidity and credit	liquidity, no credit
centrally via client account	no liquidity, credit	$\emptyset$

TABLE 11. Costs of the CCP for settling a netting set of deals of a defaulted clearing member, depending on the nature of the defaulted portfolio.

The deals corresponding to the lower right cell would entail no liquidity nor credit risk, but this cell is in fact empty, due to the nature of delta-one transactions that do not require intermediation.

Our XVA metrics are computed under the premise that the (random) loss triggered by the default of a market participant in the future is allocated between the surviving members of its CCP, pro-rata of their default fund contribution to the CCP (see e.g. after (A.8)). At time “0–”, i.e. “right before” the instant default of the clearing member  $d$  at time 0, the participants  $i$  to the exchanges charge to their clients the expectation of their ensuing losses, in the form of their  $CVA_i$ , as well as their collateral funding costs  $FVA_i$  and  $MVA_i$ , and costs of capital  $KVA_i$ . These costs sum up to  $XVA_i = CVA_i + FVA_i + MVA_i + KVA_i$ , computed for each participant  $i$  the way detailed for  $i = 0$  in Section A, based on the pre-default Radner equilibrium quantities and prices on each exchange. To quantify the XVA impact of a given default resolution procedure for a CCP portfolio of  $d$ , we also compute the time 0, post-default  $XVA'_i$ , for any participant  $i \neq d$  to the exchanges. The credit cost of the settlement of the defaulted portfolio, coming on top of the already computed MC, is

$$CC = \sum_{i \neq d} \underbrace{(XVA'_i - XVA_i)}_{\Delta XVA_i} + AC, \quad (6.2)$$

where the auctioning cost  $AC$  is another XVA incremental impact corresponding to the FTP in Bastide et al. (2023, Section 7) (which only involved credit costs), i.e. the XVA impact of auctioning any hedged (as opposed to liquidated) positions. Indeed, a CCP is not inclined to keep the defaulted portfolio and the corresponding hedge on its book, it will look at auctioning them (Ferrara and Li, 2020; Oleschak, 2019).

In the end, the full FTP (6.1) that emerges from the present paper (where both market and credit costs are involved) can be detailed as

$$FTP = \underbrace{LC + \sum_{E'} \sum_{i \in E'} \Delta \rho_i}_{MC} + \underbrace{\sum_{i \neq d} \Delta XVA_i}_{CC} + AC. \quad (6.3)$$

**Remark 6.1.** Under the pure auctioning default resolution strategy, only the last term remains. But this  $AC$  term could be very expensive in the case of an unhedged portfolio (see the first quotation in Section 3). ■

**6.3. Example 5.1 Continued.** We complete from the credit costs perspective the “liquidation on own exchange” Example 5.1. In the case of default resolution procedures implemented on the own exchange  $D$  of the CCP, (6.3) boils down to

$$FTP = \underbrace{LC_D + \sum_{i \in D \setminus \{d\}} \Delta \rho_i}_{MC} + \underbrace{\sum_{i \in D \setminus \{d\}} \Delta XVA_i}_{CC} + AC. \quad (6.4)$$

We assume risk-free OTC bilateral counterparties (end-users)  $o$  and no simple participants to the exchange  $D$ , which thus only involves proprietary trading between its clearing members. We endorse the Gaussian latent factors XVA setup of Section A.1. Table 12 provides

the resulting “time 0–”  $XVA_i$  (with all members,  $d$  included) and time 0  $XVA'_i$  (without  $d$ ), using the allocated positions of Table 7 and the XVA specifications of Table 21 along with  $IM_i = DF_i = 0$ , whilst Table 13 presents the same results for  $IM_i$  and  $DF_i$  set at the 75% and 80% confidence level. Note that the chosen period length of  $T = 5$  years covers the bulk (if not the final maturity) of most realistic CCP portfolios<sup>13</sup>. The aggregated XVA cost (6.2) of liquidating the defaulted portfolio<sup>14</sup> is  $CC = \sum_{i \in D \setminus \{d\}} \Delta XVA_i$ , namely  $-0.77$  for the case  $IM_i = DF_i = 0$  and  $-0.25$  with  $IM_i$  and  $DF_i$  set at the 75% and 80% confidence level, coming on top of the market cost of  $MC = 0.70$  already obtained in Example 5.1.

$CM_i$	1	2	3	4	5	6	7	8
$XVA_i$	0.81	1.21	1.36	1.70	1.74	2.05	2.02	2.37
$XVA'_i$	0.92	1.17	1.47	1.61	1.82	1.93	2.07	2.19
$\Delta XVA_i$	0.11	-0.04	0.12	-0.09	0.08	-0.12	0.05	-0.18
$CM_i$	9	10	11	12	13	14	15	
$XVA_i$	2.25	2.62	2.42	2.84	2.57	3.04	2.69	
$XVA'_i$	2.28	2.42	2.42	2.60	2.54	2.77		
$\Delta XVA_i$	0.03	-0.20	-0.01	-0.24	-0.03	-0.27		

TABLE 12. The pre- and post-default XVAs computed from (A.11) when the CCP liquidates  $d$  on its own exchange,  $D$ , in the expected shortfall case with  $IM_i = DF_i = 0$ .

$CM_i$	1	2	3	4	5	6	7	8
$XVA_i$	0.37	0.59	0.72	0.89	1.00	1.12	1.21	1.34
$XVA'_i$	0.42	0.56	0.80	0.84	1.07	1.05	1.28	1.22
$\Delta XVA_i$	0.05	-0.03	0.08	-0.06	0.07	-0.07	0.07	-0.11
$CM_i$	9	10	11	12	13	14	15	
$XVA_i$	1.42	1.50	1.57	1.64	1.72	1.77	1.85	
$XVA'_i$	1.49	1.37	1.63	1.49	1.78	1.60		
$\Delta XVA_i$	0.07	-0.12	0.06	-0.15	0.06	-0.17		

TABLE 13. The pre- and post-default XVAs computed from (A.11) when the CCP liquidates  $d$  on its own exchange,  $D$ , in the expected shortfall case with  $IM_i$  and  $DF_i$  set at the 75% and 80% confidence level.

Instead of liquidation on the CCP’s own exchange  $D$ , we now consider another default resolution strategy, in the form of an (idealized) auction inducing the taker giving rise to the least auction cost  $AC$  among all possible takers  $i \in D \setminus \{d\}$ <sup>15</sup>. The results are displayed in

<sup>13</sup>most OTC derivatives have a maturity of less than 5 years (Bank for International Settlements, 2022, Graphs A.2–4).

<sup>14</sup>note that there is no auction in this (liquidation) case.

<sup>15</sup>This approach developed in (Bastide et al., 2023, Section 7) can indeed be seen as rendering the output of an idealized, efficient auction used for closing out the account of a defaulted clearing member (cf. Oleschak (2019, Section 3.3)).

Table 14 for the case  $IM_i = DF_i = 0$  and in Table 15 for  $IM_i$  and  $DF_i$  set at the 75% and 80% confidence level. From Tables 14-15, participant 14 is the survivor taker leading to

$CM_i$	$\sum_{i \in D \setminus \{d\}} (CVA'_i - CVA_i)$	$\sum_{i \in D \setminus \{d\}} (KVA'_i - KVA_i)$	AC
14	-1.00 (-0.38)	-3.52 (-0.63)	-4.52 (-1.01)
12	-0.85 (-0.34)	-2.93 (-0.57)	-3.78 (-0.91)
10	-0.58 (-0.23)	-2.02 (-0.42)	-2.60 (-0.64)
8	-0.31 (-0.11)	-1.13 (-0.24)	-1.44 (-0.35)
6	-0.04 (0.02)	-0.30 (-0.04)	-0.34 (-0.02)
4	0.23 (0.15)	0.42 (0.20)	0.65 (0.35)
2	0.50 (0.29)	0.95 (0.52)	1.45 (0.81)
1	0.77 (0.43)	1.59 (0.86)	2.35 (1.29)
3	0.77 (0.39)	2.09 (0.62)	2.85 (1.01)
5	0.77 (0.36)	2.47 (0.47)	3.23 (0.82)
7	0.77 (0.32)	2.70 (0.37)	3.46 (0.70)
9	0.77 (0.29)	2.84 (0.30)	3.60 (0.60)
11	0.77 (0.26)	2.92 (0.27)	3.69 (0.53)
13	0.77 (0.23)	2.96 (0.25)	3.73 (0.48)

TABLE 14. Auctioning costs AC corresponding to the different possible takers of the portfolio of the defaulted member  $d = 15$ , ranked by increasing value, for  $IM_i = DF_i = 0$ . In parenthesis, the contributions to AC of the considered possible taker itself.

$CM_i$	$\sum_{i \in D \setminus \{d\}} (MVA'_i - MVA_i)$	$\sum_{i \in D \setminus \{d\}} (CVA'_i - CVA_i)$	$\sum_{i \in D \setminus \{d\}} (KVA'_i - KVA_i)$	AC
14	-0.06 (-0.15)	-0.49 (-0.13)	-2.33 (-0.42)	-2.88 (-0.71)
12	-0.01 (-0.11)	-0.45 (-0.12)	-2.00 (-0.39)	-2.46 (-0.63)
10	0.06 (-0.04)	-0.36 (-0.08)	-1.47 (-0.27)	-1.77 (-0.40)
8	0.13 (0.03)	-0.27 (-0.04)	-0.93 (-0.14)	-1.07 (-0.16)
6	0.20 (0.10)	-0.17 (0.00)	-0.44 (0.01)	-0.41 (0.11)
4	0.26 (0.17)	-0.07 (0.04)	0.00 (0.19)	0.20 (0.40)
2	0.33 (0.23)	0.03 (0.09)	0.38 (0.41)	0.74 (0.74)
1	0.41 (0.31)	0.11 (0.14)	0.76 (0.64)	1.28 (1.09)
3	0.42 (0.31)	0.10 (0.13)	0.92 (0.51)	1.44 (0.95)
5	0.42 (0.31)	0.09 (0.11)	1.06 (0.42)	1.58 (0.85)
7	0.43 (0.32)	0.08 (0.10)	1.16 (0.37)	1.67 (0.78)
9	0.44 (0.32)	0.07 (0.09)	1.23 (0.31)	1.74 (0.72)
11	0.45 (0.32)	0.06 (0.08)	1.28 (0.28)	1.78 (0.68)
13	0.45 (0.33)	0.05 (0.06)	1.29 (0.26)	1.80 (0.65)

TABLE 15. Auctioning costs AC corresponding to the different possible takers of the portfolio of the defaulted member  $d = 15$ , ranked by increasing value, for  $IM_i$  and  $DF_i$  set at the 75% and 80% confidence level. In parenthesis, the contributions to AC of the considered possible taker itself.

the smallest auctioning cost AC, namely  $-4.52$  and  $-2.88$ , when taking over the defaulted portfolio of  $CM_{15}$  (and there are in this case no additional costs to consider, cf. Remark 6.1). Such take-over makes intuitive sense given the pre-default position  $\mathbf{q}_{15} = -16.8$  of the defaulting member  $d = 15$ , compared with the position of member 14 at  $\mathbf{q}_{14} = 17.92$ ,

an almost offset of  $\mathbf{q}_{15}$ . Member 12 shows the second best solution with an auctioning cost AC close to zero, also justifiable by his offsetting position at  $\mathbf{q}_{12} = 15.36$ .

Finally, we consider one more default resolution strategy, where the CCP hedges (as per Section 3.3.3) the defaulted portfolio before auctioning all its positions. In the hedging case, with the CCP  $c$  contributing to the post-default quantities and price discovery ( $(\mathbf{q}'_i)_{i \in D' = (D \setminus \{d\}) \cup \{c\}}, \mathbf{p}^D$ ), resolving (3.10) (for  $E = D$ ) and (3.14)–(3.16) in the configuration of Example 5.1 (except for the new member  $c$ ) leads to  $\mathbf{q}'_c = -\mathbf{q}_d$ . Coincidentally, this hedging resolution thus leads to a perfect replication as per Section 3.3.5. The corresponding  $\sum_{i \neq d} \Delta XVA_i$ , detailed in Tables 16 and 17, is  $-14.45$  for  $IM_i = DF_i = 0$  and  $-6.80$  for  $IM_i$  and  $DF_i$  set at the 75% and 80% confidence levels. As the residual market risk is null in such a replication case, taking over the defaulted portfolio and its hedge does not generate any market risk. Hence in these cases no additional cost is generated by the auctioning process, i.e.  $AC = 0$ .

$CM_i$	1	2	3	4	5	6	7	8
$XVA_i$	0.81	1.21	1.36	1.7	1.74	2.05	2.02	2.37
$XVA'_i$	0.19	0.34	0.54	0.63	0.83	0.89	1.07	1.13
$\Delta XVA_i$	-0.62	-0.87	-0.81	-1.06	-0.91	-1.15	-0.94	-1.24
$CM_i$	9	10	11	12	13	14	15	$c$
$XVA_i$	2.25	2.62	2.42	2.84	2.57	3.04	2.69	
$XVA'_i$	1.28	1.34	1.44	1.54	1.59	1.72		2.39
$\Delta XVA_i$	-0.98	-1.27	-0.98	-1.3	-0.98	-1.32		

TABLE 16. The pre- and post-default XVAs computed from (A.11) when the CCP hedges the portfolio of the defaulted member  $d = 15$  on its own exchange D, in the expected shortfall case with  $IM_i = DF_i = 0$ .

$CM_i$	1	2	3	4	5	6	7	8
$XVA_i$	0.37	0.59	0.72	0.89	1	1.12	1.21	1.34
$XVA'_i$	0.13	0.21	0.38	0.4	0.6	0.56	0.79	0.72
$\Delta XVA_i$	-0.24	-0.38	-0.34	-0.5	-0.4	-0.56	-0.42	-0.62
$CM_i$	9	10	11	12	13	14	15	$c$
$XVA_i$	1.42	1.50	1.57	1.64	1.72	1.77	1.85	
$XVA'_i$	0.97	0.85	1.12	0.98	1.26	1.09		1.30
$\Delta XVA_i$	-0.45	-0.64	-0.45	-0.67	-0.46	-0.68		

TABLE 17. The pre- and post-default XVAs computed from (A.11) when the CCP hedges the portfolio of the defaulted member  $d = 15$  on its own exchange D, in the expected shortfall case with  $IM_i$  and  $DF_i$  at the 75% and 80% confidence level.

We sum up in Tables 18 and 19 the FTP of each considered default management scheme without and with collateral (in the sense here of initial margins and default fund contributions), from the cheapest to the dearest one (again, in this example, hedging then auctioning with  $D' \setminus D = \{c\}$  happens to coincide with replicating then auctioning with  $D' \setminus D = \emptyset$ ). The FTPs of the hedging then auctioning scheme provides much larger gains

then the pure auctioning strategy, which itself provides more gains than the full liquidation strategy. Coincidentally in this example, at least, this happens to be in line with the BIS recommendations quoted in the beginning of Section 3. However, our approach only endorses the point of view of the participants to the exchange. Indeed, our costs of settling the house portfolio of a defaulted clearing member ignore the damage of the default to the “outer” actors  $o$  (end-users external to the exchanges). From this viewpoint (compare the two panels of Figure 2), whenever available, centrally cleared trading is preferable to bilateral trading (but, as per today, centrally cleared trading can only concern the standardized half of the market, cf. Table 10).

	$LC_D$	$\sum_{i \in D'} \Delta \rho_i$	$\sum_{i \in D \setminus \{d\}} \Delta XVA_i$	AC	FTP
liquidating	0	0.70	-0.77	0	-0.08
auctioning	0	0	0	-4.52	-4.52
hedging then auctioning	-1.39	2.09	-14.45	0	-13.75

TABLE 18. FTPs of different default management schemes on  $D$  split as per (6.4) for  $IM_i = DF_i = 0$ .

	$LC_D$	$\sum_{i \in D'} \Delta \rho_i$	$\sum_{i \in D \setminus \{d\}} \Delta XVA_i$	AC	FTP
liquidating	0	0.70	-0.25	0	0.45
auctioning	0	0	0	-2.88	-2.88
hedging then auctioning	-1.39	2.09	-6.80	0	-6.11

TABLE 19. FTPs of different default management schemes on  $D$  split as per (6.4) for  $IM_i$  and  $DF_i$  set at the 75% and 80% confidence level.

In this example, hedging then auctioning with  $D' \setminus D = \{c\}$  happens to coincide with replicating then auctioning with  $D' \setminus D = \emptyset$ .

## APPENDIX A. XVA GAUSSIAN SETUP

The purpose of this part is to provide a bridge from the equilibrium setup of Sections 2–5 to the XVA setup of Bastide et al. (2023), so that we are able to provide an overall FTP (6.3) quantifying the market but also credit costs of a given default resolution strategy. We leave for future research the extension of the approach of this paper to a setup where not only the market costs, but also the credit costs, would be treated endogenously as part of a global (or perhaps two-stage<sup>16</sup>) equilibrium, ideally in the setup of a dynamic model.

We endorse the structure of exchanges  $E = A \cup B$  depicted in Section 6.1. Note that even for those clients clearing through a CCP member and also having OTC positions with that same member, the corresponding exposures must be treated separately, hence  $B \cap O = \emptyset$ . For each participant  $i$  to an exchange  $E$ , a comparison of the present setup with Bastide et al. (2023, Eqns. (15)-(16)) yields the mapping of Table 20.

Let  $J_i$  be the survival indicator of participant  $i$  to an exchange  $E$ , i.e.  $J_i = \mathbb{1}_{\{\tau_i > T\}}$ , where  $\tau_i$  is the default time of participant  $i$  over the period  $[0, T]$ , with probability  $\gamma_i = \mathbb{P}(J_i = 0)$  of default over  $[0, T]$ . We denote likewise  $J_o = \mathbb{1}_{\{\tau_o > T\}}$  for any end-user  $o \in O$ . Via the mapping of Table 20, (Bastide et al., 2023, Eqns. (15)-(16)) yield the following

<sup>16</sup>accounting for the segregation between the market and credit spheres in banks (Bellini, 2017, Section 6.1) (The European Parliament and the Council of the European Union, 2013, Article 92).



<b>(Bastide et al., 2023, Eqns. (15)-(16))</b>	<b>This paper</b>	<b>Description</b>
$\mathcal{P}_c^{ccp} - \text{MtM}_c^{ccp}$	$(\mathbf{q}_b^0)^\top (\mathbf{p}^E - P)$	Cash flows from a cleared client ( $c$ in Bastide et al. (2023), $b$ in this paper) of a CCP ( $ccp$ in Bastide et al. (2023), the CCP of exchange $E$ in this paper) to the participant 0 (if the latter is not a clearing member of the CCP, these cash flows are simply zero)
$\mathcal{P}_i^{ccp} - \text{MtM}_i^{ccp}$	$\sum_{b \in B} (\mathbf{q}_b^a)^\top (\mathbf{p}^E - P)$	Client account cash flows from a clearing member ( $i$ in Bastide et al. (2023), $a$ in this paper) to his CCP ( $ccp$ in Bastide et al. (2023), the CCP of exchange $E$ in this paper)
$\overline{\mathcal{P}}_i^{ccp} - \overline{\text{MtM}}_i^{ccp}$	$\mathbf{q}_a^\top (\mathbf{p}^E - P)$	Proprietary account cash flows from a clearing member ( $i$ in Bastide et al. (2023), $a$ in this paper) to his CCP
$\mathcal{P}_b - \text{VM}_b$	$R_0^o$	Cash flows from an end-user ( $b$ in Bastide et al. (2023), $o$ in this paper) to participant 0

TABLE 20. Some notation adaptation for the cash flows of some market participants after variation margin is subtracted, in the setup of Bastide et al. (2023) and in this paper.

(pre-default equilibrium) credit loss profile  $\mathcal{C}_0$  of a participant  $i = 0$  to the exchanges, on which we focus in what follows:  $\mathcal{C}_0 = \sum_{E=A \cup B} (\mathcal{C}_0^B + \mathcal{C}_0^A) + \mathcal{C}_0^O$ , where

$$\begin{aligned}
\mathcal{C}_0^B &= \sum_{b \in B} (1 - J_b) \left( (\mathbf{q}_b^0)^\top (\mathbf{p}^E - P) - \text{IM}_0^b \right)^+, \\
\mathcal{C}_0^A &= w_0^A \sum_{a \in A} (1 - J_a) \left[ \left( \left( \sum_{b \in B} \mathbf{q}_b^a \right)^\top (\mathbf{p}^E - P) - \text{IM}_a^A \right)^+ + \right. \\
&\quad \left. \left( \mathbf{q}_a^\top (\mathbf{p}^E - P) - \overline{\text{IM}}_a^A \right)^+ - \text{DF}_a^A \right]^+, \\
\mathcal{C}_0^O &= \sum_{o \in O} (1 - J_o) (R_0^o - \text{IM}_0^o)^+.
\end{aligned} \tag{A.1}$$

Here  $\text{IM}_0^b$  is the initial margin (IM) requested by the participant 0 to the simple participant  $b \in B$  on the cleared position  $\mathbf{q}_b^0$  (equal to 0 if  $0 \notin A$ );  $w_0^A$  (equal to 0 if  $0 \notin A$ ) is the loss allocation coefficient of the participant 0 w.r.t. the CCP of the exchange  $E = A \cup B$ ;  $\text{IM}_a^A$ ,  $\overline{\text{IM}}_a^A$ , and  $\text{DF}_a^A$  are the initial margins for the cleared clients and proprietary accounts as well as the default fund contribution requested by (the CCP of) exchange  $E$  to the clearing member  $a$ ;  $R_0^o$  is the exogenous receivable of the participant 0 from its OTC bilateral counterparty  $o$ , with corresponding initial margin  $\text{IM}_0^o$  (which can be null under OTC agreement) requested by 0 to  $o$  (cf. Figure 1).

Likewise, the post-default equilibrium<sup>17</sup> default loss profile  $\mathcal{C}'_0$  of the participant 0 is  $\mathcal{C}'_0 = \sum_{E'=A' \cup B'} (\mathcal{C}'_0{}^{B'} + \mathcal{C}'_0{}^{A'}) + \mathcal{C}_0^O$ , for  $\mathcal{C}_0^O$  as in (A.1) and (cf. Table 20 and (3.8)-(3.5))

$$\begin{aligned} \mathcal{C}'_0{}^{B'} &= \sum_{b \in B'} (1 - J_b) \left( (\mathbf{q}_b^0)^\top (\mathbf{p}'^E - P) + (\mathbf{q}_b^0 + (\Delta \mathbf{q}_b^0)^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) - \text{IM}_0^{lb} \right)^+, \\ \mathcal{C}'_0{}^{A'} &\stackrel{(3.8)}{=} w_0'^A \sum_{a \in A'} (1 - J_a) \times \\ &\quad \left( \left( \sum_{b \in B'} (\mathbf{q}_b^{a'})^\top (\mathbf{p}'^E - P) + \sum_{b \in B'} (\mathbf{q}_b^a + (\Delta \mathbf{q}_b^a)^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) - \text{IM}_a'^A \right)^+ + \right. \\ &\quad \left. \left( (\mathbf{q}_a^l)^\top (\mathbf{p}'^E - P) + (\mathbf{q}_a^l + \Delta \mathbf{q}_a^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) - \overline{\text{IM}}_a'^A \right)^+ - \text{DF}_a'^A \right)^+, \end{aligned} \quad (\text{A.2})$$

where  $w_0'^A$ ,  $\text{IM}_a'^A$ ,  $\overline{\text{IM}}_a'^A$ ,  $\text{DF}_a'^A$ , and  $\text{IM}_0^{lb}$  are the post-default analogs of  $w_0^A$ ,  $\text{IM}_a^A$ ,  $\overline{\text{IM}}_a^A$ ,  $\text{DF}_a^A$ , and  $\text{IM}_0^b$  in (A.1).

The pre- and post-default CVA of the participant 0 are given by

$$\text{CVA}_0 = \mathbb{E} [\mathcal{C}_0 | J_0 = 1] = (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}_0], \quad \text{CVA}'_0 = (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}'_0], \quad (\text{A.3})$$

by Bastide et al. (2023, Theorem 3.7). Denoting by  $\overline{\text{IM}}_0^o$  the initial margin from the participant 0 to its OTC bilateral counterparty  $o$ , based on Remark 2.2, such margin remains constant in the post-default equilibrium, hence the pre- and post-default MVA of participant 0 are given by

$$\begin{aligned} \text{MVA}_0 &= \tilde{\gamma}_0 \left( \sum_{o \in O} \overline{\text{IM}}_0^o + \sum_E \left( \text{IM}_0^E + \overline{\text{IM}}_0^E + \text{DF}_0^E \right) \right), \\ \text{MVA}'_0 &= \tilde{\gamma}_0 \left( \sum_{o \in O} \overline{\text{IM}}_0^o + \sum_E \left( \text{IM}_0^{E'} + \overline{\text{IM}}_0^{E'} + \text{DF}_0^{E'} \right) \right), \end{aligned} \quad (\text{A.4})$$

for some possibly blended funding rate  $\tilde{\gamma}_0 \leq \gamma_0$  as detailed in Albanese, Armenti, and Crépey (2020, Section 5). The pre-default KVA, defined for a hurdle rate  $h$ , is calculated based on an expected shortfall  $\mathbb{E}\mathbb{S}_{\tilde{\alpha}_0}^0$  of the participant 0 under its own survival measure,  $\mathbb{P}(\cdot | J_0) / (1 - \gamma_0)$  (with  $\tilde{\alpha}_0 >$  the confidence level  $\alpha_0$  introduced for the market cost computation in Section 5 when the risk measures used by the hedgers are expected shortfall risk measures<sup>18</sup>), as

$$\begin{aligned} \text{KVA}_0 &= \frac{h}{1 + h} \mathbb{E}\mathbb{S}_{\tilde{\alpha}_0}^0 (\mathcal{C}_0 - \text{CVA}_0) \\ &= \frac{h}{1 + h} \mathbb{E} [J_0 (\mathcal{C}_0 - \text{CVA}_0) | \mathcal{C}_0 - \text{CVA}_0 \geq \text{Va}\mathbb{R}_{\tilde{\alpha}_0}^0 (J_0 (\mathcal{C}_0 - \text{CVA}_0)), J_0 = 1], \end{aligned} \quad (\text{A.5})$$

by Bastide et al. (2023, Theorem 3.7, last row of Table 2), where  $\text{Va}\mathbb{R}_{\tilde{\alpha}_0}^0$  denotes the value-at-risk at the confidence level  $\tilde{\alpha}_0$  under the measure  $(1 - \gamma_0)^{-1} \mathbb{P}(\cdot | J_0)$ . The post-default

<sup>17</sup>post-default referring as usual in the paper to the instant default at time 0 of a clearing member  $d$ , here assumed  $\neq$  the reference clearing member 0.

<sup>18</sup>regulatory and economic capital aim at capturing extreme losses that can occur once every 1000 years (Basel Committee on Banking Supervision, 2005, paragraph 5.1), which leads to considering a much higher confidence level  $\tilde{\alpha}_0$  for economic capital calculation, such as 0.9975, from which the KVA is defined, than the  $\alpha_0$  used for market risk, set to 0.975, in line with (Bank for International Settlements, 2013, Section 1.4 (i)).

KVA has a similar expression substituting  $\mathcal{C}'_0$  to  $\mathcal{C}_0$  and  $\text{CVA}'_0$  to  $\text{CVA}_0$  in (A.5):

$$\begin{aligned} \text{KVA}'_0 &= \frac{h}{1+h} \mathbb{E} \mathbb{S}_{\tilde{\alpha}_0}^0 (\mathcal{C}'_0 - \text{CVA}'_0) \\ &= \frac{h}{1+h} \mathbb{E} [J_0 (\mathcal{C}'_0 - \text{CVA}'_0) | \mathcal{C}'_0 - \text{CVA}'_0 \geq \mathbb{V} \mathbb{a} \mathbb{R}_{\tilde{\alpha}_0}^0 (J_0 (\mathcal{C}'_0 - \text{CVA}'_0)), J_0 = 1], \end{aligned} \quad (\text{A.6})$$

Finally, by Bastide et al. (2023, Theorem 3.7, next to last row of Table 2), the pre- and post-FVA of the participant 0 is given by

$$\begin{aligned} \text{FVA}_0 &= \frac{\gamma_0}{1+\gamma_0} \left( \sum_{o \in \mathcal{O}} \mathbb{E} R_o^o - (\text{CVA}_0 + \text{MVA}_0) - \mathbb{E} \mathbb{S}_{\tilde{\alpha}_0}^0 (\mathcal{C}_0 - \text{CVA}_0) \right)^+, \\ \text{FVA}'_0 &= \frac{\gamma_0}{1+\gamma_0} \left( \sum_{o \in \mathcal{O}} \mathbb{E} R_o^o - (\text{CVA}'_0 + \text{MVA}'_0) - \mathbb{E} \mathbb{S}_{\tilde{\alpha}_0}^0 (\mathcal{C}'_0 - \text{CVA}'_0) \right)^+. \end{aligned} \quad (\text{A.7})$$

**A.1. XVA details in the setup of Section 6.3.** In the setup of Section 6.3, only clearing members  $a$  participate to the only exchange of interest D (so all participants  $i$  are clearing members  $a$  and there are no cleared clients  $b$ ) and all the end-users (OTC bilateral counterparties)  $o$  are assumed to be default risk-free, Hence the pre-default credit loss (A.1) of member  $0 \in \text{D}$  reduces to

$$\mathcal{C}_0 = w_0^{\text{D}} \sum_{j \in \text{D}} (1 - J_j) \left( \mathbf{q}_j (\mathbf{p}^{\text{D}} - P) - \overline{\text{IM}}_j^{\text{D}} - \text{DF}_j^{\text{D}} \right)^+, \quad (\text{A.8})$$

where  $w_0^{\text{D}} = \frac{\text{DF}_0^{\text{D}} J_0}{\sum_{j \in \text{D}} \text{DF}_j^{\text{D}} J_j}$ . Under the post-default equilibrium when the CCP fully liquidates on D (so that  $\Delta \mathbf{q}_i^h = 0$ ), the post-default credit loss (A.2) of member  $0 \in \text{D}$  reduces to

$$\begin{aligned} \mathcal{C}'_0 &= w_0^{\text{D}} \sum_{i \in \text{D}'} (1 - J_i) \left( \mathbf{q}'_i (\mathbf{p}^{\text{D}} - P) + \mathbf{q}'_i (\mathbf{p}^{\text{D}} - \mathbf{p}'^{\text{D}}) - \overline{\text{IM}}_i^{\text{D}} - \text{DF}_i^{\text{D}} \right)^+ \\ &= w_0^{\text{D}} \sum_{i \in \text{D}'} (1 - J_i) \left( \mathbf{q}'_i (\mathbf{p}^{\text{D}} - P) - \overline{\text{IM}}_i^{\text{D}} - \text{DF}_i^{\text{D}} \right)^+, \end{aligned} \quad (\text{A.9})$$

whereas, when the CCP fully hedges on D (so that  $\Delta \mathbf{q}_i^l = 0$ ,  $\mathbf{q}_j + \Delta \mathbf{q}_j^h = \mathbf{q}'_j$ ),

$$\mathcal{C}'_0 = w_0^{\text{D}} \sum_{i \in \text{D}'} (1 - J_i) \left( \mathbf{q}'_i (\mathbf{p}^{\text{D}} - P) + \mathbf{q}_i (\mathbf{p}^{\text{D}} - \mathbf{p}'^{\text{D}}) - \overline{\text{IM}}_i^{\text{D}} - \text{DF}_i^{\text{D}} \right)^+. \quad (\text{A.10})$$

In both cases  $w_0^{\text{D}}$ ,  $\overline{\text{IM}}_i^{\text{D}}$  and  $\text{DF}_i^{\text{D}}$ ,  $i \in \text{D}'$ , are the post-default analogs of  $w_0^{\text{D}}$ ,  $\overline{\text{IM}}_i^{\text{D}}$  and  $\text{DF}_i^{\text{D}}$ ,  $i \in \text{D}$ , based on the post-default updated portfolio positions. The pre- and post-default CVA, MVA and KVA of member  $0 \in \text{D} \setminus \{d\}$  are given by

$$\begin{aligned} \text{CVA}_0 &= (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}_0], \quad \text{CVA}'_0 = (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}'_0], \\ \text{MVA}_0 &= \tilde{\gamma}_0 (\overline{\text{IM}}_0 + \text{DF}_0), \quad \text{MVA}'_0 = \tilde{\gamma}_0 (\overline{\text{IM}}'_0 + \text{DF}'_0), \\ \text{KVA}_0 &= \frac{h}{1+h} \mathbb{E} [J_0 (\mathcal{C}_0 - \text{CVA}_0) | \mathcal{C}_0 - \text{CVA}_0 \geq \mathbb{V} \mathbb{a} \mathbb{R}_{\tilde{\alpha}_0}^0 (J_0 (\mathcal{C}_0 - \text{CVA}_0)), J_0 = 1], \\ \text{KVA}'_0 &= \frac{h}{1+h} \mathbb{E} [J_0 (\mathcal{C}'_0 - \text{CVA}'_0) | \mathcal{C}'_0 - \text{CVA}'_0 \geq \mathbb{V} \mathbb{a} \mathbb{R}_{\tilde{\alpha}_0}^0 (J_0 (\mathcal{C}'_0 - \text{CVA}'_0)), J_0 = 1]. \end{aligned} \quad (\text{A.11})$$

Moreover, in the setup of Section 6.3,  $\mathbb{E} R_o^o$  in (A.7) corresponds to  $\text{MtM}_b - \text{VM}_b$  in the setup of Bastide et al. (2023), i.e. a difference of received ( $\text{VM}_b$ ) variation margin by the member 0 and posted ( $\text{MtM}_b$ ) variation margin by the member 0 for an OTC bilateral position between the clearing member 0 and the end-user  $o$ . We assume, as it is the case in practice, that there is only marginal, if no, difference between the two quantities. Hence we

have  $\mathbb{E}R_0^0 \approx 0$  and, in any case, dominated by  $(\text{CVA}_0 + \text{MVA}_0) - \mathbb{E}\mathcal{S}_{\alpha_0}^0(\mathcal{C}_0 - \text{CVA}_0)$  and  $(\text{CVA}'_0 + \text{MVA}'_0) - \mathbb{E}\mathcal{S}_{\alpha_0}^0(\mathcal{C}'_0 - \text{CVA}'_0)$  in (A.7), leading to negligible  $\text{FVA}_0$  and  $\text{FVA}'_0$ , which we therefore simply take as 0 in our numerics (and do not report in the numerics of Section 6.3).

*Latent Factor Model.* For default modeling purposes, we introduce for each member  $i \in \mathcal{D}$  a latent variable  $X_i \sim \mathcal{N}_1(0, 1)$  such that  $\{J_i = 0\} \iff \{X_i \leq \Phi^{-1}(\gamma_i)\}$ . These default latent variables are correlated as per  $X_i = \sqrt{\rho^{cr}}\varepsilon + \sqrt{1 - \rho^{cr}}\varepsilon_i$ , where  $\varepsilon$  and  $\varepsilon_i$  are i.i.d.  $\mathcal{N}_1(0, 1)$ , while  $\rho^{cr}$  is a positive credit/credit correlation coefficient. Writing  $P = \mu + \sigma Y$  with  $Y \sim \mathcal{N}_1(0, 1)$ , the IM posted to the CCP by member  $i$ , based on the idea of a  $\mathbf{q}_i(\mathbf{p}^D - P)$  VM call not fulfilled over its corresponding time period  $T$  (versus  $\Delta_s$  in Bastide et al. (2023)), is computed by the VaR metric<sup>19</sup> at a confidence level  $\alpha_{im} \in (1/2, 1)$  as

$$\overline{\text{IM}}_i^D = \text{VaR}_{\alpha_{im}}(\mathbf{q}_i(\mathbf{p}^D - P)) = \mathbf{q}_i(\mathbf{p}^D - \mu) + |\mathbf{q}_i| \sigma \Phi^{-1}(\alpha_{im}). \quad (\text{A.12})$$

The liquidation time period  $\Delta_l$  in Bastide et al. (2023) is also taken as the one-period of time considered in the Radner equilibrium setup of the present paper, so that  $\Delta_s = \Delta_l = T$ . The default fund is calculated at the level of the considered CCP of  $d$  as the sum of the two highest stress loss over IM (SLOIM), where SLOIM is given for each member  $i$  as

$$\text{SLOIM}_i^D = \text{VaR}_{\alpha_{df}}(\mathbf{q}_i(\mathbf{p}^D - P) - \overline{\text{IM}}_i^D) = |\mathbf{q}_i| \sigma (\Phi^{-1}(\alpha_{df}) - \Phi^{-1}(\alpha_{im})),$$

for some confidence level  $\alpha_{df} > \alpha_{im}$ . The default fund contribution of member  $i$  is given as

$$\text{DF}_i^D = \frac{\text{SLOIM}_i^D}{\sum_{j \in \mathcal{D}} \text{SLOIM}_j^D} (\text{SLOIM}_{(0)}^D + \text{SLOIM}_{(1)}^D), \quad (\text{A.13})$$

based on the cover-2 amount given as the sum of two largest stressed losses over IM ( $\text{SLOIM}_i$ ) among its members, identified with subscripts (0) and (1).

Portfolios maturity $T$	5 years
Liquidation period at default $\Delta_l$	5 years
Credit factors correlation $\rho^{cr}$	20%
IM covering period (margin period of risk) $\Delta_s$	5 years
Default probabilities $\gamma_i$	39.3%
MVA funding blending ratio $\tilde{\gamma}_i/\gamma_i$	25%
Quantile levels $\tilde{\alpha}_i$ used for clearing members KVA	99.75%
Hurdle rate $h$ used for KVA computations	10.0%
Number of Monte-Carlo simulations (for CVA and KVA computations)	10M
Number of batches (for KVA computations)	100

TABLE 21. XVAs calculation configuration

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<sup>19</sup>under the member survival measure.

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