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Operations on fuzzy numbers

DIDIER DUBOIS† and HENRI PRADÉ†

A fuzzy number is a fuzzy subset of the real line whose highest membership values are clustered around a given real number called the mean value; the membership function is monotonic on both sides of this mean value. In this paper, the usual algebraic operations on real numbers are extended to fuzzy numbers by the use of a fuzzification principle. The practical use of fuzzified operations is shown to be easy, requiring no more computation than when dealing with error intervals in classic tolerance analysis. The field of applications of this approach seems to be large, since it allows many known algorithms to be fitted to fuzzy data.

1. Introduction

In many scientific areas, such as systems analysis and operations research, a model has to be set up using data which is only approximately known. Fuzzy sets theory, introduced by Zadeh (1965), makes this possible. Fuzzy numerical data can be represented by means of fuzzy subsets of the real line, known as fuzzy numbers. This approach may be of practical interest only if we can easily perform algebraic operations on them. Fuzzy sets theory generalizes tolerance analysis; it completely differs from a probabilistic approach. Jain (1976) had already attempted algebraic operations on fuzzy sets having supports. Jain's method will be shown to be inexact and rather impractical, since it often involves too many computations.

In the following sections, we set forth results for the algebraic combination of fuzzy sets with infinite supports. Paradoxically, our method can be proved to be more rapid than Jain's, and, moreover, exact. Section 2 recalls the fuzzification principle and gives the definition of a fuzzy number. Section 3 is devoted to the sum of fuzzy numbers. Section 4 deals with the fuzzy product and the quotient of fuzzy numbers. Section 5 shows how to find the fuzzy value of the 'greatest' or the 'smallest' of several fuzzy subsets of the real line. These operations are extensions of the usual max and min operations. Section 6 provides some results on the addition of fuzzy numbers of type 2 (Mizumoto and Tanaka 1976), and § 7 offers some concluding remarks.

2. Fuzzy numbers and fuzzifications

2.1. Fuzzy numbers

A real fuzzy number \tilde{n} is more precisely described as any fuzzy subset of the real line R , whose membership function $\mu_{\tilde{n}}$ is

- (i) A continuous mapping from R to the closed interval $[0, 1]$.
- (ii) Constant on $(-\infty, c]$: $\mu_{\tilde{n}}(x) = 0 \quad \forall x \in (-\infty, c]$.

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- (iii) Strictly increasing on $[c, a]$.
- (iv) Constant on $[a, b]$: $\mu_{\tilde{n}}(x) = 1 \quad \forall x \in [a, b]$.
- (v) Strictly decreasing on $[b, d]$.
- (vi) Constant on $(d, +\infty)$: $\mu_{\tilde{n}}(x) = 0 \quad \forall x \in [d, +\infty)$.

a, b, c and d are real numbers. Eventually we can have

$$c = -\infty, \quad \text{or } a = b, \quad \text{or } c = a, \quad \text{or } b = d, \quad \text{or } d = +\infty$$

We will call *mean value* of \tilde{n} an element of $[a, b]$, often $(a+b)/2$.

Comments

If $a = c = b = d$, \tilde{n} is an ordinary real number.

If $a = c$ and $b = d$, \tilde{n} is a representation of the tolerance interval $[a, b]$ of the measurement of a quantity.

If $a = b$, \tilde{n} is a representation of a fuzzy number, the value of which is approximately a .

$\mu_{\tilde{n}}(x)$ is the truth value of the assertion 'the value of \tilde{n} is x '.

Two fuzzy numbers are equal if and only if they have the same membership functions.

2.2. Fuzzifications

From any function f from R^n to R which associates a real number $y = f(x_1, \dots, x_n)$ to each (x_1, \dots, x_n) in R^n we can induce a function \tilde{f} from $\mathcal{F}(R)^n$ to $\mathcal{F}(R)$; $\mathcal{F}(R)$ is the class of fuzzy subsets of R . This can be achieved via a fuzzification principle mentioned by Gaines (1976), Negoita (1976) and others: $\forall (\tilde{x}_1, \dots, \tilde{x}_n) \in \mathcal{F}(R)^n$ whose membership functions are $\mu_{\tilde{x}_1}, \dots, \mu_{\tilde{x}_n}$, and $\tilde{y} = \tilde{f}(\tilde{x}_1, \dots, \tilde{x}_n)$ is a fuzzy subset of R whose membership function $\mu_{\tilde{y}}$ is defined as

$$\forall t \in R, \quad \mu_{\tilde{y}}(t) = \max_{i=1, \dots, n} \min(\mu_{\tilde{x}_1}(s_1), \dots, \mu_{\tilde{x}_n}(s_n)) \quad (1)$$

The membership value $\mu_{\tilde{y}}(t)$ is the greatest of the truth values of the realizations of t by (s_1, \dots, s_n) through f ; the truth value of (s_1, \dots, s_n) being the smallest value among $\mu_{\tilde{x}_1}(s_1), \dots, \mu_{\tilde{x}_n}(s_n)$.

There are other possible fuzzifications; for instance Jain (1976) used the probabilistic sum * instead of max in eqn. (1); (* is defined as $a*b = a+b-ab$). Jain (1976) proposes that since t appears more than once with different grades of membership, the membership of t in y should be considered on the basis of the membership of all elements satisfying $t = f(s_1, \dots, s_n)$. Jain's formula not take into account the number of ways in which an event can occur, only the fact that the occurrence may happen. However, Gaines' formula will be shown to involve far less computation than Jain's when performing operations on fuzzy numbers.

3. Sum of fuzzy numbers

3.1. Definition

Let \tilde{m} and \tilde{n} be two fuzzy numbers with membership functions f and g , and mean values m and n , respectively. The membership function h of their sum $\tilde{m} \oplus \tilde{n}$ is defined as

$$h(z) = \max_{x+y=z} \min (f(x), g(y)), \quad (x, y, z) \in \mathbb{R}^3$$

which can be expressed in the form

$$h(z) = \max_x \min (f(x), g(z-x))$$

It is obvious that this operation is a convolution. In order to implement it on a computer, we could think of using a sampling of f and g in the neighborhood of the mean values. Thus m and n would be approximated by fuzzy sets having finite supports, as was done by Jain (1976). If the supports of \tilde{m} and \tilde{n} have many elements, we can obtain precise results, but the computation time can be very long. If, to save computation time, we use small supports, we can be led to incorrect results (Dubois and Prade 1978 b)—it is possible to obtain the resulting mean value, even if the membership functions of m and n are strictly monotonous on each side of their mean values. Moreover, as Jain (1976) shows, if p_m and p_n are the number of elements in the supports of m and n , we have that $p_{m \oplus n} = p_m \times p_n$. So we must reduce, sometimes arbitrarily, the resulting support. All these remarks are independent of the nature of the fuzzified operation.

In the sections that follow, it will be shown that when performing usual fuzzified operations (in the sense of Gaines (1976)), exact results can be obtained very easily without sampling, with, in most cases, few hypotheses concerning the analytical expressions of the membership functions.

3.2. General case

Let \tilde{m} and \tilde{n} be two fuzzy numbers, as defined in §2.1; they are pictured in Fig. 1.

Lemma 1

Let h be the membership function of $\tilde{m} \oplus \tilde{n}$; then

- (i) $\forall z \in (-\infty, c_f + c_g] \cup [d_f + d_g, +\infty)$, $h(z) = 0$,
- (ii) $z \in [a_f + a_g, b_f + b_g]$, $h(z) = 1$.

Proof

Obvious.

Lemma 2

Let $\omega \in]0, 1[$ and $(p, r) \in \mathbb{R}^2$ such that

$$f(p) = g(r) = \omega, \quad p < a_f, r < a_g$$

Then

$$h(p+r) = \omega$$

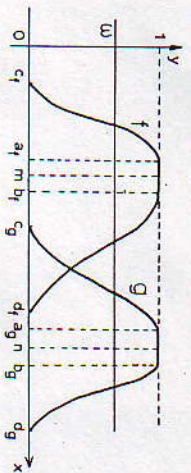


Figure 1.

Proof

Let $k = p+r$. To evaluate $h(k)$ we must consider every pair (x, y) such that $x+y=k$.

- (i) If $x < p$, then $y > r$. Since f is strictly increasing on $[c_f, a_f]$, $f(x) < \omega$, therefore $\min (f(x), g(y)) < \omega$.
- (ii) If $x > p$, then $y < r$. Since g is strictly increasing on $[c_g, a_g]$, $g(y) < \omega$, therefore $\min (f(x), g(y)) < \omega$.

Consequently, $\min (f(x), g(y))$ is at a maximum for $x=p, y=r$ and $h(k) = \omega$. Q.E.D.

Corollary

Let $\omega \in]0, 1[$ and $(p, r) \in \mathbb{R}^2$ such that

$$f(p) = g(r) = \omega, \quad p > b_f, r > b_g$$

Then $h(p+r) = \omega$.

Proof

As for Lemma 2.

Lemma 3

Let $\omega \in]0, 1[$ and (p, r) such that $\omega = f(p) = g(r)$ and $p+r < a_f + a_g$. Then $\exists (p^*, r^*)$ such that

- (i) $p^* < a_f, r^* < a_g$,
- (ii) $f(p^*) = g(r^*) = h(p+r)$,
- (iii) $p^* + r^* = p+r$.

Proof

Let f_+ and g_+ be the restrictions of f and g on $[c_f, a_f]$ and $[c_g, a_g]$, respectively; f_- and g_- are continuous and strictly increasing in those intervals. Their inverse functions, f_-^{-1} and g_-^{-1} exist, and are continuous and strictly increasing on $[0, 1]$; their values belong to $[c_f, a_f]$ and $[c_g, a_g]$, respectively. Therefore the function $l = f_-^{-1} + g_-^{-1}$ is strictly increasing and bijective from $[0, 1]$ in $[c_f + c_g, a_f + a_g]$.

Let $k = p+r$; $k > c_f + c_g$ since $\omega \in]0, 1[$. Let ω' be uniquely defined as $l(\omega') = k$, and let $p^* = f_-^{-1}(\omega')$ and $r^* = g_-^{-1}(\omega')$.

We know that $\omega' \in [0, 1]$ since $k < a_j + a_g$ by hypothesis; therefore $p^* < a_j$ and $r^* < a_g$; moreover, $f(p^*) = g(r^*) = \omega'$ and $p^* + r^* = f_{-1}^{-1}(\omega') + g_{-1}^{-1}(\omega') = l(\omega') = k = p + r$; ω' and (p^*, r^*) satisfy the hypothesis of Lemma 2. Consequently, $h(k) = \omega' = f(p^*) = g(r^*)$. Q.E.D.

Let us note that $\omega' \geq \omega$, since $\min\{f(x), g(y)\}$ is at a maximum for $x = p^*$ and $y = r^*$.

Corollary

Let $\omega \in [0, 1]$ and (p, r) such that $f(p) = g(r)$ and $p + r > b_j + b_g$. Then $\exists(p^*, r^*)$ such that

- (i) $p^* > b_j, r^* > b_g$,
- (ii) $f(p^*) = g(r^*) = k(p + r)$,
- (iii) $p^* + r^* = p + r$.

Proof

As for Lemma 3.

Theorem 1

If $\tilde{m} \oplus \tilde{n}$ is a fuzzy number as defined in §2.1; h has the same properties as f and g ; particularly:

- (i) h is continuous in R ,
- (ii) $h(z) = 0, z \in (-\infty, c_j + c_g] \cup [d_j + d_g, +\infty)$,
- (iii) $h(z)$ is strictly increasing in $]c_j + c_g, a_j + a_g[$ (and strictly decreasing in $]b_j + b_g, d_j + d_g[$),
- (iv) $h(z) = 1, z \in [a_j + a_g, b_j + b_g]$.

Proof

This comes from the method used to obtain h , according to the preceding lemmas. Particularly, h_- depends only on f_- and g_- . (The exact formula is $h_- = f_{-1}^{-1} = (f_{-1}^{-1} + g_{-1}^{-1})^{-1}$.)

Since l is continuous and strictly increasing, so is h_- .

Q.E.D.

3.3. *Practical calculus*

We now assume that \tilde{m} and \tilde{n} have membership functions of the same kind; more specifically, that f_- and g_- differ only by a translation in the Ox direction (the length of the translation vector is $|m - n|$) and an orthogonal affinity whose axis is parallel to O ; and that f_+ and g_+ differ by the same translation but perhaps by another affinity.

Let L and R be two even functions belonging to the class defined in §2.1. The fuzzy number \tilde{m} will have a membership function f such that

$$f_-(x) = L_- \left(\frac{x - m}{\alpha} \right), \quad \alpha > 0$$

$$f_+(x) = R_+ \left(\frac{x - m}{\beta} \right), \quad \beta > 0$$

In the same way, \tilde{n} will have a membership function g such that

$$g_-(x) = L_- \left(\frac{x - n}{\gamma} \right), \quad \gamma > 0$$

$$g_+(x) = R_+ \left(\frac{x - n}{\delta} \right), \quad \delta > 0$$

α and β will be respectively called the left and right spreads of \tilde{m} respectively. \tilde{m} and \tilde{n} will be referred to as ' L - R fuzzy numbers'.

Let us compute $\tilde{m} \oplus \tilde{n}$. We know, by Lemma 3, that $\omega \in [0, 1]$, $\exists(p, r)$, so that $\omega = f_-(p) = g_-(r) = h_-(z)$ where $z = p + r$. This can be expressed as

$$\omega = L_- \left(\frac{p - m}{\alpha} \right) = L_- \left(\frac{r - n}{\gamma} \right)$$

which is equivalent to $p = m - \alpha \cdot L_-^{-1}(\omega)$ and $r = n - \gamma \cdot L_-^{-1}(\omega)$; this implies that $z = p + r = m + n - (\alpha + \gamma) \cdot L_-^{-1}(\omega)$, which is the same as

$$\omega = L_- \left(\frac{z - (m + n)}{\alpha + \gamma} \right) = h_-(z)$$

In the same way we could prove that

$$h_+(z) = R_+ \left(\frac{z - (m + n)}{\beta + \delta} \right)$$

Given L and R , we can write the L - R fuzzy numbers $\tilde{m} = (m, \alpha, \beta)$ and $\tilde{n} = (n, \gamma, \delta)$. The sum $\tilde{m} \oplus \tilde{n}$ is an L - R fuzzy number, and

$$\left[(m, \alpha, \beta) \oplus (n, \gamma, \delta) \right] = (m + n, \alpha + \gamma, \beta + \delta) \tag{2}$$

3.4. *Comments*

(a) All these results remain valid if we use fuzzy numbers whose membership functions are increasing, but not strictly, in $]c, a[$ and decreasing, but not strictly, in $]b, d[$.

(b) To add two L - R fuzzy numbers requires three classic additions; this extra amount of computation seems not to be too time-consuming in practice. We have shown it to be considerably less than that involved in Jain's (1976) method.

Equation (2) does not depend on the analytical expressions of L and R , provided that all the fuzzy numbers are of the L - R type. Those expressions can be used at the end of the computation to represent the results. Choosing L and R is a subjective matter; we can modify this choice without additional computation except for the expression of the final results. Examples of L and R functions are

$$*y = \max(0, 1 - |x|) \qquad *y = \max(0, 1 - x^2)$$

$$*y = \exp(-|x|) \qquad *y = \exp(-x^2)$$

$$*y = \frac{1}{1 + |x|} \qquad *y = \frac{1}{1 + x^2}$$

$$*y = 1 \text{ in } [-1, +1], y = 0 \text{ elsewhere.}$$

It is obvious that we can use different functions for L and R in order to express a possible dissymmetry in the knowledge of the value of m : 'approximately m '. When the choice of L and R has been made, it is equivalent to know (α, β) , or two membership values, one on each side of m . The greater α and β , the wider the membership function, and the fuzzier the number. An ordinary real number m will be written $(m, 0, 0)$ (the spreads are null).

(c) If the membership function is $y = 1$ in $[-1, +1]$ and zero elsewhere, eqn. (2) means 'the absolute tolerance of a sum is the sum of the absolute tolerances'. Here α and β are respectively left and right errors.

Let us consider α -cuts of m (which are ordinary subsets), each of which can be interpreted as an error interval whose truth value is α . Using the theorem on the absolute tolerance of a sum for every α -cut, and the definition of a fuzzy subset in terms of its α -cuts, we can again find Theorem 1.

(d) The sum of fuzzy numbers is associative and commutative; $(0, 0, 0)$ is the identity. Unless its spreads are null, a fuzzy number has no symmetric element; the set of fuzzy real numbers is a semi-group for \oplus .

3.5. Subtraction of fuzzy numbers

3.5.1. Opposite of a fuzzy number

Let \tilde{m} be a fuzzy number and f its membership function. The membership function g of the opposite of \tilde{m} , written $\widetilde{-m}$, is defined as

$$g(z) = \max_{z=-x} \min f(x) = f(-z)$$

g is the symmetric function of f and Oy is the symmetry axis in Fig. 1. If \tilde{m} is an L - R fuzzy number, (m, α, β) , $\widetilde{-m}$ is an R - L fuzzy number $(-m, \beta, \alpha)$.

3.5.2. Consequence

We can easily perform a subtraction of two fuzzy numbers, only if they are of opposite types (L - R and R - L respectively) since the difference operator is defined as $\tilde{m} \ominus \tilde{n} = \widetilde{m \oplus -n}$. If $\tilde{m} = (m, \alpha, \beta)$, L - R type, and $\tilde{n} = (n, \gamma, \delta)$, R - L type, then

$$\tilde{m} \ominus \tilde{n} = (m - n, \alpha + \delta, \beta + \gamma)$$

is L - R type. Particularly, if \tilde{m} and \tilde{n} type is L - L or R - R , subtraction is easy to perform.

4. Product and quotient of fuzzy numbers

4.1. Definition

Let \tilde{m} and \tilde{n} be two fuzzy numbers; f and g are their respective membership functions. The product $\tilde{m} \odot \tilde{n}$ of these fuzzy numbers is defined by its membership function $h(z)$:

$$h(z) = \max_{xy=z} \min (f(x), g(y))$$

Here, we restrict ourselves to the case where c_f and c_g are positive. It is easy to show that lemmas analogous to Lemmas 1, 2 and 3 can be established, and we can obtain the following theorem.

Theorem 2

h defines a fuzzy number, as defined in § 2.1,

$$c_h = c_f \cdot c_g, \quad a_h = a_f \cdot a_g, \quad b_h = b_f \cdot b_g, \quad d_h = d_f \cdot d_g$$

$$h_- = (f_-^{-1} \cdot g_-^{-1})^{-1}, \quad h_+ = (f_+^{-1} \cdot g_+^{-1})^{-1}$$

4.2. Practical calculus

When \tilde{m} and \tilde{n} are L - R fuzzy numbers, we can deduce results of the same kind as for the sum; $\tilde{m} \cdot \tilde{n} = (m, \alpha, \beta)$, $\tilde{n} = (n, \gamma, \delta)$. We could show that $\forall \omega \in]0, 1[$, $\exists (p, r) \in R^{+2}$ such that

$$\omega = f_-(p) = g_-(r) = h(p \cdot r)$$

Using the function L_- , we can write

$$\omega = L_- \left(\frac{p-m}{\alpha} \right) = L_- \left(\frac{r-n}{\gamma} \right)$$

then

$$p = m - \alpha L_-^{-1}(\omega), \quad r = n - \gamma L_-^{-1}(\omega)$$

and

$$z = pr = mn - (m\gamma + n\alpha) \cdot L_-^{-1}(\omega) + \alpha \cdot \gamma \cdot L_-^{-1}(\omega)^2$$

Without any approximation, this second-order equation in $L_-^{-1}(\omega)$, whose discriminant is $(m\gamma - n\alpha)^2 + 4\alpha\gamma z \geq 0$, has always one positive root ($z \leq mn$) and we can deduce explicitly $h(z)$. Usually it will not be expressed in terms of L_- and R_+ : $\tilde{m} \odot \tilde{n}$ is not an L - R fuzzy number.

If, however, we neglect the terms $\alpha\gamma \cdot L_-^{-1}(\omega)^2$, provided that α and γ are small compared with m and n , and ω is in the neighbourhood of 1, the equation in $L_-^{-1}(\omega)$ becomes simple and we can deduce the approximation formula

$$(m, \alpha, \beta) \odot (n, \gamma, \delta) \approx (mn, m\gamma + n\alpha, m\delta + n\beta) \tag{3}$$

4.3. Comments

(a) When \tilde{m} or \tilde{n} is non-fuzzy, eqn. (3) becomes exact:

$$(m, \alpha, \beta) \cdot (n, 0, 0) = (mn, n\alpha, n\beta)$$

More generally, $h(z) = f(z/n)$.

(b) The approximate product is associative and commutative. $(1, 0, 0)$ is the identity for \odot , the set of fuzzy positive real numbers is a semi-group for \odot . Moreover, the approximate product is distributive on \oplus .

(c) Equation (3) gives for the square of \tilde{m}

$$(\tilde{m})^2 \approx (m^2, 2m\alpha, 2m\beta)$$

and more generally, for the p th power of \tilde{m} ,

$$(\tilde{m})^p \approx (m^p, pm\alpha, pm\beta)$$

Equation (3) is a generalization of the theorem concerning the error on a product of two measurements, or the p th power of a measurement.

(d) From eqn. (3), the fuzziness of a product is greater than that of each factor. Too frequent a use of this formula may lead to wrong results.

4.4. Quotient of two real positive fuzzy numbers

4.4.1. 'Inverse' of a fuzzy number \tilde{m}
 Gaines' (1976) formula applied (see § 2.2) to the inversion function is reduced here to

$$h(z) = \max_{z=1/x} f(x) = f(1/x)$$

Let us denote the inverse of \tilde{m} by $\widetilde{1/m}$. (N.B.—This is not the usual inversion in the sense of the symmetric element of \tilde{m} for the law \odot .)
 If

$$f(x) = L_-\left(\frac{x-m}{\alpha}\right), \quad x \leq m$$

we obtain

$$h(z) = f(1/z) = L_+\left(\frac{1-mz}{\alpha z}\right), \quad z \geq 1/m$$

$\widetilde{1/m}$ is not an L-R fuzzy number when \tilde{m} is so; but, in the neighbourhood of $1/m$, an approximation formula can be used:

$$h(z) \# L_+\left(\frac{1/m-z}{\alpha/m^2}\right), \quad z \geq 1/m$$

and finally

$$\widetilde{1/m} = L_+(1/m, \alpha, \beta) \# L_+(1/m, \beta/m^2, \alpha/m^2) \quad (4)$$

If \tilde{m} is of the L-R type, $\widetilde{1/m}$ is approximately of the R-L type, near $1/m$.

4.4.2. Quotient

The value of \tilde{m}/\tilde{n} can now be defined approximately as $\tilde{m}/\tilde{n} \# \tilde{n} \odot \widetilde{1/n}$.
 When \tilde{m} is an L-R fuzzy number and \tilde{n} an R-L one, from eqns. (3) and (4) we obtain

$$(m, \alpha, \beta) / (n, \gamma, \delta) \# \left(\frac{m}{n}, \frac{\delta m + \alpha n}{n^2}, \frac{\gamma m + \beta n}{n^2} \right)$$

near m/n . We have thus generalized the theorems concerning the error on the inverse of a measurement and the quotient of two measurements.

4.5. Cases in which m or n are negative

We could establish the following results. If the support of \tilde{m} is in the negative part of R , and that of \tilde{n} is in the positive one ($m < 0, n > 0$) then

$$(m, \alpha, \beta) \odot (n, \gamma, \delta) \# (mn, n\alpha - m\delta, n\beta - m\gamma) \\ = - [-(m, \alpha, \beta) \odot (n, \gamma, \delta)]$$

If \tilde{m} and \tilde{n} have their supports in the negative part of R ($m < 0, n < 0$), then

$$(m, \alpha, \beta) \odot (n, \gamma, \delta) \# (mn, -n\beta - m\delta, -n\alpha - m\gamma) \\ = -(m, \alpha, \beta) \odot (n, \gamma, \delta)$$

If f or g take high values on both sides of 0, no approximation formula of this kind is valid, since the order of magnitude of the spreads is greater or equal to the mean value of the number itself. Lastly, since $-(1/\tilde{n}) = 1/-\tilde{n}$, we can generalize the quotient of \tilde{m} by \tilde{n} , whatever the signs of m and n , provided that their support is in the positive or the negative part of R .

5. Fuzzification of 'max' and 'min' operations

Two questions arise when fuzzifying the max and min operations:
 (1) what is the value of the smaller or the greater of two fuzzy numbers;
 (2) what is the truth value of the assertion ' \tilde{m} is greater (smaller) than \tilde{n} '?
 Here we only deal with the first question; proofs will concern the max operation only.

5.1. Definition

Let \tilde{m} and \tilde{n} be two fuzzy numbers and f and g their respective membership functions. The 'fuzzy max' or \tilde{m} and \tilde{n} , which we denote by $\widetilde{\max(\tilde{m}, \tilde{n})}$, has a membership function h such that

$$h(z) = \max_{\max(x,y)=z} \min(f(x), g(y)) \\ l(z) = \max_{\min(x,y)=z} \min(f(x), g(y))$$

In the same way, $\widetilde{\min(\tilde{m}, \tilde{n})}$ has a membership function l such that

5.2. General results

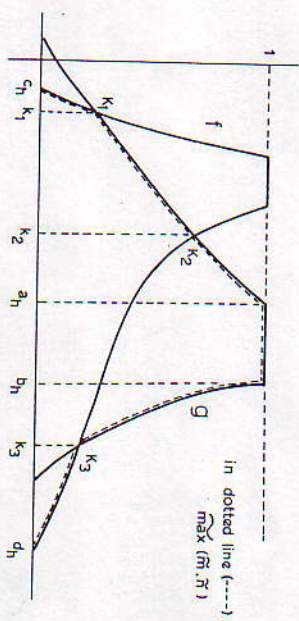


Figure 2.

The explanation will be given for the case where f and g have at most three intersection points K_1, K_2 and K_3 whose abscissae are k_1, k_2 and k_3 , where $f(k_i) \in]0, 1[$ for $i=1, 2, 3$ (see Fig. 2). This is realized, for instance, when f_- and g_- (resp. f_+ and g_+) have the same concavity. When K_1, K_2, K_3 exist, the following inequalities are always satisfied:

$$f(x) \leq g(y) \quad \forall (x, y) \in (-\infty, k_1]^2 \quad \text{and } x \leq y \quad (5)$$

$$f(x) \geq g(y) \quad \forall (x, y) \in [k_1, k_2]^2 \quad \text{and } x \geq y \quad (6)$$

$$f(x) \leq g(y) \quad \forall (x, y) \in [k_2, k_3]^2 \quad \text{and } x \geq y \tag{7}$$

$$f(x) \geq g(y) \quad \forall (x, y) \in [k_2, +\infty) \quad \text{and } x \leq y \tag{8}$$

These inequalities remain valid in the extreme cases when one of the K_i 's is such that $f(k_i) \in \{0, 1\}$, if we admit that K_i may be indeterminate in an interval I such that $\forall x \in I, f(x) = g(x) = 0$ or 1 ; this case will be detailed below. Moreover, note that the ordinate of K_2 is always greater than those of K_1 and K_3 .

Since $\max(x, y) = z$ is equivalent to $(x = z \text{ and } y \leq z)$ or $(y = z \text{ and } x \leq z)$, we can deduce that

$$h(z) = \max(\max_{x \leq z} (f(x), g(z)), \max_{y \leq z} (f(z), g(y)))$$

Lemma 4

Let us write $A(z) = \max \min_{y \leq z} (f(z), g(y))$. Then one of the two following propositions is true.

- (i) $\exists y^* \leq z, g(y^*) \geq f(z)$, and we have $A(z) = f(z)$.
- (ii) $A(z) = \max_{y \leq z} g(y)$.

In the same way, if $B(z) = \max \min_{x \leq z} (f(x), g(z))$ one of the two following propositions is true

- (i) $\exists x^* \leq z, f(x^*) \geq g(z)$, and we have $B(z) = g(z)$.
- (ii) $B(z) = \max_{x \leq z} f(x)$.

Proof

Obvious.

Theorem 3

The membership function h of $\widetilde{\max}(\tilde{m}, \tilde{n})$ is

$$h(z) = f(z), \quad z \in (-\infty, k_1] \cup [k_3, +\infty)$$

$$h(z) = g(z), \quad z \in [k_1, k_3]$$

where $f(k_i) \neq 0, 1 \quad \forall i$ and $b_j < a_j$.

Proof

- (i) $z \leq k_1$

By eqn. (5), $g(y) \geq f(x)$ for $y = z$, then, by Lemma 4, $A(z) = f(z)$.

By eqn. (5), $g(z) \geq f(x) \quad \forall x \leq z$, then, by Lemma 4, $B(z) = \max_{x \leq z} f(x)$ and $B(z) = f(z)$, since f is increasing in $(-\infty, k_1]$. Thus,

$$h(z) = \max(A(z), B(z)) = f(z)$$

- (ii) $z \in [k_1, k_3]$

By eqn. (6), $f(z) \geq g(y) \quad \forall y \leq z$, then, by Lemma 4, $A(z) = \max_{y \leq z} g(y)$, and $A(z) = g(z)$ since g is increasing in $(-\infty, k_3]$.

By eqn. (6), $f(x) \geq g(z)$ for $x = z$, then $B(z) = g(z)$ and $h(z) = g(z)$.

- (iii) $z \in [k_2, k_3]$

By eqn. (7), $f(z) \leq g(y)$ for $y = z$, then $A(z) = f(z)$. Moreover $\exists x \leq z$ and $f(x) \geq g(z)$ (we choose $x \in [a_j, b_j]$ where $f(x) = 1$), then $B(z) = g(z)$.
By eqn. (7), $h(z) = \max(f(z), g(z)) = g(z)$.

- (iv) $z \geq k_3$

$\exists (x, y)$ such that $g(y) \geq f(z)$, and $f(x) \geq g(z)$ with $x \leq z$ and $y \leq z$. We choose $(x, y) \in [a_j, b_j] \times [a_j, b_j]$ where $f(x) = g(y) = 1$. Then, by Lemma 4, $A(z) = f(z)$ and $B(z) = g(z)$, and by eqn. (8), $h(z) = \max(f(z), g(z)) = f(z)$. Q.E.D.

5.3. Summary : practical rule

Theorem 3 is equivalent to the following assertions

- (i) $\widetilde{\max}(\tilde{m}, \tilde{n})$ is a fuzzy number as defined in § 2.1.
- (ii) $a_h = \max(a_f, a_g), c_h = \max(c_f, c_g), b_h = \max(b_f, b_g)$ and $d_h = \max(d_f, d_g)$.
- (iii) The graph of h is obtained by joining the points $(c_h, 0), (a_h, 1), (b_h, 1)$ and $(d_h, 0)$, in that order, with the corresponding parts of the graph of f or g (see Fig. 2).

We could show that this rule remains valid when $\exists k_i$:

$$f(k_i) \in \{0, 1\}$$

For the operator min, the function l is obtained using the same rule but replacing max by min.

5.4. Approximation formulae

When \tilde{m} and \tilde{n} are $L-R$ fuzzy numbers, $\widetilde{\max}(\tilde{m}, \tilde{n})$ is not always such (see Fig. 2). More precisely,

- (a) If $a_h = a_g, b_h = b_g, c_h = c_g$ and $d_h = d_g$, then $h(z) = g(z) \quad \forall z$.
- (b) If a pair of elements of the set $\{a_h, b_h, c_h, d_h\}$ is composed of a point in the graph of f and another in that of g , then $\widetilde{\max}(\tilde{m}, \tilde{n})$ is not an $L-R$ fuzzy number.

However, in all cases $\widetilde{\max}(m, n)$ is always a correct mean value for $\widetilde{\max}(\tilde{m}, \tilde{n})$. Moreover, in case (a) the formula

$$\widetilde{\max}(m, \alpha, \beta), (\gamma, \delta) = (m, \gamma, \delta), \quad n > m$$

is exact. This formula is a good approximation in case (b) when the ordinates of K_1 and K_3 are low; it is exact, particularly in $[k_1, k_3]$, which is always the most significant interval, because we find there the highest truth values.

When $[k_1, k_3]$ is very narrow, and $\max(f(k_1), f(k_3))$ is close to 1 (i.e. when \tilde{m} and \tilde{n} are close to each other), we can use the formula

$$\widetilde{\max}(m, \alpha, \beta), (n, \gamma, \delta) = (\max(m, n), \min(\alpha, \gamma), \max(\beta, \delta))$$

which is exact when $m = n$, and this is the only point where f and g reach the value 1.

5.5. Case when f and g have more than three intersection points

Let K_2 be the intersection point with highest ordinate when $b_j < a_j$. K_2 is unique since f is strictly decreasing, and g strictly increasing, in $[b_j, a_j]$.

When $b_j \geq a_j$, K_2 is an indeterminate point of an interval whose ordinate is 1; let K_1 and K_3 be the first intersection points on the left and right side respectively of K_2 . (We assume that in a finite interval of R , the number of intersection points is finite (see Fig. 3).)

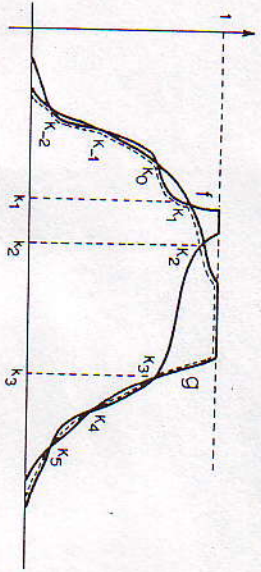


Figure 3.

We could show that $h(z) = g(z)$ in $[k_1, k_2]$, as in Theorem 3.

Let K_n and K_{n+1} be two consecutive intersection points different to K_1 , K_2 or K_3 . We can show that

(i) If K_n and K_{n+1} are on the left side of K_1 ($n < 0$), then

$$h(z) = \min(g(z), h(z))$$

(ii) If K_n and K_{n+1} are on the right side of K_3 ($n > 3$), then

$$h(z) = \max(g(z), h(z))$$

This enables us to complete the rule of § 5.3: outside $[k_1, k_3]$, $h(z)$ is composed of the rightest parts of f and g , as shown on Fig. 3, where $h(z)$ is represented by the dotted line.

5.6. Some structural properties

$\widetilde{\max}$ and $\widetilde{\min}$ are associative and commutative operators. Moreover, we have the following properties.

(i) $\widetilde{\max}(\widetilde{m}, \max(\widetilde{m}, \widetilde{n})) = \max(\widetilde{m}, \widetilde{n})$ (absorption).

(ii) $\widetilde{\max}(\widetilde{m}, \widetilde{n}) \oplus \widetilde{\min}(\widetilde{m}, \widetilde{n}) = \widetilde{m} \oplus \widetilde{n}$.

N.B.—Similar results for $\widetilde{\min}$ can be deduced from those for $\widetilde{\max}$ by exchanging $\widetilde{\min}$ and $\widetilde{\max}$ in the formulae in particular,

$$\widetilde{\min}((m, \alpha, \beta), (n, \gamma, \delta)) = (m, \alpha, \beta), \quad m < n$$

$$\widetilde{\min}((m, \alpha, \beta), (n, \gamma, \delta)) = (\min(m, n), \max(\alpha, \gamma), \min(\beta, \delta)), \quad m = n$$

6. Further results

Some refinements concerning operations on fuzzy numbers can be introduced. For instance, a fuzzy set of type 2 (Mizumoto and Tanaka 1976) X,

on a universe U is defined by the pair $(x, \mu_X(x))$, $x \in U$, where $\mu_X(x)$ is no longer a real number, but now is the membership function of a fuzzy subset of $[0, 1]$.

In practice, a fuzzy number of type two, \widetilde{m} , can be represented by a family of parametered membership functions f_λ , where λ is the truth value of the assertion ' f_λ is a good membership function for \widetilde{m} '. When f_λ is an L - R function, we can show that this definition is equivalent to assigning fuzzy spreads to the mean value m of \widetilde{m} . We can derive the relation $\widetilde{m} = (m, \widetilde{\alpha}, \widetilde{\beta})$, and can show that the sum of two L - R fuzzy numbers of type 2 is given by

$$\widetilde{m} \oplus \widetilde{n} = (m, \widetilde{\alpha}, \widetilde{\beta}) \oplus (n, \widetilde{\gamma}, \widetilde{\delta}) = (m+n, \widetilde{\alpha} \oplus \widetilde{\gamma}, \widetilde{\beta} \oplus \widetilde{\delta})$$

where \oplus is the symbol for fuzzified \oplus .

7. Conclusions

Fuzzy algebraic calculus is a generalization of the usual tolerance analysis, so it can be applied in any scientific domain where quantities which are vaguely known have to be combined, provided that this uncertainty may be quantified. Statistical data on these quantities can help us for this purpose. For instance, linear programming and shorter paths algorithms can be fuzzified in a computable manner using this approach.

A fuzzy PERT method has already been proposed (Dubois and Prade 1978 a); new principles for realistic traffic affectation can be established when speeds are not well-known (Dubois and Prade 1978 a); multicriteria aggregation using fuzzy weights and fuzzy evaluations of actions by means of criteria seems to be a promising method, as Jain (1977) has shown. These applications can be seriously considered, since performing operations on fuzzy numbers requires a moderate extra amount of computation compared to using classic numbers—in fact, the same amount as when classic error intervals are combined.

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