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Abstract. We consider the complex Ginzburg-Laudau operator on a bounded domain. We prove some estimates for the inverse of the linearized operator.

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## 1 Introduction.

We consider the Ginzburg-Landau equation on a bounded connected domain $\Omega$,

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) \text{ in } \Omega \\ u = g \text{ in } \partial\Omega \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $u$ and $g$ have complex values and degree $(g, \partial\Omega) \geq 1$. This equation has been intensively studied, in [2], and many others.
Let us denote

$$\mathcal{N}_\varepsilon(u) = \Delta u + \frac{1}{\varepsilon^2} u(1-|u|^2)$$

and let us define $f_d$ as the only solution of the differential equation

$$\begin{cases} f_d'' + \frac{f_d'}{r} - \frac{d^2}{r^2} f_d = -f_d(1-f_d^2) \\ f_d(0) = 0, \quad \lim_{r\to+\infty} f_d(r) = 1. \end{cases} \tag{1.2}$$

The equation (1.2) is completely studied in [6]. And let $u_0(x) = f_d(\frac{|x|}{\varepsilon})e^{id\theta}$. We have

$$\mathcal{N}_\varepsilon(u_0) = 0.$$

We will always denote

$$f(r) = f_d(\frac{r}{\varepsilon}).$$

The linearized operator around any function $u$ is given by

$$d\mathcal{N}_\varepsilon(u)(\omega) = \Delta\omega + \frac{\omega}{\varepsilon^2}(1-|u|^2) - \frac{2}{\varepsilon^2} u(\overline{u}.\omega + u.\omega).$$

Let us consider the linearized operator around the solution $u_0$, ie

$$d\mathcal{N}_\varepsilon(u_0)(\omega) = \Delta\omega + \frac{\omega}{\varepsilon^2}(1-f^2) - \frac{2}{\varepsilon^2}f^2 e^{id\theta}e^{id\theta}.\omega,$$

where $\omega$ is any complex valued function and $2u.\omega = \overline{u}\omega + \overline{\omega}u$.
We will use the operator

$$\mathcal{L}_\varepsilon := e^{-id\theta}d\mathcal{N}_\varepsilon(u_0)e^{id\theta}$$

instead of $d\mathcal{N}_\varepsilon(u_0)$ and we will use also the rescaled operator $\mathcal{L}_1$. These operators are of importance for some technics of construction of solutions for the equation (1.1).
The invariance of the equation (1.1) wrt the translations and the rotations gives

$$0 = d\mathcal{N}_\varepsilon(u_0)(iu_0) = d\mathcal{N}_\varepsilon(u_0)(\frac{\partial u_0}{\partial x_1}) = d\mathcal{N}_\varepsilon(u_0)(\frac{\partial u_0}{\partial x_2}).$$

A calculus gives

$$\frac{\partial u_0}{\partial \theta} = idu_0,$$

$$e^{-id\theta}\frac{\partial u_0}{\partial x_1} = \frac{1}{2}\left(\frac{1}{\varepsilon}f_d'(\frac{r}{\varepsilon}) + \frac{d}{r}f_d(\frac{r}{\varepsilon})\right)e^{-i\theta} + \frac{1}{2}\left(\frac{1}{\varepsilon}f_d'(\frac{r}{\varepsilon}) - \frac{d}{r}f_d(\frac{r}{\varepsilon})\right)e^{i\theta}$$

and

$$e^{-id\theta}\frac{\partial u_0}{\partial x_2} = \frac{i}{2}\left(\frac{1}{\varepsilon}f_d'(\frac{r}{\varepsilon}) + \frac{d}{r}f_d(\frac{r}{\varepsilon})\right)e^{-i\theta} + \frac{i}{2}\left(-\frac{1}{\varepsilon}f_d'(\frac{r}{\varepsilon}) + \frac{d}{r}f_d(\frac{r}{\varepsilon})\right)e^{i\theta}.$$

In [1], we have proved that

**Theorem 1.1** *For all $d \geq 1$ the set of the solutions of $\mathcal{L}_1\omega = 0$ which are defined at 0 and bounded at $+\infty$ is reduced to the three functions provided by the invariance of the equation (1.1) by the rotations and the translations of the coordinates, ie $if_d$, $(f_d' + \frac{d}{r}f_d)e^{-i\theta} + (f_d' - \frac{d}{r}f_d)e^{i\theta}$ and $i(f_d' + \frac{d}{r}f_d)e^{-i\theta} - i(f_d' - \frac{d}{r}f_d)e^{i\theta}$.*

The case $d = 1$ was known by [11], chapter 3. In this book, Pacard and Rivière construct some solutions for (1.1), having the degree one around each singularity. Before that, the eigenvalue problem $\mathcal{L}_\varepsilon\omega = -\lambda(\varepsilon)\omega$, had been studied in several papers, including [7], [10], [8], [9]. It is used in many others, among them [4], [5].

In the present paper, we let

$$\mathcal{H} := \{\omega : B(0,1) \to \mathbb{C}, \quad e^{id\theta}\omega \in H_0^1(B(0,1))\},$$

endowed with the scalar product

$$< \omega, \eta >= \int_{B(0,1)} \left(\nabla(e^{i\theta}\omega).\nabla(e^{-i\theta}\overline{\omega}) + \frac{f^2}{\varepsilon^2}(\omega + \overline{\omega})(\eta + \overline{\eta})\right)dx$$

that makes it a Hibert vector space. We explain in the end of this part in which sense we define an eigenvalue problem in $\mathcal{H}$, the eigenvalues being denoted by $\mu(\varepsilon)$. And we let $i\Phi_0$, $\Phi_1$ and $i\Phi_2$ be some functions in $\mathcal{H}$, associated with $ie^{-id\theta}u_0$, $e^{-id\theta}\frac{\partial u_0}{\partial x_1}$ and $e^{-id\theta}\frac{\partial u_0}{\partial x_2}$, in a sense that will be defined below in Theorem 1.4. We denote $W$ the subspace of $\mathcal{H}$ spanned by $i\Phi_0$, $\Phi_1$ and $i\Phi_2$ and we define $W^\perp$ by : $\mathcal{H} = W \oplus W^\perp$. We denote by $(W^\perp)_+$ and

$(W^\perp)_-$ the subspaces of $W^\perp$, verifying : $W^\perp = (W^\perp)_+ \oplus (W^\perp)_-$, associated respectively to the eigenvalues $\mu(\varepsilon) > 1$ and to the eigenvalues $\mu(\varepsilon) < 1$. For $h \in L^2(B(0,1))$ we consider the equation

$$\begin{cases} \mathcal{L}_\varepsilon \omega = h \\ \omega \in \mathcal{H}. \end{cases} \tag{1.3}$$

We will prove the following

**Theorem 1.2** *There exists $M > 0$ independent of $\varepsilon$ such that for all $\omega \in \left(W^\perp\right)_+ \cup \left(W^\perp\right)_-$ a solution of (1.3), we have*

$$\int_{B(0,1)} \frac{1}{\varepsilon^2 + |x|^2} |\omega|^2 dx \leq M \int_{B(0,1)} (\varepsilon^2 + |x|^2)|h|^2 dx \tag{1.4}$$

*and*

$$\int_{B(0,1)} |\nabla(e^{id\theta}\omega)|^2 dx \leq M \int_{B(0,1)} (\varepsilon^2 + |x|^2)|h|^2 dx. \tag{1.5}$$

Now, for every $\omega \in \mathcal{H}$ we consider the Fourier expansion

$$\omega(x) = \sum_{n\geq 1} (a_n(r)e^{-in\theta} + b_n(r)e^{in\theta}) + a_0(r), \quad a_n(r) \in \mathbb{C}, \quad b_n(r) \in \mathbb{C}, \quad r \in ]0,1].$$

And we denote

$$h = \sum_{n\geq 1} (\alpha_n e^{-in\theta} + \beta_n e^{in\theta}) + \alpha_0,$$

where the $\alpha_j$ and $\beta_j$ have complex values.

**Remark 1.1** *We can replace (1.4) by*

$$\begin{array}{c} \int_0^1 |a_0|^2 \frac{rdr}{\varepsilon^2+r^2} + \sum_{n\geq 1} \int_0^1 (|a_n|^2 + |b_n|^2) \frac{rdr}{\varepsilon^2+r^2} \\ \leq C \sum_{n\geq 1} \int_0^1 r(\varepsilon^2 + r^2)(|\alpha_n|^2 + |\beta_n|^2)dr + \int_0^1 r(\varepsilon^2 + r^2)|\alpha_0|^2 dr \end{array} \tag{1.6}$$

*and we can replace (1.5) by*

$$\begin{array}{c} \int_0^1 (r|a_0'|^2 + \frac{d^2}{r}|a_0|^2)dr + \\ + \sum_{n\geq 1} \int_0^1 \left( r|a_n'|^2 + r|b_n'|^2 + \frac{(n-d)^2}{r}|a_n|^2 + \frac{(n+d)^2}{r}|b_n|^2 \right) dr \\ \leq C \sum_{n\geq 1} \int_0^1 r(\varepsilon^2 + r^2)(|\alpha_n|^2 + |\beta_n|^2)dr + \int_0^1 r(\varepsilon^2 + r^2)|\alpha_0|^2 dr. \end{array} \tag{1.7}$$

Further, let us recall in which way the equation $\mathcal{L}_\varepsilon \omega = h$ is transformed into systems of ordinary equations.
Letting, for $n \geq 1$, $\omega_n(x) = a_n(r)e^{-in\theta} + b_n(r)e^{in\theta}$ and, for $n = 0$, $\omega_0(x) = a_0(r)$, we have

$$2e^{id\theta}.e^{id\theta}\omega_n = \omega_n + \overline{\omega}_n = (b_n + \overline{a}_n)e^{in\theta} + (\overline{b}_n + a_n)e^{-in\theta}.$$

$$\text{Moreover} \qquad e^{-id\theta}\Delta(e^{id\theta}\omega) = \Delta\omega - \frac{d^2}{r^2}\omega + i\frac{2d}{r^2}\frac{\partial\omega}{\partial\theta}.$$

Consequently

$$\begin{gathered} e^{-id\theta}d\mathcal{N}_\varepsilon(u_0)e^{id\theta}\omega = \\ = \sum_{n\geq 1} e^{-in\theta}\left(a_n'' + \tfrac{a_n'}{r} - \tfrac{(n-d)^2}{r^2}a_n + \tfrac{a_n}{\varepsilon^2}(1-f^2) - \tfrac{a_n}{\varepsilon^2}f^2 - \tfrac{\overline{b}_n}{\varepsilon^2}f^2\right) + \\ + \sum_{n\geq 1} e^{in\theta}\left(b_n'' + \tfrac{b_n'}{r} - \tfrac{(n+d)^2}{r^2}b_n + \tfrac{b_n}{\varepsilon^2}(1-f^2) - \tfrac{b_n}{\varepsilon^2}f^2 - \tfrac{\overline{a}_n}{\varepsilon^2}f^2\right) + \\ + a_0'' + \tfrac{a_0'}{r} - \tfrac{d^2}{r^2}a_0 + \tfrac{a_0}{\varepsilon^2}(1-f^2) - \tfrac{a_0+\overline{a}_0}{\varepsilon^2}f^2. \end{gathered} \tag{1.8}$$

For $n \geq 1$, Separating the Fourier components of $e^{-id\theta}d\mathcal{N}_\varepsilon(u_0)e^{id\theta}\omega$, we can consider the operators

$$\begin{gathered} \text{for } n\geq 1, \quad \mathcal{L}_\varepsilon(\omega_n) = \left(a_n'' + \tfrac{a_n'}{r} - \tfrac{(n-d)^2}{r^2}a_n + \tfrac{a_n}{\varepsilon^2}(1-2f^2) - \tfrac{\overline{b}_n}{\varepsilon^2}f^2\right)e^{-in\theta} + \\ + \left(b_n'' + \tfrac{b_n'}{r} - \tfrac{(n+d)^2}{r^2}b_n + \tfrac{b_n}{\varepsilon^2}(1-2f^2) - \tfrac{\overline{a}_n}{\varepsilon^2}f^2\right)e^{in\theta} \end{gathered}$$

$$\text{and, for } n=0, \quad \mathcal{L}_\varepsilon(a_0) = a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2}a_0 + \frac{a_0}{\varepsilon^2}(1-f^2) - \frac{a_0+\overline{a}_0}{\varepsilon^2}f^2.$$

Separating the real part and the imaginary part of $a_n$ and of $b_n$, we consider the following operators, where, now, $a_n$ and $b_n$ are real valued function

$$\begin{gathered} \text{for } n\geq 1 \quad \mathcal{L}_\varepsilon : (a_n e^{-in\theta} + b_n e^{in\theta}) \mapsto \\ \left(a_n'' + \tfrac{a_n'}{r} - \tfrac{(n-d)^2}{r^2}a_n + \tfrac{a_n}{\varepsilon^2}(1-2f^2) - \tfrac{b_n}{\varepsilon^2}f^2\right)e^{-in\theta} + \\ + \left(b_n'' + \tfrac{b_n'}{r} - \tfrac{(n+d)^2}{r^2}b_n + \tfrac{b_n}{\varepsilon^2}(1-2f^2) - \tfrac{a_n}{\varepsilon^2}f^2\right)e^{in\theta}; \end{gathered}$$

$$\begin{gathered} \mathcal{L}_\varepsilon : (ia_n e^{-in\theta} + ib_n e^{in\theta}) \mapsto \\ i\left(a_n'' + \tfrac{a_n'}{r} - \tfrac{(n-d)^2}{r^2}a_n + \tfrac{a_n}{\varepsilon^2}(1-2f^2) + \tfrac{b_n}{\varepsilon^2}f^2\right)e^{-in\theta} + \\ + i\left(b_n'' + \tfrac{b_n'}{r} - \tfrac{(n+d)^2}{r^2}b_n + \tfrac{b_n}{\varepsilon^2}(1-2f^2) + \tfrac{a_n}{\varepsilon^2}f^2\right)e^{in\theta}; \end{gathered}$$

$$\begin{gathered} \text{and, for } n=0, \quad \mathcal{L}_\varepsilon : ia_0 \mapsto i(a_0'' + \tfrac{a_0'}{r} - \tfrac{d^2}{r^2}a_0 + \tfrac{a_0}{\varepsilon^2}(1-f^2)); \\ \mathcal{L}_\varepsilon : a_0 \mapsto a_0'' + \tfrac{a_0'}{r} - \tfrac{d^2}{r^2}a_0 + \tfrac{a_0}{\varepsilon^2}(1-f^2) - \tfrac{2a_0}{\varepsilon^2}f^2. \end{gathered} \tag{1.9}$$

Now, let us define

$$\omega_{\mathcal{R}} := \sum_{n\geq 1}(\mathcal{R}a_n(r)e^{-in\theta} + \mathcal{R}b_n(r)e^{in\theta}) + \mathcal{R}a_0(r)$$

and

$$\omega_{\mathcal{I}} := \sum_{n\geq 1}(\mathcal{I}a_n(r)e^{-in\theta} + \mathcal{I}b_n(r)e^{in\theta}) + \mathcal{I}a_0(r)$$

where, for any $a \in \mathbb{C}$, $\mathcal{R}a$ is the real part and $\mathcal{I}a$ is the imaginary part of $a$.
We denote $\omega_n = \omega_{n,\mathcal{R}} + i\omega_{n,\mathcal{I}}$. From the equation (1.3) we are led to the equations, for each $n \in \mathbb{N}$,

$$\left\{\begin{array}{l} \mathcal{L}_\varepsilon(\omega_{n,\mathcal{R}}) = h_{n,\mathcal{R}} \\ \mathcal{L}_\varepsilon(i\omega_{n,\mathcal{I}}) = ih_{n,\mathcal{I}} \end{array}\right. \tag{1.10}$$

and, in view of (1.9), when $n \geq 1$, (1.10) gives two second order ordinary differential systems, the both of them with two equations, and with two real valued unknown functions, $a$ and $b$. For $n = 0$, we have two ordinary second order ordinary differential equations, each of them with one real valued unknown function.
In what follows in this section, unlike in Theorem 1.2, $a$ and $b$ or $a_n$ and $b_n$ will be real valued functions.
The first works on the question, quoted above, are considering the following eigenvalue problem, for $r \in [0, 1]$ and for $n = 0$

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a + \frac{1}{\varepsilon^2}(1 - f^2)a = -\lambda(\varepsilon)a, \quad a(1) = 0 \tag{1.11}$$

and

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - 2af^2 + \frac{1}{\varepsilon^2}(1 - f^2)a = -\lambda(\varepsilon)a, \quad a(1) = 0. \tag{1.12}$$

And for $r \in [0, 1]$ and for $n \geq 1$

$$\left\{\begin{array}{rl} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - \frac{1}{\varepsilon^2}f^2b + \frac{1}{\varepsilon^2}(1 - 2f^2)a & = -\lambda(\varepsilon)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - \frac{1}{\varepsilon^2}f^2a + \frac{1}{\varepsilon^2}(1 - 2f^2)b & = -\lambda(\varepsilon)b \\ a(1) = b(1) = 0 & \end{array}\right. \tag{1.13}$$

The question of whether there exist some eigenvalues such that $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$ is related to the question of the existence of bounded solutions in $[0, +\infty[$ for the following system, which is a rescaled form of (1.13), but with the interval $[0, +\infty[$ instead of $[0, \frac{1}{\varepsilon}]$ for $n \geq 1$

$$\left\{\begin{array}{rl} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - f_d^2b + (1 - 2f_d^2)a & = 0 \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - f_d^2a + (1 - 2f_d^2)b & = 0. \end{array}\right. \tag{1.14}$$

And for $n = 0$

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a + (1 - f_d^2)a = 0 \tag{1.15}$$

and

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - 2af_d^2 + (1 - f_d^2)a = 0. \tag{1.16}$$

A bounded solution means that $a$ and $b$ are defined at 0 and that the both functions have finite limits at $+\infty$.

Since the 90', some results on the behaviors of the eigenvalues $\lambda(\varepsilon)$, for $n = 0$ and $n = 1$ are known and also for $d = 1$ and all $n$.
And the following Theorem was proved in [8]

**Theorem 1.3** *For $d \geq 1$ and $n \geq 2$, If there are no bounded solution $(a, b)$ of (1.14), then $|\lambda(\varepsilon)| > C$, for some $C > 0$ independent of $\varepsilon$ and for every eigenvalue $\lambda(\varepsilon)$ of the problem (1.13).*

In [1], we proved that there are no bounded solution of (1.14) when $n \geq 2$.
We claim that, making use of Theorem 1.1, we are able to give a completely different proof of Theorem 1.3. But neither this technique nor that of [8] permits to obtain $|\varepsilon^2\lambda(\varepsilon)| \geq C$.
We associate the problem of the existence of bounded solutions of (1.14) with the following eigenvalue problem, instead of (1.13), $a$ and $b$ being real valued. For $r \in [0,1]$ and for $n \geq 1$

$$\left\{ \begin{array}{rl} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - \frac{1}{\varepsilon^2}f^2a - \frac{1}{\varepsilon^2}f^2b & = -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - \frac{1}{\varepsilon^2}f^2b - \frac{1}{\varepsilon^2}f^2a & = -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)b \\ a(1) = b(1) = 0 & \end{array} \right. \tag{1.17}$$

and for $r \in [0,1]$ and for $n = 0$

$$\left\{ \begin{array}{rl} a'' + \frac{a'}{r} - \frac{d^2}{r^2}a & = -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)a \\ a(1) = 0. & \end{array} \right. \tag{1.18}$$

and

$$\left\{ \begin{array}{rl} a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - \frac{2}{\varepsilon^2}f^2a & = -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)a \\ a(1) = 0. & \end{array} \right. \tag{1.19}$$

The more important advantage of using the eigenvalue problem (1.17) instead of (1.13) is to have in the right hand side a behavior $1/r^2$ as $r \to +\infty$. And we consider the problem $|1 - \mu(\varepsilon)| \geq C$ instead of the insolved problem $\varepsilon^2|\lambda(\varepsilon)| \geq C$.
Let us remark that a result similar to Theorem 1.3 together with Theorem 1.1 gives the following result (that we proved in [1]) :
For $d \geq 1$, for $n \geq 2$ and for any eigenvalue $\mu(\varepsilon)$ we have

$$|\frac{\mu(\varepsilon) - 1}{\varepsilon^2}| \geq C$$

for some $C$ independent of $\varepsilon$.
In the present paper, we will see that actually $|1 - \mu(\varepsilon)| \geq C$.

Let us turn first to the cases of the first eigenvalues for $n = 0$ and for $n = 1$. Let $m_{n,d}$ be defined in (1.21) and (1.22). We will prove the following properties, that are almost the analogue properties of the eigenvalues $\lambda_{n,d}(\varepsilon)$, $n = 0$ and $n = 1$, but the proofs have to be adapted. For all $d \geq 1$,

**Theorem 1.4** *(i) there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, $m_{0,d}(\varepsilon) \geq 1 + \varepsilon^2$; $m_{0,d}(\varepsilon) \to 1$, as $\varepsilon \to 0$.*
*There exists an associated eigenvector $i\Phi_0$, $\Phi_0$ being a positive solution of (1.18) with $m_{0,d}$ in place of $\mu$, such that $(\Phi_0 - f)(r) \to 0$ as $\varepsilon \to 0$, uniformly in all $[0,R]$, $R > 0$. Moreover, $\Phi_0 \leq f$. And for the problem (1.18), $\mu(\varepsilon) - 1 \geq M$, for any eigenvalue, except if $\mu = m_{0,d}$, for some $M$ independent of $\varepsilon$.*
*(ii) For $n = 0$ and for the problem (1.19), $\mu(\varepsilon) - 1 \geq C$, for every eigenvalue.*
*(iii) $m_{1,d}(\varepsilon) > 1$ and $(m_{1,d}(\varepsilon) - 1) \leq M\varepsilon^2$, with $M$ independent of $\varepsilon$.*
*(iv) $m_{n,d}(\varepsilon) \geq m_{1,d}(\varepsilon)$ for all $n \geq 2d - 1$.*

And now

**Theorem 1.5** *(i) For all $n \geq 2$, there exists $C > 0$ independent of $\varepsilon$ and there exists $\varepsilon_0$ such that $|1 - \mu(\varepsilon)| \geq C$ for every eigenvalue $\mu(\varepsilon)$ for the problem (1.20) and for every $\varepsilon < \varepsilon_0$.*
*(ii) For $n = 1$, $\mu(\varepsilon) - 1 \geq C$, except if $\mu = m_{1,d}$.*
*(iii) We can chose the same $C$ and the same $\varepsilon_0$ for all $n \in \mathbb{N}^\star$ and for all eigenvalue $\mu(\varepsilon)$.*
*(iv) For $n = 1$, there exists two functions $\Phi_1 = ae^{-i\theta} + be^{i\theta}$ and $\Phi_2 = ae^{-i\theta} - be^{i\theta}$ of $\mathcal{H}$, where $a$ and $b$ are real valued functions and associated to the eigenvalue $m_{1,d}(\varepsilon)$, ie $\mathcal{L}_\varepsilon(\Phi_1 + i\Phi_2) = (1 - m_{1,d})\mathcal{C}(\Phi_1 + i\Phi_2)$ and such that $\|e^{id\theta}\tilde{\Phi}_j - \frac{\partial \tilde{u}_0}{\partial x_j}\|_{H^1(B(0,1/\epsilon))} \to 0$, as $\varepsilon \to 0$, for $j = 1, 2$, where we use the notation $\tilde{\Phi}_j(x) = \Phi_j(x/\varepsilon)$.*

On the other hand, we have some converse to Theorem 1.5.

**Lemma 1.1** *Let $n \geq 1$. If $m_{n,d}(\varepsilon) \geq 1$ and if there exists some bounded solution $(a, b)$ of (1.14), then $m_{n,d}(\varepsilon) - 1 \leq M\varepsilon^{2n}$, with $M$ independent of $\varepsilon$.*

Now let us recall some notation for the eigenvalue problem (1.17).
We denote by $\mathcal{H}'$ the dual space of $\mathcal{H}$.
Let us define the embedding

$$\begin{array}{rl} I: & \mathcal{H} \to \mathcal{H}' \\ & \omega \mapsto (\eta \mapsto \int_0^1 r\omega\overline{\eta}dr) \end{array}$$

Since the embedding $H_0^1(B(0,1)) \times H_0^1(B(0,1)) \subset L^2(B(0,1)) \times L^2(B(0,1))$ is compact, then $I$ is a compact operator.
Let us define $\mathcal{C} = \frac{1}{\varepsilon^2}(1 - f^2)$. We define the operator $\mathcal{T}$ by

$$\mathcal{L}_\varepsilon = -\mathcal{T} + \mathcal{C}I.$$

The quantity

$$< \mathcal{T}\omega, \eta >_{\mathcal{H}',\mathcal{H}}$$

is the scalar product on $\mathcal{H}$, defined in the beginning of the present part. So, $\mathcal{T}$ is an isomorphism, by the Riesz Theorem.
Since $\mathcal{C}I$ is a compact operator and thanks to the continuity of $\mathcal{T}^{-1}$, then $\mathcal{T}^{-1}\mathcal{C}$ is a compact operator from $\mathcal{H}$ into itself, whose eigenvalues are denoted by $1/\mu$. We use a notion of $\mathcal{C}$-eigenvalue, as in [3]. It is classical that $\mu \in \mathbb{R}_+^\star$.
For $n \geq 1$, $\omega_{n,\mathcal{R}}$ and $\omega_{n,\mathcal{I}}$ belong to the set

$$\mathcal{H}_{n,d} = \{ae^{-in\theta} + be^{in\theta} \in \mathcal{H}, (a, b) : [0, 1] \mapsto \mathbb{R} \times \mathbb{R}\}.$$

and $\omega_{0,\mathcal{R}}$ and $\omega_{0,\mathcal{I}}$ belong to

$$\mathcal{H}_{0,d} = \{a \in \mathcal{H}; a : [0, 1] \mapsto \mathbb{R}\}.$$

If $ae^{-ind\theta} + be^{ind\theta} \in \mathcal{H}_{n,d}$, with $n \geq 1$, we have

$$< \mathcal{T}(ae^{-ind\theta} + be^{ind\theta}), ue^{-ind\theta} + ve^{ind\theta} >_{\mathcal{H}'_{n,d},\mathcal{H}_{n,d}}$$

$$= \frac{1}{2\pi}\int_{B(0,1)} (\nabla(e^{i(n-d)\theta}a).\nabla(e^{-i(n-d)\theta}u) + \nabla(e^{i(n+d)\theta}b).\nabla(e^{-i(n+d)\theta}v) +$$

$$+\frac{1}{\varepsilon^2}f^2(a+b)(u+v))dx$$

$$= \int_0^1 (ra'u' + rb'v' + \frac{(n-d)^2}{r}au + \frac{(n+d)^2}{r}bv + \frac{r}{\varepsilon^2}f^2(a+b)(u+v))dr.$$

And, for $a \in \mathcal{H}_{0,d}$, we have

$$\mathcal{T}(a) = -a'' - \frac{a'}{r} + \frac{d^2}{r^2}a + \frac{2a}{\varepsilon^2}f^2,$$

and

$$\mathcal{T}(ia) = i(-a'' - \frac{a'}{r} + \frac{d^2}{r^2}a).$$

We can use the following form of the system (1.17)

$$\begin{cases} \mathcal{L}_\varepsilon(\omega) = (1-\mu(\varepsilon))\mathcal{C}\omega \\ \\ \omega = ae^{-in\theta} + be^{in\theta} \in \mathcal{H}_{n,d}. \end{cases} \tag{1.20}$$

It is clear that every eigenvalue in some $\mathcal{H}_{n,d}$, $n \geq 0$ is an eigenvalue in $\mathcal{H}$. Conversely, in view of (1.10), for every eigenvalue in $\mathcal{H}$, there exists an eigenvector in $\oplus_{n\geq 0}\mathcal{H}_{n,d}$ or in $\oplus_{n\geq 0} i\mathcal{H}_{n,d}$.
In the first case, there is an eigenvector in at least a subspace $\mathcal{H}_{n,d}$, for some $n \geq 0$ and in the second case, there exists an eigenvector in some $i\mathcal{H}_{n,d}$, for at least some $n \geq 0$. In the second case, if $n \geq 1$, by (1.9), if $a$ and $b$ are real valued, we have

$$\mathcal{L}_\varepsilon(iae^{-in\theta} + ibe^{in\theta}) = i(\alpha e^{-in\theta} - \beta e^{in\theta})$$

where $\alpha$ and $\beta$ are defined by

$$\mathcal{L}_\varepsilon(ae^{-in\theta} - be^{in\theta}) = \alpha e^{-in\theta} + \beta e^{in\theta}.$$

Thus we have

$$\mathcal{L}_\varepsilon(iae^{-in\theta} + ibe^{in\theta}) = (1-\mu(\varepsilon))\mathcal{C}(iae^{-in\theta} + ibe^{in\theta})$$

$$\Leftrightarrow$$

$$\alpha e^{-in\theta} - \beta e^{in\theta} = (1-\mu(\varepsilon))\mathcal{C}(ae^{-in\theta} + be^{in\theta}).$$

But this is equivalent to

$$\alpha e^{-in\theta} + \beta e^{in\theta} = (1-\mu(\varepsilon))\mathcal{C}(ae^{-in\theta} - be^{in\theta}).$$

Finally, we have proved that

$$\left(\mathcal{L}_\varepsilon(iae^{-in\theta} + ibe^{in\theta}) = (1-\mu(\varepsilon))\mathcal{C}(iae^{-in\theta} + ibe^{in\theta})\right)$$

$$\Leftrightarrow \left(\mathcal{L}_\varepsilon(ae^{-in\theta} - be^{in\theta}) = (1-\mu(\varepsilon))\mathcal{C}(ae^{-in\theta} - be^{in\theta})\right).$$

And we conclude that the set of the eigenvalues for the problem in $\mathcal{H}$

$$\mathcal{L}_\varepsilon \omega = (1-\mu(\varepsilon))\mathcal{C}\omega$$

is the reunion of the sets of the eigenvalues for the eigenvalue problems in $\mathcal{H}_{n,d}$ for $n \geq 0$ and in $i\mathcal{H}_{0,d}$.

Last, we define $m_{n,d}(\varepsilon)$ as the first eigenvalue for the above eigenvalue problem (1.17) in $\mathcal{H}_{n,d}$, where we denote $\omega = ae^{-in\theta} + be^{in\theta}$, that is

$$m_{n,d}(\varepsilon) = \inf_{\omega \in \mathcal{H}_{n,d}\setminus\{0\}} \frac{\int_0^1 (ra'^2 + rb'^2 + \frac{(n-d)^2}{r}a^2 + \frac{(n+d)^2}{r}b^2 + \frac{r}{\varepsilon^2} f_d^2(\frac{r}{\varepsilon})(a+b)^2)dr}{\frac{1}{\varepsilon^2}\int_0^1 r(1-f_d^2(\frac{r}{\varepsilon}))(a^2+b^2)dr} \tag{1.21}$$

and $m_{0,d}(\varepsilon)$ is the first eigenvalue for the problem (1.18) in $\mathcal{H}_{0,d}$,

$$m_{0,d}(\varepsilon) = \inf_{a \in \mathcal{H}_{0,d}\setminus\{0\}} \frac{\int_0^1 (ra'^2 + \frac{d^2}{r}a^2)dr}{\frac{1}{\varepsilon^2}\int_0^1 r(1-f_d^2(\frac{r}{\varepsilon}))a^2 dr} \tag{1.22}$$

and $\tilde{m}_{0,d}(\varepsilon)$ is the first eigenvalue for the problem (1.19) in $\mathcal{H}_{0,d}$,

$$\tilde{m}_{0,d}(\varepsilon) = \inf_{a \in \mathcal{H}_{0,d}\setminus\{0\}} \frac{\int_0^1 (ra'^2 + \frac{d^2}{r}a^2 + 2rf_d^2(\frac{r}{\varepsilon})a^2)dr}{\frac{1}{\varepsilon^2}\int_0^1 r(1-f_d^2(\frac{r}{\varepsilon}))a^2 dr}. \tag{1.23}$$

It is classical that these infimum are attained. Since, for $n \geq 1$, the quantity to minimize decreases if we replace $(a,b)$ by $(\max\{|a|,|b|\}, -\min\{|a|,|b|\})$, we have some infimum such that $a \geq -b \geq 0$. And if $n = 0$, we have some infimum such that $a \geq 0$. More, considering the rescaling $(\tilde{a},\tilde{b})(r) = (a(\varepsilon r), b(\varepsilon r))$ and an extension by 0 outside $[0, 1/\varepsilon]$, we see that $\varepsilon \mapsto m_{n,d}(\varepsilon)$ decreases when $\varepsilon$ decreases. Thus $\lim_{\varepsilon\to 0} m_{n,d}(\varepsilon)$ exists.

The principal objective of the paper is the proof of Theorem 1.2. But the essential tool is the result about the eigenvalues $\mu(\varepsilon)$, in Theorem 1.5 and Theorem 1.4. We have to prove that the eigenvalues stay away from 1, as $\varepsilon$ tends to 0, with some exceptions when $n = 0$ and $n = 1$. Let us present the idea of the proof. We consider the system (1.17) involving $\mu(\varepsilon)$. Using the rescaling $(a,b)(r) = (\tilde{a},\tilde{b})(\varepsilon r)$, for $r \in ]0, \frac{1}{\varepsilon}]$ and writing $(a,b)$ instead of $(\tilde{a},\tilde{b})$ we obtain the following system for $r \in ]0, \frac{1}{\varepsilon}]$

$$\left\{\begin{array}{rl} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - f_d^2 b - f_d^2 a & = -\mu(\varepsilon)(1-f_d^2)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - f_d^2 a - f_d^2 b & = -\mu(\varepsilon)(1-f_d^2)b \\ a(\frac{1}{\varepsilon}) = b(\frac{1}{\varepsilon}) = 0. & \end{array}\right. \tag{1.24}$$

Now, we consider the following system in $]0, +\infty[$, where $\mu$ is a real parameter.

$$\left\{\begin{array}{rl} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - f_d^2 b - f_d^2 a & = -\mu(1-f_d^2)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - f_d^2 a - f_d^2 b & = -\mu(1-f_d^2)b \end{array}\right. \tag{1.25}$$

and we consider the system (1.25) as a perturbation of the system (1.14), with coefficients depending continuously of the real parameter $\mu$. When $\mu \to 1$, the coefficients of (1.25)

tend to the coefficients of (1.14). Now, it is more clear in order to apply the principles of the Cauchy-Lipschitz theory for the linear differential equations to reformulate the system (1.25) as a differential system of degree one, ie we define the vector

$$X = (a, ra', b, rb')^t$$

and we rewrite the system (1.25) as

$$X' = MX - (1-\mu)(1-f_d^2)(0, a, 0, b)^t \tag{1.26}$$

where $M$ is the matrix

$$M = \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 \\ -r(1-2f_d^2) + \frac{(n-d)^2}{r} & 0 & rf_d^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ rf_d^2 & 0 & -r(1-2f_d^2) + \frac{(n+d)^2}{r} & 0 \end{pmatrix}.$$

A supposed well known principle, from the Cauchy Lipshitz Theory, is that if $R > 0$ is given and if the initial Cauchy data at $R$, ie $(a(R), ra'(R), b(R), rb'(R))^t$ is a given vector, that depends continuously on the parameter $\mu$, then we can conclude that the vector $(a(r), ra'(r), b(r), rb'(r))^t$ depends continuously on $\mu$, for all $r > 0$ and also that the continuity in $\mu$ is uniform in all $[r_0, r_1]$, $0 < r_0 < r_1$, in the sense that if $\mu \to \mu_0$, then $(a, ra', b, rb')$ tends to a limit in $(L^\infty([r_0, r_1]))^4$.

In Part II, we give the proof of Theorem 1.4.

In Part III, we construct a base of four solutions of the system (1.25), determined by they behaviors near 0. Two of them are well defined at 0 and two of them blow up at 0. We prove the continuity of these four solutions wrt $\mu$, uniformly in all $[r_0, r_1]$, $0 < r_0 < r_1$. And we prove that when $\mu \to 1$ the four solutions tend to a base of four solutions of (1.14), determined by they behaviors near 0. The method is to construct the four solutions in a determined interval $]0, R]$ where $R$ is proved to be independent of $\mu$. By the Cauchy Lipschitz Theory, a solution defined in $]0, R]$ has a unique extension by a solution defined in $]0, +\infty[$. And by the principle recalled above, if the vector $(a, ra', b, rb')^t(r)$ depends continuously on the parameter $\mu$ for all $r \in ]0, R]$ then its extension to $]0, +\infty[$, that we denote also by $(a, ra', b, rb')^t$, depends also continuously on the parameter $\mu$, uniformly in all $[r_0, r_1]$, $0 < r_0 < r_1$. We use the same principle to construct a base of solutions of (1.25) determined by they behaviors at $+\infty$. Two of them are bounded as $r \to +\infty$ and two of them blow up as $r \to +\infty$. We construct them in an interval $[R, +\infty[$, where $R$ is proved to be independent of the parameter $\mu$, and we prove that the four solutions depend continuously on $\mu$ in this interval. Following the same principle as above, the four solutions have extensions in $]0, +\infty[$, that depend continuously on $\mu$. And finally, the both bases tend, as $\mu$ tends to 1, to bases of solutions of (1.14), whose behaviors at 0 or at $+\infty$ are determined.

In Part IV, we prove Theorem 1.5. We will use the results of Part III and two results proved in [1], that are :
(i) there are no bounded solutions for the system (1.14);
(ii) the solution of (1.14) that have the most vanishing behavior at 0 is exponentially

blowing up at $+\infty$ and the solution of (1.14) that has an exponentially vanishing behavior at $+\infty$ has the most blowing up behavior at 0.
The proof in Part IV is the following : using Part III and (ii), we prove that if there exists some eigenvalue $\mu(\varepsilon)$ tending to 1, then the associated eigenspace is one dimensional and an associated eigenvector tends to a bounded solution of (1.14). Further, (i) permits to conclude the proof of Theorem 1.5.

In Part V, we prove Theorem 1.2.

In Part VI, we prove Lemma 1.1.

And we will use the expansions of $f_d$, proved in [6].

$$f_d(r) = 1 - \frac{d^2}{2r^2} + O(\frac{1}{r^4}) \text{ near } +\infty \tag{1.27}$$

and

$$f_d(r) = A_d(r^d - \frac{1}{4(d+1)} r^{d+2}) + O(r^{d+4}) \text{ near } 0. \tag{1.28}$$

Morover, $f_d$ increases in $[0, +\infty[$ and

$$0 \leq f_d \leq 1 \quad \text{in } [0, +\infty[ \ .$$

## 2 The proof of Theorem 1.4.

**(i).** Proof of $m_{0,d}(\varepsilon) \to 1$, of $(m_{0,d}(\varepsilon) - 1) \geq \varepsilon^2$, of the existence of an eigenvector $a$ that tends to $f_d$ uniformly in each $[0, R]$, $R > 0$, that $a \leq f_d$ and that $\mu(\varepsilon) - 1 \geq C$, for any other eigenvalue.
Using the Euler equation of the infimum problem (1.22), we have

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2} a + \frac{1}{\varepsilon^2}(1 - f^2)a = -\frac{m_{0,d}(\varepsilon) - 1}{\varepsilon^2}(1 - f^2)a \tag{2.29}$$

where $r \in [0, 1]$, $f(r) = f_d(\frac{r}{\varepsilon})$ and $a(r) \geq 0$ and $a(1) = 0$.
Firstly, Multiplying the equation (2.29), by $tf$ and the equation of $f$ by $ta$ and integrating by parts on $[0, r]$, we find

$$[rf'a - ra'f]_0^r = (m_{0,d}(\varepsilon) - 1)/\varepsilon^2 \int_0^r t(1 - f^2)afdt.$$

Firstly, we let $r = 1$. Since $a > 0$ and since $a(1) = 0$, we have $a'(1) < 0$ and we are led to $(m_{0,d}(\varepsilon) - 1) > 0$.
Secondly, we deduce that for all $0 < r \leq 1$, $-rf^2(\frac{a}{f})' > 0$. And consequently, $af^{-1}$ decreases in $[0, 1]$. But we can choose $a(0) = f(0) = 1$. This proves that $af^{-1} \leq 1$ in $[0, 1]$.
Further, let us use a truncature of $f$ as a test function for the infimum $m_{0,d}(\varepsilon)$. A real number $0 < N < 1$ being given, we let

$$a^{cut} = f_d \quad \text{in } [0, \tfrac{N}{\varepsilon}], \qquad a^{cut} = f_d(\tfrac{N}{\varepsilon})h \quad \text{in } [\tfrac{N}{\varepsilon}, \tfrac{1}{\varepsilon}]$$

where

$$h(r) = ee^{-u}, \qquad u = (\tfrac{N}{\varepsilon} - \tfrac{1}{\varepsilon})/(r - \tfrac{1}{\varepsilon}).$$

Using the rescaling $\tilde{a}^{cut}(r) = a^{cut}(r/\varepsilon)$, we have $\tilde{a}^{cut} \in \mathcal{H}_{0,d}(B(0,1))$. Consequently

$$0 < m_{0,d}(\varepsilon) - 1 \leq \frac{\int_0^{\frac{1}{\varepsilon}} \left( r(a^{cut})'^2 + \frac{d^2}{r}(a^{cut})^2 - r(1-f_d^2)(a^{cut})^2 \right) dr}{\int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)(a^{cut})^2 dr}.$$

In what follows in this part, the notation $M$ means some positive real number independent of $\varepsilon$

Now, the expansion of $f_d$ at $+\infty$ gives some $M$ and some $R > 0$ such that

$$\text{for } r \geq R, \quad |1 - f_d^2 + \frac{d^2}{r^2}| \leq Mr^{-4}.$$

And $a^{cut}$ being a bounded function, we deduce that

$$|\int_0^{\frac{1}{\varepsilon}} \left( \frac{d^2}{r}(a^{cut})^2 - r(1-f_d^2)(a^{cut})^2 \right) dr| \leq M.$$

Now, we have to estimate

$$I_1 = \int_0^{\frac{1}{\varepsilon}} r(a^{cut})'^2 dr \quad \text{and} \quad I_2 = \int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)(a^{cut})^2 dr.$$

We have

$$I_1 = \int_0^{\frac{N}{\varepsilon}} r(f_d')^2 dr + f^2(\frac{N}{\varepsilon}) \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(h')^2 dr.$$

But there exists $M$ and $R$ such that for $r \geq R$

$$0 < f_d' \leq \frac{M}{r^3}.$$

We deduce that

$$\int_0^{\frac{N}{\varepsilon}} r(f_d')^2 dr \leq M.$$

Now, let us estimate $\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(h')^2 dr$. Since we denote $u = (\frac{N}{\varepsilon} - \frac{1}{\varepsilon})/(r - \frac{1}{\varepsilon})$. We have

$$u' = -u^2\varepsilon(N-1)^{-1} \quad \text{and} \quad h' = -u'h.$$

We write

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(h')^2 dr \leq \frac{1}{\varepsilon} \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} u'h^2u'dr,$$

but

$$\frac{1}{\varepsilon} \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} u'h^2u'dr = \frac{e^2}{\varepsilon} \int_1^{+\infty} u^2\varepsilon(N-1)^{-1}e^{-2u}du.$$

We have proved that

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(h')^2 dr \leq M \tag{2.30}$$

and since $f_d^2(\frac{N}{\varepsilon}) \to 1$ as $\varepsilon \to 0$, we conclude that

$$I_1 \leq M.$$

Now

$$I_2 \geq \int_0^{\frac{N}{\varepsilon}} r(1-f_d^2)f_d^2 dr.$$

But there exists $R > 0$ and $M > 0$ such that for all $r \geq R$

$$(1-f_d^2)f_d^2 \geq M/r^2.$$

We deduce that

$$I_2 \geq M \ln(\frac{N}{\varepsilon}).$$

Finally

$$0 \leq m_{0,d}(\varepsilon) - 1 \leq M \ln^{-1}(\frac{N}{\varepsilon}).$$

This gives $m_{0,d}(\varepsilon) \to 1$.
Secondly, let us prove the existence of a solution $a$ tending to $f_d$ that spans the eigenspace associated to $m_{0,d}(\varepsilon)$.
We are looking for a solution of the rescaled equation of (2.29)

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a + (1-f_d^2)a = -(m_{0,d}(\varepsilon)-1)(1-f_d^2)a \tag{2.31}$$

of the form $uf_d$. We write

$$u''f_d + 2u'f_d + \frac{u'f_d'}{r} = -(m_{0,d}(\varepsilon)-1)(1-f_d^2)f_d u,$$

that is

$$(u'(rf_d^2))' = -(m_{0,d}(\varepsilon)-1)r(1-f_d^2)f_d^2 u. \tag{2.32}$$

We define the fixed point problem

$$u = 1 - (m_{0,d}(\varepsilon)-1)\int_0^r \frac{f_d^{-2}}{t}\int_0^t s(1-f_d^2)f_d^2 u ds dt. \tag{2.33}$$

We denote it by $u = \Phi(u)$.
Each solution of this fixed point problem is a solution of (2.32).
We define the sequence $(\alpha_j)_{j\in\mathbb{N}}$ by the induction

$$\alpha_0 = 1 \quad \text{and} \quad \alpha_{k+1} = \Phi(\alpha_k)$$

and we write

$$|\alpha_1 - \alpha_0|(r) \leq (m_{0,d}(\varepsilon)-1)\int_0^r \frac{f_d^{-2}(t)}{t}\int_0^t s(1-f_d^2)f_d^2 ds dt$$

and, using $f_d^{-2}(t) \leq f_d^{-2}(s)$ and $1 - f_d^2 \leq 1$ we get

$$|\alpha_1 - \alpha_0|(r) \leq (m_{0,d}(\varepsilon) - 1)\frac{r^2}{4}.$$

And for all $k \geq 1$,

$$|\alpha_{j+1} - \alpha_j|(r) \leq (m_{0,d}(\varepsilon) - 1)\int_0^r \frac{f_d^{-2}(t)}{t}\int_0^t s f_d^2 ds dt \|\alpha_j - \alpha_{j-1}\|_{L^\infty([0,r])},$$

that gives

$$|\alpha_{j+1} - \alpha_j|(r) \leq (m_{0,d}(\varepsilon) - 1)\frac{r^2}{4}\|\alpha_j - \alpha_{j-1}\|_{L^\infty([0,r])}.$$

Consequently, if $R > 0$ is given, we have for all $0 \leq r \leq R$

$$\|\alpha_{j+1} - \alpha_j\|_{L^\infty([0,R])} \leq (\frac{(m_{0,d}(\varepsilon) - 1)R^2}{4})^{j+1}. \tag{2.34}$$

Since $m_{0,d}(\varepsilon) - 1 \to 0$, then for each $R > 0$, the sum

$$u = \sum_{j\geq 0}(\alpha_{j+1} - \alpha_j) + \alpha_0.$$

is convergent for $\varepsilon$ small enough, depending on $R$. We define $a$ by

$$a(r) = f_d(r) + f_d(r)\sum_{j\geq 0}(\alpha_{j+1} - \alpha_j).$$

It is is a solution of (2.31). Let us remark that if we fix $R > 0$ and if we fix $\varepsilon < \varepsilon(R)$ sufficiently small to define $a$ in $[0, R]$, then the Cauchy-Lipschitz Theory permits to define a prolongation of $a$, that we still name $a$, for all $r \in [0, +\infty[$, as the solution of (2.31) for the initial Cauchy data $(a(R), a'(R))$.
Moreover, due to (2.34), the convergence of the serie $u(r)$, for $r < R$ and for $\varepsilon < \varepsilon(R)$, is uniform wrt the values of the parameter $m_{0,d}(\varepsilon)$. And clearely, for each $k$, $\alpha_k$ depends continuously of the parameter $m_{0,d}(\varepsilon)$ too. For this reason, the sum $u(r)$ depends continously of the parameter $m_{0,d}(\varepsilon)$, for all $r < R$ and $\varepsilon < \varepsilon(R)$. This implies that $a(R)$, depends continously of the parameter $m_{0,d}(\varepsilon)$, for the norm $L^\infty([0, R])$.
Let us prove that $a'(R))$ depends continuously on the parameter $m_{0,d}(\varepsilon)$, too. The derivation of (2.33) gives

$$u'(r) = -(m_{0,d}(\varepsilon) - 1)r^{-1}f_d^{-2}(r)\int_0^r t(1 - f_d^2)f_d^2 u(t)dt.$$

We deduce that $u'$, and consequently $a'$, depend continuously on the parameter $m_{0,d}(\varepsilon)$. Since the Cauchy data $(a(R), a'(R))$ in the initial point $R$, together with the coefficients of the equation depend continously of the parameter $m_{0,d}(\varepsilon)$, so does also the extended solution $a(r)$, together with $a'(r)$, for all $\varepsilon < \varepsilon(R)$ and for all $r \geq 0$.
On the other hand, we can deduce from (2.34) that for all $R > 0$, for all $\varepsilon < \varepsilon(R)$

$$|a - f_d|(R) \leq f_d(R)\sum_{j\geq 0}\frac{1}{1 - K}K$$

with $K = (m_{0,d}(\varepsilon) - 1)R^2/4$. We can chose $\varepsilon(R)$ small enough to have $1/1 - K < 2$, so

$$|a - f_d|(R) \leq f_d(R) \sum_{j \geq 0} \frac{1}{2}(m_{0,d}(\varepsilon) - 1)R^2,$$

for all $R > 0$ and all $\varepsilon < \varepsilon(R)$.
This gives the behavior at 0 of $a$ and this proves again that $\tilde{a}(r) - f_d(r) \to 0$, as $m_{0,d}(\varepsilon) \to 1$, for all $r > 0$ and uniformly in each $[0, R]$.
Now, since the eigenspace associated to the first eigenvector is one dimensional, the $a$ we just constructed is a base of the eigenvector associated to $m_{0,d}(\varepsilon)$.

Now, let us prove that $m_{0,d}(\varepsilon) - 1 \geq M\varepsilon^2$.
Returning to (2.32), where $u = f_d^{-1}a$, we deduce that for all $t \in [0, \frac{1}{\varepsilon}]$,

$$tf_d^2(f_d^{-1}a)'(t) = -(m_{0,d}(\varepsilon) - 1) \int_0^t s(1 - f_d^2) f_d a ds$$

and we use the condition $a(\frac{1}{\varepsilon}) = 0$ to obtain, for $r \in [0, \frac{1}{\varepsilon}]$,

$$f_d^{-1}a(r) = -(m_{0,d}(\varepsilon) - 1) \int_{\frac{1}{\varepsilon}}^r \frac{1}{tf_d^2(t)} \int_0^t s(1 - f_d^2) f_d a ds dt.$$

And using the rescaling, this gives, for all $0 \leq r \leq 1$,

$$f^{-1}\tilde{a}(r) = \frac{m_{0,d}(\varepsilon) - 1}{\varepsilon^2} \int_r^1 \frac{1}{tf^2(t)} \int_0^t s(1 - f^2) f \tilde{a} ds dt.$$

Recalling that $\tilde{a}f^{-1} \leq 1$, we are led to

$$f^{-1}\tilde{a}(r) \leq \frac{m_{0,d}(\varepsilon) - 1}{\varepsilon^2} \int_r^1 \frac{1}{tf^2(t)} \int_0^t s(1 - f^2) f^2 ds dt.$$

But $t \mapsto tf^2(t)$ increases and $1 - f^2 \leq 1$, so we obtain, for all $r \in [0, 1]$,

$$f^{-1}\tilde{a}(r) \leq \frac{m_{0,d}(\varepsilon) - 1}{\varepsilon^2}(1 - r^2)/2.$$

But $f \leq 1$. Consequently, for all $0 \leq r \leq 1$

$$\tilde{a}(r) \leq \frac{m_{0,d}(\varepsilon) - 1}{\varepsilon^2}.$$

But $\tilde{a}(0) = 1$. We deduce that $m_{0,d}(\varepsilon) - 1 \geq \varepsilon^2$.

Now, if $\mu(\varepsilon) > m_{0,d}(\varepsilon)$ let us prove that $\mu(\varepsilon) - 1 \geq M$. Let us suppose that $\mu(\varepsilon)$ tends to 1. We can construct an eigenvector for $\mu(\varepsilon)$, exactly as we constructed $a$, an eigenvector for $m_{0,d}(\varepsilon)$, just above, but with $\mu$ instead of $m_{0,d}(\varepsilon)$. We are led to define

$$a_1 = f_d + f_d \sum_{j \geq 0} (\alpha_{j+1} - \alpha_j)$$

where the sequence $(\alpha_j)_{j\geq 0}$ verifies (2.34) and, letting $K = |1-\mu|R^2/4$, we choose $R > 0$ such that $|1-\mu|R^2/4 < 1$ and $\frac{1}{1-K} < 2$. This gives

$$|a_1(R) - f_d(R)| \leq f_d(R)|\mu - 1|R^2/2$$

and this is true for every $0 < r \leq R$, in the place of $R$.

Now, if $a_2$ is any solution of the equation (2.29), with $\mu(\varepsilon)$ instead of $m_{0,d}(\varepsilon)$, we can combine the equation of $a_1$ and the equation of $a_2$ and integrating by parts we are led to, for all $r_1 > 0$ and $r_2 > 0$

$$[r(a_1'a_2 - a_1a_2')(r)]_{r_1}^{r_2} = 0.$$

We deduce that there exists $C$ and $D$ such that $a_2(r) = Da_1(r) + Ca_1(r)\int_1^r \frac{dt}{ta_1^2(t)}$. We define

$$a_2(r) := a_1(r)\int_1^r \frac{dt}{ta_1^2(t)}.$$

We verify that $a_2$ is a solution and a calculus gives, for all $0 < r \leq R$,

$$|a_2^\varepsilon(r) - A_d^{-1}r^{-d}| \leq Cr^{-d+1}.$$

So, $a_1$ and $a_2$ form a base of solutions for (2.29), for $\mu$ instead of $m_{0,d}$. And in view of the behaviors at 0, this proves that $a_1$ spans the eigenvector space for $\mu$. But, since $\mu(\varepsilon) \to 1$, $\mu$ takes its values in the range of the map $\varepsilon \mapsto m_{0,d}(\varepsilon)$, that is an increasing continuous map. So, for $\varepsilon$ small enough $a_1$ is also an eigenvector for $m_{0,d}$, maybe for another value of $\varepsilon$. Consequently, this eigenvector associated to $\mu(\varepsilon)$ is positive, for $\varepsilon$ small enough. But this is in contradiction with the necessary condition

$$(\mu(\varepsilon) - m_{0,d}(\varepsilon))\int_0^{\frac{1}{\varepsilon}} (1-f_d^2)aa_1 ds = 0,$$

unless $\mu(\varepsilon) - m_{0,d}(\varepsilon) = 0$. We conclude that $\mu(\varepsilon) \geq 1 + M$, or $\mu(\varepsilon) = m_{0,d}(\varepsilon)$.

**(ii).** Let us prove that $\tilde{m}_{0,d}(\varepsilon) \geq 1 + M$, where $\tilde{m}_{0,d}$ is defined in (1.23). It is clear by their definitions that $m_{0,d} < \tilde{m}_{0,d}$.
Now let us consider a solution $b$ of the Euler equation

$$b'' + \frac{b'}{r} - \frac{d^2}{r^2}b + (1-f_d^2)b - 2bf_d^2 = -(\tilde{m}_{0,d}(\varepsilon) - 1)(1-f_d^2)b \tag{2.35}$$

such that its rescaling $\tilde{b}$ realizes $\tilde{m}_{0,d}$. Since $b$ is an eigenvector for a first eigenvalue, it is defined up to a multiplicative constant and we can suppose $b \geq 0$ in $[0, \frac{1}{\varepsilon}]$ and $b(0) = 1$. Since $\tilde{m}_{0,d}(\varepsilon)$ tends to a limit as $\varepsilon \to 0$, then, as for $a$ realizing $m_{0,d}(\varepsilon)$, we can construct $b$ of the form $b = uf_d$, by a fixed point method to solve

$$(u'(rf_d^2))' = -(\tilde{m}_{0,d}(\varepsilon) - 1)r(1-f_d^2)f_d^2u + 2ruf_d^3.$$

We skip the construction, since it is sufficient to follow the construction of $a$, an eigenvector for $m_{0,d}$.

And $b$ tends to a limit $\overline{b}$, as $\varepsilon \to 0$, uniformly in all $[0, R]$, $R > 0$.
Further, combining (2.31) with (2.35) we obtain

$$-2\int_0^{\frac{1}{\varepsilon}} r f_d^2 b a dr = (m_{0,d}(\varepsilon) - \tilde{m}_{0,d}(\varepsilon)) \int_0^{\frac{1}{\varepsilon}} r(1 - f_d^2) b a dr$$

that is

$$\int_0^{\frac{1}{\varepsilon}} r\left(-2f_d^2 + (\tilde{m}_{0,d}(\varepsilon) - m_{0,d}(\varepsilon))(1 - f_d^2)\right) b a dr = 0. \tag{2.36}$$

Now, since $\lim_{r\to+\infty} -2f_d^2 + (1 - f_d^2) = -2$, we chose $r_0 > 0$ such that

$$r > r_0, \qquad -2f_d^2 + (1 - f_d^2) < -1.$$

Using (2.36) we can write, if $\varepsilon < 1/r_0$,

$$0 < \int_0^{r_0} r\left(-2f_d^2 + (\tilde{m}_{0,d}(\varepsilon) - m_{0,d}(\varepsilon))(1 - f_d^2)\right) b a dr +$$

$$+ \int_{r_0}^{\frac{1}{\varepsilon}} r\left(-1 + (1 - f_d^2)(-1 + (\tilde{m}_{0,d}(\varepsilon) - m_{0,d}(\varepsilon))\right) b a dr.$$

Now, if we suppose that $(\tilde{m}_{0,d}(\varepsilon) - m_{0,d}(\varepsilon)) \to 0$, we choose $\varepsilon_0$ such that for all $\varepsilon < \varepsilon_0$, $(-1 + (\tilde{m}_{0,d}(\varepsilon) - m_{0,d}(\varepsilon)) < 0$. Consequently, the second integral is negative for $\varepsilon < \varepsilon_0$. But the first integral tends to the limit $\int_0^{r_0} r(-2f_d^2)\overline{b} f_d dr$, so, for $\varepsilon$ small enough, the second integral is negative, too.
This contradiction proves that $\tilde{m}_{0,d}(\varepsilon) \geq 1 + M$.

**(iii).** Proof of $m_{1,d}(\varepsilon) > 1$. The proof is almost the same as for $\lambda_{1,d} > 0$, in [10].
Let $(\tilde{a}, \tilde{b})$, $a \geq -b \geq 0$ that realizes $m_{1,d}(\varepsilon)$. We write $x = a + b$ and $y = a - b$. The system (1.17), gives, for $n = 1$ and $r \in [0, 1/\varepsilon]$

$$\begin{cases} x'' + \frac{x'}{r} - \frac{1+d^2}{r^2}x + \frac{2d}{r^2}y - 2f_d^2 x &= -m_{1,d}(1 - f_d^2)x \\ y'' + \frac{y'}{r} - \frac{1+d^2}{r^2}y + \frac{2d}{r^2}x &= -m_{1,d}(1 - f_d^2)y \\ x(\frac{1}{\varepsilon}) = y(\frac{1}{\varepsilon}) &= 0. \end{cases} \tag{2.37}$$

But if we let

$$x_0 = f_d' \quad \text{and } y_0 = df_d/r$$

then $x_0$ and $y_0$ are solutions of

$$\begin{cases} x'' + \frac{x'}{r} - \frac{1+d^2}{r^2}x + \frac{2d}{r^2}y - 2f_d^2 x &= -(1 - f_d^2)x \\ y'' + \frac{y'}{r} - \frac{1+d^2}{r^2}y + \frac{2d}{r^2}x &= -(1 - f_d^2)y. \end{cases} \tag{2.38}$$

We multiply the first equation of (2.37) by $rx_0$ and we multiply the first equation of (2.38) by $rx$ and we integrate the difference in $[0, 1/\varepsilon]$. We do the same thing with the

second equations of (2.37) and of (2.38). And we sum the second result and the first one. We obtain

$$\big[r(x'x_0 - x'_0x + y'x_0 - x'_0y)\big]_0^{\frac{1}{\varepsilon}} == -(m_{1,d}-1)\int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)(xx_0+yy_0)dr.$$

But $(1-f_d^2)(xx_0+yy_0) > 0$ and

$$\big[r(x'x_0 - x'_0x + y'x_0 - x'_0y)\big]_0^{\frac{1}{\varepsilon}} = \frac{1}{\varepsilon}(x'(\frac{1}{\varepsilon})x_0(\frac{1}{\varepsilon}) + y'(\frac{1}{\varepsilon})y_0(\frac{1}{\varepsilon})) < 0.$$

This proves that $m_{1,d}(\varepsilon) > 1$.

The proof of $m_{1,d}(\varepsilon) - 1 \leq M\varepsilon^2$ follows from Lemma 1.1.

**(iv).** The proof is almost the same as for $\lambda_{n,d} > \lambda_{1,d}$, for $n \geq 2d-1$ in [10]. Let us recall it.
If $ae^{-in\theta} + be^{in\theta}$ realizes $m_{n,d}$ and if $\frac{1}{\varepsilon^2}\int_0^1 r(1-f_d^2)(a^2+b^2)dr = 1$, we write

$$m_{n,d}(\varepsilon) \geq m_{1,d}(\varepsilon) + \int_0^1 (\frac{(d-n)^2-(d-1)^2}{r}a^2 + \frac{(d+n)^2-(d+1)^2}{r}b^2)dr$$

and since $a \geq -b \geq 0$, this gives, provided that $(d-n)^2-(d-1)^2 \geq 0$, ie $2d-n-1 \leq 0$,

$$m_{n,d}(\varepsilon) \geq m_{1,d}(\varepsilon) + 2\int_0^1 \frac{n^2-1}{r}b^2dr,$$

that gives $m_{n,d} \geq m_{1,d}$.

This terminates the proof of Theorem 1.4.

# 3 Two bases of solutions for (1.25), depending on the parameter $\mu$.

## 3.1 A base defined near 0.

In all this part, $d \geq 1$ and $n \geq 1$. The propositions below give a base of 4 solutions of the system (1.25), involving the parameter $\mu$, provided $\mu \leq k_0$, for some given $k_0 > 1$. These solutions are determined by they behaviors at 0. The results are true also if $\mu = 1$ and in this case, we have a base of solutions for the system (1.14). In the propositions below, we let the condition on $\mu$ be $|\mu - 1| \leq 1$, to be more clear.

**Proposition 3.1** *Let us suppose that $|\mu-1| \leq 1$. There exist 2 independent solutions of (1.25) defined at 0, that we denote by $(a_1^\mu, b_1^\mu)$ and $(a_3^\mu, b_3^\mu)$ and they are such that there exists some $R > 0$ independent of the parameter $\mu$ and some constant $M$ depending only on $R$, such that for all $r \in [0,R]$*

$$|a_1^\mu(r) - A_d^2K_1r^{n+3d+2}| \leq Mr^{n+3d+4}, \quad |b_1^\mu(r) - r^{n+d}| \leq Mr^{n+d+2},$$
$$\text{where } K_1 = ((3d+n+2)^2 - |n-d|^2)^{-1} \tag{3.39}$$

*and*

$$|a_3^\mu(r) - r^{|n-d|}| \le Mr^{|n-d|+2}, \quad |b_3^\mu(r) - A_d^2 K_3 r^{|n-d|+2d+2}| \le Mr^{|n-d|+2d+4}, \\ \text{where } K_3 = ((|n-d|+2d+2)^2 - (n+d)^2)^{-1}, \tag{3.40}$$

*$A_d$ being given in (1.28).*

**Proposition 3.2** *There exist 2 independent solutions of (1.25) blowing up at 0, that we denote by $(a_2^\mu, b_2^\mu)$ and $(a_4^\mu, b_4^\mu)$ and they are such that there exists some $R > 0$ independent of the parameter $\mu$ and some constant $M$ also independent of $\mu$ such that for all $r \in [0, R]$*

$$|a_2^\mu| \le Mr^{|n-d|} \quad \text{if } n \le d \quad \text{and} \quad |a_2^\mu| \le Mr^{-|n-d|+2}|\ln r| \quad \text{if } n \ge d+1 \\ |b_2^\mu - r^{-n-d}| \le Mr^{-n-d+2} \tag{3.41}$$

*and*

$$|a_4^\mu(r) - r^{-|n-d|}| \le Mr^{-|n-d|+2}|\ln r|, \quad |b_4^\mu| \le Mr^{n+d-1}, \quad \text{if } n \ne d \text{ and } n \le d-1,$$

$$|a_4^\mu(r) - r^{-|n-d|}| \le Mr^{-|n-d|+2}|\ln r|, \quad |b_4^\mu| \le Mr^{3d-n+2}|\ln r|, \quad \text{if } n \ge d+1,$$

$$|a_4^\mu(r) + \ln r| \le Mr^2|\ln r|, \quad |b_4^\mu - A_d^2 K_4 r^{2d+2}|\ln r|| \le Mr^{2d+4}|\ln r|, \quad \text{if } n = d,$$

$$\text{where } K_4 = ((4d+2)(2d+2))^{-1}. \tag{3.42}$$

**Proposition 3.3** *Denoting $X_i^\mu = (a_i^\mu, r(a_i^\mu)', b_i^\mu, r(b_i^\mu)')^t$, $i = 1, \dots, 4$ the solutions of (1.26), where $(a_i, b_i)$ are defined in Propositions 3.1 and 3.2, we have*

$$\lim_{\mu \to 1} X_i^\mu = X_i, \quad i = 1, \dots, 4,$$

*where the notation $X_i$ is used for $X_i^1$. And the limit is in the sense of the topology of $(L^\infty([r_0, r_1]))^4$, for all $0 < r_0 < r_1$.*

**Proof of Proposition 3.1.** In this part, we first remark that for all $\alpha \in \mathbb{R}$, $r^\alpha$ and $r^{-\alpha}$ are solutions of the equation

$$x'' + \frac{x'}{r} - \frac{\alpha^2}{r^2} x = 0.$$

And we look for solutions $(a, b)$ of (1.25) of the form $a = hr^{\pm|n-d|}$, $b = kr^{\pm(n+d)}$.
To construct $(a_1^\mu, b_1^\mu)$ and $(a_3^\mu, b_3^\mu)$ in some interval $[0, R]$, we use a fixed point problem and we prove that we can choose $R > 0$ independent of the parameter $\mu$, provided that $|\mu - 1| \le 1$. Letting $a = hr^{|n-d|}$ and $b = kr^{n+d}$, we write (1.25) on the form

$$\begin{cases} h'' + \frac{2|n-d|+1}{r} h' - f_d^2 h - f_d^2 k r^{n+d-|n-d|} = -\mu(1 - f_d^2)h \\ k'' + \frac{2(n+d)+1}{r} k' - f_d^2 k - f_d^2 h r^{|n-d|-n-d} = -\mu(1 - f_d^2)k \end{cases} \tag{3.43}$$

and we consider the following integral equation

$$\begin{cases} h &= \alpha_0 + \int_0^r t^{-2|n-d|-1} \int_0^t s^{2|n-d|+1} \left( (f_d^2 - \mu(1-f_d^2))h + f_d^2 k s^{n+d-|n-d|} \right) ds dt \\ k &= \beta_0 + \int_0^r t^{-2(n+d)-1} \int_0^t s^{2(n+d)+1} \left( (f_d^2 - \mu(1-f_d^2))k + f_d^2 h s^{|n-d|-n-d} \right) ds dt \end{cases} \tag{3.44}$$

We define $\Phi(h,k)$ as the rhs of the above integral equations and we have to solve the fixed point problem $(h,k) = \Phi(h,k)$. And we want $(h,k)$ to depend continuously on the parameter $\mu$, in the sense that $(h,h',k,k')(r)$ tends to a limit, as $\mu$ tends to a limit $\mu_0$.

First, let us construct $(a_1^\mu, b_1^\mu)$.
For this purpose, we define two sequences by

$$\begin{cases} \alpha_0 = 0 \quad \beta_0 = 1 \\ (\alpha_{j+1}, \beta_{j+1}) = \Phi(\alpha_j, \beta_j). \end{cases} \tag{3.45}$$

And using (1.28) and supposing $R > 0$ is small enough in order that $f_d^2 \leq A_d^2 s^{2d}$ for $s \in [0,R]$, and since $|f_d^2 + \mu(1-f_d^2)| \leq 3$, we obtain

$$|\alpha_1 - \alpha_0| \leq \frac{A_d^2 r^{3d+n+2-|n-d|}}{(3d+n+2-|n-d|)^2} \quad \text{and} \quad |\beta_1 - \beta_0| \leq \frac{3}{4} r^2 \tag{3.46}$$

and for all $j \geq 1$ and all $0 \leq r \leq R$,

$$|\alpha_{j+1} - \alpha_j|(r) \leq \tfrac{3}{4} r^2 \|\alpha_j - \alpha_{j-1}\|_{L^\infty([0,r])} + \tfrac{A_d^2 r^{3d+n+2-|n-d|}}{(3d+n+2-|n-d|)^2} \|\beta_j - \beta_{j-1}\|_{L^\infty([0,r])}$$

$$\text{and} \tag{3.47}$$

$$|\beta_{j+1} - \beta_j|(r) \leq \tfrac{3}{4} r^2 \|\beta_j - \beta_{j-1}\|_{L^\infty([0,r])} + \tfrac{A_d^2 r^{d-n+2+|n-d|}}{(d-n+2+|n-d|)^2} \|\alpha_j - \alpha_{j-1}\|_{L^\infty([0,r])}.$$

Clearly, by an induction, every $(\alpha_j, \beta_j)$ depends continuously on the parameter $\mu$. Now we remark that, since $n \geq 1$ and $d \geq 1$, we have

$$d - n + |n-d| + 2 \geq 2 \quad \text{and} \quad 3d + n + 2 - |n-d| \geq 2.$$

Letting

$$C = \max\{\{\frac{3}{4}, \frac{A_d^2}{4}\},$$

an induction gives, for $j \geq 0$,

$$\|\alpha_{j+1} - \alpha_j\|_{L^\infty([0,R])} \leq 2^j C^{j+1} R^{3d+n-|n-d|+2(j+1)}$$

$$\text{and} \tag{3.48}$$

$$\|\beta_{j+1} - \beta_j\|_{L^\infty([0,R])} \leq 2^j C^{j+1} R^{2(j+1)}.$$

We chose $R$ small enough to have $2CR^2 < 1/2$ and we may define

$$h_1(r) = \alpha_0 + \sum_{j \geq 0} (\alpha_{j+1} - \alpha_j)(r) \quad \text{and} \quad k_1(r) = \beta_0 + \sum_{j \geq 0} (\beta_{j+1} - \beta_j)(r). \tag{3.49}$$

The estimate (3.48) gives the convergence of the sums, for all $0 < r \leq R$.
Moreover, the convergence of the sums in $L^\infty([0,R])$ is uniform wrt the parameter $\mu$, if $|\mu - 1| \leq 1$. Thanks to the continuity of each $(\alpha_j, \beta_j)$ remarked below, this gives the continuity of $h$ and $k$ wrt $\mu$.
Last, we can derive the following estimates

$$|h(R) - \alpha_1(R)| \leq \sum_{j\geq 1} \|\alpha_{j+1} - \alpha_j\|_{L^\infty([0,R])} \quad \text{and} \quad |k(R) - 1| \leq \sum_{j\geq 0} \|\beta_{j+1} - \beta_j\|_{L^\infty([0,R])},$$

First, use (3.48) to obtain

$$|k(R) - 1| \leq MR^2 \quad \text{and} \quad |h(R) - \alpha_1(R)| \leq \sum_{j\geq 1} 2^j C^{j+1} R^{2(j-1)} R^{3d+n-|n-d|+4},$$

or

$$|h(R) - \alpha_1(R)| \leq MR^{3d+n-|n-d|+4}.$$

for some $M$ independent of $\mu$.
Now, returning to the definition of $\alpha_1$ in (3.44), we get

$$\alpha_1(r) = \int_0^r t^{-2|n-d|-1} \int_0^t s^{2|n-d|+1} \left( f_d^2 s^{n+d-|n-d|} \right) ds dt.$$

Using (1.28) there exists $M$ depending only on $R$ such that

$$|f_d^2 - A_d^2 s^{2d}| \leq M s^{2d+2}$$

and consequently, for $0 \leq r \leq R$,

$$|\alpha_1(r) - \frac{A_d^2 r^{3d+n-|n-d|+2}}{(3d+n+2+|n-d|)(3d+n+2-|n-d|)}| \leq M r^{3d+n-|n-d|+4} \tag{3.50}$$

for some other $M$ independent of $\mu$, provided $|\mu - 1| \leq 1$.
We deduce that

$$|h(R) - \frac{A_d^2 R^{3d+n-|n-d|+2}}{(3d+n+2+|n-d|)(3d+n+2-|n-d|)}| \leq M R^{3d+n-|n-d|+4}.$$

This is true for the $R$ chosen below and also for all real number smaller than it, with the same $M$.
Letting $a_1^\mu(r) = r^{|n-d|} h(r)$ and $b_1^\mu(r) = r^{n+d} k(r)$ and returning to (3.50), we obtain the desired property (3.39).
And we deduce from (3.44) that

$$\begin{cases} h' &= r^{-1} \int_0^r s \left( f_d^2 - \mu(1 - f_d^2) h + f_d^2 k s^{n+d-|n-d|} \right) ds \\ k' &= r^{-1} \int_0^r s \left( f_d^2 - \mu(1 - f_d^2) k + f_d^2 h s^{|n-d|-n-d} \right) ds. \end{cases} \tag{3.51}$$

We infer that $h'(r)$ and $k'(r)$ depend continously on $\mu$ for all $r \in ]0, R]$.
Consequently $X_1^\mu(r)$ depends continuously on $\mu$, for all $r \in ]0, R]$.

Secondly, let us construct $(a_3^\mu, b_3^\mu)$.
We use (3.44) again and we construct $(\alpha_j, \beta_j)$ by the induction

$$\begin{cases} \alpha_0 = 1 \quad \beta_0 = 0 \\ (\alpha_{j+1}, \beta_{j+1}) = \Phi(\alpha_j, \beta_j). \end{cases} \tag{3.52}$$

We follow the same construction as for $(a_1^\mu, b_1^\mu)$ above. With the same condition on $R$, we estimate

$$|\alpha_1 - \alpha_0| \leq \frac{3}{4} r^2 \quad \text{and} \quad |\beta_1 - \beta_0| \leq \frac{A_d^2 r^{d-n+|n-d|+2}}{(d-n+|n-d|+2)^2}.$$

And the estimate (3.47) is true again. And (3.48) is replaced by

$$\begin{gathered} \|\alpha_{j+1} - \alpha_j\|_{L^\infty([0,R])} \leq 2^j C^{j+1} R^{2(j+1)} \\ \text{and} \\ \|\beta_{j+1} - \beta_j\|_{L^\infty([0,R])} \leq 2^j C^{j+1} R^{d-n+|n-d|+2(j+1)}. \end{gathered} \tag{3.53}$$

Thus we can conclude to the construction of $h_3$ and $k_3$ exactly as in (3.49). And we have, with $C$ defined above

$$|h_3(R) - 1| \leq \sum_{j \geq 0} 2^j C^{j+1} R^{2j+2} \quad \text{and} \quad |k_3(R) - \beta_1(R)| \leq \sum_{j \geq 1} \|\beta_{j+1} - \beta_j\|_{L^\infty([0,R])}.$$

So, we use (3.53) to have some $M$ such that

$$|h_3(R) - 1| \leq M R^2 \quad \text{and} \quad |k_3(R) - \beta_1(R)| \leq M R^{d-n+|n-d|+4},$$

where $M$ is independent of $\mu$ and is valid for the $R$ chosen above, and also for any smaller positive real number.
Now, as for $\alpha_1$ above, we compute $\beta_1$. Since

$$\beta_1 = \int_0^r t^{-2(n+d)-1} \int_0^t s^{2(n+d)+1} f_d^2 s^{|n-d|-n-d} ds dt,$$

we have that for $0 < r \leq R$

$$|\beta_1 - \frac{A_d^2 r^{d-n+|n-d|+2}}{(3d+n+|n-d|+2)(d-n+|n-d|+2)}| \leq M r^{d-n+|n-d|+4}$$

with $M$ independent on $\mu$. We deduce that

$$|k_3(R) - \frac{A_d^2 R^{d-n+|n-d|+2}}{(3d+n+|n-d|+2)(d-n+|n-d|+2)}| \leq M R^{d-n+|n-d|+4},$$

for some $M$ independent of $\mu$ and independent of $R$ small enough.
Now we define

$$a_3^\mu = r^{|n-d|} h \quad \text{and} \quad b_3^\mu = r^{n+d} k.$$

And we have proved the estimates (3.40).
The proof of the continuity of $X_3^\mu(r)$ wrt $\mu$, for $r \in ]0, R]$, works exactly as for $X_1^\mu$.

**Proof of Proposition 3.2**. To construct $(a_2^\mu, b_2^\mu)$ and $(a_4^\mu, b_4^\mu)$ in some interval $]0, R]$, we are looking for a solution $(a, b)$ of (1.25) of the form

$$a(r) = h(r)r^{-|n-d|} \quad \text{and} \quad b(r) = k(r)r^{-n-d}.$$

That gives

$$\begin{cases} h'' + \frac{-2|n-d|+1}{r}h' - f_d^2 h - f_d^2 k r^{-n-d+|n-d|} = -\mu(1-f_d^2)h \\ k'' + \frac{-2(n+d)+1}{r}k' - f_d^2 k - f_d^2 h r^{-|n-d|+n+d} = -\mu(1-f_d^2)k \end{cases} \tag{3.54}$$

We will choose $(h, k)(0) = (0, 1)$ for $(a_2^\mu, b_2^\mu)$ and $(h, k)(0) = (1, 0)$ for $(a_4^\mu, b_4^\mu)$.

First, let us construct $(a_2^\mu, b_2^\mu)$. We have to distinguish the cases $n \leq d$ and $n \geq d+1$.

(i) For $n \leq d$. We chose some $R > 0$. We will have to choose $R$ small enough, but in a way valid for all values of the parameter $\mu$, belonging to the desired range. We write (3.54) on the form, for all $0 < r < R$,

$$\begin{cases} h &= \alpha_0 + \int_0^r t^{2|n-d|-1} \int_0^t s^{-2|n-d|+1} \left( (f_d^2 - \mu(1-f_d^2))h + f_d^2 k s^{-n-d+|n-d|} \right) ds dt \\ \\ k &= \beta_0 + \int_0^r t^{2(n+d)-1} \int_R^t s^{-2(n+d)+1} \left( (f_d^2 - \mu(1-f_d^2))k + f_d^2 h s^{-|n-d|+n+d} \right) ds dt \end{cases} \tag{3.55}$$

We define

$$\begin{cases} \alpha_0 = 0 \quad \beta_0 = 1 \\ (\alpha_{j+1}, \beta_{j+1}) = \Phi(\alpha_j, \beta_j). \end{cases} \tag{3.56}$$

As above, for $R$ small enough, we obtain that for all $0 < r \leq R$

$$|\alpha_1 - \alpha_0|(r) \leq \frac{A_d^2 r^{d-n+|n-d|+2}}{(d-n+|n-d|+2)^2} \quad \text{and} \quad |\beta_1 - \beta_0|(r) \leq \frac{3}{4}r^2. \tag{3.57}$$

And for all $j \geq 1$

$$|\alpha_{j+1} - \alpha_j|(r) \leq \int_0^r t^{2|n-d|-1} \int_0^t (3s \| r^{-2|n-d|}(\alpha_j - \alpha_{j-1}) \|_{L^\infty([0,R])} +$$

$$+ A_d^2 s^{-|n-d|-n+d+3} \| r^{-2}(\beta_j - \beta_{j-1}) \|_{L^\infty([0,R])}) ds dt$$

that gives

$$|\alpha_{j+1} - \alpha_j|(r) \leq \int_0^r \left( \frac{3}{2} t^{2|n-d|+1} \| r^{-2|n-d|}(\alpha_j - \alpha_{j-1}) \|_{L^\infty([0,R])} + \frac{A_d^2}{4} t^{2|n-d|+3} \| r^{-2}(\beta_j - \beta_{j-1}) \|_{L^\infty([0,R])} \right) dt$$

and we are led to

$$|\alpha_{j+1} - \alpha_j|(r) \leq 3 \frac{r^{2|n-d|+2}}{2(2|n-d|+2)} \| r^{-2|n-d|}(\alpha_j - \alpha_{j-1}) \|_{L^\infty([0,R])} +$$

$$+ \frac{A_d^2 r^{2|n-d|+4}}{4(2|n-d|+4)} \| r^{-2}(\beta_j - \beta_{j-1}) \|_{L^\infty([0,R])}. \tag{3.58}$$

And

$$|\beta_{j+1}-\beta_j|(r)\leq \int_0^r t^{2(n+d)-1}\int_t^R [3s^{-2(n+d)+3}\|r^{-2}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+$$

$$+A_d^2 s^{-n+d+|n-d|+1}\|r^{-2|n-d|}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R])}]dsdt,$$

that gives

$$|\beta_{j+1}-\beta_j|(r)\leq \int_0^r t^{2(n+d)-1}[3D_0(t)\|r^{-2}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+$$

$$+A_d^2 R^{2|n-d|+2}\|r^{-2|n-d|}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R])}]dt$$

where

$$D_0(t)=\begin{cases}\frac{t^{-2(n+d)+4}}{(2(n+d)-4)} & \text{if } (n,d)\neq(1,1)\\ \\ |\ln t| & \text{if } (n,d)=(1,1).\end{cases}$$

And, since $R\leq 1$, we are led to

$$\begin{aligned}|\beta_{j+1}-\beta_j|(r)&\leq 3D(r)\|r^{-2}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+\\ &+A_d^2R^2\tfrac{r^{2(n+d)}}{2(n+d)}\|r^{-2|n-d|}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R])}\end{aligned}\tag{3.59}$$

$$D(r)=\begin{cases}\frac{r^4}{4(2(n+d)-4)} & \text{if } (n,d)\neq(1,1)\\ \\ \frac{r^{2(n+d)}}{2(n+d)}|\ln r| & \text{if } (n,d)=(1,1).\end{cases}$$

We remark that in any case $D(r)r^{-2}\leq r^2|\ln r|$. So an induction gives, with $C=\max\{3,\frac{A_d^2}{2}\}$ defined as above, for $j\geq 0$,

$$\begin{aligned}&\|r^{-2|n-d|}(\alpha_{j+1}-\alpha_j)\|_{L^\infty([0,R])}\leq 2^jC^{j+1}(R^2|\ln R|)^j\\ \text{and}\quad &\|r^{-2}(\beta_{j+1}-\beta_j)\|_{L^\infty([0,R])}\leq 2^jC^{j+1}(R^2|\ln R|)^j.\end{aligned}\tag{3.60}$$

We choose $R$ small enough to have $2CR^2|\ln R|<\frac{1}{2}$. We define $h$ and $k$ by

$$h(r)=r^{2|n-d|}\sum_{j\geq 0}r^{-2|n-d|}(\alpha_{j+1}-\alpha_j)\quad\text{and}\quad k(r)=\beta_0+r^2\sum_{j\geq 0}r^{-2}(\beta_{j+1}-\beta_j)$$

and we obtain, for all $0\leq r\leq R$,

$$|h(r)|\leq C(1-2CR^2|\ln R|)^{-1}r^{2|n-d|}\quad\text{and}\quad |k(r)-1|\leq C(1-2CR^2|\ln R|)^{-1}r^2$$

We define $a_2^\mu=r^{-|n-d|}h$ and $b_2^\mu=r^{-n-d}k$. We deduce the proof of (3.41), when $n\leq d$. And since for the same reason as in the construction of $(a_1^\mu,b_1^\mu)$, $(h,h',k,k')(r)$ is continuous wrt $\mu$, we have that $X_2^\mu(r)$ is continuous wrt $\mu$, too, for all $0<r\leq R$.
This terminates the construction of $(a_2^\mu,b_2^\mu)$, when $n\leq d$.

(ii) $n \geq d+1$. Now, let us examine the construction of $(a_2^\mu, b_2^\mu)$, when $n \geq d+1$. We define again $a = r^{-|n-d|}h$ and $b = r^{-n-d}k$, so we solve the system (3.54) again, but not in the same Banach space. Instead of (3.55), we consider the following fixed point problem, when $R > 0$ is chosen.

$$\left\{ \begin{array}{ll} h & = \alpha_0 + \int_0^r t^{2|n-d|-1} \int_R^t s^{-2|n-d|+1} \left( (f_d^2 - \mu(1-f_d^2))h + f_d^2 k s^{-n-d+|n-d|} \right) dsdt \\ \\ k & = \beta_0 + \int_0^r t^{2(n+d)-1} \int_R^t s^{-2(n+d)+1} \left( (f_d^2 - \mu(1-f_d^2))k + f_d^2 h s^{-|n-d|+n+d} \right) dsdt. \end{array} \right. \tag{3.61}$$

We define the sequence $(\alpha_j, \beta_j)_{j \geq 0}$ by the same induction as for $n \leq d$, ie $\alpha_0 = 0$ and $\beta_0 = 1$. We find the estimates

$$|\alpha_1 - \alpha_0|(r) \leq A_d^2 \int_0^r t^{2|n-d|-1} \int_t^R s^{-2|n-d|+1} s^{2d} s^{-n-d+|n-d|} dsdt.$$

But $-2|n-d| + 1 + 2d - n - d + |n-d| = -2n + 2d + 1$ can be equal to $-1$, or is $\leq -3$, that gives

$$|\alpha_1 - \alpha_0|(r) \leq A_d^2 D_1(r) \tag{3.62}$$

where

$$D_1(r) = \left\{ \begin{array}{ll} r^2/2(2n-2d-2) & \text{if } n \geq d+2 \\ r^{2(n-d)} |\ln r|/2(n-d) & \text{if } n = d+1. \end{array} \right.$$

while

$$|\beta_1 - \beta_0|(r) \leq \frac{3}{2(2(n+d)-2)} r^2. \tag{3.63}$$

And, for $j \geq 1$,

$$|\alpha_{j+1} - \alpha_j|(r) \leq \int_0^r t^{2|n-d|-1} \int_t^R (3s^{3-2|n-d|} |\ln s| \| r^{-2} |\ln r|^{-1} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([0,R])} +$$

$$+ A_d^2 s^{-2(n-d)+3} \| r^{-2} (\beta_j - \beta_{j-1}) \|_{L^\infty([0,R])}) dsdt,$$

that gives

$$|\alpha_{j+1} - \alpha_j|(r) \leq \int_0^r t^{2|n-d|-1} [3C_0(t) \| r^{-2} |\ln r|^{-1} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([0,R])} +$$

$$+ A_d^2 C_0(t) \| r^{-2} (\beta_j - \beta_{j-1}) \|_{L^\infty([0,R])}] dt$$

where

$$C_0(t) = \left\{ \begin{array}{c} R^2/2|\ln R| \quad \text{if } n = d+1 \\ |\ln t|^2 \quad \text{if } n = d+2 \\ \frac{t^{-2|n-d|+4}}{2|n-d|-4} |\ln t| \quad \text{if } n \geq d+3 \end{array} \right. \tag{3.64}$$

And we obtain

$$\begin{array}{c} |\alpha_{j+1} - \alpha_j|(r) \leq 3C(r) \| r^{-2} |\ln r|^{-1} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([0,R])} + \\ \\ + A_d^2 C(r) \| r^{-2} (\beta_j - \beta_{j-1}) \|_{L^\infty([0,R])} \end{array} \tag{3.65}$$

where

$$C(r) = \begin{cases} R^2|\ln R|r^2/4 & \text{if } n = d+1 \\ \frac{r^4}{4}|\ln r|^2 & \text{if } n = d+2 \\ \frac{r^4}{4(2|n-d|-4)}|\ln r| & \text{if } n \geq d+3. \end{cases} \tag{3.66}$$

Now

$$|\beta_{j+1} - \beta_j|(r) \leq \int_0^r t^{2(n+d)-1} \int_t^R [3s^{-2(n+d)+3}\|r^{-2}(\beta_j - \beta_{j-1})\|_{L^\infty([0,R])} +$$

$$+A_d^2 s^{3-2|n-d|}|\ln s|\||r^{-2}|\ln r|^{-1}(\alpha_j - \alpha_{j-1})\|_{L^\infty([0,R])}ds]dt,$$

that gives

$$|\beta_{j+1} - \beta_j|(r) \leq \int_0^r t^{2(n+d)-1}[3\frac{t^{-2(n+d)+4}}{2(n+d)-4}\|r^{-2}(\beta_j - \beta_{j-1})\|_{L^\infty([0,R])} +$$

$$+A_d^2 C_0(t)\|r^{-2}|\ln r|(\alpha_j - \alpha_{j-1})\|_{L^\infty([0,R])}]dt,$$

where $C_0(t)$ is defined in (3.64) above. And we obtain

$$\begin{aligned} |\beta_{j+1} - \beta_j|(r) &\leq 3\tfrac{r^4}{4(2(n+d)-4)}\|r^{-2}(\beta_j - \beta_{j-1})\|_{L^\infty([0,R])} + \\ &+A_d^2 D(r)\|r^{-2}|\ln r|^{-1}(\alpha_j - \alpha_{j-1})\|_{L^\infty([0,R])}, \end{aligned} \tag{3.67}$$

where

$$D(r) = \begin{cases} R^2|\ln R|\frac{r^{2(n+d)}}{4(n+d)} & \text{if } n = d+1 \\ \frac{r^{2(n+d)}}{2(n+d)}|\ln r|^2 & \text{if } n = d+2 \\ \frac{r^4}{4(2|n-d|-4)}|\ln r| & \text{if } n \geq d+3. \end{cases}$$

We define $C = \max\{3, A_d^2\}$. We remark that in any case, for $n \geq d+1$, for $0 \leq r \leq R < 1$,

$$r^{-2}D(r) \leq R^2 \quad \text{and} \quad r^{-2}|\ln r|^{-1}C(r) \leq R^2|\ln R|.$$

Considering (3.62), (3.63), (3.65) and (3.67), an induction gives

$$\begin{aligned} &\|r^{-2}|\ln r|^{-1}(\alpha_{j+1} - \alpha_j)\|_{L^\infty([0,R])} \leq 2^j C^{j+1}(R^2 \ln R)^j \\ &\qquad\qquad \text{and} \\ &\|r^{-2}(\beta_{j+1} - \beta_j)\|_{L^\infty([0,R])} \leq 2^j C^{j+1}(R^2 \ln R)^j. \end{aligned} \tag{3.68}$$

We chose $R$ such that $2CR^2|\ln R| < 1/2$ and we define $h$ and $k$ for $r \in [0, R]$ by

$$h(r) = r^2|\ln r|\sum_{j\geq 0} r^{-2}|\ln r|^{-1}(\alpha_{j+1} - \alpha_j) \quad \text{and} \quad k(r) = \beta_0 + r^2 \sum_{j\geq 0} r^{-2}(\beta_{j+1} - \beta_j).$$

The convergence of the sums, for $R > 0$ small enough follows from (3.68) and we have the continuity of $h$, $h'$, $k$, $k'$ wrt $\mu$, too. And we obtain, for all $0 < r \leq R$,

$$|h(r)| \leq Mr^2|\ln r| \quad \text{and} \quad |k(r) - 1| \leq Mr^2.$$

We define $a_2^\mu = r^{-|n-d|}h$ and $b_2^\mu = r^{-n-d}k$ and we can recapitulate the cases $n \leq d$ and $n \geq d+1$ by (3.41).

Now, let us construct $(a_4^\mu, b_4^\mu)$. We distinguish the cases $n \neq d$ and $n = d$.

(i) $n \neq d$.
We use the fixed point problem (3.61), that is

$$\begin{cases} h &= \alpha_0 + \int_0^r t^{2|n-d|-1} \int_R^t s^{-2|n-d|+1} \left( (f_d^2 - \mu(1-f_d^2))h + f_d^2 k s^{-n-d+|n-d|} \right) dsdt \\ k &= \beta_0 + \int_0^r t^{2(n+d)-1} \int_R^t s^{-2(n+d)+1} \left( (f_d^2 - \mu(1-f_d^2))k + f_d^2 h s^{-|n-d|+n+d} \right) dsdt. \end{cases}$$

and we define

$$\begin{cases} \alpha_0 = 1 \quad \beta_0 = 0 \\ (\alpha_{j+1}, \beta_{j+1}) = \Phi(\alpha_j, \beta_j). \end{cases} \tag{3.69}$$

For $R$ small enough as above, we obtain that for all $0 < r \leq R$

$$|\alpha_1 - \alpha_0|(r) \leq 3 \int_0^r t^{2|n-d|-1} \int_t^R s^{-2|n-d|+1} dsdt,$$

that gives

$$|\alpha_1 - \alpha_0|(r) \leq \frac{3}{2} D_1(r) \tag{3.70}$$

where

$$D_1(r) = \begin{cases} \frac{r^2}{2|n-d|-2} & \text{if } |n-d| \geq 2 \\ r^2(-\ln r) & \text{if } |n-d| = 1. \end{cases}$$

And

$$|\beta_1 - \beta_0| \leq \int_0^r t^{2(n+d)-1} \int_t^R A_d^2 s^{-(n+d)-|n-d|+2d+1} dsdt,$$

that gives

$$|\beta_1 - \beta_0| \leq A_d^2 D_2(r), \tag{3.71}$$

where

$$D_2(r) = \begin{cases} \frac{r^{4d+2}}{(2|n-d|-2)(4d+2)} & \text{if } n \geq d+2 \\ \frac{r^{4d+2}(-\ln r)}{4d+2} & \text{if } n = d+1 \\ \frac{R^2}{2} \frac{r^{2(n+d)}}{2(n+d)} & \text{if } n \leq d-1. \end{cases}$$

In view of (3.71), we have to distinguish the cases $n \leq d-1$ and $n \geq d+1$.
(a) $n \leq d-1$.
For all $j \geq 1$ we write

$$|\alpha_{j+1} - \alpha_j| \leq \int_0^r t^{2|n-d|-1} \int_t^R [3s^{-2|n-d|+3} |\ln s| \| r^{-2} |\ln r|^{-1} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([0,R]} +$$

$$+ A_d^2 s^{-|n-d|+2d+n+d} \| r^{-2(n+d)+1} (\beta_j - \beta_{j-1}) \|_{L^\infty([0,R])}] dsdt,$$

that gives

$$|\alpha_{j+1}-\alpha_j| \le \int_0^r t^{2|n-d|-1}[3C_0(t)\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}+$$

$$+A_d^2\frac{t^{3d+n+1-|n-d|}}{3d+n+1-|n-d|}\|r^{-2(n+d)-1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}]dsdt,$$

where $C_0(t)$ is defined in (3.64) above. And we are led to

$$|\alpha_{j+1}-\alpha_j| \le 3C(r)\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}+$$

$$+A_d^2R\frac{r^{2|n-d|}}{2|n-d|}\|r^{-2(n+d)+1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])},$$

where $C(r)$ is defined in (3.66). But $C(r)\le r^2R^2|\ln R|$. And since $r\le R<1$, we use $|\ln R|\le|\ln r|$ to obtain $C(r)\le Rr^2|\ln r|$. Moreover $2|n-d|\ge 2$. We are led to

$$\begin{aligned}|\alpha_{j+1}-\alpha_j| &\le 3Rr^2|\ln r|\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}+\\ &+A_d^2Rr^2|\ln r|\|r^{-2(n+d)+1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}.\end{aligned}\tag{3.72}$$

And in the other hand

$$|\beta_{j+1}-\beta_j| \le \int_0^r t^{2(n+d)-1}\int_t^R[3\|r^{-2(n+d)+1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+$$

$$+A_d^2s^{-(n+d)-|n-d|+3+2d}|\ln s|\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}]dsdt$$

and since $-(n+d)-|n-d|+3+2d=3$, we deduce that

$$|\beta_{j+1}-\beta_j| \le \int_0^r t^{2(n+d)-1}[3R\|r^{-2(n+d)+1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+$$

$$+A_d^2\frac{R^4}{4}|\ln R|\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}]dsdt$$

And we are led to

$$\begin{aligned}|\beta_{j+1}-\beta_j| &\le 3Rr^{2(n+d)-1}\|r^{-2(n+d)+1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+\\ &+A_d^2Rr^{2(n+d)-1}\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]},\end{aligned}\tag{3.73}$$

Letting $C=\max\{3,A_d^2\}$, we deduce from (3.70), (3.71), (3.72) and (3.73) that, for all $j\ge 0$,

$$|\ln r|^{-1}r^{-2}|\alpha_{j+1}-\alpha_j|\le 2^jC^{j+1}R^j \text{ and } r^{-2(n+d)+1}|\beta_{j+1}-\beta_j|\le 2^jC^{j+1}R^j.\tag{3.74}$$

We chose $R$ small enough to have $2CR<\frac{1}{2}$ and we define

$$h=\alpha_0+|\ln r|r^2\textstyle\sum_{j\ge0}|\ln r|^{-1}r^{-2}(\alpha_{j+1}-\alpha_j)$$

$$\text{and}\quad k=\beta_0+r^{2(n+d)-1}\textstyle\sum_{j\ge0}r^{-2(n+d)+1}(\beta_{j+1}-\beta_j).$$

We conclude as usual to the continuity of $h$ and $k$ wrt to $\mu$, when $r \leq R$ and we estimate, for $r \in [0, R]$,

$$|h(r) - 1| \leq Mr^2 |\ln r| \quad \text{and} \quad |k(r)| \leq Mr^{2(n+d)-1}.$$

Letting $a_4^\mu = hr^{-|n-d|}$ and $b_4^\mu = kr^{-n-d}$, we have proved (3.42), when $n \leq d-1$.

(b) Let us examine the case $n \geq d+1$.
For all $j \geq 1$ we write

$$|\alpha_{j+1} - \alpha_j| \leq \int_0^r t^{2|n-d|-1} \int_t^R [3s^{-2|n-d|+3} |\ln s| \| r^{-2} |\ln r|^{-1} (\alpha_j - \alpha_{j-1})\|_{L^\infty([0,R]} +$$

$$+ A_d^2 s^{-|n-d|+6d-n-d+3} |\ln s| \| r^{-4d-2} |\ln r|^{-1} (\beta_j - \beta_{j-1})\|_{L^\infty([0,R])}] ds dt,$$

that gives

$$|\alpha_{j+1} - \alpha_j| \leq \int_0^r t^{2|n-d|-1} [3C_0(t) \| r^{-2} |\ln r|^{-1} (\alpha_j - \alpha_{j-1})\|_{L^\infty([0,R]} +$$

$$+ A_d^2 C_2(t) \| r^{-4d-2} |\ln r|^{-1} (\beta_j - \beta_{j-1})\|_{L^\infty([0,R])}] ds dt,$$

where $C_0(t)$ is defined above in (3.64] and

$$C_2(t) = \begin{cases} R|\ln t| & \text{if } -2n+6d+4 \geq 0 \\ |\ln t|^2 & \text{if } -2n+6d+4 = -1 \\ \frac{t^{-2n+6d+4}}{2n-6d+4} |\ln t| & \text{if } -2n+6d+4 < -1. \end{cases}$$

And we are led to

$$|\alpha_{j+1} - \alpha_j| \leq 3C(r) \| r^{-2} || \ln r|^{-1} (\alpha_j - \alpha_{j-1})\|_{L^\infty([0,R]} +$$

$$+ A_d^2 C_3(r) \| r^{-4d-2} |\ln r|^{-1} (\beta_j - \beta_{j-1})\|_{L^\infty([0,R])},$$

where $C(r)$ is defined in (3.66) and

$$C_3(r) = \begin{cases} R \frac{r^{2|n-d|}}{2|n-d|} |\ln r| & \text{if } -2n+6d+4 \geq 0 \\ \frac{r^{2|n-d|}}{2|n-d|} |\ln r|^2 & \text{if } -2n+6d+4 = -1 \\ \frac{r^{4d+4}}{(4d+4)(2n-6d+4)} |\ln r| & \text{if } -2n+6d+4 < -1. \end{cases}$$

But $C(r) \leq r^2 R^2 |\ln R|$. And since $R < 1$, we use $|\ln R| \leq |\ln r|$ to obtain $C(r) \leq Rr^2 |\ln r|$. Moreover, if $n = d+1$, we have $-2n+6d+4 \geq 0$ and if $n \geq d+2$, we have $2|n-d| \geq 4$. We deduce, for all $r \leq R$ and since $n \geq d+1$,

$$\begin{aligned} |\alpha_{j+1} - \alpha_j| &\leq 3R^2 r^2 |\ln r| \| r^{-2} |\ln r|^{-1} (\alpha_j - \alpha_{j-1})\|_{L^\infty([0,R]} + \\ &+ A_d^2 Rr^2 |\ln r| \| r^{-4d-2} |\ln r|^{-1} (\beta_j - \beta_{j-1})\|_{L^\infty([0,R])}. \end{aligned} \tag{3.75}$$

And in the other hand

$$|\beta_{j+1} - \beta_j| \leq \int_0^r t^{2(n+d)-1} \int_t^R [3s^{-2n+2d+3} |\ln s| \| r^{-4d-2} |\ln r|^{-1} (\beta_j - \beta_{j-1})\|_{L^\infty([0,R])} +$$

$$+A_d^2 s^{-(n+d)-|n-d|+3+2d}|\ln s|\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}]dsdt$$

and since $-(n+d)-|n-d|+3+2d=-2n+2d+3$ can be equal to 1, $-1$ or is $\leq -3$, we deduce that

$$|\beta_{j+1}-\beta_j|\leq \int_0^r C_4(t)\|r^{-4d-2}|\ln r|^{-1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+$$

$$+A_d^2 C_4(t)\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}]dsdt$$

where

$$C_4(t)=\begin{cases}\frac{R^{-2n+2d+3}}{-2n+2d+3}|\ln R| & \text{if } n=d+1\\ |\ln R|^2 & \text{if } n=d+2\\ \frac{t^{-2n+2d+4}}{(2n-2d-4)}|\ln t| & \text{if } n\geq d+3.\end{cases}$$

And we are led to

$$|\beta_{j+1}-\beta_j|\leq 3C_5(r)\|r^{-4d-2}|\ln r|^{-1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+$$

$$+A_d^2 C_5(r)\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]}]dsdt$$

where

$$C_5(r)=\begin{cases}R|\ln R|r^{\frac{2(n+d)}{2(n+d)}} & \text{if } n=d+1\\ |\ln R|^2 r^{\frac{2(n+d)}{2(n+d)}} & \text{if } n=d+2\\ \frac{r^{4d+4}}{(2n-2d-4)(4d+4)}|\ln r| & \text{if } n\geq d+3.\end{cases}$$

In any case, we have $C_5(r)\leq r^{4d+2}|\ln r|R$. We deduce that

$$\begin{aligned}|\beta_{j+1}-\beta_j|&\leq 3Rr^{4d+2}|\ln r|\|r^{-4d-2}|\ln r|^{-1}(\beta_j-\beta_{j-1})\|_{L^\infty([0,R])}+\\ &+A_d^2 Rr^{4d+2}|\ln r|\|r^{-2}|\ln r|^{-1}(\alpha_j-\alpha_{j-1})\|_{L^\infty([0,R]},\end{aligned}\tag{3.76}$$

Letting $C=\max\{3,A_d^2\}$, we deduce from (3.70), (3.71), (3.75) and (3.76) that, for all $j\geq 0$

$$r^{-2}|\ln r|^{-1}|\alpha_{j+1}-\alpha_j|\leq 2^j C^{j+1}R^j \text{ and } r^{-4d-2}|\ln r|^{-1}|\beta_{j+1}-\beta_j|\leq 2^j C^{j+1}R^j.\tag{3.77}$$

We chose $R$ small enough to have $2CR<\frac{1}{2}$ and we define

$$h=\alpha_0+r^2|\ln r|\textstyle\sum_{j\geq 0} r^{-2}|\ln r|^{-1}(\alpha_{j+1}-\alpha_j)$$

$$\text{and}\quad k=\beta_0+r^{4d+2}|\ln r|\textstyle\sum_{j\geq 0} r^{-4d-2}|\ln r|^{-1}(\beta_{j+1}-\beta_j).$$

We conclude as usual to the continuity of $h$ and $k$ wrt to $\mu$, when $r\leq R$ and we estimate, for $r\in[0,R]$,

$$|h(r)-1|\leq Mr^2|\ln r| \quad\text{and}\quad |k(r)|\leq Mr^{4d+2}|\ln r|.$$

Letting $a_4^\mu=hr^{-|n-d|}$ and $b_4^\mu=kr^{-n-d}$, we have proved (3.42), when $n\geq d+1$.

(ii) Now, if $n = d$ we verify that $\ln r$ verifies the equation

$$a'' + \frac{a'}{r} = 0.$$

Consequently if we define $a = h|\ln r|$ and $b = kr^{-2d}$, then $(a, b)$ is a solution of (1.25), with $n = d$, if and only if $(h, k)$ verifies

$$\left\{ \begin{array}{c} h'' + \frac{2|\ln r|^{-1}+1}{r} h' - f_d^2 h - f_d^2 k r^{-2d} (-\ln r)^{-1} = -\mu(1 - f_d^2)h \\ k'' + \frac{-2d+1}{r} k' - f_d^2 k - f_d^2 h r^{2d} (-\ln r) = -\mu(1 - f_d^2)k \end{array} \right. \tag{3.78}$$

We denote $\tau(r) = |\ln r|$. Since

$$h'' + \frac{2|\ln r|^{-1}+1}{r} h' = (r\tau^2 h')' \tau^{-2} r^{-1} \quad \text{and} \quad k'' + \frac{-2d+1}{r} k' = (r^{-2d+1} k')' r^{2d-1},$$

we consider the following fixed point problem

$$\left\{ \begin{array}{ll} h & = \alpha_0 + \int_0^r t^{-1}\tau^{-2} \int_0^t s\tau^2 \left( (f_d^2 - \mu(1 - f_d^2))h + f_d^2 k s^{-2d} \tau^{-1} \right) ds dt \\ \\ k & = \beta_0 + \int_0^r t^{2d-1} \int_0^t s^{-2d+1} \left( (f_d^2 - \mu(1 - f_d^2))k + f_d^2 h s^{2d} \tau \right) ds dt \end{array} \right. \tag{3.79}$$

We define $(\alpha_j, \beta_j)$ by the induction (3.69). Thus $\alpha_0 = 1$ and $\beta_0 = 0$. With the same estimates for $f_d$ a above, there exists $R$ small enough such that we have for all $0 < r \leq R$

$$|\alpha_1 - \alpha_0|(r) \leq \frac{3}{4} r^2 \quad \text{and} \quad |\beta_1 - \beta_0|(r) \leq \frac{A_d^2 r^{4d+2}}{(2d+2)(4d+2)} (-\ln r).$$

Now, for all $j \geq 1$

$$|\alpha_{j+1} - \alpha_j|(r) \leq$$

$$\leq \tfrac{3}{4} r^2 \|\alpha_j - \alpha_{j-1}\|_{L^\infty([0,r])} + \int_0^r t^{-1}\tau^{-2} \int_0^t A_d^2 s(-\ln s) s^{2d} ds dt \| r^{-2d} (\beta_j - \beta_{j-1}) \|_{L^\infty([0,r])}.$$

We use $(-\ln s)^{-1} \leq 1$ to obtain

$$|\alpha_{j+1} - \alpha_j|(r) \leq \frac{3}{4} r^2 \|\alpha_j - \alpha_{j-1}\|_{L^\infty([0,r])} + \ + A_d^2 \frac{r^{2d+2}}{(2d+2)^2} \| r^{-2d} (\beta_j - \beta_{j-1}) \|_{L^\infty([0,r])}.$$

And

$$|\beta_{j+1} - \beta_j|(r) \leq \int_0^r t^{2d-1} \int_0^t \left( 3s \| r^{-2d} (\beta_j - \beta_{j-1}) \|_{L^\infty[0,r]} + A_d^2 s^{2d+1} \tau \| \alpha_j - \alpha_{j-1} \|_{L^\infty[0,r]} \right) ds dt,$$

that gives

$$|\beta_{j+1} - \beta_j|(r) \leq \frac{3}{4} r^{2d+2} \| r^{-2d} (\beta_j - \beta_{j-1}) \|_{L^\infty([0,r])} + \frac{A_d^2 r^{4d+2}}{(4d+2)(2d+2)} \tau \| \alpha_j - \alpha_{j-1} \|_{L^\infty([0,r])}.$$

We define $C = \max\{\frac{3}{4}, \frac{A_d^2}{4}\}$. And finally, for all $j \geq 0$ and for all $r \in [0, R]$,

$$\begin{aligned} &\|\alpha_{j+1} - \alpha_j\|_{L^\infty([0,R])} \leq 2^j C^{j+1} R^{2(j+1)} \\ \text{and} \quad &\|r^{-2d}(\beta_{j+1} - \beta_j)\|_{L^\infty([0,R])} \leq 2^j C^{j+1}(-\ln R) R^{2d+2(j+1)}. \end{aligned} \tag{3.80}$$

Now, we chose $R$ small enough to have $2CR^2|\ln R| < 1$ and we define $(h, k)$ by

$$h = \alpha_0 + \sum_{j\geq 0}(\alpha_{j+1} - \alpha_j) \quad \text{and} \quad k = r^{2d}\sum_{j\geq 0} r^{-2d}(\beta_{j+1} - \beta_j).$$

The convergence of the sums in $L^\infty([0, R])$, uniform wrt $\mu$ is given by the estimate (3.80) and we conclude to the continuity of $h$ and $k$ wrt $\mu$ as above. Now we estimate

$$|h(R)-1| \leq \sum_{j\geq 0} \|\alpha_{j+1}-\alpha_j\|_{L^\infty([0,R])} \quad \text{and} \quad |k(R)-\beta_1(R)| \leq R^{2d}\sum_{j\geq 1} \|r^{-2d}(\beta_{j+1}-\beta_j)\|_{L^\infty([0,R])}.$$

We deduce from (3.80) that

$$|h(R) - 1| \leq MR^2 \quad \text{and} \quad |k(R) - \beta_1(R)| \leq MR^{2d+4}(-\ln R),$$

and this estimate is valid for all $r \leq R$, with the same $M$.
Now we estimate

$$\beta_1 = \int_0^r t^{2d-1} \int_0^t s^{-2d+1} f_d^2 s^{2d}(-\ln s) ds dt.$$

We have

$$|\beta_1 - \frac{A_d^2 r^{4d+2}}{(4d+2)^2}(-\ln r)| \leq Mr^{4d+4}.$$

We infer that

$$|k(R) - \frac{A_d^2 R^{4d+2}}{(4d+2)(2d+2)}(-\ln R)| \leq MR^{4d+4}(-\ln R),$$

for another $M$ independent of $\mu$ and independent of $R$ small enough. Now we define

$$a_4^\mu = h|\ln r| \quad \text{and} \quad b_4^\mu = r^{-2d}k$$

and as for $n \neq d$, we conclude to (3.42). And we conclude as above to the continuity of $X_4^\mu$ wrt $\mu$, for all $0 < r \leq R$.

## 3.2 A base defined near infinity.

Now let us turn to a base of solutions of (1.25) defined at $+\infty$.
The propositions 3.4 and 3.5 below are valid in particular for $\mu = 1$, and in this case they give a base of solutions of (1.14). In all what follows, we suppose that $d \geq 1$, $n \geq 1$. We distinguish two independent solutions having exponential behaviors at $+\infty$ and two independent solutions having polynomial behavior at $+\infty$. The four of them form a base

of solutions.

We will denote

$$J_+ := \frac{e^{\sqrt{2}r}}{\sqrt{r}}, \quad J_- := \frac{e^{-\sqrt{2}r}}{\sqrt{r}}.$$

We will prove that $J_+$ and $J_-$ are the exponentional behaviors at $+\infty$. Let us enonce

**Proposition 3.4** *If $|1-\mu| \leq 1$, there exist two solutions of (1.25) defined by there exponentional behaviors at $+\infty$, that we denote by $(u_i^\mu, v_i^\mu)$, $i=1$ and $i=2$. They are such that they exists some $R>0$ independent of the parameter $\mu$ and such that in $[R,+\infty[$*

$$|u_1^\mu - J_+| + |v_1^\mu - J_+| \leq Mr^{-1}J_+, \qquad |u_2^\mu - J_-| + |v_2^\mu - J_-| \leq Mr^{-1}J_-, \tag{3.81}$$

*where $M$ is independent of $\mu$, too.*

Now, we will suppose that $\mu$ is sufficiently closed to 1, ie $-\frac{1}{2} \leq d^2(1-\mu) \leq \frac{1}{2}$ and for all $n \geq 1$ we denote

$$n_\mu := \sqrt{n^2 + d^2(1-\mu)}.$$

Let us enonce

**Proposition 3.5** *If $-\frac{1}{2} \leq d^2(1-\mu) \leq \frac{1}{2}$, there exists two solutions of (1.25) defined by there polynomial behaviors at $+\infty$ and denoted by $(u_i^\mu, v_i^\mu)$, $i=3$ and $i=4$. They are such that, for some $R>0$ and some $M>0$ independent of $\mu$, they verify, in $[R,+\infty[$*

$$|u_3^\mu - r^{n_\mu}| + |v_3^\mu + r^{n_\mu}| \leq Mr^{-2}r^{n_\mu}, \qquad |u_4^\mu - r^{-n_\mu}| + |v_4^\mu + r^{-n_\mu}| \leq Mr^{-2}r^{-n_\mu} \tag{3.82}$$

**Proposition 3.6** *Denoting $Y_i^\mu = (u_i^\mu, r(u_i^\mu)', v_i^\mu, r(v_i^\mu)')^t$ the associated solution of (1.26), we have that $Y_i^\mu \to Y_i$ as $\mu \to 1$, in $(L^\infty([r_0,r_1]))^4$, for all $0<r_0<r_1$, where the notation $Y_i$ is used in place of $Y_i^1$.*

Before proving the first two propositions, we let $x=a+b$ and $y=a-b$ and we replace the system (1.25) by the following system verified by $(x,y)$

$$\begin{cases} x'' + \frac{x'}{r} - \frac{n^2+d^2}{r^2}x + \frac{2nd}{r^2}y - 2f_d^2 x + \mu(1-f_d^2)x &= 0 \\ y'' + \frac{y'}{r} - \frac{n^2+d^2}{r^2}y + \frac{2nd}{r^2}x + \mu(1-f_d^2)y &= 0. \end{cases} \tag{3.83}$$

We let

$$\tilde{x}(r) := r^{\frac{1}{2}}x(r).$$

Thanks to

$$x'' + \frac{x'}{r} - \frac{1}{4r^2}x = r^{-\frac{1}{2}}\tilde{x}'',$$

we can replace the first equation of (3.83) by

$$\tilde{x}'' - 2\tilde{x} + \frac{-n^2-d^2+\frac{1}{4}}{r^2}\tilde{x} + \frac{2nd}{r^{\frac{3}{2}}}y + 2(1-f_d^2)\tilde{x} + \mu(1-f_d^2)\tilde{x} = 0,$$

that can be written as

$$(e^{2\sqrt{2}r}(r^{\frac{1}{2}}xe^{-\sqrt{2}r})')' = e^{\sqrt{2}r}\left((\frac{n^2+d^2-\frac{1}{4}}{r^2} - (\mu+2)(1-f_d^2))r^{\frac{1}{2}}x - \frac{2nd}{r^{\frac{3}{2}}}y\right)$$

$$\text{or} \tag{3.84}$$

$$(e^{-2\sqrt{2}r}(r^{\frac{1}{2}}xe^{\sqrt{2}r})')' = e^{-\sqrt{2}r}\left((\frac{n^2+d^2-\frac{1}{4}}{r^2} - (\mu+2)(1-f_d^2))r^{\frac{1}{2}}x - \frac{2nd}{r^{\frac{3}{2}}}y\right).$$

The second equation of the system (3.83) can be written as

$$y'' + \frac{y'}{r} - \frac{n^2+d^2(1-\mu)}{r^2}y + \frac{2nd}{r^2}x + \mu(1-f_d^2 - \frac{d^2}{r^2})y = 0,$$

that gives

$$(r^{2n_\mu+1}(r^{-n_\mu}y)')' = r^{n_\mu+1}(-\tfrac{2nd}{r^2}x - \mu(1-f_d^2-\tfrac{d^2}{r^2})y)$$

$$\text{or} \tag{3.85}$$

$$(r^{-2n_\mu+1}(r^{n_\mu}y)')' = r^{-n_\mu+1}(-\tfrac{2nd}{r^2}x - \mu(1-f_d^2-\tfrac{d^2}{r^2})y).$$

To prove Proposition 3.4 and Proposition 3.5, we are looking for solutions of the system (3.83) of the form

$$x = hJ_+ \text{ and } y = kr^{n_\mu} \text{ or } x = hJ_- \text{ and } y = kr^{-n_\mu}$$

and we will let $\lim_{+\infty}(h,k) = (1,0)$ in Proposition 3.4 and $\lim_{+\infty}(h,k) = (0,1)$ in Proposition 3.5.
We use (3.84) and (3.85) to infer that $(h,k)$ must verify the following systems,
when $(x,y) = (hJ_+, kr^{n_\mu})$

$$\begin{cases} (e^{2\sqrt{2}r}h')' = e^{\sqrt{2}r}\left(q(r)e^{\sqrt{2}r}h - \frac{2nd}{r^{3/2}}kr^{n_\mu}\right) \\ (r^{2n_\mu+1}k')' = r^{n_\mu+1}(-\frac{2nd}{r^2}J_+h - \mu(1-f_d^2-\frac{d^2}{r^2})kr^{n_\mu}) \end{cases} \tag{3.86}$$

or, when $(x,y) = (hJ_-, kr^{-n_\mu})$

$$\begin{cases} (e^{-2\sqrt{2}r}h')' = e^{-\sqrt{2}r}\left(q(r)e^{-\sqrt{2}r}h - \frac{2nd}{r^{3/2}}kr^{-n_\mu}\right) \\ (r^{-2n_\mu+1}k')' = r^{-n_\mu+1}(-\frac{2nd}{r^2}J_-h - \mu(1-f_d^2-\frac{d^2}{r^2})kr^{-n_\mu}) \end{cases} \tag{3.87}$$

where

$$q(r) = (n^2+d^2-\frac{1}{4})r^{-2} - (\mu+2)(1-f_d^2).$$

In what follows, we will need the following estimates, obtained by an integration by part. Let $\gamma \in \mathbb{R}$ and $\delta > 0$ be given. Then

$$\int_t^{+\infty} s^\gamma e^{-\delta s}ds \le \frac{2}{\delta}t^\gamma e^{-\delta t} \quad \text{for all } t \ge \frac{2\gamma}{\delta} \tag{3.88}$$

and

$$\int_R^t s^\gamma e^{\delta s}ds \le \begin{cases} \frac{2}{\delta}t^\gamma e^{\delta t} & \text{for all } t \ge R \ge \frac{-2\gamma}{\delta} \quad \text{if } \gamma < 0 \\ \frac{1}{\delta}t^\gamma e^{\delta t} & \text{for all } t \ge R > 0 \quad \text{if } \gamma \ge 0. \end{cases} \tag{3.89}$$

And we will use, for $s \geq R$ and $t \geq R$

$$|\mu|(1-f_d^2) \leq \frac{C}{r^2}, \quad |1-f_d^2-\frac{d^2}{r^2}| \leq \frac{C}{r^4} \quad \text{and} \quad |q(s)| \leq \frac{C}{r^2} \tag{3.90}$$

where $C$ is independent of $R > 1$ and independent of $\mu$, provided $|\mu - 1| \leq 1$.
We will fix $R > 1$, large enough to have (3.88), (3.89) and (3.90), for the desired $\gamma$ and $\delta$ we will have to use.

### 3.2.1 Proof of Proposition 3.4.

We devide the proof into 3 steps. In Step 1, we construct solutions of (3.83) of the form $(x, y) = (hJ_+, kr^n)$. In Step 2, we construct solutions of the form $(x, y) = (hJ_-, kr^{-n})$. In the both steps, $\lim_{+\infty}(h, k) = (1, 0)$. In step 3, we conclude the proof of the proposition.

**Step 1**. The exponentional blowing up behavior.
We prove the following

**Lemma 3.2** *For all $\mu$ verifying $|\mu - 1| \leq 1$, there exists a solution of (3.86), denoted by $(h_1, k_1)$, that is defined in some interval $[R, +\infty[$, where $R$ is independent of the parameter $\mu$. It is continuous wrt the parameter $\mu$ in the sense that $(h, h', k, k')(r)$ tends to a limit as $\mu$ tends to a limit $\mu_0$, for all $r \geq R$. Moreover, the behaviors at $+\infty$ are given by*

$$|h_1(r) - 1| \leq Mr^{-1} \quad \textit{and} \quad |k_1(r)| \leq Me^{\sqrt{2}r}r^{-n_\mu - \frac{5}{2}} \tag{3.91}$$

*for all $r \in [R, +\infty[$ and for some $M$ independent of $\mu$.*

**Proof** Let $R > 0$ be given. We rewrite (3.86) as the fixed point problem

$$\begin{cases} h &= \alpha_0 + \int_{+\infty}^r e^{-2\sqrt{2}t} \int_R^t e^{\sqrt{2}s}(-\frac{2nd}{s^{3/2}}s^{n_\mu}k + q(s)e^{\sqrt{2}s}h)dsdt \\ k &= \beta_0 + \int_R^r t^{-2n_\mu - 1} \int_R^t s^{n_\mu+1}(-\frac{2nd}{s^2}s^{-1/2}e^{\sqrt{2}s}h - \mu(1-f_d^2-\frac{d^2}{s^2})s^{n_\mu}k)dsdt. \end{cases} \tag{3.92}$$

In what follows, we may choose $R$ as large as necessary, but this choice will be always valid for all $\mu$ belonging to the desired range. We will denote each fixed point equation by

$$(h, k) = \Phi(h, k), \text{ for } (h, k) \text{ defined in } [R, +\infty[.$$

We define a sequence $(\alpha_j, \beta_j)$ by the induction

$$\begin{array}{c} (\alpha_{j+1}, \beta_{j+1}) = \Phi(\alpha_j, \beta_j) \\ (\alpha_0, \beta_0) = (1, 0). \end{array} \tag{3.93}$$

Using (3.88), (3.89) and (3.90), we estimate, for $r \geq R_0$,

$$|\alpha_1 - \alpha_0| \leq C \int_r^{+\infty} e^{-2\sqrt{2}t} \int_R^t e^{2\sqrt{2}s}s^{-2}dsdt \leq C \int_r^{+\infty} e^{-2\sqrt{2}t}\frac{1}{\sqrt{2}}t^{-2}e^{2\sqrt{2}t}dt$$

that gives

$$|\alpha_1 - \alpha_0| \leq Cr^{-1}. \tag{3.94}$$

In the same way,

$$|\beta_1-\beta_0|\leq\int_R^r t^{-2n_\mu-1}\int_R^t 2nds^{n_\mu-3/2}e^{\sqrt{2}s}dsdt\leq\int_R^r t^{-2n_\mu-1}\frac{4nd}{\sqrt{2}}t^{n_\mu-3/2}e^{\sqrt{2}t}dt$$

$$\leq\frac{4nd}{\sqrt{2}}\int_R^r t^{-n_\mu-5/2}e^{\sqrt{2}t}dt$$

that gives

$$|\beta_1-\beta_0|\leq 4ndr^{-n_\mu-5/2}e^{\sqrt{2}r}. \tag{3.95}$$

And we estimate, for all $j\geq 1$

$$|\alpha_{j+1}-\alpha_j|\leq\int_r^{+\infty}e^{-2\sqrt{2}t}\int_R^t e^{\sqrt{2}s}[Ce^{\sqrt{2}s}s^{-3}\|r(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+$$

$$+2nds^{-4}e^{\sqrt{2}s}\|r^{n_\mu+\frac{5}{2}}e^{-\sqrt{2}r}(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}]dsdt$$

$$\leq\int_r^{+\infty}\left(\frac{C}{\sqrt{2}}t^{-3}\|r(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+\sqrt{2}2ndt^{-4}e^{-\sqrt{2}t}\|r^{n_\mu+\frac{5}{2}}e^{-\sqrt{2}r}(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}\right)dt$$

that gives

$$\begin{aligned}|\alpha_{j+1}-\alpha_j|&\leq Cr^{-2}\|r(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+\\&+4ndr^{-4}e^{-\sqrt{2}r}\|r^{n_\mu+\frac{5}{2}}e^{-\sqrt{2}r}(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}.\end{aligned} \tag{3.96}$$

And we estimate

$$|\beta_{j+1}-\beta_j|\leq\int_R^r t^{-2n_\mu-1}\int_R^t[2nds^{n_\mu-\frac{5}{2}}e^{\sqrt{2}s}\|r(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+$$

$$+Cs^{n_\mu-3-\frac{5}{2}}e^{\sqrt{2}s}\|r^{n_\mu+\frac{5}{2}}e^{-\sqrt{2}r}(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}]dsdt.$$

That gives

$$|\beta_{j+1}-\beta_j|\leq\int_R^r t^{-2n_\mu-1}(2nd\frac{2}{\sqrt{2}}t^{n_\mu-\frac{5}{2}}e^{\sqrt{2}t}\|r(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+$$

$$+C\frac{1}{\sqrt{2}}t^{n_\mu-\frac{11}{2}}e^{\sqrt{2}t}\|r^{n_\mu+\frac{5}{2}}e^{-\sqrt{2}r}(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)})dt.$$

And we obtain

$$\begin{aligned}|\beta_{j+1}-\beta_j|&\leq 4nde^{\sqrt{2}r}r^{-n_\mu-7/2}\|r(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+\\&+Cr^{-n_\mu-\frac{13}{2}}e^{\sqrt{2}r}\|r^{n_\mu+\frac{5}{2}}e^{-\sqrt{2}r}(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}.\end{aligned} \tag{3.97}$$

We let $K=\max\{2nd,C\}$ and we deduce from (3.94), (3.95), (3.96) and (3.97) that for all $j\geq 0$

$$\begin{aligned}&\|r(\alpha_{j+1}-\alpha_j)\|_{L^\infty([R,+\infty[)}\leq 2^jK^{j+1}R^{-j}\\&\text{and}\quad\|r^{n_\mu+\frac{5}{2}}e^{-\sqrt{2}r}(\beta_{j+1}-\beta_j)\|_{L^\infty([R,+\infty[)}\leq 2^jK^{j+1}R^{-j}.\end{aligned} \tag{3.98}$$

We chose $R$ large enough to have $2KR^{-1} < 1/2$ and we define

$$\begin{aligned} & h(r) = \alpha_0 + r^{-1} \textstyle\sum_{j\geq 0} r(\alpha_{j+1} - \alpha_j)(r) \\ \text{and} \qquad & k(r) = \beta_0 + e^{\sqrt{2}r} r^{-n_\mu + \frac{5}{2}} \textstyle\sum_{j\geq 0} e^{-\sqrt{2}r} r^{n_\mu - \frac{5}{2}} (\beta_{j+1} - \beta_j)(r). \end{aligned} \tag{3.99}$$

Thanks to (3.98), the sums converge in $L^\infty([R, +\infty[)$. Morover, by its construction, $(h, k)$ is a solution of the fixed point problem (3.92). Let us remark that an induction together with (3.96) and (3.97) leads to the continuity wrt the parameter $\mu$ of $\alpha_j(r)$ and $r^{n_\mu + \frac{5}{2}} e^{-\sqrt{2}r} \beta_j(r)$, for all $j \geq 1$ and for all $r \geq R$, by the Lebesgue Theorem. And since $K$ is independent of $\mu$, we deduce that $h(r)$ and $k(r)$ depend continuously on $\mu$, for all $r \geq R$.
We can compute the behaviors at $+\infty$. We deduce from (3.98) that for $r \geq R$

$$|h(r) - 1| \leq r^{-1} K (1 - 2KR^{-1})^{-1}$$

and that

$$|k(r)| \leq e^{\sqrt{2}r} r^{-n_\mu - \frac{5}{2}} K (1 - 2KR^{-1})^{-1}.$$

This gives the proof of (3.91).
Now we can compute

$$\begin{cases} h' &= e^{-2\sqrt{2}r} \int_R^r e^{\sqrt{2}s} (-\frac{2nd}{s^{3/2}} s^n k + q(s) e^{\sqrt{2}s} h) ds \\ \\ k' &= r^{-2n-1} \int_R^r s^{n+1} (\frac{2nd}{s^2} s^{-1/2} e^{\sqrt{2}s} h - \mu(1 - f_d^2) s^n k) ds. \end{cases}$$

So, $h'(r)$ and $k'(r)$ depend continuously on $\mu$, too, for all $r \geq R$.
This terminates the proof of Lemma 3.2.

**Step 2.** The exponentional vanishing behavior.
We prove the following

**Lemma 3.3** *For all $\mu$ verifyng $|\mu - 1| \leq 1$, there exists a solution of (3.87), denoted by $(h_2, k_2)$, that is defined in some interval $[R, +\infty[$, where $R$ is independent of the parameter $\mu$. It is continuous wrt the parameter $\mu$ in the sense that $(h, h', k, k')(r)$ tends to a limit as $\mu$ tends to a limit $\mu_0$. Moreover, its behavior at $+\infty$ is given by*

$$|h_2(r) - 1| \leq M r^{-1} \quad \textit{and} \quad |k_2(r)| \leq M e^{-\sqrt{2}r} r^{n_\mu - \frac{5}{2}} \tag{3.100}$$

*for all $r \in [R, +\infty[$ and for some $M$ independent of $\mu$.*

**Proof** We return to the system (3.87) and we rewrite it as the fixed point problem

$$\begin{cases} h &= \alpha_0 + \int_{+\infty}^r e^{2\sqrt{2}t} \int_{+\infty}^t e^{-\sqrt{2}s} (-\frac{2nd}{s^{3/2}} s^{-n_\mu} k + q(s) e^{-\sqrt{2}s} h) ds dt \\ \\ k &= \beta_0 + \int_{+\infty}^r t^{2n_\mu - 1} \int_{+\infty}^t s^{-n_\mu + 1} (-\frac{2nd}{s^2} s^{-1/2} e^{-\sqrt{2}s} h - \mu(1 - f_d^2 - \frac{d^2}{r^2}) s^{-n_\mu} k) ds dt. \end{cases} \tag{3.101}$$

We define $(\alpha_j, \beta_j)$ for all $j \geq 0$ by the induction (3.93) again.
Using (3.88), (3.89) and (3.90), we estimate

$$|\alpha_1 - \alpha_0| \leq \int_r^{+\infty} e^{2\sqrt{2}t} \int_t^{+\infty} Cs^{-2} e^{-2\sqrt{2}s} ds dt \leq \frac{C}{\sqrt{2}} \int_r^{+\infty} t^{-2} dt$$

that gives

$$|\alpha_1 - \alpha_0| \leq Cr^{-1}. \tag{3.102}$$

And

$$\begin{aligned} |\beta_1 - \beta_0| &\leq \int_r^{+\infty} t^{2n_\mu - 1} \int_t^{+\infty} s^{-n_\mu - 3/2} 2nde^{-\sqrt{2}s} ds dt \\ &\leq \frac{2}{\sqrt{2}} \int_r^{+\infty} t^{n_\mu - 5/2} 2nde^{-\sqrt{2}t} dt \end{aligned}$$

that gives

$$|\beta_1 - \beta_0| \leq 4nde^{-\sqrt{2}r} r^{n_\mu - 5/2}. \tag{3.103}$$

And for all $j \geq 1$

$$\begin{aligned} |\alpha_{j+1} - \alpha_j| \leq \int_r^{+\infty} e^{2\sqrt{2}t} \int_t^{+\infty} (2nds^{-4} e^{-2\sqrt{2}s} \| r^{-n_\mu + \frac{5}{2}} e^{\sqrt{2}r} (\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)} + \\ + Cs^{-3} e^{-2\sqrt{2}s} \| r(\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)}) ds dt \\ \leq \int_r^{+\infty} e^{2\sqrt{2}t} \frac{4nd}{2\sqrt{2}} t^{-4} e^{-2\sqrt{2}t} \| r^{-n_\mu + \frac{5}{2}} e^{\sqrt{2}r} (\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)} dt + \\ + \int_r^{+\infty} e^{2\sqrt{2}t} \frac{C}{\sqrt{2}} t^{-3} e^{-2\sqrt{2}t} \| r(\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)}) dt \end{aligned}$$

and we are led to

$$\begin{aligned} |\alpha_{j+1} - \alpha_j| \leq \tfrac{4nd}{\sqrt{2}} r^{-3} \| r^{-n_\mu + \frac{5}{2}} e^{\sqrt{2}r} (\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)} + \\ + \tfrac{C}{\sqrt{2}} r^{-2} \| r(\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)}. \end{aligned} \tag{3.104}$$

Now, we estimate

$$\begin{aligned} |\beta_{j+1} - \beta_j| \leq \int_r^{+\infty} t^{2n_\mu - 1} \int_t^{+\infty} s^{-n_\mu + 1} (2nds^{-7/2} e^{-\sqrt{2}s} \| r(\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} + \\ + Cs^{-4-\frac{5}{2}} e^{-\sqrt{2}s} \| r^{-n_\mu + \frac{5}{2}} e^{\sqrt{2}r} (\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)}) ds dt, \end{aligned}$$

that gives

$$\begin{aligned} |\beta_{j+1} - \beta_j| \leq \int_r^{+\infty} t^{2n_\mu - 1} (\frac{4nd}{\sqrt{2}} t^{-n_\mu - \frac{5}{2}} e^{-\sqrt{2}t} \| r(\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} + \\ + \frac{2C}{\sqrt{2}} t^{-n_\mu - \frac{11}{2}} e^{-\sqrt{2}t} \| r^{-n_\mu + \frac{5}{2}} e^{\sqrt{2}r} (\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)}) dt. \end{aligned}$$

We are led to

$$|\beta_{j+1}-\beta_j| \le 4ndr^{n_\mu-7/2}e^{-\sqrt{2}r}\|r(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+ \\ +2Cr^{n_\mu-\frac{13}{2}}e^{-\sqrt{2}r}\|r^{-n_\mu+\frac{5}{2}}e^{\sqrt{2}r}(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}. \tag{3.105}$$

Letting $R>1$ and $K=\max\{4nd,2C\}$, we deduce from (3.102), (3.103), (3.104) and (3.105) that for all $j\ge 0$,

$$\|r(\alpha_{j+1}-\alpha_j)\|_{L^\infty([R,+\infty[)} \le 2^jK^{j+1}R^{-j} \\ \text{and} \qquad \|r^{-n_\mu+\frac{5}{2}}e^{\sqrt{2}r}(\beta_{j+1}-\beta_j)\|_{L^\infty([R,+\infty[)} \le 2^jK^{j+1}R^{-j}. \tag{3.106}$$

We chose $R$ large enough to have $2KR^{-1}<1/2$ and we define

$$h = 1+r^{-1}\textstyle\sum_{j\ge 0} r(\alpha_{j+1}-\alpha_j) \\ \text{and} \qquad k = e^{-\sqrt{2}r}r^{n_\mu-\frac{5}{2}}\textstyle\sum_{j\ge 0} e^{\sqrt{2}r}r^{-n_\mu+\frac{5}{2}}(\beta_{j+1}-\beta_j). \tag{3.107}$$

Moreover, by its construction, $(h,k)$ is a solution of (3.101). As in Step 1, an induction together with (3.104) and (3.105) leads to the continuity wrt the parameter $\mu$ of $\alpha_j$ and of $r^{-n_\mu}e^{\sqrt{2}r}\beta_j$, for all $j\ge 1$. And (3.106) gives the continuity of $h(r)$ and $k(r)$ wrt $\mu$, for all $r\ge R$. Now let us verify the behaviors at $+\infty$. We use (3.106) to obtain, for all $r\ge R$,

$$|h(R)-1| \le r^{-1}K(1-2KR^{-1})^{-1} \text{ and } |k(R)| \le e^{-\sqrt{2}r}r^{n_\mu-\frac{5}{2}}K(1-2KR^{-1})^{-1}.$$

This gives (3.100).
Now we write

$$\begin{cases} h' &= e^{2\sqrt{2}r}\int_{+\infty}^r e^{-\sqrt{2}s}(-\frac{2nd}{s^{3/2}}s^{-n}k+q(s)e^{-\sqrt{2}s}h)dsdt \\ k' &= r^{2n-1}\int_{+\infty}^r s^{-n+1}(\frac{2nd}{s^2}s^{-1/2}e^{-\sqrt{2}s}h-\mu(1-f_d^2)s^{-n}k)dsdt \end{cases}$$

and we deduce that $h'$ and $k'$ are continuous wrt $\mu$, too.
This terminates the proof of Lemma 3.3.

**Step 3** Let us conclude the proof of Proposition 3.4.
To construct the solution $(u_1^\mu,v_1^\mu)$, we first denote, for $r\in[R,+\infty[$,

$$x_1 = h_1J_+ \quad \text{and} \quad y_1 = k_1r^{n_\mu}$$

where $R$, $(h_1,k_1)$ are defined in Lemma 3.2. Thus, $(x_1,y_1)$ is a solution of (3.83). By (3.91), we obtain, with the same $M$ independent of $\mu$ and for all $r\in[R,+\infty[$

$$|x_1-J_+| \le Mr^{-1}J_+ \quad \text{and} \quad |y_1| \le MJ_+r^{-2}. \tag{3.108}$$

that there is a unique prolongation of $(x_1,y_1)$ as a solution of (3.83) defined in $]0,+\infty[$. And denoting also this prolongation by $(x_1,y_1)$ we have that $(x_1,x_1',y_1,y_1')$ is continuous

wrt $\mu$, in $(L^\infty([r_0, r_1]))^4$, for all $r_1 > r_0 > 0$.
Now we define

$$u_1^\mu = (x_1 + y_1)/2 \quad \text{and} \quad v_1^\mu = (x_1 - y_1)/2$$

to obtain that $(u_1^\mu, v_1^\mu)$ is a solution of (1.25) and we use (3.108) to infer that in $[R, +\infty[$

$$|u_1^\mu - J_+| + |v_1^\mu - J_+| \leq Cr^{-1}J_+$$

with $C$ independent of $\mu$. and we have that $Y_1^\mu(r) \to Y_1^1(r)$ as $\mu \to 1$, for all $r > 0$ in $(L^\infty([r_0, r_1]))^4$, for all $0 < r_0 < r_1$.

Now, to construct the solution $(u_2^\mu, v_2^\mu)$, we first denote, for $r \in [R, +\infty[$,

$$x_2 = h_2 J_- \quad \text{and} \quad y_2 = k_2 r^{-n_\mu}$$

where $R$, $(h_2, k_2)$ are defined in Lemma 3.3. By (3.100), we obtain, with the same $M$ independent of $\mu$ and for all $r \in [R, +\infty[$

$$|x_2 - J_-| \leq Mr^{-1}J_- \quad \text{and} \quad |y_2| \leq Me^{-\sqrt{2}r}r^{-2}. \tag{3.109}$$

We follow the same pattern of proof as for $(u_1^\mu, v_1^\mu)$. We denote

$$u_2^\mu = (x_2 + y_2)/2 \quad \text{and} \quad v_2^\mu = (x_2 - y_2)/2$$

and we deduce that, in $[R, +\infty[$

$$|u_2^\mu - J_-| \leq MJ_-r^{-1} \quad \text{and} \quad |v_2^\mu - J_-| \leq MJ_-r^{-1}.$$

The prolongation of $(u_2^\mu, v_2^\mu)$ as a solution of (1.25) in $]0, +\infty[$, and the continuity of this solution wrt $\mu$ follow from the same principle as above.
This terminates the proof of Proposition 3.4.

### 3.2.2 Proof of Proposition 3.5.

We are looking for solutions of the system (3.83) of the form

$$x = hJ_+ \text{ and } y = kr^{n_\mu} \text{ or } x = hJ_- \text{ and } y = kr^{-n_\mu}, \text{ with } \lim_{+\infty}(h, k) = (0, 1).$$

**Step 1** the polynomial blowing up behavior at $+\infty$.
We are looking for $(x, y) = (hJ_+, kr^{n_\mu})$, with $\lim_{+\infty} h = 0$ and $\lim_{+\infty} k = 1$. We prove the following

**Lemma 3.4** *For all $\mu$ verifyng $-\frac{1}{2} \leq d^2(1 - \mu) \leq \frac{1}{2}$, there exists a solution of (3.86), denoted by $(h_3, k_3)$, that is defined in some interval $[R, +\infty[$, where $R$ is independent of the parameter $\mu$. It is continuous wrt the parameter $\mu$ in the sense that $(h, h', k, k')(r)$ tends to a limit as $\mu$ tends to a limit $\mu_0$, for all $r \in [R, +\infty[$. Moreover, its behavior at $+\infty$ is given by*

$$|h_3(r)| \leq Me^{-\sqrt{2}r}r^{n_\mu - \frac{3}{2}} \quad \textit{and} \quad |k_3(r) - 1| \leq Mr^{-1} \tag{3.110}$$

*for all $r \in [R, +\infty[$ and for some $M$ independent of $\mu$.*

**Proof** First, let us consider the case $n \geq 2$.
Let $R > 0$ be given. We rewrite (3.86) as the fixed point problem

$$\begin{cases} h &= \alpha_0 + \int_{+\infty}^{r} e^{-2\sqrt{2}t} \int_R^t e^{\sqrt{2}s}(-\frac{2nd}{s^{3/2}} s^{n_\mu} k + q(s) e^{\sqrt{2}s} h) ds dt \\ k &= \beta_0 + \int_{+\infty}^{r} t^{-2n_\mu - 1} \int_R^t s^{n_\mu+1}(-\frac{2nd}{s^2} s^{-1/2} e^{\sqrt{2}s} h - \mu(1 - f_d^2 - \frac{d^2}{s^2}) s^{n_\mu} k) ds dt. \end{cases} \tag{3.111}$$

We define a sequence $(\alpha_j, \beta_j)$ by the induction

$$\begin{array}{c} (\alpha_{j+1}, \beta_{j+1}) = \Phi(\alpha_j, \beta_j) \\ (\alpha_0, \beta_0) = (0, 1). \end{array} \tag{3.112}$$

Using (3.88), (3.89) and (3.90), we estimate, for $r \geq R$,

$$|\alpha_1 - \alpha_0| \leq \int_r^{+\infty} e^{-2\sqrt{2}t} \int_R^t e^{\sqrt{2}s} 2nd s^{n_\mu - 3/2} ds dt$$

$$\leq 2nd \int_r^{+\infty} e^{-2\sqrt{2}t} \frac{2}{\sqrt{2}} t^{n_\mu - 3/2} e^{\sqrt{2}t} dt$$

that gives

$$|\alpha_1 - \alpha_0| \leq 2nd r^{n_\mu - 3/2} e^{-\sqrt{2}r}. \tag{3.113}$$

Now, using (3.90), we obtain

$$|\beta_1 - \beta_0| \leq \int_r^{+\infty} t^{-2n_\mu - 1} \int_R^t s^{n_\mu + 1} C s^{n_\mu - 4} ds dt.$$

Let us remark that when $n \geq 2$, we have $2n_\mu - 3 > -1$. Indeed, we have supposed that $d^2|1 - \mu| < 1/2$. So, when $n \geq 2$, we have $4n_\mu^2 \geq 14$, so $2n_\mu > 3$ and we obtain

$$|\beta_1 - \beta_0| \leq \frac{C}{2n_\mu - 2} \int_r^{+\infty} t^{-2n_\mu - 1} t^{2n_\mu - 2} dt \leq \frac{C}{4n_\mu - 4} r^{-2}.$$

that gives, when $n \geq 2$

$$|\beta_1 - \beta_0| \leq \frac{C}{2} r^{-2}. \tag{3.114}$$

We estimate, for $j \geq 1$

$$|\alpha_{j+1} - \alpha_j| \leq \int_r^{+\infty} e^{-2\sqrt{2}t} \int_R^t e^{\sqrt{2}s} (2nd s^{n_\mu - \frac{3}{2} - 1} \|r(\beta_j - \beta_{j-1})\|_{L^\infty([R,+\infty[)} ds +$$

$$+ C s^{n_\mu - 2 - \frac{3}{2}} \|r^{-n_\mu + \frac{3}{2}} e^{\sqrt{2}r} (\alpha_j - \alpha_{j-1})\|_{L^\infty([R,+\infty[)} ds) dt$$

$$\leq \int_r^{+\infty} e^{-\sqrt{2}t} (\frac{4nd}{\sqrt{2}} t^{n_\mu - 5/2} \|r(\beta_j - \beta_{j-1})\|_{L^\infty([R,+\infty[)} + \frac{2C}{\sqrt{2}} t^{n_\mu - \frac{7}{2}} \|r^{-n_\mu + \frac{3}{2}} e^{\sqrt{2}r} (\alpha_j - \alpha_{j-1})\|_{L^\infty([R,+\infty[)}) dt.$$

And we obtain

$$\begin{array}{c} |\alpha_{j+1} - \alpha_j| \leq 4nd r^{n_\mu - 5/2} e^{-\sqrt{2}r} \|r(\beta_j - \beta_{j-1})\|_{L^\infty([R,+\infty[)} + \\ + 2C e^{-\sqrt{2}r} r^{n_\mu - \frac{7}{2}} \|r^{-n_\mu + \frac{3}{2}} e^{\sqrt{2}r} (\alpha_j - \alpha_{j-1})\|_{L^\infty([R,+\infty[)}. \end{array} \tag{3.115}$$

Now we estimate

$$|\beta_{j+1}-\beta_j| \le \int_r^{+\infty} t^{-2n_\mu-1}\int_R^t s^{n_\mu+1}(2nds^{n_\mu-4}\|r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+$$

$$+2Cs^{n_\mu-5}\|r(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)})dsdt.$$

Since $2n_\mu-3>0$, we obtain

$$|\beta_{j+1}-\beta_j| \le \int_r^{+\infty} t^{-2n_\mu-1}2nd\frac{t^{2n_\mu-2}}{2n_\mu-2}\|r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}dt+$$

$$+2C\int_r^{+\infty} t^{-2n_\mu-1}\frac{t^{2n_\mu-3}}{2n_\mu-3}\|r(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[}dt$$

that gives

$$|\beta_{j+1}-\beta_j| \le \frac{nd}{(2n_\mu-2)}r^{-2}\|r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)}+\frac{2C}{3(2n_\mu-3)}r^{-3}\|r(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}).$$

We use $n_\mu^2 \ge 7/2$ to obtain

$$\begin{aligned}|\beta_{j+1}-\beta_j| &\le \tfrac{nd}{2\sqrt{3}-2}r^{-2}\|r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[}+\\ &+\tfrac{2C}{3(2\sqrt{3}-3)}r^{-3}\|r(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)}.\end{aligned} \tag{3.116}$$

We let $K=\max\{\frac{nd}{2\sqrt{3}-2},\frac{2C}{3(2\sqrt{3}-3)},4nd,2C\}$ and we deduce from (3.113), (3.114), (3.115) and (3.116) that for all $r\ge R$ and for all $j\ge 0$, we have

$$\begin{aligned}&\|r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_{j+1}-\alpha_j)\|_{L^\infty([R,+\infty[)} \le 2^jK^{j+1}R^{-j}\\ \text{and}\quad &\|r(\beta_{j+1}-\beta_j)\|_{L^\infty([R,+\infty[)} \le 2^jK^{j+1}R^{-j}.\end{aligned} \tag{3.117}$$

We chose $R$ large enough to have also $2KR^{-1}<\frac{1}{2}$.
Now we may define $(h_3,k_3)$ by

$$h_3(r)=e^{-\sqrt{2}r}r^{n_\mu-\frac{3}{2}}\sum_{j\ge 0}r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_{j+1}-\alpha_j) \quad\text{and}\quad k_3(r)=1+r^{-1}\sum_{j\ge 0}r(\beta_{j+1}-\beta_j).$$

To prove (3.110), we write, for $r\in[R,+\infty[$

$$e^{\sqrt{2}r}r^{-n_\mu+\frac{3}{2}}|h_3(r)| \le \sum_{j\ge 0}\|r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_j-\alpha_{j-1})\|_{L^\infty([R,+\infty[)},$$

that gives, for $r\ge R$,

$$|h_3(r)| \le Me^{-\sqrt{2}r}r^{n_\mu-\frac{3}{2}}.$$

And we write

$$r|k_3(r)-1| \le K(1-2KR^{-1})^{-1},$$

that gives

$$|k_3(R) - 1| \leq Mr^{-1}.$$

Thus, we obtain (3.110).
The continuity of $\alpha_j$, $\beta_j$, of $h_3$ and $k_3$ and further of $h_3'$ and $k_3'$ wrt the parameter $\mu$ follows as in the proof of Lemma 3.2. This terminates the proof of Lemma 3.4, for $n \geq 2$.

Now, when $n = 1$, we use Theorem 1.4 (iii) to see that only the case $\mu \geq 1$ and $\mu$ closed to 1 is of interest. In this case, letting $0 < d^2(\mu - 1) < 1/2$, we have $\sqrt{1/2} < n_\mu \leq 1$. Let us indicate what changes in the proof above. We keep using the fixed point problem (3.111). The inequalities (3.113) and (3.115) remain inchanged. And since now

$$\sqrt{1/2} < n_\mu \leq 1,$$

we estimate

$$|\beta_1 - \beta_0| \leq \int_r^{+\infty} t^{-2n_\mu - 1} \int_R^t 2Cs^{2n_\mu - 3} ds dt$$

and since $2n_\mu - 3 < -1$ and since $R > 1$, that gives (we use $s^{2n_\mu - 3} \leq s^{-1}$, to avoid to divise by $2n_\mu - 2$, for the sake of the continuity when $\mu \to 1$, that implies $n_\mu \to 1$)

$$|\beta_1 - \beta_0| \leq \frac{C}{n_\mu} r^{-2n_\mu} \ln r$$

and that gives

$$|\beta_1 - \beta_0| \leq C\sqrt{2} r^{-2n_\mu} \ln r. \tag{3.118}$$

And

$$|\beta_{j+1} - \beta_j| \leq \int_r^{+\infty} t^{-2n_\mu - 1} \int_R^t s^{n_\mu + 1} (2nds^{n_\mu - 4} \| r^{-n_\mu + \frac{3}{2}} e^{\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} +$$

$$+ 2Cs^{n_\mu - 5} \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)}) ds dt.$$

As above, we use $R > 1$ and we obtain

$$|\beta_{j+1} - \beta_j| \leq \int_r^{+\infty} t^{-2n_\mu - 1} (2nd \ln t \| r^{-n_\mu + \frac{3}{2}} e^{\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} dt +$$

$$+ 2C \ln t \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)} dt$$

that gives

$$|\beta_{j+1} - \beta_j| \leq \frac{nd}{n_\mu} r^{-2n_\mu} \ln r \| r^{-n_\mu + \frac{3}{2}} e^{\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} +$$

$$+ \frac{C}{n_\mu} r^{-2n_\mu} \ln r \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)}).$$

We use $2n_\mu \geq \sqrt{2}$ to obtain

$$\begin{aligned} |\beta_{j+1} - \beta_j| &\leq nd\sqrt{2} r^{-\sqrt{2}} \ln r \| r^{-n_\mu + 1} e^{\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} + \\ &+ C\sqrt{2} r^{-\sqrt{2}} \ln r \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)}). \end{aligned} \tag{3.119}$$

We let $K = \max\{C\sqrt{2}, nd\sqrt{2}\}$. We deduce from (3.113), (3.115), (3.118) and (3.119) that

$$\begin{aligned} &\|r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_{j+1}-\alpha_j)\|_{L^\infty([R,+\infty[)} \le 2^j K^{j+1}(R^{-\sqrt{2}+1}\ln R)^j \\ \text{and} \quad &\|r(\beta_j-\beta_{j-1})\|_{L^\infty([R,+\infty[)} \le 2^j K^{j+1}(R^{-\sqrt{2}+1}\ln R)^j \end{aligned} \tag{3.120}$$

We chose $R$ large enough to have $2KR^{-\sqrt{2}+1}\ln R < \frac{1}{2}$ and we define

$$h_3(r) = e^{-\sqrt{2}r}r^{n_\mu-\frac{3}{2}}\sum_{j\ge 0} r^{-n_\mu+\frac{3}{2}}e^{\sqrt{2}r}(\alpha_{j+1}-\alpha_j) \quad \text{and} \quad k_3(r) = 1 + r^{-1}\sum_{j\ge 0} r(\beta_{j+1}-\beta_j).$$

and we conclude that the proof of Lemma 3.4, we have done above for $n \ge 2$, works also when $n = 1$.
This terminates the proof of Lemma 3.4.

**Step 2** The polynomial vanishing behavior at $+\infty$.
We are looking for $(x, y) = (hJ_-, kr^{-n_\mu})$, with $\lim_{+\infty} h = 0$ and $\lim_{+\infty} k = 1$. We prove the following

**Lemma 3.5** *For all $\mu$ verifyng $-\frac{1}{2} \le d^2(1-\mu) \le \frac{1}{2}$, there exists a solution of (3.87), denoted by $(h_4, k_4)$, that is defined in some interval $[R, +\infty[$, where $R$ is independent of the parameter $\mu$. It is continuous wrt the parameter $\mu$ in the sense that $(h, h', k, k')(r)$ tends to a limit as $\mu$ tends to a limit $\mu_0$. Moreover, its behavior at $+\infty$ are given by*

$$|h_4(r)| \le Me^{\sqrt{2}r}r^{-n_\mu-\frac{3}{2}} \quad \textit{and} \quad |k_4(r)-1| \le Mr^{-1} \tag{3.121}$$

*for all $r \in [R, +\infty[$ and for some $M$ independent of $\mu$.*

**Proof** Let $R > 0$ be given. We rewrite (3.87) as the fixed point problem

$$\begin{cases} h &= \alpha_0 + \int_R^r e^{2\sqrt{2}t}\int_{+\infty}^t e^{-\sqrt{2}s}(-\frac{2nd}{s^{3/2}}s^{-n_\mu}k + q(s)e^{-\sqrt{2}s}h)dsdt \\ k &= \beta_0 + \int_{+\infty}^r t^{2n_\mu-1}\int_{+\infty}^t s^{-n_\mu+1}(\frac{2nd}{s^2}s^{-1/2}e^{-\sqrt{2}s}h - \mu(1-f_d^2-\frac{d^2}{r^2})s^{-n_\mu}k)dsdt. \end{cases} \tag{3.122}$$

We define a sequence $(\alpha_j, \beta_j)$ by the induction (3.112) again. Using (3.88), (3.89) and (3.90), we estimate, for $r \ge R$,

$$|\alpha_1-\alpha_0| \le \int_R^r e^{2\sqrt{2}t}\int_t^{+\infty} e^{-\sqrt{2}s}2nds^{-n_\mu-3/2}dsdt \le 2nd\int_R^r e^{2\sqrt{2}t}\frac{2}{\sqrt{2}}t^{-n_\mu-3/2}e^{-\sqrt{2}t}dt$$

that gives

$$|\alpha_1-\alpha_0| \le 4ndr^{-n_\mu-3/2}e^{\sqrt{2}r}. \tag{3.123}$$

Now, using (3.90), we obtain

$$|\beta_1-\beta_0| \le \int_r^{+\infty} t^{2n_\mu-1}\int_t^{+\infty} s^{-2n_\mu-3}Cdsdt \le \int_r^{+\infty} t^{-3}\frac{C}{2n_\mu+2}dt.$$

that gives

$$|\beta_1 - \beta_0| \leq \frac{C}{4} r^{-2}. \tag{3.124}$$

We estimate, for $j \geq 1$

$$|\alpha_{j+1} - \alpha_j| \leq \int_R^r e^{2\sqrt{2}t} ( \int_t^{+\infty} e^{-\sqrt{2}s} (2nds^{-n_\mu - 5/2} \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)} +$$

$$+ Cs^{-n_\mu - \frac{7}{2}} e^{-\sqrt{2}s} \| r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} ds) dt$$

$$\leq \int_R^r e^{\sqrt{2}t} t^{-n_\mu - 5/2} \left( \frac{4nd}{\sqrt{2}} \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)} + \frac{2C}{\sqrt{2}} \| r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} \right) dt.$$

And we obtain

$$\begin{aligned} |\alpha_{j+1} - \alpha_j| &\leq 4ndr^{-n_\mu - 5/2} e^{\sqrt{2}r} \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)} + \\ &+ 2Ce^{\sqrt{2}r} r^{-n_\mu - 5/2} \| r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)}. \end{aligned} \tag{3.125}$$

Now we estimate

$$|\beta_{j+1} - \beta_j| \leq \int_r^{+\infty} t^{2n_\mu - 1} \int_t^{+\infty} (2nds^{-2n_\mu - 3} \| r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} +$$

$$+ Cs^{-2n_\mu - 4} \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)}) ds dt.$$

We deduce that

$$|\beta_{j+1} - \beta_j| \leq \int_r^{+\infty} t^{2n_\mu - 1} (2nd \frac{t^{-2n_\mu - 2}}{2n_\mu + 2} dt \| r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} +$$

$$+ C \int_r^{+\infty} t^{2n_\mu - 1} \frac{t^{-2n_\mu - 3}}{2n_\mu + 3} dt \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[)}.$$

And, since $n_\mu > 0$,

$$\begin{aligned} |\beta_{j+1} - \beta_j| &\leq ndr^{-2} \| r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \|_{L^\infty([R,+\infty[)} + \\ &+ \tfrac{C}{4} r^{-3} \| r(\beta_j - \beta_{j-1}) \|_{L^\infty([R,+\infty[}. \end{aligned} \tag{3.126}$$

We let $K = \max\{4nd, 2C\}$ and (3.123), (3.124), (3.125) and (3.126) give that for all $r \geq R$

$$\begin{aligned} \| r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_{j+1} - \alpha_j) \|_{L^\infty([R,+\infty[)} &\leq 2^j K^{j+1} R^{-j} \\ \text{and} \qquad \| r(\beta_{j+1} - \beta_j) \|_{L^\infty([R,+\infty[)} &\leq 2^j K^{j+1} R^{-j} \end{aligned} \tag{3.127}$$

We chose $R$ large enough to have $2KR^{-1} < 1/2$ and we define, for $r \geq R$

$$h_4(r) = r^{-n_\mu - \frac{3}{2}} e^{\sqrt{2}r} \sum_{j \geq 0} r^{n_\mu + \frac{3}{2}} e^{-\sqrt{2}r} (\alpha_j - \alpha_{j-1}) \quad \text{and} \quad k_4(r) = 1 + r^{-1} \sum_{j \geq 0} r(\beta_{j+1} - \beta_j).$$

Now we deduce that for all $r \geq R$,

$$r^{n_\mu+\frac{3}{2}} e^{-\sqrt{2}r} |h_4(r)| \leq K(1 - 2KR^{-1/2})^{-1}$$

and

$$r|k_4(r) - 1| \leq K(1 - 2KR^{-1/2})^{-1}.$$

We deduce the estimate (3.121), where $M$ is independent of $\mu$ and $r$.
Moreover, the continuity of $(h_4, h_4', k_4, k_4')(r)$ wrt $\mu$, for all $r \geq R$ follows from (3.127) by the same proof as for Lemma 3.2.
This terminates the proof of Lemma 3.5.

**Step 3** The proof of Proposition 3.5.
To construct the solution $(u_3^\mu, v_3^\mu)$, we first denote, for $r \in [R, +\infty[$,

$$x_3 = h_3 J_+ \quad \text{and} \quad y_3 = k_3 r^{n_\mu}$$

where $R$, $(h_3, k_3)$ are defined in Lemma 3.4. Thus, $(x_3, y_3)$ is a solution of (3.83). By (3.110), we obtain, with the same $M$ independent of $\mu$ and for all $r \in [R, +\infty[$

$$|x_3| \leq M r^{n_\mu - 2} \quad \text{and} \quad |y_3 - r^{n_\mu}| \leq M r^{n_\mu - 1}. \tag{3.128}$$

Moreover, in view of the properties of $(h, k)$, we have that $(x_3, x_3', y_3, y_3')(r)$ is continuous wrt $\mu$, for all $r \geq R$.
Now, exactly as in the proof of Proposition 3.4, we have a extension of of $(x_3, y_3)$ as a solution of (3.83) defined in $]0, +\infty[$. And denoting also this extension by $(x_3, y_3)$ we have that $(x_3, x_3', y_3, y_3')$ is continuous wrt $\mu$, in $(L^\infty([r_0, r_1]))^4$, for all $r_1 > r_0 > 0$.
Now we define

$$u_3^\mu = (x_3 + y_3)/2 \quad \text{and} \quad v_3^\mu = (x_3 - y_3)/2$$

to obtain that $(u_3^\mu, v_3^\mu)$ is a solution of (1.25) and to prove that in $[R, +\infty[$

$$|u_3^\mu - r^{n_\mu}| \leq C r^{n_\mu - 1} \quad \text{and} \quad |v_1^\mu + r^{n_\mu}| \leq C r^{n_\mu - 1}$$

with $C$ independent of $\mu$ and that $Y_3^\mu \to Y_3^1$ as $\mu \to 1$ in $(L^\infty([r_0, r_1]))^4$, for all $0 < r_0 < r_1$.

Now, to construct the solution $(u_4^\mu, v_4^\mu)$, we first denote, for $r \in [R, +\infty[$,

$$x_4 = h_4 J_- \quad \text{and} \quad y_4 = k_4 r^{-n_\mu}$$

where $R$, $(h_4, k_4)$ is defined in Lemma 3.5. By (3.121), we obtain, with the same $M$ independent of $\mu$ and for all $r \in [R, +\infty[$

$$|x_4| \leq M r^{-n_\mu - 2} \quad \text{and} \quad |y_4 - r^{-n_\mu}| \leq M r^{-n_\mu - 1}. \tag{3.129}$$

We follow the same pattern of proof as for $(u_1^\mu, v_1^\mu)$. We denote

$$u_4^\mu = (x_4 + y_4)/2 \quad \text{and} \quad v_4^\mu = (x_4 - y_4)/2$$

and we deduce that, in $[R, +\infty[$

$$|u_4^\mu - r^{-n_\mu}| \leq Mr^{-n_\mu - 1} \quad \text{and} \quad |v_4^\mu + r^{-n_\mu}| \leq Mr^{-n_\mu - 1}.$$

The prolongation of $(u_4^\mu, v_4^\mu)$ as a solution of (1.25) in $]0, +\infty[$, and the continuity of this solution wrt $\mu$ follow from the same principle as above.
This terminates the proof of Proposition 3.5.

**Proof of Proposition 3.6.** The arguments are exactly the same as in the proof of Proposition 3.3.

# 4 Proof of Theorem 1.5.

Concerning the solutions of (1.14), we have proved in [1] that the least behavior at 0 is related to the exponentially increasing behavior at $+\infty$ and that the exponentially decreasing behavior at $+\infty$ is related to the stronger blowing up behavior at 0. Let us quote it.

**Theorem 4.6** *(Theorem 1.6 in [1]). The solution $(a_1, b_1)$, that is the solution of (1.14) defined in Proposition 3.2, for $\mu = 1$, has the exponentially blowing up behavior at $+\infty$, ie $C(J_+, J_+)$, for some $C > 0$. And the solution $(u_2, v_2)$ of (1.14), that is defined in Proposition 3.4, for $\mu = 1$, has the most blowing up behavior at 0, ie $b \geq Cr^{-n-d}$, for some $C > 0$.*

We will need also Theorem 1.1. Let us prove Theorem 1.5.

**Proof of (i).** Let $d \geq 1$ and $n \geq 1$. Let us suppose that $\mu(\varepsilon) \to 1$, as $\varepsilon \to 0$. We will denote $\mu$ instead of $\mu(\varepsilon)$ and we will use the solutions $X_i^\mu$, $i = 1, \ldots, 4$ defined near 0 in Propositions 3.2 and 3.1 and the solutions $Y_i^\mu$, $i = 1, \ldots, 4$ defined near $+\infty$ in Propositions 3.4 and 3.5. Recall that they tend respectively to $X_i$ and $Y_i$, as $\mu \to 1$.
Firstly, since $Y_i^\mu$, $i = 1, \ldots, 4$ form a base of solutions, we can write

$$X_1^\mu = \sum_{j=1}^{4} C_j^\mu Y_j^\mu.$$

Each real number $C_j^\mu$ can be computed by means of a $4 \times 4$ determinant, for any fixed $r > 0$, and consequently, since $X_1^\mu(r)$ and each $Y_j^\mu(r)$ has a limit as $\mu \to 1$, then, each $C_j^\mu$ has a limit too. And by Theorem 4.6, $X_1$ has the exponentional blowing up behavior at $+\infty$ and we deduce that $C_1 \neq 0$, where $C_1 = \lim_{\mu \to 1} C_1^\mu$. Consequently, $C_1^\mu \neq 0$, when $\mu$ is closed to 1, and we can choose $X_1^\mu$ to represent a solution of (1.24) having the exponentially blowing up behavior at $+\infty$ instead of $Y_1^\mu$.
So, we can write $X_3^\mu$ as a combination of $X_1^\mu$ and $Y_i^\mu$, $i = 2, 3, 4$. And consequently, we write

$$X_3^\mu = D_1^\mu X_1^\mu + \sum_{j=2}^{4} D_j^\mu Y_j^\mu \tag{4.130}$$

and, as explained above, each real number $D_j^\mu$ has a limit as $\mu \to 1$. We are going to prove that $D_3^\mu \to 0$ as $\mu \to 1$.
Let $(a, b)$ be an eigenvector associated to $\mu$, ie $(a, b)$ is any solution of (1.24) that is defined at 0. We let $X^\mu = (a, ra', b, rb')^t$ the associated solution of (1.26). Since $(a, b)$ is defined at 0, there exists two real numbers $(A_1^\mu, A_3^\mu) \neq (0, 0)$ such that

$$X^\mu = A_1^\mu X_1^\mu + A_3^\mu X_3^\mu. \tag{4.131}$$

Now, the condition $a(\frac{1}{\varepsilon}) = b(\frac{1}{\varepsilon}) = 0$ leads us to the system

$$\begin{cases} A_1^\mu a_1^\mu(\frac{1}{\varepsilon}) + A_3^\mu a_3^\mu(\frac{1}{\varepsilon}) = 0 \\ \\ A_1^\mu b_1^\mu(\frac{1}{\varepsilon}) + A_3^\mu b_3^\mu(\frac{1}{\varepsilon}) = 0. \end{cases}$$

Thus if we denote by $\Delta$ the determinant

$$\Delta := a_1^\mu(\frac{1}{\varepsilon})b_3^\mu(\frac{1}{\varepsilon}) - b_1^\mu(\frac{1}{\varepsilon})a_3^\mu(\frac{1}{\varepsilon}),$$

we have, $\Delta = 0$.
We obtain firstly that the corresponding real eigenspace is one dimensional.

Now, we deduce from (4.130) that the determinant $\Delta$ defined above is

$$\Delta = \begin{vmatrix} a_1^\mu(\frac{1}{\varepsilon}) & \sum_{j=2}^4 D_j^\mu u_j^\mu(\frac{1}{\varepsilon}) \\ \\ b_1^\mu(\frac{1}{\varepsilon}) & \sum_{j=2}^4 D_j^\mu v_j^\mu(\frac{1}{\varepsilon}) \end{vmatrix}.$$

And we have proved above that $\Delta = 0$. This implies either that there exists some real number $k_\varepsilon$ such that the first column of $\Delta$ is obtained by multiplying the second column by $k_\varepsilon$, either the second column is null. Let us prove that the first possibility cannot be true.
In view of Propositions 3.4 and 3.5, there exists some $R > 0$ and some $M > 0$, the both being independent of $\mu$, such that for all $r \geq R$

$$|a_1^\mu(r) - C_1^\mu J_+| \leq M J_+ r^{-\frac{1}{2}}, \quad |b_1^\mu(r) - C_1^\mu J_+| \leq M J_+ r^{-\frac{1}{2}},$$

$$|\sum_{j=2}^4 D_j^\mu u_j^\mu(r) - D_2^\mu J_- - D_3^\mu r^{n_\mu} - D_4^\mu r^{-n_\mu}| \leq M r^{n_\mu - 1}$$

and

$$|\sum_{j=2}^4 D_j^\mu v_j^\mu(r) - D_2^\mu J_- + D_3^\mu r^{n_\mu} + D_4^\mu r^{-n_\mu}| \leq M r^{n_\mu - 1}.$$

And we apply these estimates for $r = 1/\varepsilon$, to obtain, for all $\varepsilon$ small enough

$$|a_1^\mu(\tfrac{1}{\varepsilon}) - C_1^\mu \varepsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}}| \leq M\varepsilon e^{\frac{\sqrt{2}}{\varepsilon}}, \quad |b_1^\mu(\tfrac{1}{\varepsilon}) - C_1^\mu \varepsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}}| \leq M\varepsilon e^{\frac{\sqrt{2}}{\varepsilon}},$$

$$\sum_{j=2}^4 D_j^\mu u_j^\mu(\tfrac{1}{\varepsilon}) - D_2^\mu e^{\frac{-\sqrt{2}}{\varepsilon}} \varepsilon^{\frac{1}{2}} - D_3^\mu \varepsilon^{-n_\mu} - D_4^\mu \varepsilon^{n_\mu}| \leq M\varepsilon^{-n_\mu + 1}$$

and

$$\sum_{j=2}^4 D_j^\mu u_j^\mu(\tfrac{1}{\varepsilon}) - D_2^\mu e^{\frac{-\sqrt{2}}{\varepsilon}} \varepsilon^{\frac{1}{2}} + D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu}| \leq M\varepsilon^{-n_\mu + 1},$$

M being independent of $\mu$ and of $\varepsilon$. And, as seen above $C_1^\mu \to C_1 > 0$, as $\varepsilon \to 0$.
Thus, the first possibility to have $\Delta = 0$ implies that for $\varepsilon$ small enough and for some $M$ independent of $\mu$ and $\varepsilon$ and for some real number $k_\varepsilon$,

$$\begin{cases} |C_1^\mu \varepsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} - k_\varepsilon (D_2^\mu e^{\frac{-\sqrt{2}}{\varepsilon}} \varepsilon^{\frac{1}{2}} + D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu})| \leq M\epsilon^{\frac{1}{2}} (\epsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} + k_\varepsilon (D_2^\mu e^{\frac{-\sqrt{2}}{\varepsilon}} \varepsilon^{\frac{1}{2}} + D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu})) \\ |C_1^\mu \varepsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} - k_\varepsilon (D_2^\mu e^{\frac{-\sqrt{2}}{\varepsilon}} \varepsilon^{\frac{1}{2}} - D_3^\mu \varepsilon^{-n_\mu} - D_4^\mu \varepsilon^{n_\mu})| \leq M\epsilon^{\frac{1}{2}} (\epsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} + k_\varepsilon (D_2^\mu e^{\frac{-\sqrt{2}}{\varepsilon}} \varepsilon^{\frac{1}{2}} + D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu})) \end{cases}$$

Let us prove that this cannot occur.
Recall that $\mu$ stands for $\mu(\varepsilon)$ and that it tends to 1, as $\varepsilon$ tends to 0, and that each $D_i^\mu$ tends to a real limit as $\mu$ tends to 1. Thus, if $(D_3^\mu, D_4^\mu) \not\to (0,0)$, the term $D_2^\mu e^{\frac{-\sqrt{2}}{\varepsilon}} \varepsilon^{\frac{1}{2}}$ is very small behind $D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu}$ and consequently we can write

$$\begin{cases} |C_1^\mu \varepsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} - k_\varepsilon (D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu})| \leq M\epsilon^{\frac{1}{2}} (\epsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} + k_\varepsilon (D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu})) \\ |C_1^\mu \varepsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} + k_\varepsilon (D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu})| \leq M\epsilon^{\frac{1}{2}} (\epsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} + k_\varepsilon (D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu})), \end{cases} \tag{4.132}$$

for another $M$ independent of $\varepsilon$ and of $\mu$.
Soustraying the both double inequations (4.132) we obtain

$$|k_\varepsilon|(1 - M\varepsilon^{\frac{1}{2}})|D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu}| \leq M\varepsilon e^{\frac{\sqrt{2}}{\varepsilon}}.$$

Now we use this inequality into (4.132) and, since $C_1^\mu \to C_1 > 0$, we deduce that

$$\varepsilon^{\frac{1}{2}} e^{\frac{\sqrt{2}}{\varepsilon}} \leq M_1 \varepsilon e^{\frac{\sqrt{2}}{\varepsilon}},$$

for some $M_1$ independent of $\varepsilon$ and of $\mu$. This contradiction proves that

$$(D_3^\mu, D_4^\mu) \to (0,0) \quad \text{as} \quad \mu \to 1.$$

Further, we return to (4.130) and we let $\mu \to 1$. We obtain

$$X_3 = D_1 X_1 + D_2 Y_2.$$

But we know, by Proposition 3.1 (that we use for $\mu = 1$), that

$$|a_3(r) - r^{|n-d|}| \leq M r^{|n-d|+2}, \qquad |b_3(r) - \frac{A_d^2 r^{|n-d|+2d+2}}{(d-n+|n-d|+2)^2}| \leq M r^{|n-d|+2d+4},$$

$$|a_1(r) - \frac{A_d^2}{(3d+n+2-|n-d|)^2} r^{n+3d+2}| \leq M r^{n+3d+4}, \qquad |b_1(r) - r^{n+d}| \leq M r^{n+d+2}$$

and consequently

$$|b_3(r) - D_1 b_1(r)| \leq M r^{n+d}.$$

But we know by Proposition 4.6 that $(u_2, v_2)$ have the most blowing up behavior at 0, ie there exists $C > 0$ such that

$$|b_2(r)| \geq C r^{-n-d}.$$

So we cannot have $X_3 = D_1 X_1 + D_2 Y_2$. We have proved that the first column of $\Delta$ cannot be obtained by multiplying the second column by a real number $k_\varepsilon$.

So we are led to examine the second possibility to have $\Delta = 0$, that is

$$\sum_{j=2}^{4} D_j^\mu u_j^\mu(\frac{1}{\varepsilon}) = \sum_{j=2}^{4} D_j^\mu v_j^\mu(\frac{1}{\varepsilon}) = 0.$$

In view of Propositions 3.4 and 3.5 we can deduce from $\sum_{j=2}^{4} D_j^\mu u_j^\mu(\frac{1}{\varepsilon}) = 0$ that

$$|D_2^\mu J_-(\frac{1}{\varepsilon}) + D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu}| \leq M\varepsilon^{\frac{1}{2}} |D_2^\mu J_-(\frac{1}{\varepsilon}) + D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu}|,$$

where $M$ is independent of $\mu$ and of $\varepsilon$.
This condition implies that $D_3^\mu \to 0$ as $\varepsilon \to 0$. And, more precisely, we can give the estimate

$$|D_3^\mu| \leq M\varepsilon^{n_\mu}(\varepsilon^{n_\mu} + J_-(\frac{1}{\varepsilon})), \tag{4.133}$$

where $M$ is independent of $\varepsilon$ and $\mu$.
Returning to (4.130) and letting $\mu \to 1$, we are led to

$$X_3 - D_1 X_1 = D_2 Y_2 + D_4 Y_4, \tag{4.134}$$

that is bounded at $+\infty$ and bounded at 0. We deduce that if $\mu(\varepsilon) \to 1$, then there exists a bounded solution of (1.14). If $n \geq 2$, by Theorem 1.1. This terminates the proof of (i).

**Proof of (iv).** Let us prove (iv) and we will be able to deduce (ii) from (iv). Let us suppose that there exists a bounded solution of (1.14). It is true for $n = 1$. From Lemma 1.1, we know that the first eigenvalue tends to 1. We proved in (i) just above that if $\mu(\varepsilon) \to 1$, then the eigenspace associated to $\mu(\varepsilon)$ is one dimensional and that a base of this eigenspace, given by (4.131) can be written as

$$X^\mu = (A_1^\mu + A_3^\mu D_1^\mu)X_1^\mu + A_3^\mu(X_3^\mu - D_1^\mu X_1^\mu). \tag{4.135}$$

Denoting $\tilde{A}_1^\mu = A_1^\mu + A_3^\mu D_1^\mu$, we may suppose that

$$\max\{\tilde{A}_1^\mu, A_3^\mu\} = 1$$

and, since $(\tilde{A}_1^\mu, A_3^\mu)$ tends to a limit denoted by $(\tilde{A}_1, A_3)$ as $\mu \to 1$, we have that

$$X^\mu \to \tilde{A}_1 X_1 + A_3(X_3 - D_1 X_1), \quad \max\{\tilde{A}_1, A_3\} = 1. \tag{4.136}$$

The condition $a^\mu(\frac{1}{\varepsilon}) = 0$ leads to

$$\tilde{A}_1^\mu a_1^\mu(\frac{1}{\varepsilon}) = -A_3^\mu(a_3^\mu(\frac{1}{\varepsilon}) - D_1^\mu a_1^\mu(\frac{1}{\varepsilon})).$$

We use (4.130) and in view of the behaviors at $+\infty$ proved in Propositions 3.4, 3.5 and Theorem 4.6, we infer that the principal parts as $\varepsilon \to 0$ are

$$\tilde{A}_1^\mu C_1^\mu J_+(\frac{1}{\varepsilon}) = -A_3^\mu(D_2^\mu J_-(\frac{1}{\varepsilon}) + D_3^\mu \varepsilon^{-n_\mu} + D_4^\mu \varepsilon^{n_\mu}).$$

Since $|A_3^\mu| \leq 1$ and since each $D_j^\mu$ has a finite limit, we deduce that $\tilde{A}_1^\mu \to 0$, as $\varepsilon \to 0$. And we can even give this precision, in view of (4.133)

$$|\tilde{A}_1^\mu| \leq M(J_+(\frac{1}{\varepsilon}))^{-1}(J_-(\frac{1}{\varepsilon}) + \varepsilon^{n_\mu}) \tag{4.137}$$

for some $M$ idependent of $\varepsilon$ and $\mu$.
So, $\tilde{A}_1 = 0$ and we can conclude from (4.136) that

$$X^\mu \to X_3 - D_1 X_1 \quad \text{as} \quad \varepsilon \to 0$$

and since we know from (4.134) that $X_3 - D_1 X_1$ is bounded at $+\infty$, we have proved that the limit of the eigenvector is bounded at $+\infty$ and we have also that

$$X^\mu \to D_2 Y_2 + D_4 Y_4 \quad \text{as} \quad \varepsilon \to 0. \tag{4.138}$$

Now, let us denote the eigenvector

$$X^\mu = (a^\mu, r(a^\mu)', b^\mu, r(b^\mu)')^t \quad \text{and} \quad X_3 - D_1 X_1 = (a, ra', b, rb')^t$$

and let us prove that

$$\int_0^{\frac{1}{\varepsilon}} r(1 - f_d^2)(a^\mu - a)^2 dr + \int_0^{\frac{1}{\varepsilon}} r(1 - f_d^2)(b^\mu - b)^2 dr \to 0, \quad \text{as } \varepsilon \to 0. \tag{4.139}$$

The convergence of the integral on $[0, R]$, for all $R > 0$ is clear. And let us recall that for $r$ large, $0 < 1 - f_d^2 < Cr^{-2}$. More, we use (4.135) to write

$$a^\mu = \tilde{A}_1^\mu a_1^\mu + A_3^\mu (a_3^\mu - D_1^\mu a_1^\mu)$$

and we use (4.130) to deduce that

$$a^\mu = \tilde{A}_1^\mu a_1^\mu + A_3^\mu (D_2^\mu u_2^\mu + D_3^\mu u_3^\mu + D_4^\mu u_4^\mu).$$

In view of (4.133) and of (4.137) and in view of the behaviors at $+\infty$ of $a_1^\mu$ and $u_3^\mu$, we deduce that, for $R \leq r \leq 1/\varepsilon$, $R$ large and valid for all $\mu$,

$$|\tilde{A}_1^\mu a_1^\mu + A_3^\mu D_3^\mu u_3^\mu| \leq M \left( (J_+(\frac{1}{\varepsilon}))^{-1} \varepsilon^{n_\mu} J_+(r) + \varepsilon^{2n_\mu} r^{n_\mu} \right)$$

for some $M$ independent of $\varepsilon$ and $\mu$. We deduce that for $R$ large enough and for $\varepsilon < 1/R$,

$$\int_R^{\frac{1}{\varepsilon}} r(1 - f_d^2) |\tilde{A}_1^\mu a_1^\mu + A_3^\mu D_3^\mu u_3^\mu|^2 dr \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.140}$$

And (4.138) gives $a = D_2 u_2 + D_4 u_4$. We write

$$a^\mu - a = \tilde{A}_1^\mu a_1^\mu + A_3^\mu D_3^\mu a_3^\mu + A_3^\mu \left( D_2^\mu (u_2^\mu - u_2) + D_4^\mu (u_4^\mu - u_4) + (D_2^\mu - D_2) u_2 + (D_4^\mu - D_4) u_4) \right)$$

Since $D_2^\mu - D_2 \to 0$, $D_4^\mu - D_4 \to 0$, $u_4^\mu - u_4 \to 0$, $u_2^\mu - u_2 \to 0$ and $|u_2^\mu| \leq MJ_-$, $|u_4^\mu| \leq Mr^{-n_\mu}$, $|u_2| \leq MJ_-$, $|u_4| \leq Mr^{-n_\mu}$, we conclude by the Lebesgue Theorem that

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_R^{\frac{1}{\varepsilon}} r(1 - f_d^2) |\tilde{A}_3^\mu| |D_2^\mu (u_2^\mu - u_2) + D_4^\mu (u_4^\mu - u_4) + (D_2^\mu - D_2) u_2 + \\ + (D_4^\mu - D_4) u_4|^2 dr = 0. \end{aligned} \tag{4.141}$$

And we have exactly the same estimate for $b^\mu - b$.
We deduce (4.139).
But combining the system (1.14) verifyed by $(a,b)$ and the system (1.25) verified by $(a^\mu, b^\mu)$ we obtain (letting $n=1$)

$$\int_0^{\frac{1}{\varepsilon}} r[((a^\mu)' - a')^2 + ((b^\mu)' - b')^2 + \tfrac{(1-d)^2}{r^2}(a^\mu - a)^2 + \tfrac{(1+d)^2}{r^2}(b^\mu - b)^2 +$$

$$+ f_d^2(a^\mu - a)^2 + f_d^2(b^\mu - b)^2]dr = \mu \int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)\left((a^\mu - a)^2 + (b^\mu - b)^2\right) dr +$$

$$+ (\mu - 1)\int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)\left((a^\mu - a)a + (b^\mu - b)b\right) dr.$$

We use (4.139) and the Lebesgue Theorem as above to deduce that the rhs tends to 0, as $\varepsilon$ tends to 0. Consequently, the lhs tends to 0, too. Combining this result with (4.139) we infer that

$$\begin{aligned} &\int_0^{\frac{1}{\varepsilon}} r\left(((a^\mu)' - a')^2 + ((b^\mu)' - b')^2 + (a^\mu - a)^2 + (b^\mu - b)^2\right) dr + \\ &+ \int_0^{\frac{1}{\varepsilon}} \left(\tfrac{(a^\mu - a)^2}{r} + \tfrac{(b^\mu - b)^2}{r}\right) dr \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{4.142}$$

If we return to the complex notation, ie

$$\omega^\mu := e^{i(n-d)\theta}a^\mu + e^{i(n+d)\theta}b^\mu \quad \text{and} \quad \omega := e^{i(n-d)\theta}a + e^{i(n+d)\theta}b$$

we just proved that

$$\|\omega^\mu - \omega\|_{H^1(B(0,\frac{1}{\varepsilon}))} \to 0 \quad \text{as } \varepsilon \to 0,$$

that terminates the proof of (iv).

**Proof of (ii)**. Let us prove that at most one eigenvalue $\mu$ can tend to 1. If $\mu$ and $\tilde{\mu}$ are two eigenvalues, then we have

$$(\mu - \tilde{\mu})\int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)(a^\mu \tilde{a}^\mu + b^\mu \tilde{b}^\mu)dr = 0$$

where $(a,b)$ and $(\tilde{a},\tilde{b})$ are solutions of (1.24), respectively for $\mu$ and for $\tilde{\mu}$, and $(a,b)$ and $(\tilde{a},\tilde{b})$ are defined at $r=0$. Since $r(1-f_d^2)(|a|+|b|) \le Mr^{-2}$, (4.139) gives the convergence of $(r(1-f_d^2))^{\frac{1}{2}}a^\mu \chi_{[0,\frac{1}{\varepsilon}]}$ to $(r(1-f_d^2))^{\frac{1}{2}}a$ and of $(r(1-f_d^2))^{\frac{1}{2}}b^\mu \chi_{[0,\frac{1}{\varepsilon}]}$ to $(r(1-f_d^2))^{\frac{1}{2}}b$ in $L^2([0,+\infty[)$. So we obtain that

$$\lim_{\varepsilon \to 0}\int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)(a^\mu \tilde{a}^\mu + b^\mu \tilde{b}^\mu)dr = \int_0^{+\infty} r(1-f_d^2)(a^2+b^2)dr. \tag{4.143}$$

Since the limit is not zero, we deduce that for $\varepsilon$ small enough

$$\int_0^{\frac{1}{\varepsilon}} r(1-f_d^2)(a^\mu \tilde{a}^\mu + b^\mu \tilde{b}^\mu)dr \neq 0.$$

So if $\mu$ and $\tilde{\mu}$ tend to 1 as $\varepsilon \to 0$, then $\mu = \tilde{\mu}$.
We have proved (ii).

**Proof of (iii).** Let $n$ be given. Let us recall that the operator $\mathcal{T}^{-1}\mathcal{C}$ is compact and its restriction to $\mathcal{H}_{n,d}$ is compact too. By the standard theory of compact operators, the eigenvalues $1/\mu$ are bounded and they cannot have any accumulation point except 0. We can deduce there is at most a finite number of eigenvalues $\mu$ in $]0,1[$. So we can choose the same $C$ and the same $\varepsilon_0$ for all these eigenvalues, in order to have $0 < 1-\mu(\varepsilon) < C$ for all $\varepsilon < \varepsilon_0$. By Theorem 1.4 (iv), there exists a finite number of $n$ for which we have some eigenvalues less than 1. So we can chose the same $\varepsilon_0$ and the same $C$ for all the eigenvalues less than 1 and for all $n$. Now, for $n \geq d$ and for a given $\varepsilon$, $n \mapsto m_{n,d}(\varepsilon)$ is clearly increasing. So, we chose the same $C$ and the same $\varepsilon_0$ for every $n \geq 2d-1$ and very eigenvalue $\mu(\varepsilon)$. Finally, we can chose some $C$ and some $\varepsilon_0$ independent of $n$ and independent of the eigenvalue $\mu(\varepsilon)$.
This terminates the proof of Theorem 1.5.

# 5 The proof of Theorem 1.2.

There exists a Hilbertian base $(\zeta_j)_{j\in J}$ of $\mathcal{H}$, such that

$$< \mathcal{C}\zeta_i, \zeta_j >_{(L^2\times L^2)(B(0,1))} = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad < \mathcal{C}\zeta_j, \zeta_j >_{(L^2\times L^2)(B(0,1))} = 1.$$

If we denote by $\mu_j$ the eigenvalue associated to $\zeta_j$, by Theorem 1.5 there exists some $C > 0$ independent of $\varepsilon$ and independent of the eigenvalue $\mu_j$ and some $\varepsilon_0$ also independent of $\mu_j$ such that

$$1 - \mu_j(\varepsilon) < -C \quad \text{or} \quad 1 - \mu_j(\varepsilon) > C \quad \text{for all } \varepsilon < \varepsilon_0.$$

We define the subsets $I_+$ and $I_-$ of $J$ by $j \in I_+$ if $\mu_j > 1$ and $j \in I_-$, if $\mu_j < 1$. We define $(W^\perp)_+$ the subspace of $W^\perp$ spanned by $(\zeta_j)_{j\in I_+}$ and $(W^\perp)_-$ spanned by $(\zeta_j)_{j\in I_-}$.
We divide the proof of Theorem 1.2 into two steps.
In Step 1, we prove the following inequality, for $\omega \in \mathcal{H} \cap \left((W^\perp)_+ \cup (W^\perp)_-\right)$,

$$| < \mathcal{L}_\varepsilon \omega, \omega >_{L^2(B(0,1))} | \geq M < \mathcal{C}\omega, \omega >_{L^2(B(0,1))} \tag{5.144}$$

In Step 2, we prove (1.4), (1.6), (1.5) and (1.7).

**Step 1**
Let $\omega = \sum_{n\geq 1}(a_n e^{-in\theta} + b_n e^{in\theta}) + a_0$, belonging to $\mathcal{H} \cap ((W^\perp)_+ \cup W^\perp)_-)$.
We write

$$\omega = \sum_{i\in J} \alpha_j \zeta_j.$$

From the definition of $(\zeta_j)_{j\in J}$, we infer that

$$\mathcal{T}\omega = \sum_{j\in J} \alpha_j \mu_j \mathcal{C}\zeta_j$$

and consequently

$$\mathcal{L}_\varepsilon \omega = \sum_{j\in J} \alpha_j (1-\mu_j)\mathcal{C}\zeta_j,$$

that gives

$$< \mathcal{L}_\varepsilon \omega, \omega >_{L^2(B(0,1))} = \sum_{j\in J} \alpha_j^2 (1-\mu_j) < \mathcal{C}\zeta_j, \zeta_j > .$$

If $\omega \in (W^\perp)_+$, $\alpha_j = 0$ unless $j \in I_+$. We deduce that

$$< \mathcal{L}_\varepsilon \omega, \omega >_{L^2(B(0,1))} \leq -C < \mathcal{C}\omega, \omega >_{L^2(B(0,1))} .$$

If $\omega \in (W^\perp)_-$, $\alpha_j = 0$ unless $j \in I_-$. We deduce that

$$< \mathcal{L}_\varepsilon \omega, \omega >_{L^2(B(0,1))} \geq C < \mathcal{C}\omega, \omega >_{L^2(B(0,1))} .$$

In any case, we have (5.144).

**Step 2**

We consider that $\omega \in (W^\perp)_+$ or $\omega \in (W^\perp)_-$ and by Step 1 we have (5.144). But we have

$$< \mathcal{L}_\varepsilon \omega, \omega >_{L^2(B(0,1))} = < h, \omega >_{L^2(B(0,1))} . \tag{5.145}$$

First, we write the Cauchy-Schwartz inequality

$$| < h, \omega >_{L^2(B(0,1))} | \leq \|\varepsilon^{-1}\sqrt{1-f^2}\omega\|_{L^2(B(0,1))} \|\varepsilon\sqrt{(1-f^2)^{-1}}h\|_{L^2(B(0,1))}. \tag{5.146}$$

We deduce from (5.145) together with (5.144) that

$$| < h, \omega >_{L^2(B(0,1))} | \geq M < \mathcal{C}\omega, \omega >_{L^2(B(0,1))} .$$

We use (5.146) to deduce that

$$< \mathcal{C}\omega, \omega >_{L^2(B(0,1))} \leq \frac{1}{M^2} \int_0^1 r \frac{\varepsilon^2}{1-f^2} |h|^2 dr. \tag{5.147}$$

But since $\mathcal{L} = -\mathcal{T} + \mathcal{C}$, we use (5.145) to write

$$< \mathcal{T}\omega, \omega >_{L^2(B(0,1))} = - < h, \omega >_{L^2(B(0,1)} + < \mathcal{C}\omega, \omega >_{L^2(B(0,1))} .$$

Using (5.146) again, that gives

$$< \mathcal{T}\omega, \omega >_{L^2(B(0,1))} \leq < \mathcal{C}\omega, \omega >^{\frac{1}{2}}_{L^2(B(0,1))} \left( \int_0^1 r \frac{\varepsilon^2}{1-f^2} |h|^2 dr \right)^{\frac{1}{2}} + < \mathcal{C}\omega, \omega >_{L^2(B(0,1))}$$

and using (5.147), we obtain

$$< \mathcal{T}\omega, \omega >_{L^2(B(0,1))} \leq (M^{-1} + M^{-2}) \int_0^1 r \frac{\varepsilon^2}{1-f^2} |h|^2 dr.$$

Now in view of (1.28) and (1.27), we have that $\lim_{r\to 0}(1-f_d^2(r))(1+r^2)=1$ and $\lim_{r\to+\infty}(1-f_d^2(r))(1+r^2)=d^2$, and this is a positive function in $[0,+\infty[$. We deduce that there exists some $C>0$ such that $1-f_d^2(r)\geq C/(1+r^2),$ for all $r\geq 0$. So we use

$$1-f^2\geq C\frac{\varepsilon^2}{\varepsilon^2+r^2}\quad \text{for all } r\in[0,1] \tag{5.148}$$

with $C$ independent of $\varepsilon$ to obtain

$$<\mathcal{T}\omega,\omega>_{L^2(B(0,1))}\leq M\int_0^1 r(\varepsilon^2+r^2)|h|^2dr, \tag{5.149}$$

for some other $M$, independent of $\varepsilon$. That gives in particular

$$\int_{B(0,1)}|\nabla(e^{id\theta}\omega)|^2dx\leq M\int_0^1 r(\varepsilon^2+r^2)|h|^2dr.$$

We have proved (1.5).

Now, we use (5.148) in (5.147) to obtain

$$\int_0^1\frac{r}{\varepsilon^2+r^2}|\omega|^2dr\leq M\int_0^1 r(\varepsilon^2+r^2)|h|^2dr. \tag{5.150}$$

Thus we have proved (1.4).

Now, let us recall that

$$<\mathcal{T}\omega_{\mathcal{R}},\omega_{\mathcal{R}}>_{L^2(B(0,1)}=\sum_{n\geq 1}\int_0^1\left(r|a_n'|^2+r|b_n'|^2+\frac{(n-d)^2}{r}|a_n|^2+\frac{(n+d)^2}{r}|b_n|^2+\frac{|a_n+b_n|^2}{r}f^2\right)dr+$$

$$+\int_0^1(r|a_0'|^2+\frac{d^2}{r}|a_0|^2+2rf_d|a_0|^2)dr$$

and that

$$<\mathcal{T}i\omega_{\mathcal{I}},i\omega_{\mathcal{I}}>_{L^2(B(0,1)}=\sum_{n\geq 1}\int_0^1\left(r|a_n'|^2+r|b_n'|^2+\frac{(n-d)^2}{r}|a_n|^2+\frac{(n+d)^2}{r}|b_n|^2+\frac{|a_n-b_n|^2}{r}f^2\right)dr+$$

$$+\int_0^1(r|a_0'|^2+\frac{d^2}{r}|a_0|^2)dr$$

We conclude that (1.5) and (1.7) are valid when $a_n$ and $b_n$ are complex valued.

# 6 The proof of Lemma 1.1.

We suppose that $d\geq 1$ and $n\geq 1$. Let us suppose that $(a,b)$ is a bounded solution of (1.14). Let $N$ be chosen and $0<N<1$. Let us define $(a^{cut},b^{cut})$ by

$$(a^{cut},b^{cut})(r)=\begin{cases}(a,b)(r) & \text{for } 0\leq r\leq\frac{N}{\varepsilon}\\ (a,b)(\frac{N}{\varepsilon})h(r) & \text{for } \frac{N}{\varepsilon}\leq r\leq\frac{1}{\varepsilon}\end{cases}$$

where, as in the beginning of the proof of Theorem 1.4, Part 2,

$$h(r) = e^{-u}e, \quad u = \frac{\frac{N}{\varepsilon} - \frac{1}{\varepsilon}}{r - \frac{1}{\varepsilon}}.$$

We have $a^{cut}e^{id\theta} \in H_0^1(B(0, \frac{1}{\varepsilon}))$ and $b^{cut}e^{id\theta} \in H_0^1(B(0, \frac{1}{\varepsilon}))$. We denote

$$\omega = ae^{-in\theta} + be^{in\theta} \quad \text{and} \quad \omega^{cut} = a^{cut}e^{-in\theta} + b^{cut}e^{in\theta}.$$

Using the usual rescaling, we have $\tilde{\omega}^{cut} \in \mathcal{H}_{n,d}$.
We have supposed that $m_{n,d}(\varepsilon) \geq 1$, so we deduce from the definition of $m_{n,d}(\varepsilon)$ that

$$0 \leq m_{n,d}(\varepsilon) - 1 \leq \frac{- < \mathcal{L}_\varepsilon \tilde{\omega}^{cut}, \tilde{\omega}^{cut} >_{L^2(B(0,1))}}{< \mathcal{C}\tilde{\omega}^{cut}, \tilde{\omega}^{cut} >_{L^2(B(0,1))}}.$$

Now we have

$$< \mathcal{C}\tilde{\omega}^{cut}, \tilde{\omega}^{cut} >_{L^2(B(0,1))} = < (1 - f_d^2)\omega^{cut}, \omega^{cut} >_{L^2(B(0,1/\varepsilon))} .$$

But, in view of the possible bounded behaviors at $+\infty$, (in Proposition 3.5 in which $\mu = 1$), there exists $M$ and $R$ such that for all $r > R$,

$$|a| + |b| \leq Mr^{-n}, \quad |a'| + |b'| \leq Mr^{-n-1} \quad \text{and} \quad 1 - f_d^2 \leq Mr^{-2}. \tag{6.151}$$

We use the Lebesgue Theorem to deduce that

$$< \mathcal{C}\tilde{\omega}^{cut}, \tilde{\omega}^{cut} >_{L^2(B(0,N))} \to < (1 - f_d^2)\omega, \omega >_{L^2(\mathbb{R}^2)} \quad \text{as } \varepsilon \to 0. \tag{6.152}$$

Now let us estimate $- < \mathcal{L}_\varepsilon \tilde{\omega}^{cut}, \tilde{\omega}^{cut} >_{L^2(B(0,1))}$. Since $(a, b)$ verifies the system (1.14), we have

$$- < \mathcal{L}_\varepsilon \tilde{\omega}^{cut}, \tilde{\omega}^{cut} >_{L^2(B(0,1))} =$$

$$= - < \mathcal{L}_\varepsilon \omega, \omega >_{L^2(B(0, \frac{N}{\varepsilon}))} + [ra'a + b'b]_0^{\frac{N}{\varepsilon}} +$$

$$+ \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} [(a^2(\tfrac{N}{\varepsilon}) + b^2(\tfrac{N}{\varepsilon}))(r(h')^2 - r(1 - f_d^2)h^2) + \tfrac{(n-d)^2}{r}a^2(\tfrac{N}{\varepsilon}) + \tfrac{(n+d)^2}{r}b^2(\tfrac{N}{\varepsilon}) +$$

$$+ rf_d^2h^2(a(\tfrac{N}{\varepsilon}) + b(\tfrac{N}{\varepsilon}))^2]dr$$

We deduce by (6.151) that

$$\big[ra'a + b'b\big]_0^{\frac{N}{\varepsilon}} \leq M\varepsilon^{2n}.$$

Now we use the estimate (2.30) and (6.151) to obtain

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} (a^2(\frac{N}{\varepsilon}) + b^2(\frac{N}{\varepsilon}))r(h')^2 dr \leq M\varepsilon^{2n}.$$

On the other hand, by Proposition 3.5, we have

$$|a + b| \leq Mr^{-n-1}$$

and consequently

$$(a(\frac{N}{\varepsilon})+b(\frac{N}{\varepsilon}))^2 \leq M\varepsilon^{2n+2}.$$

We deduce that

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r f_d^2 h^2 (a(\frac{N}{\varepsilon})+b(\frac{N}{\varepsilon}))^2 dr \leq M\varepsilon^{2n+2} \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r h^2 dr.$$

But

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r h^2 dr \leq \varepsilon^{-1} e^2 \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} e^{-2u} dr$$

and

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} e^{-2u} dr = \int_1^{+\infty} e^{-2u} u^{-2} (\frac{1}{\varepsilon} - \frac{N}{\varepsilon}) du.$$

And finally

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r f_d^2 h^2 (a(\frac{N}{\varepsilon})+b(\frac{N}{\varepsilon}))^2 dr \leq M\varepsilon^{2n}.$$

Now

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(1-f_d^2) h^2 dr \leq M \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r^{-1} h^2 dr$$

that gives, using the estimate just above,

$$(a^2(\frac{N}{\varepsilon})+b^2(\frac{N}{\varepsilon})) \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(1-f_d^2) h^2 dr \leq M\varepsilon^{2n}.$$

And we conclude that

$$- < \mathcal{L}_\varepsilon \omega^{cut}, \omega^{cut} >_{L^2(B(0,\frac{1}{\varepsilon}))} \leq M\varepsilon^{2n},$$

while by (6.152)

$$< \mathcal{C}\omega^{cut}, \omega^{cut} >_{L^2(B(0,\frac{1}{\varepsilon}))} \geq M.$$

This gives

$$0 < m_{n,d}(\varepsilon) - 1 \leq M\varepsilon^{2n}.$$

We have proved Lemma 1.1.

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