



HAL
open science

Fuzzy real algebra: some results

Didier Dubois, Henri Prade

► **To cite this version:**

Didier Dubois, Henri Prade. Fuzzy real algebra: some results. *Fuzzy Sets and Systems*, 1979, 2 (4), pp.327–348. <10.1016/0165-0114(79)90005-8>. <hal-04215228>

HAL Id: hal-04215228

<https://hal.science/hal-04215228v1>

Submitted on 22 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

FUZZY REAL ALGEBRA: SOME RESULTS

Didier DUBOIS

School of Electrical Engineering, Purdue University, West Lafayette, IN 47905, U.S.A.

and

Henri PRADE

Stanford Artificial Intelligence Lab., Stanford University, 94305, U.S.A.

Received 12 June 1978

Revised July 1978

In this paper, the possibility to perform easily most of the extended n -ary operations on fuzzy subsets of the real line is shown. A general algorithm is given. These results are particularized for usual operations such as addition, subtraction, multiplication, division, 'max' and 'min' operations for normalized convex fuzzy subsets of the real line, i.e. fuzzy numbers. A three parameters representation for fuzzy numbers is shown to be very convenient to perform usual operations. Lastly, interpretative comments about fuzzy real algebra are given and possible applications pointed out.

Keywords: Fuzzy numbers, Extension principle, Fuzzy tolerance analysis.

0. Introduction

One of the most appealing potentiality of fuzzy set theory is to make possible computation on ill-known real numbers, modeled as fuzzy subsets of the real line¹ \mathbf{R} . A preceding paper [2] has already given results for fast performance of the extended addition, subtraction and approximate product of normal convex fuzzy numbers.

We intend here to generalize the previous results to a very large class of real operations on any continuous fuzzy subset of \mathbf{R} .

A first section is devoted to general theorems on the extension of n -ary, continuous increasing, real operations. A computation algorithm is described in Section 2. In Section 3 these results are particularized for usual operations such as addition, multiplication, max and min operations on convex fuzzy numbers; subtraction and division will be shown as easy to perform as the preceding operations. In Section 4, the three parameters representation, already introduced in [2], will be recalled, which allows an even faster computation; corresponding

The authors are supported by scholarships of the Institut de Recherche en Informatique et Automatique. Rocquencourt 78150 Le Chesnay France.

¹ Some previous works related to this topic are those of R. Jain [5], S. Nahmias [7], M. Mizumoto and K. Tanaka [6].

formulae will be stated and generalized. Section 5 provides some interpretative comments about the meaning and the possible applications of fuzzy real algebra.

1. General theorems

1.1. Prerequisites

In the whole paper we are interested in continuous fuzzy subsets of the real line \mathbf{R} , in short CFS i.e. whose membership functions are continuous. However some results will only assume them piecewise continuous, the corresponding fuzzy subset will be noted PCFS.

The name 'fuzzy number' is reserved to normal convex fuzzy sets. A fuzzy set A whose membership function is μ_A is normal if and only if $\max_{x \in \mathbf{R}} \mu_A(x) = 1$. A is said convex if and only if $\forall (x, y, z) \in \mathbf{R}^3$ such that $y \in [x, z]: \mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$. Informally a fuzzy number is a bell-shaped fuzzy set. The term flat fuzzy number (FFN) refers to a fuzzy number such that $\exists (x, y) \in \mathbf{R}^2, x \neq y: \forall z \in [x, y], \mu_A(x) = 1$.

Given a real function φ of n real variables (x_1, x_2, \dots, x_n) (it can be also considered as a n -ary operation), the principle of extension, stated by L.A. Zadeh, allows us to extend this function to fuzzy variables $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$; $\tilde{\varphi}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ is a fuzzy set \tilde{y} of membership function

$$\mu_{\tilde{y}}(t) = \sup_{t = \varphi(s_1, s_2, \dots, s_n)} \min(\mu_{\tilde{x}_1}(s_1), \mu_{\tilde{x}_2}(s_2), \dots, \mu_{\tilde{x}_n}(s_n))$$

or

$$= 0 \quad \text{if } \varphi^{-1}(t) = \emptyset.$$

where $\mu_{\tilde{x}_i}$ is the membership function of \tilde{x}_i ($i = 1, n$).

Alternative extension principles have been considered in the literature, especially using probabilistic sum ($\#$) instead of max; $(a \# b = a + b - ab)$ [5]. ' $\#$ -min' extension principle was shown to lead to erratic results [1].

1.2. n -ary operations on continuous fuzzy numbers

Definition 1. A n -ary operation φ from \mathbf{R}^n to \mathbf{R} is said *increasing* if and only if

$$\begin{cases} x_1 > y_1 \\ x_2 > y_2 \\ \vdots \\ x_n > y_n \end{cases} \rightarrow \varphi(x_1, x_2, \dots, x_n) > \varphi(y_1, y_2, \dots, y_n)$$

Lemma 1. Let f_1, f_2, \dots, f_n be n convex functions from \mathbf{R} to $[0, 1]$ such that f_i is increasing on $[a_i, b_i]$ and decreasing on $[b_i, c_i] \forall i$.² (We can have $a_i = -\infty, b_i = \pm\infty, c_i = +\infty$). Let $p_i, i = 1, \dots, n$ such that $p_i \in [a_i, b_i]$ and $f_i(p_i) = \omega \in [0, 1], \forall i$. Then, if φ is increasing

$$\mu_{\tilde{\varphi}(f_1, f_2, \dots, f_n)}(\varphi(p_1, p_2, \dots, p_n)) = \omega,$$

where $\mu_{\tilde{\varphi}(f_1, f_2, \dots, f_n)}$ is the membership function of the extension of φ .

² And null outside.

Proof. Let $k = \varphi(p_1, p_2, \dots, p_n)$. Consider (x_1, x_2, \dots, x_n) such that $\varphi(x_1, x_2, \dots, x_n) = k$, $(x_1, x_2, \dots, x_n) \neq (p_1, p_2, \dots, p_n) \rightarrow x_i > p_i$ or $x_j < p_j$.

$\exists i, x_i > p_i \rightarrow \exists l x_l \leq p_l < b_l$ because φ is increasing. Then $\min(f_1(x_1), f_2(x_2), \dots, f_n(x_n)) \leq \omega$ because $f_l(x_l) \leq \omega$.

$\exists j, x_j < p_j < b_j$, then $\min(f_1(x_1), f_2(x_2), \dots, f_n(x_n)) \leq \omega$.

Thus $\forall (x_1, x_2, \dots, x_n)$ such that $\varphi(x_1, x_2, \dots, x_n) = \varphi(p_1, p_2, \dots, p_n) = k$ we have $\min(f_1(x_1), f_2(x_2), \dots, f_n(x_n)) \leq \omega$ and the maximum is reached for (p_1, p_2, \dots, p_n) , which yields the result.

The same result holds for $p_i \geq b_i, \forall i$.

This first lemma has shown that if we consider (p_1, p_2, \dots, p_n) on only increasing parts of f_1, f_2, \dots, f_n (or on the decreasing parts only) and provided that the membership values $f_i(p_i)$ of p_i are equal to ω , we could assign the same membership value ω to $\varphi(p_1, p_2, \dots, p_n)$. Lemma 2 shows that it is always possible, given any n -uple (p_1, p_2, \dots, p_n) on the convex functions f_1, f_2, \dots, f_n , to restore the hypothesis of Lemma 1, i.e. to find $(p_1^*, p_2^*, \dots, p_n^*)$ with $p_i^* \leq b_i, \forall i$ or $p_i^* \geq b_i, \forall i$ (respectively if $\varphi(p_1, p_2, \dots, p_n) \leq \varphi(b_1, b_2, \dots, b_n)$ or $\varphi(p_1, p_2, \dots, p_n) \geq \varphi(b_1, b_2, \dots, b_n)$ and with the same membership value $\omega = f_i(p_i^*)$).

Lemma 2. Let φ be an increasing continuous function from \mathbb{R}^n to \mathbb{R} , and f_1, f_2, \dots, f_n be n convex continuous functions; f_i is supposed strictly increasing on $(-\infty, b_i]$, strictly decreasing on $[b_i, +\infty)$ and $f_i(\mathbb{R}) = [0, 1], \forall i$. Let p_1, p_2, \dots, p_n be an element of \mathbb{R}^n ; assume for instance

$$\varphi(p_1, p_2, \dots, p_n) \leq \varphi(b_1, b_2, \dots, b_n).$$

Then $\exists p_1^*, p_2^*, \dots, p_n^*$ such that:

- (i) $p_i^* < b_i, \forall i$
- (ii) $f_1(p_1^*) = f_2(p_2^*) \dots = f_n(p_n^*)$,
- (iii) $\varphi(p_1^*, p_2^*, \dots, p_n^*) = \varphi(p_1, p_2, \dots, p_n)$,
- (iv) $\mu_{\varphi(f_1, f_2, \dots, f_n)}(\varphi(p_1, p_2, \dots, p_n)) = f_1(p_1^*) = \dots = f_n(p_n^*)$.

Proof Let \hat{f}_i be the restriction of f_i on $(-\infty, b_i], \forall i$. \hat{f}_i is continuous and strictly increasing on that interval; the reciprocal function \hat{f}_i^{-1} exists and is strictly increasing on $[0, 1]$ $\hat{f}_i^{-1}(0) = -\infty$ and $\hat{f}_i^{-1}(1) = b_i$.

Since φ is increasing and continuous, the function $\varphi(\hat{f}_1^{-1}, \hat{f}_2^{-1}, \dots, \hat{f}_n^{-1}) = g$ is strictly increasing from $[0, 1]$ to $[\inf \varphi, \varphi(b_1, b_2, \dots, b_n)]$; g is 1-1.

Let $P = \varphi(p_1, p_2, \dots, p_n)$. Let $\omega = g^{-1}(P)$, $p_i^* = \hat{f}_i^{-1}(\omega), \forall i$.

$$\varphi(p_1^*, p_2^*, \dots, p_n^*) = P = \varphi(p_1, p_2, \dots, p_n)$$

Moreover $p_i^* = \hat{f}_i^{-1}(\omega) \leq b_i, \forall i$.

ω and $(p_1^*, p_2^*, \dots, p_n^*)$ satisfy the hypothesis of Lemma 1, thus

$$\mu_{\varphi(f_1, f_2, \dots, f_n)}(P) = \omega = \hat{f}_i(p_i^*), \quad \forall i.$$

Note that $\mu_{\bar{\varphi}(f_1, f_2, \dots, f_n)} = g^{-1} = \varphi(\hat{f}_1^{-1}, \hat{f}_2^{-1}, \dots, \hat{f}_n^{-1})^{-1}$ on

$$[\inf \varphi, \varphi(b_1, b_2, \dots, b_n)]$$

and $\mu_{\check{\varphi}(f_1, f_2, \dots, f_n)} = \varphi(\check{f}_1^{-1}, \check{f}_2^{-1}, \dots, \check{f}_n^{-1})^{-1}$ on

$$[\varphi(b_1, b_2, \dots, b_n), \sup \varphi]$$

where f_i is the strictly decreasing part of f_i , $\forall i$.

Lemma 3. Let f_1, f_2, \dots, f_n be the membership functions of n fuzzy numbers. Let $[\lambda_i, \rho_i]$ $i = 1, \dots, n$ be n intervals on the increasing parts of the functions f_i and $\forall x_i \in [\lambda_i, \rho_i], f_i(x_i) = \omega$. Then $\forall t \in [\varphi(\lambda_1, \lambda_2, \dots, \lambda_n), \varphi(\rho_1, \rho_2, \dots, \rho_n)]$, where φ is increasing, $\mu_{\bar{\varphi}(f_1, f_2, \dots, f_n)}(t) = \omega$. (Possibly $\lambda_i = \rho_i$ for some i).

Proof. Let (p_1, p_2, \dots, p_n) such that $\forall i, p_i \in [\lambda_i, \rho_i]$ then $\min(f_1(p_1), f_2(p_2), \dots, f_n(p_n)) = \omega$. Let $\Lambda = \varphi(\lambda_1, \lambda_2, \dots, \lambda_n)$ and (p_1, p_2, \dots, p_n) such that $\varphi(p_1, p_2, \dots, p_n) = \Lambda$.

If $\exists i, p_i > \lambda_i$ then $\exists j, p_j \leq \lambda_j$ because φ is increasing; hence $\min(f_1(p_1), f_2(p_2), \dots, f_n(p_n)) \leq \omega$ because f_j is increasing.

If $\exists i, p_i < \lambda_i$ it is evident that $\min(f_1(p_1), f_2(p_2), \dots, f_n(p_n)) \leq \omega$. The same reasoning holds for $\varphi(p_1, p_2, \dots, p_n) = \varphi(\rho_1, \rho_2, \dots, \rho_n) = P$. Thus ω is the value of $\mu_{\bar{\varphi}(f_1, f_2, \dots, f_n)}(t)$, for $t \in [\Lambda, P]$. Lemma 3 holds if $\forall i, [\lambda_i, \rho_i]$ is in the decreasing part of f_i .

Note. Lemma 1 is a particular case of Lemma 3 when $\lambda_i = \rho_i, \forall i$. We can now conclude:

Theorem 1. Let $\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_n$ be continuous fuzzy numbers whose surjective membership functions are f_1, f_2, \dots, f_n from \mathbb{R} to $[0, 1]$. Let φ be a continuous increasing n -ary operation. Then the extension $\bar{\varphi}(\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_n)$ is a continuous fuzzy number whose membership function is surjective from \mathbb{R} to $[0, 1]$. The construction of this fuzzy number can be performed by applying the procedure stated in Lemmas 1 and 3 on respectively increasing and decreasing parts of f_i . This decomposition is authorized by Lemma 2.

Note. When φ is a decreasing continuous n -ary operation the same result holds; but the construction is slightly different: we must use the decreasing parts of f_i to build the increasing part of $\mu_{\bar{\varphi}(f_1, f_2, \dots, f_n)}$ and vice-versa.

1.3. Distributivity of $\bar{\varphi}$ on the union of fuzzy sets

Let φ be an n -ary real operation; let $A_1, A_2, \dots, A_n, A'_j$ be $n + 1$ fuzzy subsets of \mathbb{R} . φ can be extended to $\bar{\varphi}$, n -ary operation on the fuzzy subsets of \mathbb{R} :

$$\begin{aligned} \mu_{\bar{\varphi}}(t) &= \sup_{t = \varphi(s_1, s_2, \dots, s_n)} \min(\mu_{A_1}(s_1), \mu_{A_2}(s_2), \dots, \mu_{A_n}(s_n)) \\ \text{or} & \\ &= 0, \quad \text{if } \varphi^{-1}(t) = \varphi. \end{aligned}$$

Theorem 2. $\bar{\varphi}$ is distributive on \cup (\cup denotes union operation on fuzzy sets), i.e. let $A_1, A_2, \dots, A_n, A'_j, 1 \leq j \leq n$, be $(n+1)$ PCFS:

$$\begin{aligned} \bar{\varphi}(A_1, \dots, A_{j-1}, A_j \cup A'_j, A_{j+1}, \dots, A_n) \\ = \bar{\varphi}(A_1, A_2, \dots, A_j, \dots, A_n) \cup \bar{\varphi}(A_1, A_2, \dots, A'_j, \dots, A_n). \end{aligned}$$

Proof. For more simplicity φ is supposed monotonic and continuous.

$$\begin{aligned} \mu_{\varphi(A_1, \dots, A_j \cup A'_j, \dots, A_n)}(t) \\ = \sup_{t = \varphi(s_1, s_2, \dots, s_n)} \min(\mu_{A_1}(s_1), \dots, \max(\mu_{A_j}(s_j), \mu_{A'_j}(s_j)), \dots, \mu_{A_n}(s_n)). \end{aligned}$$

Since $\forall (p_1, p_2, \dots, p_j, \dots, p_n, q_j) \in [0, 1]^{n+1}$:

$$\begin{aligned} \min(p_1, \dots, \max(p_j, q_j), \dots, p_n) \\ = \max[\min(p_1, \dots, p_j, \dots, p_n), \min(p_1, \dots, q_j, \dots, p_n)], \end{aligned}$$

hence

$$\begin{aligned} \mu_{\bar{\varphi}(A_1, \dots, A_j \cup A'_j, \dots, A_n)}(t) \\ = t = \sup_{\varphi(s_1, s_2, \dots, s_n)} \max[\min(\mu_{A_1}(s_1), \dots, \mu_{A_j}(s_j), \dots, \mu_{A_n}(s_n)), \\ \min(\mu_{A_1}(s_1), \dots, \mu_{A'_j}(s_j), \dots, \mu_{A_n}(s_n))] \end{aligned}$$

'sup' and 'max' can commute and we get the union of the two fuzzy sets $\bar{\varphi}(A_1, \dots, A_j, \dots, A_n)$ and $\bar{\varphi}(A_1, \dots, A'_j, \dots, A_n)$.

Note. (1) This result is false when we change \cup into \cap (intersection), except if φ is 1-1.

(2) \cup is not distributive on $\bar{\varphi}$.

Let $*$ be a binary operation and \otimes its extended form:

$$\begin{aligned} (A \otimes B) \cup C \neq (A \cup C) \otimes (B \cup C) \\ = (A \otimes B) \cup (A \otimes C) \cup (B \otimes C) \cup (C \otimes C) \end{aligned}$$

(owing to Theorem 2.).

1.4. Non normalized convex fuzzy sets : flattening effect

Let A_1, A_2, \dots, A_n be n convex continuous fuzzy sets of \mathbf{R} and assume that $\forall i, \mu_{A_i}(\mathbf{R}) = [0, m_i]$.

Note that $\forall (x_1, x_2, \dots, x_n) \in \mathbf{R}^n, \min_i \mu_{A_i}(x_i) \leq \min_i m_i$.

Hence given a n -ary continuous and monotonic operation φ ,

$$\mu_{\bar{\varphi}(A_1, A_2, \dots, A_n)}(t) \leq \min_i m_i.$$

This upper bound is reached for $(x_1^*, x_2^*, \dots, x_n^*)$ such that

$$\forall i, \mu_{A_i}(x_i^*) = m_i,$$

owing to Lemma 3. Thus, let A'_i be a convex continuous fuzzy set of \mathbf{R} such that

$\mu_{A_i}(x) = \min(\mu_{A_i}(x), \min_j m_j)$, we can infer that $\bar{\varphi}(A_1, A_2, \dots, A_n) = \bar{\varphi}(A'_1, A'_2, \dots, A'_n)$.

Note that all the A_i have been flattened in A'_i and all the A'_i have the same height $\min_j m_j$.

1.5. Commutativity and associativity

1.5.1. Commutativity

If $\varphi(s_1, s_2, \dots, s_n)$ is independent of the order of the s_i 's, it is easy to check that $\bar{\varphi}(A_1, A_2, \dots, A_n)$ is independent of the order of the fuzzy sets A_i .

1.5.2. Associativity

Assume for convenience $n = 2$. If $(s_1 * s_2) * s_3 = s_1 * (s_2 * s_3) = s_1 * s_2 * s_3$, then $\varphi(s_1, s_2, s_3) = s_1 * s_2 * s_3$ is a ternary operation, thus

$$(A_1 \circledast A_2) \circledast A_3 = A_1 \circledast (A_2 \circledast A_3) = \bar{\varphi}(A_1, A_2, A_3).$$

1.6. Link between extended operation and composition rule for fuzzy relation

Assume $n = 2$. Consider the two expressions

$$h(z) = \sup_{z = \varphi(x, y)} \min(f(x), g(y)),$$

$$h'(z) = \sup_x \min(f(x), R(x, z)).$$

where f and g are continuous functions from \mathbf{R} to $[0, 1]$ and R is a fuzzy relation on \mathbf{R}^2 . Given (g, φ) , is it possible to find R such that $h = h'$ and vice versa?

If $\varphi(x, y) = x * y$ where $(\mathbf{R}, *)$ is a group:

$$z = x * y \leftrightarrow y = x^{-1} * z \quad \text{and} \quad R(x, z) = g(x^{-1} * z).$$

More generally if several y can be found such that $z = \varphi(x, y)$, several values of $g(y)$ must be considered and we are interested only in the greatest one:

$$R(x, z) = \sup_{\substack{y \\ \varphi(x, y) = z}} g(y).$$

Conversely, given R we must find g and φ such that

$$g(y) = R(x, z) \quad \text{when} \quad z = \varphi(x, y),$$

i.e. find φ such that $R(x, \varphi(x, y))$ does not depend on x . This condition allows us to find a convenient φ , and then of course $g(y) = R(x, \varphi(x, y))$. The solutions of both problems are generally not unique. If given R , it is possible to find a monotonic continuous φ , the computation of max-min composition rule will be greatly accelerated by using the general algorithm on PCFS's described in Section 2.

2. Outline of a general algorithm for the computation of extended operations

2.1.

We are now able to perform the exact computation of any monotonic continuous extended n -ary operation on PCFS's within the framework of the same algorithm.

Any PCFS can be decomposed into the union of convex eventually non normalized fuzzy sets whose membership functions f_i are such that f_i is strictly increasing or decreasing or constant in the only interval where f_i is not zero. See Fig. 1.

Owing to Theorem 2 we can perform the operation on each part separately and perform the union of the results. For each convex piece of PCFS the computation will be made easier thanks to the flattening properties (Section 1.4), and performed according to Theorem 1.

Theorem 1 was demonstrated for continuous fuzzy numbers. It can be extended to normalized convex fuzzy sets whose membership functions are piecewise continuous. The practical method is the following: consider infinite slope segment as increasing or decreasing part of the fuzzy number according to the context.

2.2. Description of the algorithm:

Each PCFS is supposed discretized into a finite number of membership levels ω_i , $i = 0, M$ ($\omega_0 = 0$, $\omega_M = 1$).

To each level is assigned a set $P_i^k = \{p_{i1}^k, \dots, p_{ij}^k\}$ of real values such $\mu_{A_k}(p_{ij}^k) = \omega_i$, $j = 1, J$ where A_k , $k = 1, \dots, n$, are the n considered PCFS's, and J is a function of i and k .

2.2.1.

Example. Two fuzzy numbers; * monotonic. In Zadeh's notations, with $M = 3$:

$$A_1 = \omega_1/p_{11}^1 + \omega_2/p_{21}^1 + \omega_3/p_{31}^1 + \omega_2/p_{22}^1 + \omega_1/p_{12}^1,$$

$$A_2 = \omega_1/p_{11}^2 + \omega_2/p_{21}^2 + \omega_3/p_{31}^2 + \omega_2/p_{22}^2 + \omega_1/p_{12}^2,$$

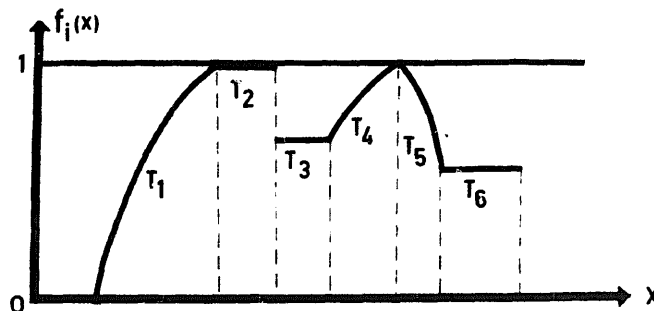


Fig. 1.

Then

$$A_1 \circledast A_2 = \omega_1/p_{11}^1 * p_{11}^2 + \omega_2/p_{21}^1 * p_{21}^2 + \omega_3/p_{31}^1 * p_{31}^2 + \omega_2/p_{22}^1 * p_{22}^2 + \omega_1/p_{12}^1 * p_{12}^2.$$

The operation is then easy to perform in the case of fuzzy numbers. It is clear that when discrete representations are available it is not good to use max-min composition on the data; the exact result is got by a direct performance of the operation φ using Theorem 1.

Example $(0.5/4 + 1/5 + 0.5/6) \oplus (0.5/1 + 1/2 + 0.5/3)$

$$= 0.5/5 + 1/7 + 0.5/9$$

$$\neq 0.5/5 + 0.5/6 + 1/7 + 0.5/8 + 0.5/9$$

\oplus is the extended addition; $+$ is of course an increasing operation.

2.2.2. Procedure for operation on PCFS

The algorithm will proceed in four steps.

Step 1. Flattening: the n PCFS's A_1, A_2, \dots, A_n will be changed in PCFS's having the same height.

Step 2. Decomposition of each fuzzy set as described above into two sets of pieces: the set of non-decreasing 'parts' and the set of non-increasing ones. The constant 'parts' between two non-decreasing ones (resp. nonincreasing) belong to the non-decreasing (resp. non-increasing) set. The constant 'parts', which are between 'parts' of different kinds, belong to both.

In Fig. 1: non-decreasing set: $\{T_1, T_2, T_3, T_4\}$,
non-increasing set: $\{T_2, T_3, T_5, T_6\}$.

Step 3. Operation φ : the operation φ is performed as in the above example for every n -tuple of 'parts' (one "part" for each fuzzy number) all belonging to the same kind of sets (non-decreasing or non-increasing). The flattening effect may be used.

Step 4. Union: for each n -tuple of parts a fuzzy set was built in Step 3. The union of these fuzzy sets is the final result.

Example. Consider the two fuzzy numbers A and B pictured on Fig. 2, we want to calculate $C = A \oplus B$. \oplus denotes the extended sum

Step 1. A and B are normalized, no flattening is necessary.

Step 2. A: non-decreasing set $\{A_1, A_2\}$

non-increasing set $\{A_2, A_3, A_4\}$

B: non-decreasing set $\{B_1, B_3, B_4, B_5\}$

non-increasing set $\{B_2, B_3, B_5, B_6\}$

Step 3 and 4. C_{ij} denote $A_i \oplus B_j$. The C_{ij} 's will be calculated in lexicographic order:

$$C_{11} = A_1 \oplus B_1; \quad C \leftarrow C_{11}.$$

C_{12} is not considered, because A_1 and B_2 do not belong to the same kind of set.

$$C_{13} = A_1 \oplus B_3.$$

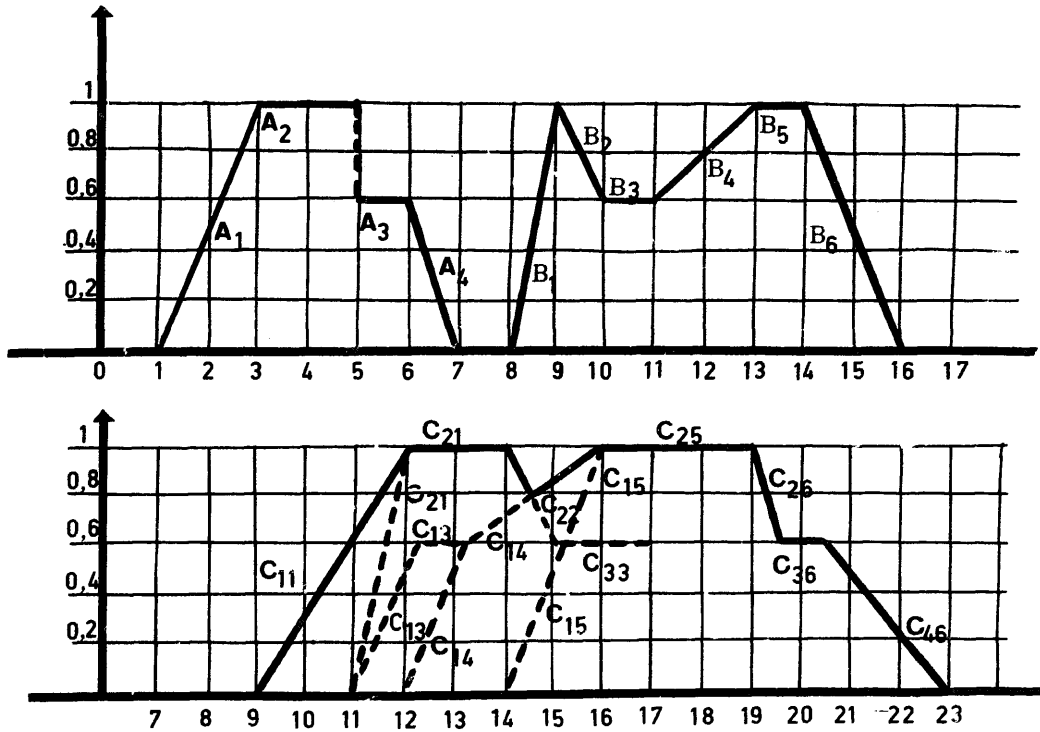


Fig. 2.

Perform $C \leftarrow C_{11} \cup C_{13}$, the part of C_{13} between abscissae 11 and 12 is dropped.

$$C_{14} = A_1 \oplus A_4; \quad C \leftarrow C \cup C_{14}.$$

$$C_{15} = A_1 \oplus B_5; \quad C \leftarrow C \cup C_{15}.$$

(The part of C_{15} between abscissae 14 and 16 is dropped). C_{16} is not considered.

$$C_{21} = A_2 \oplus B_1; \quad C \leftarrow C \cup C_{21}.$$

(The remaining part of C_{13} is dropped, the part of C_{14} between abscissae 12 and 14 is dropped.)

and so on

The final result is pictured in Fig. 2.

The above procedure is certainly not the most efficient one—a lot of redundancies remain which could be avoided by a more careful analysis. We only intend here to indicate that the algebraic calculus on rather general fuzzy sets on \mathbf{R} is practically possible.

3. Usual operations on fuzzy numbers

3.1. Unary operations

3.1.1. Opposite, inverse, exponential of a fuzzy number

Zadeh's extension principle for unary operations reduces to

$$\mu_{\varphi(\tilde{m})}(t) = \sup_{t = \varphi(x)} \mu_{\tilde{m}}(x).$$

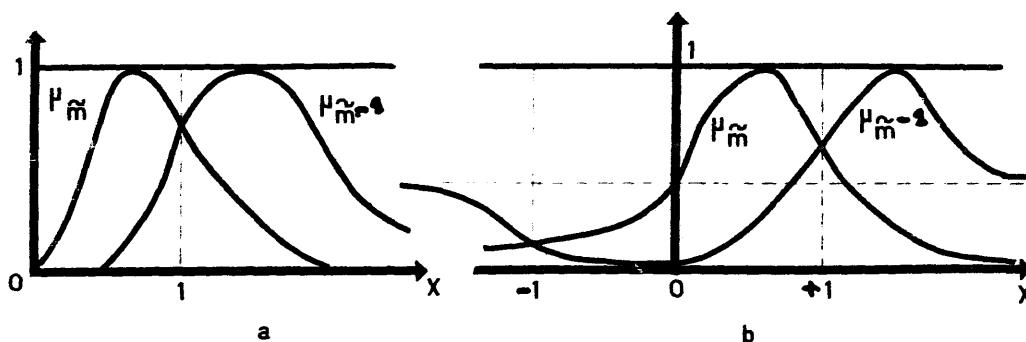


Fig. 3.

$\varphi(x) = -x\varphi(\tilde{m})$ is denoted $-\tilde{m}$; $\mu_{-\tilde{m}}(x) = \mu_{\tilde{m}}(-x)$; \tilde{m} and $-\tilde{m}$ are symmetrical with respect to the axis $x = 0$.

$\varphi(x) = (1/x)$, $x \neq 0$, $\varphi(\tilde{m})$ is denoted \tilde{m}^{-1} ; $\mu_{\tilde{m}^{-1}}(x) = \mu_{\tilde{m}}(1/x)$. If \tilde{m} is not a positive or negative fuzzy number (\tilde{m} is positive $\leftrightarrow \mu_{\tilde{m}}(x) = 0, x \leq 0$),³ \tilde{m}^{-1} is no more convex (See Fig. 3b) and generally does not vanish when $x \rightarrow \infty$. However when, for example, \tilde{m} is positive, \tilde{m}^{-1} is convex (See Fig. 3a.) and positive.

$\varphi(x) = e^x$, $\varphi(\tilde{m})$ is denoted $e^{\tilde{m}}$; $\mu_{e^{\tilde{m}}}(x) = \mu_{\tilde{m}}(\log x)$, $x > 0$; $e^{\tilde{m}}$ is a positive fuzzy number.

3.1.2. Absolute value

The absolute value of \tilde{m} is denoted $|\tilde{m}|$

$$\begin{aligned} \mu_{|\tilde{m}|}(x) &= \max(\mu_{\tilde{m}}(x), \mu_{\tilde{m}}(-x)) & \text{if } x \geq 0, \\ &= 0 & \text{if } x < 0. \end{aligned}$$

$|\tilde{m}|$ is a positive fuzzy number!

3.2. Extended addition and extended multiplication

3.2.1. General properties

\oplus denotes the extended addition and \odot the extended product. The addition is always increasing and Theorem 1 can always be applied to it. The addition of fuzzy numbers gives a fuzzy number. Note that

$$-(\tilde{m} \oplus \tilde{n}) = (-\tilde{m}) \oplus (-\tilde{n}).$$

From Section 1.5. extended addition is commutative and associative, but has no group structure: $\tilde{m} \oplus 0 = \tilde{m}$ but \tilde{m} has no opposite in the sense of a group-structure: in particular

$$\tilde{m} \oplus -\tilde{m} \neq 0.$$

The multiplication of positive numbers is increasing and that of negative numbers decreasing, so that Theorem 1 can only be applied for the product of positive or negative fuzzy numbers. The product of fuzzy numbers of different signs can be solved by $-(\tilde{m} \odot \tilde{n}) = (-\tilde{m}) \odot \tilde{n}$, so that providing a slight manipulation on the signs, Theorem 1 applies anyway. Thus the multiplication of fuzzy numbers of a

³ Notation $\tilde{m} > 0$ and $\tilde{m} < 0$ if $x \geq 0, \mu_{\tilde{m}}(x) = 0$

given sign gives a fuzzy number of a determined sign. Multiplication of ordinary fuzzy numbers can be performed in the following way:

- decompose \tilde{m} and \tilde{n} in negative and positive parts.
- perform the four operations between each part of \tilde{m} and \tilde{n} , using eventually the flattening effect.
- Using Theorem 2, the union of all the products obtained above is the final result.

So it is always easy to perform exact multiplication between fuzzy numbers. Note that extended multiplication is commutative and associative. In Section 3.2.2 we deal with the distributivity of extended multiplication on extended addition in the case when the terms are either positive or negative, which is the most practical situation.

Note. The set of positive fuzzy numbers is not a group for \odot . Although $\tilde{m} \odot 1 = \tilde{m}$, $\tilde{m} \odot \tilde{m}^{-1} \neq 1$, and \tilde{m} has no inverse in the sense of the group structure.

3.2.2. Distributivity of \odot on \oplus

Let \tilde{m} , \tilde{n} , \tilde{p} be three continuous fuzzy numbers. We want to find conditions on \tilde{m} , \tilde{n} , \tilde{p} so that $\tilde{m} \odot (\tilde{n} \oplus \tilde{p}) = (\tilde{m} \odot \tilde{n}) \oplus (\tilde{m} \odot \tilde{p})$,

$$\begin{aligned} \mu_{\tilde{m} \odot (\tilde{n} \oplus \tilde{p})}(z) &= \sup_{z=xu} \min(\mu_{\tilde{m}}(x), \sup_{u=y+t} \min(\mu_{\tilde{n}}(y), \mu_{\tilde{p}}(t))) \\ &= \sup_{z=x(y+t)} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y), \mu_{\tilde{p}}(t)), \\ \mu_{(\tilde{m} \odot \tilde{n}) \oplus (\tilde{m} \odot \tilde{p})}(z) &= \sup_{z=v+w} \min(\sup_{xy=v} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)), \sup_{tu=w} \min(\mu_{\tilde{m}}(u), \mu_{\tilde{p}}(t))) \\ &= \sup_{z=xy+tu} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y), \mu_{\tilde{m}}(u), \mu_{\tilde{p}}(t)). \end{aligned}$$

Theorem 3. *Provided that \tilde{m} is either a positive or a negative fuzzy number and that \tilde{n} and \tilde{p} are together either positive or negative fuzzy numbers, then*

$$\tilde{m} \odot (\tilde{n} \oplus \tilde{p}) = (\tilde{m} \odot \tilde{n}) \oplus (\tilde{m} \odot \tilde{p})$$

Proof. Let $\varphi(x, y, t, u) = xy + tu$:

when $\tilde{m} > 0$, $\tilde{n} > 0$, $\tilde{p} > 0$: φ is increasing,

when $\tilde{m} < 0$, $\tilde{n} < 0$, $\tilde{p} < 0$: φ is decreasing.

Thus using Theorem 1, the maximum for $\min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y), \mu_{\tilde{m}}(u), \mu_{\tilde{p}}(t))$ is reached for $\mu_{\tilde{m}}(x) = \mu_{\tilde{n}}(y) = \mu_{\tilde{m}}(u) = \mu_{\tilde{p}}(t)$ either on the increasing parts or on the decreasing parts of the functions, so that $x = u$ and

$$\mu_{\tilde{m} \odot (\tilde{n} \oplus \tilde{p})}(z) = \mu_{(\tilde{m} \odot \tilde{n}) \oplus (\tilde{m} \odot \tilde{p})}(z)$$

When \tilde{m} and the pair (\tilde{n}, \tilde{p}) have opposite signs, we consider

$$-[-(\tilde{m} \odot (\tilde{n} \oplus \tilde{p}))] = -[(-\tilde{m}) \odot (\tilde{n} \oplus \tilde{p})]$$

And since $-\tilde{m}$, \tilde{n} , \tilde{p} have the same signs:

$$\begin{aligned} -[(-\tilde{m}) \odot (\tilde{n} \oplus \tilde{p})] &= -[(-\tilde{m}) \odot \tilde{n}] \oplus [(-\tilde{m}) \odot \tilde{p}] \\ &= [-((-\tilde{m}) \odot \tilde{n})] \oplus [-((-\tilde{m}) \odot \tilde{p})], \\ &= [\tilde{m} \odot \tilde{n}] \oplus [\tilde{m} \odot \tilde{p}]. \end{aligned}$$

Note. In Theorem 3, when \tilde{n} and \tilde{p} have opposite signs, the distributivity formula no longer holds.

A counter-example can be easily provided (See Section 5.4.).

A demonstration of the non distributivity of \odot on \oplus in the general case can be found in [9, Part 1].

3.3. Extended subtraction and division

These two operations are respectively denoted by \ominus and \oplus

3.3.1. Subtraction

Subtraction is neither increasing nor decreasing. Theorem 1 cannot be directly applied. However the operation $\tilde{m} \ominus \tilde{n}$ can always be written $\tilde{m} \ominus \tilde{n} = \tilde{m} \oplus (-\tilde{n})$:

$$\begin{aligned} \mu_{\tilde{m} \ominus \tilde{n}}(z) &= \sup_{z=x-y} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)) = \sup_{z=x+y} \min(\mu_{\tilde{m}}(x), \mu_{-\tilde{n}}(-y)) \\ &= \sup_{z=x+y} \min(\mu_{\tilde{m}}(x), \mu_{-\tilde{n}}(y)) \end{aligned}$$

3.3.2. Division

Division is neither increasing nor decreasing. However, if \tilde{m} and \tilde{n} are strictly positive numbers:

$$\begin{aligned} \mu_{\tilde{m} \oplus \tilde{n}}(z) &= \sup_{z=x/y} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)) = \sup_{z=xy} \min\left(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}\left(\frac{1}{y}\right)\right) \\ &= \sup_{z=xy} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}^{-1}}(y)). \end{aligned}$$

So that $\tilde{m} \oplus \tilde{n} = \tilde{m} \odot (\tilde{n}^{-1})$.

Since \tilde{n}^{-1} is a positive fuzzy number, Theorem 1 can now be applied. If \tilde{m} and \tilde{n} are both strictly negative, the same results holds. If \tilde{m} and \tilde{n} have opposite signs, owing to $-(\tilde{n}^{-1}) = (-\tilde{n})^{-1}$, $\tilde{m} \oplus \tilde{n}$ can be still easily performed.

Division of two ordinary fuzzy numbers can be performed with an analogous procedure as that of multiplication (See Section 3.2.1.).

3.4. Extended max and min operations

These two operations are respectively denoted $\tilde{\text{māx}}$ and $\tilde{\text{mīn}}$; max and min are increasing operations on \mathbf{R} , so that Theorem 1 can be applied.

A direct application of Theorem 1 gives the practical rule of construction of $\tilde{\text{māx}}(\tilde{m}, \tilde{n})$ and $\tilde{\text{mīn}}(\tilde{m}, \tilde{n})$ (and more generally $\tilde{\text{māx}}(\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_k)$ and $\tilde{\text{mīn}}(\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_k)$, already stated in [2]; take the left-most parts (for the $\tilde{\text{mīn}}$) and the rightmost parts (for the $\tilde{\text{māx}}$) of the k -uple of increasing parts of $\tilde{m}_1, \dots, \tilde{m}_k$ and do the same for the decreasing parts. (See Fig. 4.). For intervals

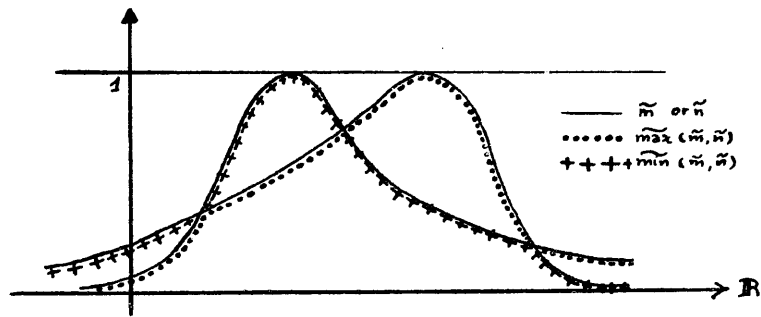


Fig. 4.

of constant membership values, perform the max (resp the min) operation, in the case of $\tilde{m}\tilde{x}$ (resp. $\tilde{m}\tilde{i}\tilde{n}$) in the left and right extremities of those intervals.

Let $\tilde{m}, \tilde{n}, \tilde{p}$ three fuzzy numbers, it is easy to check that we have the following properties.

$\tilde{m}\tilde{x}, \tilde{m}\tilde{i}\tilde{n}$ are commutative and associative operations,

De Morgan laws

$$1 \ominus \tilde{m}\tilde{i}\tilde{n}(\tilde{m}, \tilde{n}) = \tilde{m}\tilde{x}(1 \ominus \tilde{m}, 1 \ominus \tilde{n})$$

$$1 \ominus \tilde{m}\tilde{x}(\tilde{m}, \tilde{n}) = \tilde{m}\tilde{i}\tilde{n}(1 \ominus \tilde{m}, 1 \ominus \tilde{n})$$

Distributivity

$$\tilde{m}\tilde{i}\tilde{n}(\tilde{m}, \tilde{m}\tilde{x}(\tilde{n}, \tilde{p})) = \tilde{m}\tilde{x}(\tilde{m}\tilde{i}\tilde{n}(\tilde{m}, \tilde{n}), \tilde{m}\tilde{i}\tilde{n}(\tilde{m}, \tilde{p}))$$

$$\tilde{m}\tilde{x}(\tilde{m}, \tilde{m}\tilde{i}\tilde{n}(\tilde{n}, \tilde{p})) = \tilde{m}\tilde{i}\tilde{n}(\tilde{m}\tilde{x}(\tilde{m}, \tilde{n}), \tilde{m}\tilde{x}(\tilde{m}, \tilde{p}))$$

Absorption

$$\tilde{m}\tilde{x}(\tilde{m}, \tilde{m}\tilde{i}\tilde{n}(\tilde{m}, \tilde{n})) = \tilde{m}$$

$$\tilde{m}\tilde{i}\tilde{n}(\tilde{m}, \tilde{m}\tilde{x}(\tilde{m}, \tilde{n})) = \tilde{m}$$

$$\tilde{m}\tilde{x}(\tilde{m}, \tilde{n}) \oplus \tilde{m}\tilde{i}\tilde{n}(\tilde{m}, \tilde{n}) = \tilde{m} \oplus \tilde{n}$$

3.5. Other operations

3.5.1. Bounded sum, bounded difference

Zadeh has introduced the bounded sum and the bounded difference of two numbers belonging to $[0, 1]$ respectively as

$$x \hat{+} y = \min(1, x + y) \quad x \hat{-} y = \max(0, x - y).$$

It is easy to check that

$$\tilde{m} \hat{\oplus} \tilde{n} = \tilde{m}\tilde{i}\tilde{n}(1, \tilde{m} \oplus \tilde{n}).$$

If $\tilde{m} \oplus \tilde{n}$ is such that $\mu_{\tilde{m} \oplus \tilde{n}}(x) = 0$ for $x \in (-\infty, 1]$, then $\tilde{m} \hat{\oplus} \tilde{n} = 1$. If $\tilde{m} \oplus \tilde{n}$ is such that $\mu_{\tilde{m} \oplus \tilde{n}}(x) = 0$ for $x \in [1, +\infty)$, then $\tilde{m} \hat{\oplus} \tilde{n} = \tilde{m} \oplus \tilde{n}$.

When $\tilde{m} \oplus \tilde{n}$ is ordinary

$$\begin{aligned} \mu_{\tilde{m} \hat{\oplus} \tilde{n}}(x) &= \mu_{\tilde{m} \oplus \tilde{n}}(x) & \text{for } x \in (-\infty, 1] \\ &= 0 & \text{for } x > 1. \end{aligned}$$

It is easy to check also that $\tilde{m} \hat{\ominus} \tilde{n} = \tilde{m}\tilde{x}(0, \tilde{m} \ominus \tilde{n})$

3.5.2. Probabilistic sum

We consider now $x \# y = x + y - xy$ is increasing for $(x, y) \in (-\infty, 1]^2$ and decreasing for $(x, y) \in [1, +\infty)^2$. Theorem 1 can be applied on fuzzy numbers both smaller than 1 or both greater than 1.

Particularly, Theorem 1 is applicable for fuzzy numbers of $[0, 1]$ which is the ordinary used interval.

Note that

$$\mu_{\tilde{m} \oplus \tilde{n} \ominus \tilde{m} \ominus \tilde{n}}(z) = \sup_{z=x+y-tu} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y), \mu_{\tilde{m}}(t), \mu_{\tilde{n}}(u))$$

is generally different from $\sup_{z=x+y-xy} \min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y))$ the membership function of the extended \oplus .

However, when t and u are negative, $x + t - tu$ is an increasing function; owing to Theorem 1, we can assume $x = t$ and $y = u$. So that when \tilde{m} and \tilde{n} are negative, we have

$$\tilde{m} \oplus \tilde{n} = \tilde{m} \oplus \tilde{n} \ominus \tilde{m} \cdot \tilde{n}$$

4. Fast approximate formulae for usual operations

4.1. L-R representation of fuzzy members

A function, denoted L or R usually, is said a reference function of fuzzy numbers if and only if

$$\begin{aligned} L(-x) &= L(x) \\ L(0) &= 1 \\ L &\text{ is decreasing on } [0, +\infty). \end{aligned}$$

Examples:

$$\begin{aligned} y &= 1 \quad \text{for } x \in [-1, +1] \text{ and } 0 \text{ outside,} \\ y &= \max(0, 1 - |x|^p) \quad p > 1, \\ y &= e^{-|x|^p} \quad p > 1, \\ y &= \frac{1}{1 + |x|^p} \quad p > 1. \end{aligned}$$

Definition. A fuzzy number \tilde{m} is said a L-R type fuzzy number if and only if

$$\begin{aligned} \mu_{\tilde{m}}(x) &= L\left(\frac{m-x}{\alpha}\right) \quad \text{for } x \leq m; \quad \alpha > 0 \\ &= R\left(\frac{x-m}{\beta}\right) \quad \text{for } x \geq m; \quad \beta > 0 \end{aligned}$$

'L' is for left reference and 'R' for right reference. m is a real number, call the mean value of \tilde{m} ($\mu_{\tilde{m}}(m) = 1$). α and β are called left and right spreads respectively; when spreads are zero \tilde{m} is a nonfuzzy number; when spreads are infinite $\tilde{m} = \mathbf{R}$; e.g. $\alpha = +\infty$ then $\mu_{\tilde{m}}(x) = 1, x \leq m$. \tilde{m} is denoted $(m, \alpha, \beta)_{LR}$. This a rather convenient representation to model a number approximately equal to m .

4.2. Formulae for \oplus , \ominus , \odot , \oplus , $\text{m}\bar{\alpha}\text{x}$, $\text{m}\bar{\alpha}\text{i}\text{n}$

Note. The following formulae were already demonstrated in [2].

4.2.1. \oplus and \ominus

$$(m, \alpha, \beta)_{\text{LR}} \oplus (n, \gamma, \delta)_{\text{LR}} = (m+n, \alpha+\gamma, \beta+\delta)_{\text{LR}}$$

$$-(m, \alpha, \beta)_{\text{LR}} = (-m, \beta, \alpha)_{\text{RL}}$$

$$(m, \alpha, \beta)_{\text{LR}} \ominus (n, \gamma, \delta)_{\text{RL}} = (m-n, \alpha+\delta, \beta+\gamma)_{\text{LR}}$$

These formulae are without any approximation.

4.2.2. \odot

— $\bar{m} > 0, \bar{n} > 0$

$$(m, \alpha, \beta)_{\text{LR}} \odot (n, \gamma, \delta)_{\text{LR}} = (mn, m\gamma + n\alpha, m\delta + n\beta)_{\text{LR}}$$

— $\bar{m} < 0, \bar{n} > 0$

$$(m, \alpha, \beta)_{\text{RL}} \odot (n, \gamma, \delta)_{\text{LR}} = (mn, n\alpha - m\delta, n\beta - m\gamma)_{\text{RL}}$$

— $\bar{m} < 0, \bar{n} < 0$

$$(m, \alpha, \beta)_{\text{LR}} \odot (n, \gamma, \delta)_{\text{LR}} = (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{\text{RL}}$$

The smallest the spreads relatively to the mean values, the more valid these three formulae. They are particularly valid for the greatest membership values. Another approximation formula when spreads cannot be neglected compared with mean values is, for $\bar{m} > 0$ and $\bar{n} > 0$:

$$(m, \alpha, \beta)_{\text{LR}} \odot (n, \gamma, \delta)_{\text{LR}} = (mn, m\gamma + n\alpha - \alpha\gamma, m\delta + n\beta + \beta\delta)_{\text{LR}}$$

4.2.3. \oplus

— $\bar{m} > 0$

$$(m, \alpha, \beta)_{\text{LR}}^{-1} = \left(\frac{1}{m}, \frac{\beta}{m^2}, \frac{\alpha}{m^2} \right)_{\text{RL}}$$

— $\bar{m} > 0$ and $n > 0$

$$(m, \alpha, \beta)_{\text{LR}} \oplus (n, \gamma, \delta)_{\text{RL}} = \left(\frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2} \right)_{\text{LR}}$$

These formulae are approximate in the same sense as the first ones for the multiplication. When $\bar{m} < 0$ or $\bar{n} < 0$, $\bar{m} \oplus \bar{n}$ can be computed using corresponding formulae for multiplication.

4.2.4. $\text{m}\bar{\alpha}\text{x}$ and $\text{m}\bar{\alpha}\text{i}\text{n}$

If $m \gg n$.

$$\text{m}\bar{\alpha}\text{x}((m, \alpha, \beta)_{\text{LR}}, (n, \gamma, \delta)_{\text{LR}}) = (m, \alpha, \beta)_{\text{LR}}$$

$$\text{m}\bar{\alpha}\text{i}\text{n}((m, \alpha, \beta)_{\text{LR}}, (n, \gamma, \delta)_{\text{LR}}) = (n, \gamma, \delta)_{\text{LR}}$$

If m is close to n :

$$\begin{aligned} \max((m, \alpha, \beta)_{LR}, (n, \gamma, \delta)_{LR}) &\approx (\max(m, n), \min(\alpha, \gamma), \max(\beta, \delta))_{LR}, \\ \min((m, \alpha, \beta)_{LR}, (n, \gamma, \delta)_{LR}) &\approx (\min(m, n), \max(\alpha, \gamma), \min(\beta, \delta))_{LR}. \end{aligned}$$

The two first formulae are exact when $\mu_{\tilde{m}}$ and $\mu_{\tilde{n}}$ have only one intersection point.

The two last ones are exact if $m = n$ and if they have no intersection point whose membership value is strictly smaller than 1.

4.3. FFS

A flat fuzzy number (FFN) \tilde{m} (See Fig. 5.) has a membership function

$$\begin{aligned} \mu_{\tilde{m}}(x) &= L\left(\frac{a-x}{\alpha}\right) & x \leq a, \\ &= 1 & a \leq x \leq b, \\ &= R\left(\frac{x-b}{\beta}\right) & x \geq b. \end{aligned}$$

and will be denoted by

$$\tilde{m} = (a, b, \alpha, \beta)_{LR}.$$

The two following formulae can easily be checked using Theorem 1 and results of Sections 4.2.2 and 4.2.3.

$$(a, b, \alpha, \beta)_{LR} \oplus (c, d, \gamma, \delta)_{LR} = (a + c, b + d, \alpha + \gamma, \beta + \delta)_{LR}.$$

— $\tilde{m} > 0$ and $\tilde{n} > 0$

$$(a, b, \alpha, \beta)_{LR} \odot (c, d, \gamma, \delta)_{LR} \approx (ac, bd, a\gamma + c\alpha, b\delta + d\beta)_{LR}$$

this last formula is approximate.

The extension to the subtraction, the product of various signs numbers, and the division is straightforward.

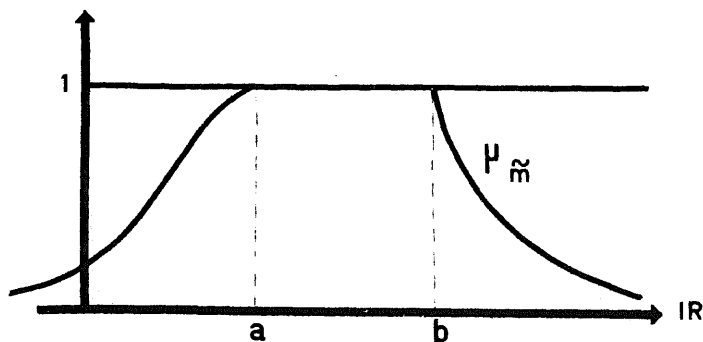


Fig. 5.

Note. Given a FFN, it would be possible to define a three parameters representation for it (L or R are flat); however this representation can be less convenient than the four parameters one; given a set of FFN, it is far more easy to find for them a L-R four parameters representation than a L-R three parameters one which often does not exist when the former does.

4.4. Zadeh's representation

Zadeh introduced the following piecewise quadratic function defined by

$$\begin{aligned}
 S(x; \alpha, \beta, \gamma) &= 0 && \text{for } x \leq \alpha, \\
 &= 2 \left(\frac{x - \alpha}{\gamma - \alpha} \right)^2 && \text{for } \alpha \leq x \leq \beta, \\
 &= 1 - 2 \left(\frac{x - \gamma}{\gamma - \alpha} \right)^2 && \text{for } \beta \leq x \leq \gamma, \\
 &= 1 && \text{for } x \geq \gamma \quad \text{with } \beta = \frac{\alpha + \gamma}{2}.
 \end{aligned}$$

It can be easily checked that

$$\begin{aligned}
 S(x; \alpha, \beta, \gamma) &= (\gamma, \gamma - \alpha, \infty)_L && \text{with } L(x) = 2(1 - |x|)^2 \\
 &&& \text{for } x \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, +1], \\
 &&& = 1 - 2x^2 \text{ for } x \in [-\frac{1}{2}, \frac{1}{2}], \\
 &&& = 0 \text{ outside.}
 \end{aligned}$$

So that the formulae of Section 4.2 can be used in the framework of Zadeh's representation.

4.5. Towards operations on fuzzy numbers of type 2: the addition

4.5.1. Fuzzy set of type 2

It is a fuzzy set whose membership values are fuzzy. More specifically it is an application from a universe U to the set of applications from $[0, 1]$ to $[0, 1]$. If A is a fuzzy subset of type 2, $\mu_A(x)$ is a fuzzy subset of $[0, 1]$ in the usual sense. Here $U = \mathbf{R}$.

4.5.2. A special case of addition

As a hint for further research consider the following fuzzy number of type 2 denoted $\bar{m} : \mu_{\bar{m}}$ is a fuzzy family of membership functions, i.e., $\{\bar{f}_\lambda, f_\lambda \mid \lambda \in [0, 1]\}$ such that

$$(1) \quad \exists m \in \mathbf{R}, \forall \lambda, \forall \lambda', \bar{f}_\lambda(m) = f_{\lambda'}(m) = 1$$

(Here for convenience m is unique, but \bar{m} could as well be flat.),

$$(2) \quad \forall x, \bar{f}_1(x) = f_1(x) = f_1(x),$$

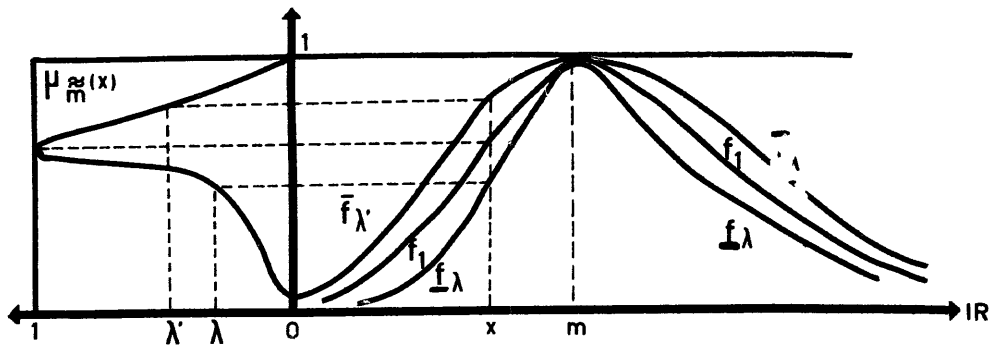


Fig. 6.

- (3) each \underline{f}_λ or \bar{f}_λ defines a continuous fuzzy number,
- (4) $\forall \lambda, \forall \lambda', \lambda < \lambda' : \underline{f}_\lambda(x) \leq \underline{f}_{\lambda'}(x) \leq \underline{f}_1(x) \leq \bar{f}_{\lambda'}(x) \leq \bar{f}_\lambda(x)$.

λ is the truth value of “ f_λ is the membership function of \tilde{m} ”. To perform the addition, consider a level λ and the two fuzzy numbers \tilde{m}_λ and $\bar{\tilde{m}}_\lambda$ whose membership functions are \underline{f}_λ and \bar{f}_λ respectively. Assume them to be of the L-R type for convenience.

$$\tilde{m}_\lambda = (m, \underline{\alpha}_\lambda, \underline{\beta}_\lambda) \text{ and } \bar{\tilde{m}}_\lambda = (m, \bar{\alpha}_\lambda, \bar{\beta}_\lambda).$$

We have

$$\begin{aligned} \bar{\alpha} &= \left(\int_{[0,1]} \lambda / \underline{\alpha}_\lambda \right) \cup \left(\int_{[0,1]} \lambda / \bar{\alpha}_\lambda \right) \\ \bar{\beta} &= \left(\int_{[0,1]} \lambda / \underline{\beta}_\lambda \right) \cup \left(\int_{[0,1]} \lambda / \bar{\beta}_\lambda \right) \end{aligned}$$

$\bar{\alpha}$ and $\bar{\beta}$ are convex and normalized fuzzy sets, i.e. fuzzy numbers. Let \tilde{n} another fuzzy number of type 2 as defined above:

$$\tilde{n}_\lambda = (n, \underline{\gamma}_\lambda, \underline{\delta}_\lambda) \text{ and } \bar{\tilde{n}}_\lambda = (n, \bar{\gamma}_\lambda, \bar{\delta}_\lambda),$$

then

$$\begin{aligned} \tilde{m}_\lambda \oplus \tilde{n}_\lambda &= (m+n, \underline{\alpha}_\lambda + \underline{\gamma}_\lambda, \underline{\beta}_\lambda + \underline{\delta}_\lambda), \\ \bar{\tilde{m}}_\lambda \oplus \bar{\tilde{n}}_\lambda &= (m+n, \bar{\alpha}_\lambda + \bar{\gamma}_\lambda, \bar{\beta}_\lambda + \bar{\delta}_\lambda). \end{aligned}$$

Let $\bar{f}_{R\lambda}$ and $\bar{f}_{L\lambda}$ be the decreasing and increasing parts of \bar{f}_λ and $\underline{f}_{R\lambda}$, $\underline{f}_{L\lambda}$ be the decreasing and increasing parts of \underline{f}_λ , respectively. $\bar{g}_{R\lambda}$, $\bar{g}_{L\lambda}$, $\underline{g}_{L\lambda}$, $\underline{g}_{R\lambda}$ are similarly defined for \bar{g}_λ and \underline{g}_λ which are current membership functions of the family $\mu_{\tilde{n}}$. The reason why it is not necessary to consider $\tilde{m}_\lambda \oplus \bar{\tilde{n}}_\lambda$ or $\bar{\tilde{m}}_\lambda \oplus \tilde{n}_\lambda$ is:

$$\forall \omega \in [0, 1], \exists \lambda^* \geq \lambda, \bar{f}_{L\lambda^*}^{-1}(\omega) + \underline{g}_{L\lambda^*}^{-1}(\omega) = \bar{f}_{L\lambda^*}^{-1}(\omega) + \bar{g}_{L\lambda^*}^{-1}(\omega)$$

or

$$= \underline{f}_{L\lambda^*}^{-1}(\omega) + \underline{g}_{L\lambda^*}^{-1}(\omega)$$

Owing to Theorem 1, we know that we have obtained $\bar{\alpha} \oplus \bar{\gamma}$ and $\bar{\beta} \oplus \bar{\delta}$ so that we can write $(m, \bar{\alpha}, \bar{\beta}) \oplus (n, \bar{\gamma}, \bar{\delta}) = (m+n, \bar{\alpha} \oplus \bar{\gamma}, \bar{\beta} \oplus \bar{\delta}) = \bar{\tilde{m}} \oplus \bar{\tilde{n}}$.

Note. To take into account the fact that a fuzzy membership function can be illdefined due to subjectivity, a fuzzy number of type 2 can be an alternative representation to the probabilistic set representation [4].

5. Interpretation of operations in fuzzy subsets of R

5.1. Interpretation of fuzzy subsets of R

A fuzzy number \tilde{m} is interpreted as: a number whose value is approximately m ; this representation implicitly assume that the transition between the values which can be m and those which cannot is progressive rather than abrupt. More specifically a flat fuzzy number can be viewed as a tolerance interval whose boundaries are ill-defined. As a matter of fact, the usual binary definition may seem somehow excessive.

A fuzzy number with several distinct maxima whose membership values are 1 can model the set of several imprecise measures of a given phenomenon.

Consider now a fuzzy real subset having maxima whose membership are not all 1; the membership value of a maximum can express the degree of quality of the information which lies in this maximum while the spreads model the imprecision (i.e., the word “approximately” in ‘approximately m ’). The degree of quality of a measure estimates the reliability of the measure (related to the conditions of the experiment, whereas precision is more related to the measurement device).

The distributivity of any extended operation on the union of fuzzy sets is easily interpreted: for instance, it seems quite natural that ‘approximately 2 or approximately 3’ plus ‘approximately one’ give ‘approximately 3 or approximately 4’.

On the contrary, a number whose value is ‘approximately 1 and approximately 3’ has less significance; this is consistent with the non-distributivity of any extended operation on the intersection operator. Note that in the case of a ‘min-max’ extension, an extended operation will be distributive on the intersection and not on the union. This last remark is a clue in favor of the ‘max-min’ composition.

The tolerance interval of a measure is the interval where it is possible that the real value lies. This interpretation is consistent with possibility theory [8].

5.2. Normalization of a fuzzy subset of R

Normaliaization assumes the existence, in the universe of discourse, of an element totally compatible with the predicate attached to the fuzzy subset. If this reference element is outside the universe the normalization assumption cannot be taken for granted. For example consider a finite set $M = \{m_1, \dots, m_n\}$ of real numbers, and a fuzzy number \tilde{n} . The fuzzy subset of numbers of M which are equal to \tilde{n} will not generally be normalized, except if $\exists i : \mu_{\tilde{n}}(m_i) = 1$.

5.3. Identification of membership functions

The problem of the estimation of the membership function is often pointed out. Even if fuzzy sets are not statistical in nature this does not prevent us from using

statistics for this estimation. But it does not require necessarily so big a number of experiments as the estimation of a probability density. The knowledge of a possibility distribution (which is the interpretation of a membership function) can require a learning stage.

Given a set of statistical data in the form of an histogram, the induction, from it, of a possibility distribution is different from that of a probability density: in the first case the histogram is normalized through an affinity which brings back to 1 the highest ordinate, in the second case the surface of the histogram is normalized. Thus, in determining in such a way a possibility distribution, we postulate that from a global precise knowledge about a population of events, we can infer a local imprecise knowledge about any element of this population.

5.4. Tolerance analysis using fuzzy sets

As indicated by the formulae of Section 4.2 the operations on fuzzy numbers generalize the operations on tolerance intervals, which is consistent with the proposed interpretation of fuzzy numbers. For example, the well-known theorem: 'The absolute error on a sum is the sum of the absolute elementary errors' becomes 'The spreads of the sum of fuzzy numbers are the sums of the spreads.' Note that when tolerance interval are modelled using the references function $y = 1$ for x belonging to $[-1, +1]$ and zero outside, absolute errors are indeed spreads. In the approximate formula of the product of two fuzzy numbers, the theorem on the additivity of relative errors for a product can be easily recognized.

A counter example of the non distributivity of \odot on \ominus for positive fuzzy numbers can easily be found in the framework of classical tolerance analysis:

$$\frac{\Delta(a(b-c))}{a(b-c)} = \frac{\Delta a}{a} + \frac{\Delta(b-c)}{b-c} = \frac{\Delta a}{a} + \frac{\Delta b + \Delta c}{b-c}.$$

Hence, $\Delta(a(b-c)) = (b-c)\Delta a + a\Delta b + a\Delta c$.

But

$$\frac{\Delta(ab-ac)}{a(b-c)} = \frac{\Delta ab + \Delta ac}{a(b-c)} = \frac{a\Delta b + b\Delta a + a\Delta c + c\Delta a}{a(b-c)},$$

where Δx denotes the absolute symmetrical error on x .

The non existence of a group structure for fuzzy numbers with respect to \oplus or \odot is related to the fact that the results of those operations are always relatively fuzzier than the numbers on which they act.

To perform the computation of an algebraic expression of fuzzy numbers, we must check that there is no intermediary variables but only elementary variables in the considered expression, in the same way as in classical tolerance analysis. Otherwise the expression is redundant in the sense that an elementary variable is used several times without necessity, and this improper use leads to a useless increase of the resulting fuzziness. The structure of the expression must be the same as the direct logical chain of inferences which produced it from elementary constituents. The conclusion cannot be more precise than the hypothesis.

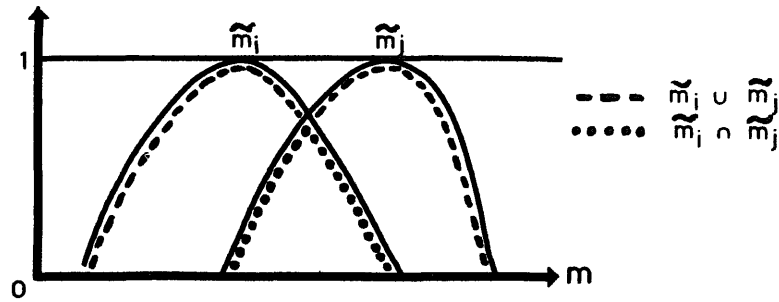


Fig. 7.

The use of fuzzy concepts in tolerance analysis can throw some light on a well known problem in measure processing: Let $\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_n$ be n approximate measures obtained by means of several devices in the same operating conditions, and which evaluate some quantifiable property of a given phenomenon. The \tilde{m}_i 's are n normalized fuzzy sets of \mathbf{R} . How to process them, especially in the case when n is too small so that probability theory cannot be applied? To estimate the real measured value, we can perform $\cup_i \tilde{m}_i$, $\cap_i \tilde{m}_i$, eventually $(1/n) \sum_{i=1}^n \tilde{m}_i$, and other operations.

$\cap_i \tilde{m}_i$ is optimistic: it assumes the invariance of the phenomenon, the accuracy of all measurement devices, and the closeness of the mean values of the m_i 's. When these assumptions are lacking $\cup_i \tilde{m}_i$ is the most valid conclusion. $(1/n) \sum \tilde{m}_i$ is a trade-off between \cap and \cup , and is justifiable only if the flat parts of the \tilde{m}_i 's have a non-empty intersection (i.e. $\cap_i \tilde{m}_i$ is normal). However, when $\cap_i \tilde{m}_i$ is not normal, the gain in precision is counterbalanced by a loss in quality, which make somewhat delusive the precision. Using $\cup_i \tilde{m}_i$ the quality is that of the best measure (when the operating conditions do not fluctuate, all measures have the same degree of quality) but precision diminishes.

This balance between precision and quality, which is not so clear using usual tolerance intervals, is a concrete illustration of Zaheh's incompatibility principle;" As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance become almost mutually exclusive characteristics."

6. Conclusion

Algebraic calculus on real fuzzy sets is devoted to play a central role in the development of the theory of fuzzy sets and its applications to systems science. The possibility of easy realization of this calculus in practical situations is of crucial importance. Applications in the fields of operations research and management science, where data usually lack precision and where this lack seems mainly irreducible, have already been pointed out by the authors [3]. Lastly the equivalence between a fuzzy relation and a fuzzy operation will make possible a faster computation for max-min compositional rule, which begins to be applied in control and learning theory.

References

- [1] Didier Dubois and Henri Prade, Comment on 'Tolerance analysis using fuzzy sets' and 'A procedure for multiple-aspect decision making', *Internat. J. Systems Sci.* 9 (3) (1978) 357–360.
- [2] Didier Dubois and Henri Prade, Operations on fuzzy numbers, *Internat. J. Systems Sci.* 9 (6) (1978) 613–626.
- [3] Didier Dubois and Henri Prade, Algorithmes de plus courts chemins pour traiter des données floues, *R.A.I.R.O. Op. Res.* 12, (2) (1978) 213–227.
- [4] Kaoru Hirota, Concepts of probabilistic sets, *Proc. 1977 IEEE Conf. Decision and Control*, (1977) 1361–1366.
- [5] Ramesh Jain Tolerance analysis using fuzzy sets, *Internat. J. System Sci.* 7 (12) 1393–1401.
- [6] M. Mizumoto and K. Tanaka, Algebraic properties of fuzzy numbers, *Proc. Internat. Conf. Cybernetics and Society*, Washington, DC (1976)
- [7] Steven Nahmias, Fuzzy variables, *Int. J. Fuzzy Sets and Systems*, 1 (2) (1978) 97–110.
- [8] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Int. J. Fuzzy Sets and Systems* 1 (1) 3–28.
- [9]. L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part 1, *Information Sci.* 8 (1975) 199–249, Part 2, *Information Sci.* 8 (1975) 301–357, Part 3, *Information Sci.* 9 (1975) 43–80.