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A lower bound for a quadratic form related to the Ginzburg-Landau equation.

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Abstract. We consider the complex Ginzburg-Landau operator on a bounded domain. We prove some estimates for the inverse of the linearized operator.

AMS classification : 34B40: Ordinary Differential Equations, Boundary value problems on infinite intervals. 35J60: Nonlinear PDE of elliptic type. 35P15: Estimation of eigenvalues, upper and lower bound.

1 Introduction.

We consider the Ginzburg-Landau equation on a bounded connected domain Ω ,

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2}u(1 - |u|^2) & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases} \quad (1.1)$$

where $\varepsilon > 0$ is a small parameter, u and g have complex values and $\text{degree}(g, \partial\Omega) \geq 1$. This equation has been intensively studied, in [4], and many others.

Let us denote

$$\mathcal{N}_\varepsilon(u) = \Delta u + \frac{1}{\varepsilon^2}u(1 - |u|^2)$$

and let us define f_d as the only solution of the differential equation

$$\begin{cases} f_d'' + \frac{f_d'}{r} - \frac{d^2}{r^2}f_d = -f_d(1 - f_d^2) \\ f_d(0) = 0, \quad \lim_{r \rightarrow +\infty} f_d(r) = 1. \end{cases} \quad (1.2)$$

The equation (1.2) is completely studied in [8]. And let $u_0(x) = f_d(\frac{|x|}{\varepsilon})e^{id\theta}$. We have

$$\mathcal{N}_\varepsilon(u_0) = 0.$$

We will always denote

$$f(r) = f_d\left(\frac{r}{\varepsilon}\right).$$

The linearized operator around any function u is given by

$$d\mathcal{N}_\varepsilon(u)(\omega) = \Delta\omega + \frac{\omega}{\varepsilon^2}(1 - |u|^2) - \frac{2}{\varepsilon^2}u(\bar{u}\omega + u\omega).$$

Let us consider the linearized operator around the solution u_0 , ie

$$d\mathcal{N}_\varepsilon(u_0)(\omega) = \Delta\omega + \frac{\omega}{\varepsilon^2}(1 - f^2) - \frac{2}{\varepsilon^2}f^2 e^{id\theta} e^{id\theta} \omega,$$

where ω is any complex valued function and $2u.\omega = \bar{u}\omega + \bar{\omega}u$.

We will use the operator

$$\mathcal{L}_\varepsilon := e^{-id\theta} d\mathcal{N}_\varepsilon(u_0) e^{id\theta}$$

instead of $d\mathcal{N}_\varepsilon(u_0)$ and we will use also the rescaled operator \mathcal{L}_1 . These operators are of importance for some technics of construction of solutions for the equation (1.1).

The invariance of the equation (1.1) wrt the translations and the rotations gives

$$0 = d\mathcal{N}_\varepsilon(u_0)(iu_0) = d\mathcal{N}_\varepsilon(u_0)\left(\frac{\partial u_0}{\partial x_1}\right) = d\mathcal{N}_\varepsilon(u_0)\left(\frac{\partial u_0}{\partial x_2}\right).$$

A calculus gives

$$\frac{\partial u_0}{\partial \theta} = idu_0,$$

$$e^{-id\theta} \frac{\partial u_0}{\partial x_1} = \frac{1}{2} \left(\frac{1}{\varepsilon} f'_d\left(\frac{r}{\varepsilon}\right) + \frac{d}{r} f_d\left(\frac{r}{\varepsilon}\right) \right) e^{-i\theta} + \frac{1}{2} \left(\frac{1}{\varepsilon} f'_d\left(\frac{r}{\varepsilon}\right) - \frac{d}{r} f_d\left(\frac{r}{\varepsilon}\right) \right) e^{i\theta}$$

and

$$e^{-id\theta} \frac{\partial u_0}{\partial x_2} = \frac{i}{2} \left(\frac{1}{\varepsilon} f'_d\left(\frac{r}{\varepsilon}\right) + \frac{d}{r} f_d\left(\frac{r}{\varepsilon}\right) \right) e^{-i\theta} + \frac{i}{2} \left(-\frac{1}{\varepsilon} f'_d\left(\frac{r}{\varepsilon}\right) + \frac{d}{r} f_d\left(\frac{r}{\varepsilon}\right) \right) e^{i\theta}.$$

In [3], we have proved that

Theorem 1.1 *For all $d \geq 1$ the set of the solutions of $\mathcal{L}_1\omega = 0$ which are defined at 0 and bounded at $+\infty$ is reduced to the three functions provided by the invariance of the equation (1.1) by the rotations and the translations of the coordinates, ie if_d , $(f'_d + \frac{d}{r}f_d)e^{-i\theta} + (f'_d - \frac{d}{r}f_d)e^{i\theta}$ and $i(f'_d + \frac{d}{r}f_d)e^{-i\theta} + i(f'_d - \frac{d}{r}f_d)e^{i\theta}$.*

The case $d = 1$ was known by [13], chapter 3. In this book, Pacard and Rivière construct some solutions for (1.1), having the degree one around each singularity. Before that, the eigenvalue problem $\mathcal{L}_\varepsilon\omega = -\lambda(\varepsilon)\omega$, had been studied in several papers, including [9], [12], [10], [11]. It is used in [6], [7]...

In the present paper, we let

$$\mathcal{H} := \{\omega : B(0,1) \rightarrow \mathbb{C}, \quad e^{-id\theta}\omega \in H_0^1(B(0,1))\}.$$

And we let Φ_0 , Φ_1 and Φ_2 be functions in \mathcal{H} , associated with $e^{-id\theta}u_0$, $e^{-id\theta}\frac{\partial u_0}{\partial x_1}$ and $e^{-id\theta}\frac{\partial u_0}{\partial x_2}$, in a sense that will be defined below in Theorem 1.5. We denote W the subspace of \mathcal{H} spanned by Φ_0 , Φ_1 and Φ_2 and we define W^\perp by $\mathcal{H} = W \oplus W^\perp$. Last, for every $\omega \in \mathcal{H}$ we consider the Fourier expansion

$$\omega(x) = \sum_{n \geq 1} (a_n(r)e^{-in\theta} + b_n(r)e^{in\theta}) + a_0(r), \quad a_n(r) \in \mathbb{C}, \quad b_n(r) \in \mathbb{C}, \quad r \in]0,1]$$

and we denote

$$\omega_{\mathcal{R}} := \sum_{n \geq 1} (\mathcal{R}a_n(r)e^{-in\theta} + \mathcal{R}b_n(r)e^{in\theta}) + \mathcal{R}a_0(r)$$

and

$$\omega_{\mathcal{I}} := \sum_{n \geq 1} (\mathcal{I}a_n(r)e^{-in\theta} + \mathcal{I}b_n(r)e^{in\theta}) + \mathcal{I}a_0(r)$$

where, for any $a \in \mathbb{C}$, $\mathcal{R}a$ is the real part and $\mathcal{I}a$ is the imaginary part of a . For $h \in L^2(B(0,1))$ we consider the equation

$$\mathcal{L}_\varepsilon \omega = h. \quad (1.3)$$

We will prove the following

Theorem 1.2 *There exists $M > 0$ independent of ε such that for all $\omega \in W^\perp$, and denoting $\omega = \omega_{\mathcal{R}} + i\omega_{\mathcal{I}}$, we have*

$$| \langle \mathcal{L}_\varepsilon(\omega_{\mathcal{R}}), \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} | \geq M \langle \mathcal{C}\omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} \quad (1.4)$$

and

$$| \langle \mathcal{L}_\varepsilon(i\omega_{\mathcal{I}}), i\omega_{\mathcal{I}} \rangle_{L^2(B(0,1))} | \geq M \langle \mathcal{C}\omega_{\mathcal{I}}, \omega_{\mathcal{I}} \rangle_{L^2(B(0,1))}$$

where $\mathcal{C} = \frac{1}{\varepsilon^2}(1 - f^2)$,

that gives, when ω is a solution of (1.3), verifying $\omega = 0$ in $\partial B(0,1)$,

$$\int_{B(0,1)} \frac{1}{\varepsilon^2 + |x|^2} |\omega|^2 dx \leq M \int_{B(0,1)} (\varepsilon^2 + |x|^2) |h|^2 dx \quad (1.5)$$

and

$$\int_{B(0,1)} |\nabla(e^{id\theta}\omega)|^2 dx \leq M \int_{B(0,1)} (\varepsilon^2 + |x|^2) |h|^2 dx. \quad (1.6)$$

Now, if we denote

$$h = \sum_{n \geq 1} (\alpha_n e^{-in\theta} + \beta_n e^{in\theta}) + \alpha_0,$$

where the α_j and β_j have complex values, let us remark that we can replace (1.5) by

$$\begin{aligned} & \int_0^1 |a_0|^2 \frac{rdr}{\varepsilon^2 + r^2} + \sum_{n \geq 1} \int_0^1 (|a_n|^2 + |b_n|^2) \frac{rdr}{\varepsilon^2 + r^2} \\ & \leq C \sum_{n \geq 1} \int_0^1 r(\varepsilon^2 + r^2)(|\alpha_n|^2 + |\beta_n|^2) dr + \int_0^1 r(\varepsilon^2 + r^2) |\alpha_0|^2 dr \end{aligned} \quad (1.7)$$

and we can replace (1.6) ,

$$\begin{aligned} & \int_0^1 (r|a'_0|^2 + \frac{d^2}{r}|a_0|^2) dr + \sum_{n \geq 1} \int_0^1 \left(r|a'_n|^2 + r|b'_n|^2 + \frac{(n-d)^2}{r}|a_n|^2 + \frac{(n+d)^2}{r}|b_n|^2 \right) dr \\ & \leq C \sum_{n \geq 1} \int_0^1 r(\varepsilon^2 + r^2)(|\alpha_n|^2 + |\beta_n|^2) dr + \int_0^1 r(\varepsilon^2 + r^2) |\alpha_0|^2 dr. \end{aligned} \quad (1.8)$$

Now let us recall in which way the equation $\mathcal{L}_\varepsilon \omega = h$ is transformed into systems of ordinary equations.

Letting, for $n \geq 1$ $\omega_n(x) = a_n(r)e^{-in\theta} + b_n(r)e^{in\theta}$, and for $n = 0$, $\omega_0(x) = a_0(r)$, we have

$$2e^{id\theta} \cdot e^{id\theta} \omega_n = \omega_n + \bar{\omega}_n = (b_n + \bar{a}_n)e^{in\theta} + (\bar{b}_n + a_n)e^{-in\theta}.$$

$$\text{Moreover} \quad e^{-id\theta} \Delta(e^{id\theta} \omega) = \Delta \omega - \frac{d^2}{r^2} \omega + i \frac{2d}{r^2} \frac{\partial \omega}{\partial \theta}.$$

Consequently

$$\begin{aligned} e^{-id\theta} d\mathcal{N}_\varepsilon(u_0) e^{id\theta} \omega &= \sum_{n \geq 1} e^{-in\theta} \left(a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2} a_n + \frac{a_n}{\varepsilon^2} (1-f^2) - \frac{a_n}{\varepsilon^2} f^2 - \frac{\bar{b}_n}{\varepsilon^2} f^2 \right) + \\ &+ \sum_{n \geq 1} e^{in\theta} \left(b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2} b_n + \frac{b_n}{\varepsilon^2} (1-f^2) - \frac{b_n}{\varepsilon^2} f^2 - \frac{\bar{a}_n}{\varepsilon^2} f^2 \right) + \\ &+ a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2} a_0 + \frac{a_0}{\varepsilon^2} (1-f^2) - \frac{a_0 + \bar{a}_0}{\varepsilon^2} f^2. \end{aligned} \quad (1.9)$$

For $n \geq 1$, Separating the Fourier components of $e^{-id\theta} d\mathcal{N}_\varepsilon(u_0) e^{id\theta} \omega$, we can consider the operators

$$\begin{aligned} \text{for } n \geq 1, \quad \mathcal{L}_\varepsilon(\omega_n) &= \left(a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2} a_n + \frac{a_n}{\varepsilon^2} (1-2f^2) - \frac{\bar{b}_n}{\varepsilon^2} f^2 \right) e^{-in\theta} + \\ &+ \left(b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2} b_n + \frac{b_n}{\varepsilon^2} (1-2f^2) - \frac{\bar{a}_n}{\varepsilon^2} f^2 \right) e^{in\theta} \end{aligned}$$

$$\text{and, for } n = 0, \quad \mathcal{L}_\varepsilon(a_0) = a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2} a_0 + \frac{a_0}{\varepsilon^2} (1-f^2) - \frac{a_0 + \bar{a}_0}{\varepsilon^2} f^2.$$

Separating the real part and the imaginary part of a_n and of b_n , we consider the following operators, where a_n and b_n are real valued function

$$\begin{aligned} \text{for } n \geq 1 \quad \mathcal{L}_\varepsilon : (a_n e^{-in\theta} + b_n e^{in\theta}) &\mapsto \begin{cases} \left(a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2} a_n + \frac{a_n}{\varepsilon^2} (1-2f^2) - \frac{b_n}{\varepsilon^2} f^2 \right) e^{-in\theta} + \\ \left(b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2} b_n + \frac{b_n}{\varepsilon^2} (1-2f^2) - \frac{a_n}{\varepsilon^2} f^2 \right) e^{in\theta}; \end{cases} \\ \mathcal{L}_\varepsilon : (i a_n e^{-in\theta} + i b_n e^{in\theta}) &\mapsto \begin{cases} i \left(a_n'' + \frac{a_n'}{r} - \frac{(n-d)^2}{r^2} a_n + \frac{a_n}{\varepsilon^2} (1-2f^2) + \frac{b_n}{\varepsilon^2} f^2 \right) e^{-in\theta} + \\ i \left(b_n'' + \frac{b_n'}{r} - \frac{(n+d)^2}{r^2} b_n + \frac{b_n}{\varepsilon^2} (1-2f^2) + \frac{a_n}{\varepsilon^2} f^2 \right) e^{in\theta}; \end{cases} \\ \text{and, for } n = 0, \quad \mathcal{L}_\varepsilon : i a_0 &\mapsto i \left(a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2} a_0 + \frac{a_0}{\varepsilon^2} (1-f^2) \right) ; \\ \mathcal{L}_\varepsilon : a_0 &\mapsto a_0'' + \frac{a_0'}{r} - \frac{d^2}{r^2} a_0 + \frac{a_0}{\varepsilon^2} (1-f^2) - \frac{2a_0}{\varepsilon^2} f^2. \end{aligned} \quad (1.10)$$

Let us consider the equation (1.3). We denote $\omega_n = \omega_{n,\mathcal{R}} + i\omega_{n,\mathcal{I}}$.

For $n \geq 1$, $\omega_{n,\mathcal{R}}$ and $\omega_{n,\mathcal{I}}$ belong to the set

$$\mathcal{H}_{n,d} = \{ a e^{-in\theta} + b e^{in\theta}, (a, b) : [0, 1] \mapsto \mathbb{R} \times \mathbb{R}; a e^{i(d-n)\theta} + b e^{i(n+d)\theta} \in H_0^1(B(0, 1)) \}.$$

and $\omega_{0,\mathcal{R}}$ and $\omega_{0,\mathcal{I}}$ belong to

$$\mathcal{H}_{0,d} = \{ a : [0, 1] \mapsto \mathbb{R}, e^{id\theta} a \in H_0^1(B(0, 1)) \}.$$

We endow $\mathcal{H}_{n,d}$ with the scalar product

$$\langle (a, b) | (u, v) \rangle = \int_0^1 (r a' u' + r b' v' + \frac{(n-d)^2}{r} a u + \frac{(n+d)^2}{r} b v + \frac{f^2}{\varepsilon^2} (a+b)(u+v)) dr$$

and we endow $\mathcal{H}_{0,d}$ with the scalar product

$$\langle a|u \rangle = \int_0^1 (ra'u' + \frac{d^2}{r}au)dr.$$

From (1.3), we are led to the equations , for $n \in \mathbb{N}$

$$\begin{cases} \mathcal{L}_\varepsilon(\omega_{n,\mathcal{R}}) = h_{n,\mathcal{R}} \\ \mathcal{L}_\varepsilon(i\omega_{n,\mathcal{I}}) = ih_{n,\mathcal{I}} \end{cases} \quad (1.11)$$

and, in view of (1.10), when $n \geq 1$, (1.11) gives two second order ordinary diifferential systems, the both of them with two equations, and with two real valued unknown functions. For $n = 0$, we have two ordinary second order ordinary diifferential equations, each of them with one real valued unknown function.

In what follows in this section, unlike in Theorem 1.2, a and b or a_n and b_n will be real valued functions.

The first works on the question, quoted above, are considering the following eigenvalue problem in each Hilbert space $\mathcal{H}_{n,d}$, that is, for $r \in [0, 1]$ and for $n = 0$

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a + \frac{1}{\varepsilon^2}(1 - f^2)a = -\lambda(\varepsilon)a, \quad a(1) = 0 \quad (1.12)$$

and

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - 2af^2 + \frac{1}{\varepsilon^2}(1 - f^2)a = -\lambda(\varepsilon)a, \quad a(1) = 0. \quad (1.13)$$

And for $r \in [0, 1]$ and for $n \geq 1$

$$\begin{cases} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - \frac{1}{\varepsilon^2}f^2b + \frac{1}{\varepsilon^2}(1 - 2f^2)a = -\lambda(\varepsilon)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - \frac{1}{\varepsilon^2}f^2a + \frac{1}{\varepsilon^2}(1 - 2f^2)b = -\lambda(\varepsilon)b \\ a(1) = b(1) = 0 \end{cases} \quad (1.14)$$

The question of whether there exist some eigenvalues such that $\lambda(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ is related to the question of the existence of bounded solutions in $[0, +\infty[$ for the following system, which is a rescaled form of (1.14), but with the domain $[0, +\infty[$ instead of $[0, \frac{1}{\varepsilon}]$ for $n \geq 1$

$$\begin{cases} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - f_d^2b + (1 - 2f_d^2)a = 0 \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - f_d^2a + (1 - 2f_d^2)b = 0. \end{cases} \quad (1.15)$$

And for $n = 0$

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a + (1 - f_d^2)a = 0 \quad (1.16)$$

and

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - 2af_d^2 + (1 - f_d^2)a = 0. \quad (1.17)$$

A bounded solution means that a and b are defined in $[0, +\infty[$ and that the both functions have finite limits at $+\infty$.

Let us quote what is known untill the 90'

Theorem 1.3 (i) For any $d \geq 1$ and for $n = 1$ the system (1.15) has a one dimensional real vector space of bounded solutions, spanned by $(f'_d + \frac{d}{r}f_d, f'_d - \frac{d}{r}f_d)$.

(ii) For $d \geq 1$ and $n = 1$, the first eigenvalue $\lambda_1(\varepsilon)$, for the eigenvalue problem (1.14) verifies $\lambda_1(\varepsilon) > 0$ and $\lambda_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and there are no other eigenvalue tending to 0.

(iii) For $d \geq 1$ and $n = 0$, there exists $C > 0$ independent of ε such that for any eigenvalue of the problems (1.12) and (1.13) we have $\lambda(\varepsilon) \geq C$, for some $C > 0$ independent of ε .

And the following Theorem was proved in [10]

Theorem 1.4 For $d \geq 1$ and $n \geq 2$, If there are no bounded solution (a, b) of (1.15), then $|\lambda(\varepsilon)| > C$, for some $C > 0$ independent of ε and for every eigenvalue $\lambda(\varepsilon)$ of the problem (1.14).

In [1], we proved that the converse of Theorem 1.4 is true.

In [3], we proved that there are no bounded solution of (1.15) when $n \geq 2$.

We claim that, making use of Theorem 1.1, we are able to give a completely different proof of Theorem 1.4. But neither this technique nor that of [10] permits to obtain $|\varepsilon^2 \lambda(\varepsilon)| \geq C$.

In [3], we associated the problem of the existence of bounded solutions of (1.15) with the following eigenvalue problem, for $r \in [0, 1]$ and for $n \geq 1$

$$\begin{cases} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - \frac{1}{\varepsilon^2}f^2a - \frac{1}{\varepsilon^2}f^2b &= -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - \frac{1}{\varepsilon^2}f^2b - \frac{1}{\varepsilon^2}f^2a &= -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)b \\ a(1) = b(1) = 0 \end{cases} \quad (1.18)$$

and for $r \in [0, 1]$ and for $n = 0$

$$\begin{cases} a'' + \frac{a'}{r} - \frac{d^2}{r^2}a &= -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)a \\ a(1) = 0. \end{cases} \quad (1.19)$$

and

$$\begin{cases} a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - \frac{2}{\varepsilon^2}f^2a &= -\frac{1}{\varepsilon^2}\mu(\varepsilon)(1-f^2)a \\ a(1) = 0. \end{cases} \quad (1.20)$$

We can also use the following form of the system (1.18)

$$\begin{cases} \mathcal{L}_\varepsilon(\omega) = (1 - \mu(\varepsilon))\mathcal{C}\omega \\ \omega = ae^{-in\theta} + be^{in\theta} \in \mathcal{H}_{n,d}, \end{cases} \quad (1.21)$$

where

$$\mathcal{C} := \frac{1}{\varepsilon^2}(1 - f^2).$$

We use a notion of \mathcal{C} -eigenvalue, as in [5].

If $ae^{-in\theta} + be^{in\theta} \in \mathcal{H}_{n,d}$, we remark that

$$\left(\mathcal{L}_\varepsilon(ae^{-in\theta} + be^{in\theta}) = (1 - \mu(\varepsilon))\mathcal{C}(ae^{-in\theta} + be^{in\theta}) \right)$$

$$\Leftrightarrow \left(\mathcal{L}_\varepsilon(iae^{-in\theta} - ibe^{in\theta}) = (1 - \mu(\varepsilon))\mathcal{C}(iae^{-in\theta} - ibe^{in\theta}) \right).$$

let us recall the cases of the first eigenvalues, for $n = 0$ and $n = 1$. Let $m_{n,d}$ be defined in (1.22) and (1.23). We have

Theorem 1.5 For all $d \geq 1$,

(i) there exists $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, $m_{0,d}(\varepsilon) \geq 1 + C\varepsilon^2$; $m_{0,d}(\varepsilon) \rightarrow 1$ and there exists an associated eigenvector Φ_0 , solution of (1.19) with $m_{0,d}$ in place of μ , such that $(\Phi_0 - f)(r) \rightarrow 0$ as $\varepsilon \rightarrow 0$, for all $r \geq 0$.

(ii) $m_{1,d}(\varepsilon) > 1$ and $\frac{m_{1,d}(\varepsilon)-1}{\varepsilon^2} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

(iii) There exists an eigenvector $\Phi_1 = ae^{-i\theta} + be^{i\theta} \in \mathcal{H}_{n,d}$ associated to the eigenvalue $m_{1,d}(\varepsilon)$, ie $\mathcal{L}_\varepsilon(\Phi_1) = \frac{1-m_{1,d}(\varepsilon)-1}{\varepsilon^2}\mathcal{C}\Phi_1$, such that $\|\frac{(1-f^2)^{\frac{1}{2}}}{\varepsilon}(\Phi_1 - e^{-id\theta}\frac{\partial u_0}{\partial x_1})\|_{L^2(B(0,1))} \rightarrow 0$, as $\varepsilon \rightarrow 0$ and $\Phi_2 = iae^{-i\theta} - ibe^{i\theta}$ is also an eigenvector, ie $\mathcal{L}_\varepsilon(\Phi_2) = \frac{1-m_{1,d}(\varepsilon)-1}{\varepsilon^2}\mathcal{C}\Phi_2$, and $\|\frac{(1-f^2)^{\frac{1}{2}}}{\varepsilon}(\Phi_2 - e^{-id\theta}\frac{\partial u_0}{\partial x_2})\|_{L^2(B(0,1))} \rightarrow 0$, as $\varepsilon \rightarrow 0$.

In the present paper, we prove the following

Theorem 1.6 For any $d \geq 1$, there exists $C > 0$ independent of ε such that

(i) for any $n \geq 2$, $|1 - \mu(\varepsilon)| \geq C$ for every eigenvalue $\mu(\varepsilon)$ for the problem (1.21) and for every ε .

(ii) For $n = 0$ and for the problem (1.19), $\mu(\varepsilon) - 1 \geq C$, for any eigenvalue, except if $\mu = m_{0,d}$ and for the problem (1.20), $\mu(\varepsilon) - 1 \geq C$, for every eigenvalue. For $n = 1$, $\mu(\varepsilon) - 1 \geq C$ except if $\mu = m_{1,d}$.

More generally, Theorem 1.6 is valid for every function $g > 0$ defined in $[0, +\infty[$ in place of $1 - f_d^2$, as long as $g(t) \leq \frac{K}{1+t^2}$ for some K and for any $\varepsilon^2\gamma(\varepsilon)$ instead of $\mu(\varepsilon) - 1$.

On the other hand, Lemma 5.2 of our unpublished preprint [2] gives some converse to Theorem 1.6. Indeed, we have

Lemma 1.1 (Lemma 5.2 in [2]). If there exists some bounded solution (a, b) of (1.15), then there exists an eigenvalue $\mu(\varepsilon)$ verifying $\mu(\varepsilon) - 1 \rightarrow 0$.

Last, let us recall that we defined $m_{n,d}(\varepsilon)$ as the first eigenvalue for the above eigenvalue problem (1.18) in $\mathcal{H}_{n,d}$, that is

$$m_{n,d}(\varepsilon) = \inf_{(a,b) \in \mathcal{H}_{n,d} \times \mathcal{H}_{n,d} \setminus \{(0,0)\}} \frac{\int_0^1 (ra'^2 + rb'^2 + \frac{(n-d)^2}{r}a^2 + \frac{(n+d)^2}{r}b^2 + \frac{r}{\varepsilon^2}f_d^2(\frac{r}{\varepsilon})(a+b)^2)dr}{\frac{1}{\varepsilon^2} \int_0^1 r(1 - f_d^2(\frac{r}{\varepsilon}))(a^2 + b^2)dr} \quad (1.22)$$

and $m_{0,d}(\varepsilon)$ is the first eigenvalue for the problem (1.19) in $\mathcal{H}_{0,d}$

$$m_{0,d}(\varepsilon) = \inf_{a \in \mathcal{H}_{0,d} \setminus \{0\}} \frac{\int_0^1 (ra'^2 + \frac{d^2}{r}a^2)dr}{\frac{1}{\varepsilon^2} \int_0^1 r(1 - f_d^2(\frac{r}{\varepsilon}))a^2dr} \quad (1.23)$$

and $\tilde{m}_{0,d}(\varepsilon)$ is the first eigenvalue for the problem (1.20) in $\mathcal{H}_{0,d}$

$$\tilde{m}_{0,d}(\varepsilon) = \inf_{a \in \mathcal{H}_{0,d} \setminus \{0\}} \frac{\int_0^1 (ra'^2 + \frac{d^2}{r}a^2 + 2rf_d(\frac{r}{\varepsilon})a^2)dr}{\frac{1}{\varepsilon^2} \int_0^1 r(1 - f_d^2(\frac{r}{\varepsilon}))a^2dr}. \quad (1.24)$$

And that the eigenvalue problem (1.14) leads to the following definitions

$$\lambda_{n,d}(\varepsilon) = \inf_{(a,b) \in \mathcal{H}_{n,d} \times \mathcal{H}_{n,d} \setminus \{(0,0)\}} \frac{\int_0^1 (ra'^2 + rb'^2 + \frac{(n-d)^2}{r}a^2 + \frac{(n+d)^2}{r}b^2 + \frac{r}{\varepsilon^2}f_d^2(\frac{r}{\varepsilon})(a+b)^2 - \frac{r}{\varepsilon^2}(1 - f_d^2(\frac{r}{\varepsilon}))(a^2 + b^2))dr}{\int_0^1 r(a^2 + b^2)dr}$$

and

$$\lambda_{0,d}(\varepsilon) = \inf_{a \in \mathcal{H}_{0,d} \setminus \{0\}} \frac{\int_0^1 (ra'^2 + \frac{d^2}{r}a^2 - \frac{r}{\varepsilon^2}(1 - f_d^2(\frac{r}{\varepsilon}))a^2)dr}{\int_0^1 ra^2dr}.$$

These infimum are attained. Considering the rescaling $(\tilde{a}, \tilde{b})(r) = (a(\varepsilon r), b(\varepsilon r))$ and an extension by 0 outside $[0, 1/\varepsilon]$, we see that $\varepsilon \mapsto m_{\gamma_1, \gamma_2}(\varepsilon)$ decreases when ε decreases. Then $\lim_{\varepsilon \rightarrow 0} m_{\gamma_1, \gamma_2}(\varepsilon)$ exists.

What is new is essentially Theorem 1.6 (i) and its consequence Theorem 1.2. We claim that we are able to prove Theorem 1.6 (i) as well as Theorem 1.4 by use of a resolvent matrix for the system (1.15) with known behaviors at 0 and at $+\infty$ and by considering the system (1.18) as a "second member" system. But we choose to present a proof of Theorem 1.6 that does not involve to inverse two 4×4 matrices. But the other point of view is to construct two bases of solutions for the linear system (1.18) involving the parameter ε . It is what we do in the second part. In the third part, we prove Theorem 1.6. This third part contains some parts of the proof of Theorem 1.5. In the fourth part, we prove Lemma 1.1 and in the fifth part we prove Theorem 1.2.

In all the paper, we use the following notation, if $x \mapsto f(x)$ and $x \mapsto g(x)$ are given function, $f = O(g)$ if $|f(x)| \leq M|g(x)|$, with M independent of x .

2 Two bases of solutions for the system (1.18) and (1.19).

First let us recall the expansions of f_d

$$f_d(r) = 1 - \frac{d^2}{2r^2} + O(\frac{1}{r^4}) \text{ near } +\infty \quad (2.25)$$

and

$$f_d(r) = A_d(r^d - \frac{1}{4(d+1)}r^{d+2}) + O(r^{d+4}) \text{ near } 0. \quad (2.26)$$

We suppose that $d \geq 1$ and that $n \geq 1$.

In [3], we gave two independent solutions bounded at 0 and two independent solutions that blow up at 0, and the same thing at $+\infty$, for the system (1.15).

Let us recall that we can rewrite the system (1.15) as

$$X' = MX \quad (2.27)$$

with

$$M = \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 \\ -r(1 - 2f_d^2) + \frac{(n-d)^2}{r} & 0 & rf_d^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ rf_d^2 & 0 & -r(1 - 2f_d^2) + \frac{(n+d)^2}{r} & 0 \end{pmatrix}.$$

and that a base of solutions is formed by four vector solutions of the form $X = (a, ra', b, rb')^t$. In [3] and in [2], there is γ_1 instead of $|n-d|$ and γ_2 instead of $n+d$. The continuity and the derivability of the solutions of the differential system wrt (γ_1, γ_2) was important and the determination of some suitable ranges for the parameters was necessary. Here, n and d are integers, so the continuity wrt to (γ_1, γ_2) doesn't exist anymore. The Theorem 1.4 and the Theorem 1.5 in [3] give, when $n \geq 1$ and $d \geq 1$ are some integers

Proposition 2.1 *For $d \geq 1$ and $n \geq 1$, there exist four independent solutions of (1.15) having the following behavior at 0 :*

$$(a_1(r), b_1(r)) \sim_0 \left(O(r^{n+3d+2}), r^{n+d} \right), \quad (a_3(r), b_3(r)) \sim_0 \left(r^{|n-d|}, O(r^{|n-d|+2d+2}) \right)$$

$$(a_2(r), b_2(r)) \sim_0 \begin{cases} (O(r^{2+d-n}), r^{-n-d}) & n \geq d+2 \\ (O(r \ln r), r^{-n-d}) & n = d+1 \\ (O(r^2), r^{-n-d}) & n = d \\ (O(r^{d-n}), r^{-n-d}) & 1 \leq n \leq d-1 \end{cases}$$

and

$$(a_4(r), b_4(r)) \sim_0 \begin{cases} (r^{-n+d}, O(r^{-n+3d+2})) & n \geq d+2 \\ (r^{-n+d}, O(-r^{n+d} \log r)) & n = d+1 \\ (-\ln r, O(r^{2d+2} \ln r)) & n = d \\ (r^{n-d}, O(r^{n+d})) & 1 \leq n \leq d-1. \end{cases}$$

We have four independent solutions having the following behaviors at $+\infty$:

$$\begin{aligned} (u_1(r), v_1(r)) &\sim_{+\infty} (J_+(r), J_+(r)) (1 + O(r^{-2})), \\ (u_2(r), v_2(r)) &\sim_{+\infty} (J_-(r), J_-(r)) (1 + O(r^{-2})), \\ (u_3(r), v_3(r)) &\sim_{+\infty} (r^n, -r^n) (1 + O(r^{-2})), \\ (u_4(r), v_4(r)) &\sim_{+\infty} (r^{-n}, -r^{-n}) (1 + O(r^{-2})), \end{aligned}$$

with the notation

$$J_+(r) = \frac{e^{\sqrt{2}r}}{\sqrt{r}}, \quad J_-(r) = \frac{e^{-\sqrt{2}r}}{\sqrt{r}}.$$

The proof of Proposition 2.1 for the behaviors at 0 is contained in the proof of Proposition 2.2 below and the proof for the behaviors at $+\infty$ is contained in the proof of Proposition 2.3 below. The only difference is that in Proposition 2.1, $\mu = 1$ and consequently there is no dependence anymore of the solutions wrt ϵ .

Let us denote by X_i , $i = 1, \dots, 4$ the four independent solutions defined near 0 and by Y_i , $i = 1, \dots, 4$ the four solutions defined near $+\infty$, for the system (1.15), which are defined in Proposition 2.1. We have proved in [3], Theorem 1.6 that the least behavior at 0 is related to the exponentially increasing behavior at $+\infty$ and that the exponentially decreasing behavior at $+\infty$ is related to the stronger blowing up behavior at 0. Let us quote it.

Theorem 2.7 (Theorem 1.6 in [3]). *The solution X_1 has the exponentially blowing up behavior at $+\infty$, like the solution Y_1 . And the solution Y_2 has the more blowing up behavior at 0, like the solution X_2 .*

Now we define a base of four solutions at 0 for the perturbed system

$$\begin{cases} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - f_d^2 b + (1 - 2f_d^2)a &= (1 - \mu(\varepsilon))(1 - f_d^2)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - f_d^2 a + (1 - 2f_d^2)b &= (1 - \mu(\varepsilon))(1 - f_d^2)b. \end{cases} \quad (2.28)$$

Proposition 2.2 *Let us suppose that $d \geq 1$ and $n \geq 1$ and that $\mu(\varepsilon) \rightarrow 1$, (i) there exist four independent solutions of (1.18), having the behaviors at 0 enounced in Proposition 2.1, more precisely, for all $r \in [0, R]$*

$$\begin{aligned} |a_1^\varepsilon(r)| &\leq Cr^{n+3d+2}, & |b_1^\varepsilon(r) - r^{n+d}| &\leq Cr^{n+d+2}, \\ |a_3^\varepsilon(r) - r^{|n-d|}| &\leq Cr^{n-d+2}, & |b_3^\varepsilon(r)| &\leq Cr^{|n-d|+2d+2}, \\ |a_2^\varepsilon(r)| &\leq Cr^2 \zeta_1^{(j=2)}(r), & |b_2^\varepsilon(r) - r^{-n-d}| &\leq Cr^2 r^{-n-d}, \\ |a_4^\varepsilon(r) - r^{|n-d|}| &\leq Cr^{|n-d|} r, & |b_4^\varepsilon(r)| &\leq Cr \zeta_2^{(j=4)}(r), \quad \text{if } n \neq d, \\ |a_4^\varepsilon(r) + \ln r| &\leq Cr^2(-\ln r), & |b_4^\varepsilon(r)| &\leq Cr^{2d+2}(-\ln r), \quad \text{if } n = d \end{aligned} \quad (2.29)$$

for some $C > 0$ and some $0 < R < 1$, the both being independent of ε .

and

(ii) denoting X_i^ε , $i = 1, \dots, 4$ these solutions, we have $\lim_{\varepsilon \rightarrow 0} X_i^\varepsilon = X_i$, $i = 1, \dots, 4$, where X_i is defined in Proposition 2.1.

Let us indicate what are the suitable maps (ζ_1, ζ_2) for each of the four solutions $(a_j^\varepsilon, b_j^\varepsilon)$.

$$\begin{aligned} (\zeta_1, \zeta_2)(r) &= \begin{cases} (r^{n+3d}, r^{n+d}) & \text{for } j = 1 \\ (r^{|n-d|}, r^{|n-d|+2d}) & \text{for } j = 3. \end{cases} \\ \text{For } j = 2, \quad \zeta_1(r) &= \begin{cases} r^{-|n-d|} & \text{if } n \geq d+2 \\ r^{|n-d|-2} & \text{if } n \leq d-1 \\ -r^{-1} \ln r & \text{if } n = d+1 \\ 1 & \text{if } n = d \end{cases} & \zeta_2(r) = r^{-n-d} \\ \text{and for } j = 4, \quad \zeta_1(r) &= \begin{cases} r^{-|n-d|} & \text{if } n \neq d \\ -\ln r & \text{if } n = d \end{cases} & \zeta_2(r) = \begin{cases} r^{-n+3d} & \text{if } n \geq d+2 \\ r^{n+d-2} & \text{if } n \leq d-1 \\ -r^{n+d-2} \ln r & \text{if } n = d+1 \\ -r^{2d} \ln(r) & \text{if } n = d. \end{cases} \end{aligned}$$

Proof of Proposition 2.2. We use the same proof as in [3] and for more details, in [2], but we involve the term $(1 - \mu(\varepsilon))(1 - f_d^2)$.

The proof below is valid for $\mu = 1$ and gives the proof of the first part of Proposition 2.1. For $(a_1^\varepsilon, b_1^\varepsilon)$ and $(a_3^\varepsilon, b_3^\varepsilon)$ we use the following form of the system (2.28)

$$\begin{cases} (r^{2|n-d|+1}(ar^{-|n-d|})')' = r^{|n-d|+1}(f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2)a) \\ (r^{2(n+d)+1}(br^{-(n+d)})')' = r^{n+d+1}(f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2)b). \end{cases} \quad (2.30)$$

To construct $(a_1^\varepsilon, b_1^\varepsilon)$ and $(a_3^\varepsilon, b_3^\varepsilon)$, we consider the following integral equation

$$\begin{cases} a &= \alpha r^{|n-d|} + r^{|n-d|} \int_0^r t^{-2(|n-d|-1)} \int_0^t s^{|n-d|+1} (f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2) a) ds dt \\ b &= \beta r^{n+d} + r^{n+d} \int_0^r t^{-2(n+d)-1} \int_0^t s^{n+d+1} (f_d^2 a + f_d^2 b - \mu(\varepsilon)(1 - f_d^2) b) ds dt \end{cases} \quad (2.31)$$

with $(\alpha, \beta) = (0, 1)$ for $(a_1^\varepsilon, b_1^\varepsilon)$ and with $(\alpha, \beta) = (1, 0)$ for $(a_3^\varepsilon, b_3^\varepsilon)$.

For $(a_2^\varepsilon, b_2^\varepsilon)$ and $(a_4^\varepsilon, b_4^\varepsilon)$ for $n \neq d$ we consider the following form of the system (2.28), when $n \neq d$.

$$\begin{cases} (r^{-2|n-d|+1} (ar^{|n-d|})')' &= r^{-|n-d|+1} (f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2) a) \\ (r^{-2(n+d)+1} (br^{n+d})')' &= r^{-(n+d)+1} (f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2) b). \end{cases} \quad (2.32)$$

But for $(a_4^\varepsilon, b_4^\varepsilon)$ and for $n = d$, we let

$$\tau(r) - \ln r.$$

and consider the system

$$\begin{cases} (r\tau^2(\tau^{-1}a)')' &= r\tau(f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2) a) \\ (r^{-2(n+d)+1} (br^{n+d})')' &= r^{-(n+d)+1} (f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2) b). \end{cases} \quad (2.33)$$

To construct $(a_2^\varepsilon, b_2^\varepsilon)$ and $(a_4^\varepsilon, b_4^\varepsilon)$ for $n \neq d$ we consider the following integral equation

$$\begin{cases} a &= \alpha r^{-|n-d|} + r^{-|n-d|} \int_0^r t^{|n-d|-1} \int_R^t s^{-|n-d|+1} (f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2) a) ds dt \\ b &= \beta r^{-(n+d)} + r^{-(n+d)} \int_0^r t^{2(n+d)-1} \int_R^t s^{-(n+d)+1} (f_d^2 b + f_d^2 a - \mu(\varepsilon)(1 - f_d^2) b) ds dt \end{cases} \quad (2.34)$$

where $(\alpha, \beta) = (0, 1)$ for $(a_2^\varepsilon, b_2^\varepsilon)$ and $(\alpha, \beta) = (1, 0)$ for $(a_4^\varepsilon, b_4^\varepsilon)$, $n \neq d$.

Here $0 < R < 1$ is a real number, but we replace R by 0 in the first equation, for $(a_2^\varepsilon, b_2^\varepsilon)$, when $1 \leq n \leq d$.

And for $(a_4^\varepsilon, b_4^\varepsilon)$, $n = d$, we consider the following integral equation

$$\begin{cases} a &= \tau(r) + \tau(r) \int_0^r \frac{1}{t} \tau^{-2}(t) \int_0^t s \tau(s) (f_d^2 b + f_d^2 a - \mu(1 - f_d^2) a) ds dt \\ b &= r^{-n-d} \int_0^r t^{2(n+d)-1} \int_0^t s^{-n-d+1} (f_d^2 a + f_d^2 b - \mu(1 - f_d^2) b) ds dt. \end{cases} \quad (2.35)$$

Let us explain the pattern of proof.

We define $\Phi(a, b)$ as the rhs of the above integral equations and we consider the two maps $r \mapsto \zeta_1(r)$ and $r \mapsto \zeta_2(r)$ defined above for each j . We want to construct solutions (a_j, b_j) , $j = 1, \dots, 4$, verifying, for some R independent of ε and some C independent of ε

$$\begin{aligned} \text{for } j = 1 \text{ and } j = 2 \quad & |a(r)\zeta_1^{-1}(r)| + |b(r)\zeta_2^{-1}(r) - 1| \leq Cr^2, \\ \text{for } j = 3 \text{ and } j = 4 \quad & |a(r)\zeta_1^{-1}(r) - 1| + |b(r)\zeta_2^{-1}(r)| \leq Cr^2 \end{aligned} \quad (2.36)$$

for all $0 < r \leq R$. For $j = 4$ and $n \neq d$, we replace r^2 in the rhs by r .

For this purpose, we define two sequences whose initial data depends of the desired solution (a_j, b_j) .

$$\text{For } j = 1, 2 \quad \alpha_0 = 0 \quad \beta_0 = \zeta_2 \quad \text{and for } j = 3, 4 \quad \alpha_0 = \zeta_1 \quad \beta_0 = 0 \quad (2.37)$$

and in any case $(\alpha_{k+1}, \beta_{k+1}) = \Phi(\alpha_k, \beta_k)$. We will denote $\nu : r \mapsto r$.

Now we will prove that for a given $0 < R < 1$ all $0 < r < R$ we have

$$\zeta^{-1} \nu^{-2} |\alpha_{k+1} - \alpha_k|(r) \leq Mr^2 (\|\nu^{-2} \zeta_1^{-1} (\alpha_k - \alpha_{k-1})\|_{L^\infty([0, R])} + \|\zeta_2^{-1} \nu^{-2} (\beta_k - \beta_{k-1})\|_{L^\infty([0, R])}), \quad (2.38)$$

$$\zeta_2^{-1} \nu^{-2} |\beta_{k+1} - \beta_k|(r) \leq Mr^2 (\|\zeta_1^{-1} \nu^{-2} (\alpha_k - \alpha_{k-1})\|_{L^\infty([0, R])} + \|\zeta_2^{-1} \nu^{-2} (\beta_k - \beta_{k-1})\|_{L^\infty([0, R])}) \quad (2.39)$$

and

$$|\alpha_1 - \alpha_0|(r) \leq Mr^2 \zeta_1(r), \quad |\beta_1 - \beta_0|(r) \leq Mr^2 \zeta_2(r) \quad (2.40)$$

where M does not depend on ε and does not depend on R .

For $j = 4$ and $n \neq d$, we have to replace ν^{-2} by ν^{-1} and r^2 in the rhs by r .

Next, we choose $R < 1$ such that $CR < 1$ and we define, for all $0 < r < R$

$$a_j^\varepsilon(r) = \zeta_1(r) r^2 \sum_{k=0}^{+\infty} (\nu^{-2} \zeta_1^{-1} (\alpha_{k+1} - \alpha_k))(r) + \alpha_0(r), \quad (2.41)$$

$$b_j^\varepsilon(r) = \zeta_2(r) r^2 \sum_{k=0}^{+\infty} (\nu^{-2} \zeta_2^{-1} (\beta_{k+1} - \beta_k))(r) + \beta_0(r),$$

$j = 1, 2, 3$ and the suitable adaptation for $j = 4$.

Thus we have $(a_j^\varepsilon, b_j^\varepsilon) = \Phi(a_j^\varepsilon, b_j^\varepsilon)$ and consequently we have defined a solution X_j^ε defined in $]0, R]$.

By the Cauchy-Lipschitz Theorem for the linear ordinary equations, the solution is defined in $]0, +\infty[$. By induction, α_k and β_k have limits as $\varepsilon \rightarrow 0$. Moreover, thanks to the inequalities just above, the sums in (2.41) are convergent uniformly wrt ε . Consequently, $a_1^\varepsilon(r)$ and $b_1^\varepsilon(r)$ have limits as $\varepsilon \rightarrow 0$, for every $r \in]0, R]$, and, in view of the integral equation, also $ra_1^\varepsilon(r)$ and $rb_1^\varepsilon(r)$ have limits as $\varepsilon \rightarrow 0$. So we can consider $(a_1^\varepsilon(R), Ra_1^\varepsilon(R), b_1^\varepsilon(R), Rb_1^\varepsilon(R))^t$ as an initial value for the solution X_1^ε , and a supposed well known principle in the Cauchy-Lipschitz Theory gives that a continuous initial value wrt ε , together with the continuous dependence of the coefficients of the equation wrt ε lead to a continuous solution wrt ε . And we deduce that for all $r \in]0, +\infty[$, $\lim_{\varepsilon \rightarrow 0} X_j^\varepsilon(r) = X_j(r)$, where $X_j(r)$ is defined in Proposition 2.1. And by the definition of a_j^ε and b_j^ε in (2.41), we have the desired behavior at 0 (2.36) for $(a_j^\varepsilon, b_j^\varepsilon)$.

For the estimates, we use

$$f_d^2(t) \leq Mt^{2d} \quad \text{and} \quad |1 - f_d^2 - \mu(\varepsilon)| \leq M,$$

This terminates the proof of Proposition 2.2.

Now let us turn to a base of solutions defined at $+\infty$. To make the proof clear, let us suppose that μ is sufficiently closed to 1 to have $-\frac{1}{2} \leq d^2(1 - \mu) \leq \frac{1}{2}$ and, for $n \geq 1$, let us define

$$n_\varepsilon = \sqrt{n^2 + d^2(1 - \mu(\varepsilon))}.$$

We note that for $n = 1$ we have $\mu(\varepsilon) > 1$ for every eigenvalue μ , so, $n_\varepsilon \geq 1$ in any case.

Proposition 2.3 *Let us suppose that $d \geq 1$ and $n \geq 1$ and that $\mu(\varepsilon) - 1 \rightarrow 0$, there exists a base of four solutions of (2.28) defined by their behaviors at $+\infty$ and denoted by Y_i^ε , $i = 1, \dots, 4$. They verify, in $[R, +\infty[$*

$$|u_1^\varepsilon - J_+| + |v_1^\varepsilon - J_+| \leq Cr^{-1}J_+, \quad |u_2^\varepsilon - J_-| + |v_2^\varepsilon - J_-| \leq Cr^{-1}J_- \quad (2.42)$$

and

$$|u_3^\varepsilon - r^{n\varepsilon}| + |v_3^\varepsilon + r^{n\varepsilon}| \leq Cr^{-1}r^{n\varepsilon}, \quad |u_4^\varepsilon - r^{-n\varepsilon}| + |v_4^\varepsilon + r^{-n\varepsilon}| \leq Cr^{-1}r^{-n\varepsilon}$$

and

$$|u_3^\varepsilon + v_3^\varepsilon| \leq Cr^{n\varepsilon}r^{-1}, \quad |u_4^\varepsilon + v_4^\varepsilon| \leq Cr^{-n\varepsilon}r^{-1}$$

where the both constants $R > 0$ and $C > 0$ are independent of ε . Moreover $Y_i^\varepsilon(r) \rightarrow Y_i(r)$ as $\varepsilon \rightarrow 0$, for all $r > 0$, where Y_i are defined in Proposition 2.1.

Proof The proof below is valid also when $\mu = 1$ and that gives the proof of the second part of Proposition 2.1.

Let us recall that we let $x = a + b$ and $y = a - b$. The system (2.28) becomes the following system verified by (x, y)

$$\begin{cases} x'' + \frac{x'}{r} - \frac{n^2+d^2}{r^2}x + \frac{2nd}{r^2}y + (1 - 3f_d^2)x + (\mu - 1)(1 - f_d^2)x = 0 \\ y'' + \frac{y'}{r} - \frac{n^2+d^2}{r^2}y + \frac{2nd}{r^2}x + (1 - f_d^2)y + (\mu - 1)(1 - f_d^2)y = 0. \end{cases} \quad (2.43)$$

We let

$$J_+ := \frac{e^{\sqrt{2}r}}{\sqrt{r}}, \quad J_- := \frac{e^{-\sqrt{2}r}}{\sqrt{r}} \quad \text{and} \quad \tilde{x}(r) := r^{\frac{1}{2}}x(r).$$

Thanks to

$$x'' + \frac{x'}{r} - \frac{1}{4r^2}x = r^{-\frac{1}{2}}\tilde{x}'' ,$$

we can replace the first equation of (2.43) by

$$\tilde{x}'' - 2\tilde{x} + \frac{-n^2 - d^2 + \frac{1}{4}}{r^2}\tilde{x} + 3(1 - f_d^2)\tilde{x} + \frac{2nd}{r^{\frac{3}{2}}}y + (\mu - 1)(1 - f_d^2)\tilde{x} = 0,$$

that is

$$(e^{2\sqrt{2}r}(\tilde{x}e^{-\sqrt{2}r}))' = e^{\sqrt{2}r} \left(q(r)\tilde{x} - \frac{2nd}{r^{\frac{3}{2}}}y \right)$$

or

$$(e^{-2\sqrt{2}r}(\tilde{x}e^{\sqrt{2}r}))' = e^{-\sqrt{2}r} \left(q(r)\tilde{x} - \frac{2nd}{r^{\frac{3}{2}}}y \right),$$

where

$$q(r) = \frac{n^2 + d^2 - \frac{1}{4}}{r^2} - 3(1 - f_d^2) - (\mu - 1)(1 - f_d^2).$$

The second equation of the system (2.43) can be written as

$$y'' + \frac{y'}{r} - \frac{n^2 + d^2(1 - \mu)}{r^2}y + \frac{2nd}{r^2}x + \mu(1 - f_d^2 - \frac{d^2}{r^2})y = 0,$$

that is also

$$(r^{2n_\varepsilon+1}(r^{-n_\varepsilon}y)')' = r^{n_\varepsilon+1}\left(-\frac{2nd}{r^2}x - \mu(1 - f_d^2 - \frac{d^2}{r^2})y\right)$$

or

$$(r^{-2n_\varepsilon+1}(r^{n_\varepsilon}y)')' = r^{-n_\varepsilon+1}\left(-\frac{2nd}{r^2}x - \mu(1 - f_d^2 - \frac{d^2}{r^2})y\right).$$

Eventually, the system (2.43) can be written as

$$\begin{cases} (e^{\pm 2\sqrt{2}r}(r^{\frac{1}{2}}e^{\mp\sqrt{2}r}x)')' = r^{\frac{1}{2}}e^{\pm\sqrt{2}r}\left(q(r)x - \frac{2nd}{r^2}y\right) \\ (r^{\pm 2n_\varepsilon+1}(r^{\mp n_\varepsilon}y)')' = r^{\pm n_\varepsilon+1}\left(-\frac{2nd}{r^2}x - \mu(1 - f_d^2 - \frac{d^2}{r^2})y\right) \end{cases} \quad (2.44)$$

We will construct four independent solutions (x_j, y_j) , $j = 1, \dots, 4$. Let us indicate the four fixed point equations we have to solve.

The exponential blowing up behavior at $+\infty$: the solution (x_1, y_1) .

We consider the fixed point problem

$$\begin{cases} x = J_+ + J_+ \int_{+\infty}^r e^{-2\sqrt{2}t} \int_R^t e^{\sqrt{2}s} s^{\frac{1}{2}} \left(-\frac{2nd}{s^2}y + q(s)x\right) ds dt \\ y = r^{n_\varepsilon} \int_R^r t^{-2n_\varepsilon-1} \int_R^t s^{n_\varepsilon+1} \left(\frac{2nd}{s^2}x - \mu(\varepsilon)(1 - f_d^2 - \frac{d^2}{s^2})y\right) ds dt. \end{cases}$$

The intermediate blowing up behavior at $+\infty$: the solution (x_3, y_3) . We consider the fixed point problem

$$\begin{cases} x = J_+ \int_{+\infty}^r e^{-2\sqrt{2}t} \int_R^t e^{\sqrt{2}s} s^{\frac{1}{2}} \left(-\frac{2nd}{s^2}y + q(s)x\right) ds dt \\ y = r^{n_\varepsilon} + r^{n_\varepsilon} \int_{+\infty}^r t^{-2n_\varepsilon-1} \int_R^t s^{n_\varepsilon+1} \left(\frac{2nd}{s^2}x - \mu(\varepsilon)(1 - f_d^2 - \frac{d^2}{s^2})y\right) ds dt. \end{cases}$$

The least behavior at $+\infty$: the solution (x_2, y_2) . We consider the fixed point problem

$$\begin{cases} x = J_- + J_- \int_{+\infty}^r e^{2\sqrt{2}t} \int_{+\infty}^t e^{-\sqrt{2}s} s^{\frac{1}{2}} \left(-\frac{2nd}{s^2}y + q(s)x\right) ds dt \\ y = r^{-n_\varepsilon} \int_{+\infty}^r t^{2n_\varepsilon-1} \int_{+\infty}^t s^{-n_\varepsilon+1} \left(\frac{2nd}{s^2}x - \mu(\varepsilon)(1 - f_d^2 - \frac{d^2}{s^2})y\right) ds dt. \end{cases}$$

The intermediate vanishing behavior at $+\infty$: the solution (x_4, y_4) . We consider the fixed point problem

$$\begin{cases} x = J_- \int_{R_0}^r e^{2\sqrt{2}t} \int_{+\infty}^t e^{-\sqrt{2}s} s^{\frac{1}{2}} \left(-\frac{2nd}{s^2}y + q(s)x\right) ds dt \\ y = r^{-n_\varepsilon} + r^{-n_\varepsilon} \int_{+\infty}^r t^{2n_\varepsilon-1} \int_{+\infty}^t s^{-n_\varepsilon+1} \left(\frac{2nd}{s^2}x - \mu(\varepsilon)(1 - f_d^2 - \frac{d^2}{s^2})y\right) ds dt. \end{cases}$$

Let us explain the pattern of proof. We denote each fixed point equation by

$$(x, y) = \Phi(x, y), \text{ for } (x, y) \text{ defined in } [R, +\infty[.$$

$$\text{Then we give a map } \zeta, \text{ that will be } \zeta(r) = \begin{cases} J_+(r) & \text{for } (x_1, y_1) \\ r^{n_\varepsilon} & \text{for } (x_3, y_3) \\ J_-(r) & \text{for } (x_2, y_2) \\ r^{-n_\varepsilon} & \text{for } (x_4, y_4). \end{cases}$$

And we want to prove, for a chosen R large enough and independent of ε , the existence of

a fixed point (x_j, y_j) verifying the estimate, for all $r \in]R, +\infty[$ and for some C depending only of R ,

$$|x_j(r) - \zeta(r)| + |y_j(r)| \leq C\zeta(r)r^{-1} \quad \text{if } j = 1, 3, \quad (2.45)$$

or

$$|x_j(r)| + |y_j(r) - \zeta(r)| \leq C\zeta(r)r^{-1} \quad \text{if } j = 2, 4. \quad (2.46)$$

For this purpose, we define by induction, for (x_1, y_1) and for (x_3, y_3)

$$(\alpha_0, \beta_0) = (\zeta, 0) \quad \text{and} \quad (\alpha_{k+1}, \beta_{k+1}) = \Phi(\alpha_k, \beta_k). \quad (2.47)$$

For (x_2, y_2) and for (x_4, y_4) , we exchange the role of x and y , that gives

$$(\alpha_0, \beta_0) = (0, \zeta) \quad \text{and} \quad (\alpha_{k+1}, \beta_{k+1}) = \Phi(\alpha_k, \beta_k). \quad (2.48)$$

We denote $\nu : r \mapsto r$.

Now we prove that there exists $M > 0$ independent of ε , independent of k , and independent of $R > 1$, such that for all $r \geq R$ and all $k \geq 1$, for $j = 1, 2$

$$\begin{aligned} |(\alpha_{k+1} - \alpha_k)\zeta^{-1}\nu|(r) &\leq Mr^{-1}(\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\nu\|_{\infty, [R, +\infty[} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\nu\|_{\infty, [R, +\infty[}) \end{aligned} \quad (2.49)$$

and

$$\begin{aligned} |(\beta_{k+1} - \beta_k)\zeta^{-1}\nu|(r) &\leq Mr^{-1}(\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\nu\|_{\infty, [R, +\infty[} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\nu\|_{\infty, [R, +\infty[}) \end{aligned} \quad (2.50)$$

and for $j = 3, 4$

$$\begin{aligned} |(\alpha_{k+1} - \alpha_k)\zeta^{-1}\nu^2|(r) &\leq Mr^{-1}(\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\nu^2\|_{\infty, [R, +\infty[} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\nu\|_{\infty, [R, +\infty[}) \end{aligned} \quad (2.51)$$

and

$$\begin{aligned} |(\beta_{k+1} - \beta_k)\zeta^{-1}\nu|(r) &\leq Mr^{-1}(\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\nu^2\|_{\infty, [R, +\infty[} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\nu\|_{\infty, [R, +\infty[}). \end{aligned} \quad (2.52)$$

We have to verify also, for $j = 1, 2$

$$|\alpha_1 - \alpha_0| \leq Mr^{-1}\zeta(r) \quad \text{and} \quad |\beta_1 - \beta_0| \leq Mr^{-1}\zeta(r) \quad (2.53)$$

and for $j = 3, 4$

$$|\alpha_1 - \alpha_0| \leq Mr^{-2}\zeta(r) \quad \text{and} \quad |\beta_1 - \beta_0| \leq Mr^{-1}\zeta(r). \quad (2.54)$$

with M independent of R and independent of ε .

Next we choose $R > 1$ such that $MR^{-1} < 1$ and we are allowed to define $x_j(r)$ and $y_j(r)$, for $j = 1, 2$ by

$$x_j(r) = \alpha_0(r) + r^{-1}\zeta \sum_{k \geq 0} r\zeta^{-1}(\alpha_{k+1} - \alpha_k)(r) \quad (2.55)$$

and

$$y_j(r) = \beta_0(r) + r^{-1} \zeta \sum_{k \geq 0} r \zeta^{-1} (\beta_{k+1} - \beta_k)(r)$$

and, for $j = 3, 4$, $x_j(r) = \alpha_0(r) + r^{-2} \zeta \sum_{k \geq 0} r^2 \zeta^{-1} (\alpha_{k+1} - \alpha_k)(r)$.

The sums converge, uniformly wrt ε , for all $r > R_0$. Consequently, arguing as for the solutions near 0, we get the existence of a solution (x, y) having the desired behavior (2.45) or (2.46) at $+\infty$ and we get also the limit of $(x, y)(r)$ as $\varepsilon \rightarrow 0$ to a solution of the same system as (2.43) but with $\mu = 1$, for each $r \in]0, +\infty[$ and having the same behavior (2.45) or (2.46) at $+\infty$. The proof of Proposition 2.3 follows, with $u^\varepsilon = x + y$ and $v^\varepsilon = x - y$.

For the estimates above, we need the following estimates, obtained by an integration by part. Let $\alpha \in \mathbb{R}$ and $\beta > 0$ be given. Then

$$\int_t^{+\infty} s^\alpha e^{-\beta s} ds \leq \frac{2}{\beta} t^\alpha e^{-\beta t} \quad \text{for all } t \geq \frac{2\alpha}{\beta} \quad (2.56)$$

and

$$\int_R^t s^\alpha e^{\beta s} ds \leq \begin{cases} \frac{2}{\beta} t^\alpha e^{\beta t} & \text{for all } t \geq R \geq \frac{-2\alpha}{\beta} \quad \text{if } \alpha < 0 \\ \frac{1}{\beta} t^\alpha e^{\beta t} & \text{for all } t \geq R > 0 \quad \text{if } \alpha \geq 0. \end{cases} \quad (2.57)$$

We will fix $R > 1$, large enough to have (2.57) for the desired α and β and we will use, for $s \geq R$,

$$|1 - \mu|(1 - f_d^2) \leq \frac{C}{r^2}, \quad |1 - f_d^2 - \frac{d^2}{r^2}| \leq \frac{C}{r^4} \quad \text{and} \quad |q(s)| \leq \frac{C}{r^2} \quad (2.58)$$

where C is independent of $R > 1$.

This terminates the proof of Proposition 2.3.

We turn now to the case $n = 0$, ie to the equation (1.20) and to the equation (1.19). We write the rescaled form of (1.19) as

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2} a + (1 - f_d^2) a + (\mu(\varepsilon) - 1)(1 - f_d^2) a = 0. \quad (2.59)$$

Proposition 2.4 (i) *There exists a function g_d such that f_d and g_d form a base of solutions of the equation (1.16) and we have*

$$|g_d(r) + \frac{1}{2dA_d} r^{-d}| \leq Cr^{-d+2} \quad \text{at } r = 0 \quad \text{and} \quad |g_d(r) - \log r| \leq Cr^{-2} \log r \quad \text{at } r = +\infty \quad (2.60)$$

where $A_d > 0$ is defined by $f_d = A_d r^d + O(r^{d+2})$ near $r = 0$.

(ii) *If $\mu(\varepsilon) \rightarrow 1$, there exists a base of two solutions a_1^ε and a_2^ε of (2.59) with the following behaviors at 0 : there exist $R > 0$ and $C > 0$ verifying for all $0 < r < R$*

$$|a_1^\varepsilon(r) - f_d(r)| \leq C|1 - \mu(\varepsilon)|r^2 f_d(r) \quad \text{and} \quad |a_2^\varepsilon(r) - r^{-d}| \leq Cr^{-d+1} \quad (2.61)$$

where R and C are independent of ε . Moreover, letting $\varepsilon \rightarrow 0$, for all $r > 0$ $a_1^\varepsilon(r) \rightarrow f_d(r)$ and $a_2^\varepsilon(r) \rightarrow -\frac{1}{2dA_d}g_d + Af_d$, for some $A \in \mathbb{R}$.

Proof (i) If g is any solution of the equation (1.16) we can combine the equation of g and the equation of f_d and integrating by parts we are led to, for all $r_1 > 0$ and $r_2 > 0$

$$[r(f_d'g - f_dg')(r)]_{r_1}^{r_2} = 0.$$

We deduce that there exists C such that $rf_d^2 \left(\frac{g_d}{f_d}\right)' = C$. This gives, for some $D \in \mathbb{R}$,

$$g(r) = Df_d(r) + Cf_d(r) \int_1^r \frac{dt}{tf_d^2(t)}.$$

We define

$$g_d(r) := f_d(r) \int_1^r \frac{dt}{tf_d^2(t)}.$$

A calculus gives (2.60).

(ii) We take advantage of the identity, valid for any function u

$$(ru^2(au^{-1})')' = (ra')'u - a(ru')'. \quad (2.62)$$

Firstly, we choose $u = f_d$.

Letting $g = af_d^{-1}$, we infer that (2.59) is equivalent to

$$(rf_d^2g')' = (1 - \mu)r(1 - f_d^2)f_d^2g.$$

We define the fixed point problem, for g

$$g = 1 + (1 - \mu) \int_0^r \frac{f_d^{-2}}{t} \int_0^t s(1 - f_d^2)f_d^2gdsdt. \quad (2.63)$$

We denote it by $g = \Phi(g)$.

Considering that $\frac{t^{2d}}{1+t^{2d}}f_d^{-2}$, $(1 - f_d^2)(1 + t^2)$ and $f_d^2(1 + t^{2d})t^{-2d}$ have positive limits together at 0 and at $+\infty$, we will use the following estimates, in $]0, +\infty[$, for some $M > 0$

$$f_d^{-2} \leq M \frac{1 + t^{2d}}{t^{2d}}, \quad 1 - f_d^2 \leq M \frac{1}{1 + t^2} \quad \text{and} \quad f_d^2 \leq M \frac{t^{2d}}{1 + t^{2d}}.$$

We define by induction

$$\alpha_0 = 1, \quad \alpha_{k+1} = \Phi(\alpha_k).$$

We define $\zeta(r) := r^2$. A calculus gives, for $r > 0$

$$|\alpha_1 - \alpha_0| \leq C|1 - \mu|r^2 \quad \text{and} \quad r^{-2}|\alpha_{k+1} - \alpha_k| \leq C|1 - \mu|r^2 \|\zeta^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r]}$$

where C is independent of r and independent of ε . We choose $R > 0$ such that $CR^2 < 1$ and we define, for all $r \in [0, R]$

$$g = 1 + \zeta \sum_{k \geq 0} \zeta^{-1} (\alpha_{k+1} - \alpha_k).$$

Thus $a_1^\varepsilon = f_d g$ verifies (2.61).

Secondly, we choose $u(r) = r^{-d}$.

We compute

$$(r^{-2d+1}(ar^{-d})')' = -r^{-d+1}\mu(1 - f_d^2)a$$

We let $g(r) := a(r)r^d$ and we solve the fixed point problem

$$g(r) = 1 + \int_0^r t^{2d-1} \int_R^t -\mu(1 - f_d^2)s^{-2d+1}g ds$$

where $R > 0$. The same method as above, with $\alpha_0 = 1$ and $\zeta(r) = r$ leads to

$$r^{-1}|\alpha_{k+1} - \alpha_k| \leq Cr \|\zeta^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0, R])} \quad \text{and} \quad r^{-1}|\alpha_1 - \alpha_0| \leq C \begin{cases} r \ln r & \text{if } d = 1 \\ r & \text{if } d \geq 2 \end{cases}$$

with C independent of ε and independent of R . Choosing R such that $CR < 1$, we conclude as above. This terminates the proof of (2.61).

We turn now to the equation (1.17), for $n = 0$. Its rescaled form is

$$a'' + \frac{a'}{r} - \frac{d^2}{r^2}a - 2f_d a + \mu(\varepsilon)(1 - f_d^2)a = 0. \quad (2.64)$$

Proposition 2.5 (i) *There exists a base of two solutions a_1 and a_2 defined near 0, for (1.17) verifying respectively*

$$|a_1 - f_d| \leq Cr^2 f_d \quad \text{and} \quad |a_2 - r^{-d}| \leq Cr^{-d+2}.$$

And if $\mu(\varepsilon) \rightarrow 1$, there exists a solution a_1^ε of (2.64) such that

$$|a_1^\varepsilon(r) - f_d(r)| \leq Cr^2 f_d(r)$$

for all $r \in [0, R]$ and for some R and some C independent of ε . Moreover $a_1^\varepsilon \rightarrow a_1$ as $\varepsilon \rightarrow 0$. And there exists a second solution that blows up at 0.

(ii) *There exists a base of two solutions of (1.17), b_1 and b_2 , defined near $+\infty$ and there exists some $R > 0$ verifying*

$$|b_1 - e^{\frac{\sqrt{2}r}{\sqrt{r}}}| \leq Cr^{-1}e^{\frac{\sqrt{2}r}{\sqrt{r}}} \quad \text{and} \quad |b_1 - e^{\frac{-\sqrt{2}r}{\sqrt{r}}}| \leq Cr^{-1}e^{\frac{-\sqrt{2}r}{\sqrt{r}}},$$

for all $r \in [R, +\infty[$.

And if $\mu(\varepsilon) \rightarrow 1$, there exists a base of two solutions b_1^ε and b_2^ε of (2.64) verifying

$$|b_1^\varepsilon - e^{\frac{\sqrt{2}r}{\sqrt{r}}}| \leq Cr^{-1}e^{\frac{\sqrt{2}r}{\sqrt{r}}} \quad \text{and} \quad |b_1^\varepsilon - e^{\frac{-\sqrt{2}r}{\sqrt{r}}}| \leq Cr^{-1}e^{\frac{-\sqrt{2}r}{\sqrt{r}}}$$

for all $r \in [R, +\infty[$, R and C being independent of ε .

(iii) *The solution a_1 defined at 0 blows up exponentially at $+\infty$, like Ab_1 , for some $A > 0$.*

Proof (i) Let us give a sketch of the proof. We can adapt the proof of (ii) in Proposition 2.4. Using (2.62) again, we choose firstly $u = f_d$ and secondly $u = r^{-d}$. We are led successively to the following forms of (2.64)

$$(rf_d^2(af_d^{-1})')' = 2rf_d^2a + (1 - \mu)af_d(1 - f_d^2)$$

and

$$(r^{-2d+1}(ar^d)')' = 2f_dar^{1-d} - \mu ar^{-d+1}(1 - f_d^2).$$

And we solve the both integral equations, where $g = af_d^{-1}$ and next $g = ar^d$,

$$g = 1 + \int_0^r \frac{f_d^{-1}}{t} \int_0^t ((1 - \mu)(1 - f_d^2) + 2f_d)s f_d^2 g \, ds dt.$$

and

$$g(r) = 1 + \int_0^r t^{2d-1} \int_R^t (-\mu(1 - f_d^2) + 2f_d) s^{-2d+1} g ds.$$

And we let $\mu = 1$, for the equation (1.17).

(ii) The same trick as above, in the proof of Proposition 2.3 leads to replace (2.64) by

$$(e^{2\sqrt{2}r}(r^{\frac{1}{2}}ae^{-\sqrt{2}r})')' = e^{\sqrt{2}r}r^{\frac{1}{2}}\left(\frac{1}{4r^2} - \mu(1 - f_d^2)\right)$$

or

$$(e^{-2\sqrt{2}r}(r^{\frac{1}{2}}ae^{\sqrt{2}r})')' = e^{-\sqrt{2}r}r^{\frac{1}{2}}\left(-\frac{1}{4r^2} + \frac{d^2}{r^2} - \mu(1 - f_d^2)\right).$$

And as above we solve the following fixed point problems, for $\mu = 1$ or for μ closed to 1 and for R large enough

$$a = J_+ + J_+ \int_{+\infty}^r e^{-2\sqrt{2}t} \int_R^t e^{\sqrt{2}s} s^{\frac{1}{2}} a \left(-\frac{1}{4r^2} + \frac{d^2}{r^2} - \mu(1 - f_d^2)\right) ds$$

and

$$a = J_- + J_- \int_{+\infty}^r e^{2\sqrt{2}t} \int_{+\infty}^t e^{-\sqrt{2}s} s^{\frac{1}{2}} a \left(-\frac{1}{4r^2} + \frac{d^2}{r^2} - \mu(1 - f_d^2)\right) ds.$$

And the method for the construction is the same as for Proposition 2.3.

(iii) Multiplying the equation of a_1 and the equation of f_d and integrating by parts, we obtain

$$r(a_1' f_d - f_d' a_1)(r) = \int_0^r 2t f_d^2(t) a_1(t) dt.$$

This gives

$$r f_d^2 \left(\frac{a_1}{f_d} \right)' (r) = \int_0^r 2t f_d^3(t) \frac{a_1}{f_d}(t) dt$$

and consequently $a_1(r) > 0$ for all r and the only convenient behavior at $+\infty$ for $\frac{a_1}{f_d}$ is the blowing up one.

3 Proof of Theorem 1.6.

Let $n \geq 1$. Let us suppose that $\mu(\varepsilon) \rightarrow 1$. Let (a, b) be any solution of (1.18), we let $X = (a, ra', b, rb')^t$. Considering the behavior of (a, b) at 0, there exists two real numbers $(A_1^\varepsilon, A_3^\varepsilon) \neq (0, 0)$ such that

$$X = A_1^\varepsilon X_1^\varepsilon + A_3^\varepsilon X_3^\varepsilon.$$

Now, the condition $a(\frac{1}{\varepsilon}) = b(\frac{1}{\varepsilon}) = 0$ leads us to the system

$$\begin{cases} A_1^\varepsilon a_1^\varepsilon(\frac{1}{\varepsilon}) + A_3^\varepsilon a_3^\varepsilon(\frac{1}{\varepsilon}) = 0 \\ A_1^\varepsilon b_1^\varepsilon(\frac{1}{\varepsilon}) + A_3^\varepsilon b_3^\varepsilon(\frac{1}{\varepsilon}) = 0. \end{cases}$$

Thus if we denote the determinant

$$\Delta := a_1^\varepsilon(\frac{1}{\varepsilon})b_3^\varepsilon(\frac{1}{\varepsilon}) - b_1^\varepsilon(\frac{1}{\varepsilon})a_3^\varepsilon(\frac{1}{\varepsilon}),$$

then $\Delta = 0$.

We obtain firstly that the corresponding real eigenspace is one dimensional.

Now, we can write

$$X_1^\varepsilon = \sum_{j=1}^4 C_j^\varepsilon Y_j^\varepsilon.$$

Each real number C_j^ε can be computed by means of a 4×4 determinant, for any fixed $r > 0$ and consequently, since $X_1^\varepsilon(r)$ and each $Y_j^\varepsilon(r)$ has a limit as $\varepsilon \rightarrow 0$, then, each C_j^ε has a limit too, denoted by C_j . By Theorem 2.7, X_1 has the exponential blowing up behavior at $+\infty$ and we deduce that $C_1 \neq 0$. Consequently, we can choose X_1^ε to represent a solution of (2.28) having the exponentially blowing up behavior at $+\infty$ instead of Y_1^ε . We can write

$$X_3^\varepsilon = D_1^\varepsilon X_1^\varepsilon + \sum_{j=2}^4 D_j^\varepsilon Y_j^\varepsilon.$$

As explained above, each real number D_j^ε has a limit as $\varepsilon \rightarrow 0$. Moreover,

$$\Delta = \begin{vmatrix} a_1^\varepsilon(\frac{1}{\varepsilon}) & \sum_{j=2}^4 D_j^\varepsilon u_j^\varepsilon(\frac{1}{\varepsilon}) \\ b_1^\varepsilon(\frac{1}{\varepsilon}) & \sum_{j=2}^4 D_j^\varepsilon v_j^\varepsilon(\frac{1}{\varepsilon}) \end{vmatrix}.$$

In view of Proposition 2.3, $\Delta \neq 0$, unless

$$\sum_{j=2}^4 D_j^\varepsilon u_j^\varepsilon(\frac{1}{\varepsilon}) = \sum_{j=2}^4 D_j^\varepsilon v_j^\varepsilon(\frac{1}{\varepsilon}) = 0.$$

This condition implies that $D_3^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. We are led to $X_3 - D_1 X_1 = D_2 Y_2 + D_4 Y_4$, that is bounded at $+\infty$ and bounded at 0. We deduce that if $\mu(\varepsilon) \rightarrow 1$, then there exists a bounded solution. Moreover, since we have proved just above that the eigenspace associated to $\mu(\varepsilon)$ is one dimensional, a base of this eigenspace is $\sum_{j=2}^4 D_j^\varepsilon Y_j^\varepsilon = X_3^\varepsilon - D_1^\varepsilon X_1^\varepsilon$, that tends to $D_2 Y_2 + D_4 Y_4$. If $n \geq 2$, by Theorem 1.1, we have the proof of (i). Now, when there exists some bounded solution of (2.1) and if $\mu(\varepsilon) \rightarrow 1$, let us denote $\omega_\varepsilon = a^\varepsilon e^{-in\theta} + b^\varepsilon e^{in\theta}$ a given associated eigenvector and ω the bounded solution such

that $\omega^\varepsilon \rightarrow \omega$, as $\varepsilon \rightarrow 0$. We denote $\omega^\varepsilon = \sum_{j=2}^4 D_j^\varepsilon \omega_j^\varepsilon = \omega_3^\varepsilon - D_1^\varepsilon \omega_1^\varepsilon$. We have, for a given $A > 0$,

$$\begin{aligned} \|\omega^\varepsilon - \omega\|_{L^2(B(0, \frac{1}{\varepsilon}))} &\leq \left\| \sum_{j=2}^4 D_j^\varepsilon \omega_j^\varepsilon - \sum_{j=2}^4 D_j \omega_j \right\|_{L^2(B(0, \frac{1}{\varepsilon}) \setminus B(0, A))} + \\ &+ \|\omega_3^\varepsilon - D_1^\varepsilon \omega_1^\varepsilon - \omega_3 + D_1 \omega_1\|_{L^2(B(0, A))}. \end{aligned}$$

The second term of the rhs tends obviously to 0, and, since $D_3 = 0$, the first term can be estimated as

$$\begin{aligned} \left\| \sum_{j=2}^4 D_j^\varepsilon \omega_j^\varepsilon - \sum_{j=2}^4 D_j \omega_j \right\|_{L^2(B(0, \frac{1}{\varepsilon}) \setminus B(0, A))} &\leq \|D_2^\varepsilon \omega_2^\varepsilon - D_2 \omega_2\|_{L^2(B(0, \frac{1}{\varepsilon}) \setminus B(0, A))} + \\ &+ \|D_4^\varepsilon \omega_4^\varepsilon - D_4 \omega_4\|_{L^2(B(0, \frac{1}{\varepsilon}) \setminus B(0, A))} + \|D_3^\varepsilon \omega_3^\varepsilon\|_{L^2(B(0, \frac{1}{\varepsilon}) \setminus B(0, A))}. \end{aligned}$$

The first two terms tend to 0 by the Lebesgue Theorem and for the third term we estimate

$$\|\omega_3^\varepsilon\|_{L^2(B(0, \frac{1}{\varepsilon}) \setminus B(0, A))} \leq C \varepsilon^{-n+1}$$

with C independent of ε , while the condition $\sum_{j=2}^4 D_j^\varepsilon u_j^\varepsilon(\frac{1}{\varepsilon}) = 0$ gives

$$D_3^\varepsilon = O(\varepsilon^{2n}).$$

Eventually, we conclude that

$$\|\omega^\varepsilon - \omega\|_{L^2(B(0, \frac{1}{\varepsilon}))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Now, to complete the proof of (ii), we consider $n = 0$ and $n = 1$. The proof of $m_{0,d} > 1$ and the proof of $m_{1,d} > 1$ are the same as for $\lambda_{0,d} > 0$ and $\lambda_{1,d} > 0$, in [12]. Indeed, multiplying the equation (1.19), by f and integrating by parts on $[0, 1]$, we obtain $m_{0,d} > 1$. Then, using a truncation of f , with value 0 for $r \geq 1$, as a test function for the infimum $m_{0,d}$, and since we know the existence of the limit, we obtain that $\lim_{\varepsilon \rightarrow 0} m_{0,d}(\varepsilon) \leq 1$. And the proof of $m_{1,d} > 1$ uses a trick involving the system (2.43). The only difference is the positive factor $1 - f_d^2$. The proof of $\lim_{\varepsilon \rightarrow 0} m_{1,d}(\varepsilon) = 1$ follows from Lemma 1.1. Indeed, it must exist $\mu(\varepsilon) \rightarrow 1$, but $m_{1,d}(\varepsilon) > 1$ is the least eigenvalue, so $m_{1,d}(\varepsilon) \rightarrow 1$. And thanks to the proof above, the limit of an eigenvector for $\mu(\varepsilon) \rightarrow 1$ has to be the bounded solution of (1.15) and the norm $\|\omega^\varepsilon - \omega\|_{L^2([0, \frac{1}{\varepsilon}])} \rightarrow 0$. But since the eigenvalue problem is, with the suitable rescaling, $\mathcal{T}\tilde{\omega}^\varepsilon = \mu(\varepsilon)\mathcal{C}\tilde{\omega}^\varepsilon$ and in view of the scalar product for $\mathcal{H}_{n,d}$, we have $\langle \mathcal{C}\tilde{\omega}^\varepsilon, \tilde{\eta}^\varepsilon \rangle_{L^2([0,1])} = 0$ when $\tilde{\eta}^\varepsilon$ and $\tilde{\omega}^\varepsilon$ are two independent eigenvectors that gives $\langle (1 - f_d)\omega^\varepsilon, \eta^\varepsilon \rangle_{L^2([0, \frac{1}{\varepsilon}])} = 0$, this is in contradiction with $\|\omega^\varepsilon - \eta^\varepsilon\|_{L^2([0, \frac{1}{\varepsilon}])} \rightarrow 0$. This proves that there is at most one eigenvalue tending to 1. We deduce (ii) for $n = 1$. Now, for $n = 0$, if $\mu(\varepsilon) > m_{0,d}(\varepsilon)$ and if $\mu(\varepsilon) \rightarrow 1$, we consider a^ε a solution of (2.59) with $\mu(\varepsilon) = m_{0,d}(\varepsilon)$ and we consider b^ε a solution for $\mu(\varepsilon)$. Thanks to Proposition 2.4, we can chose $a^\varepsilon(0) = b^\varepsilon(0) = 1$ and combining the equations of a^ε and of b^ε , we find that for all $r > 0$

$$r((a^\varepsilon)'b^\varepsilon - a^\varepsilon(b^\varepsilon)')(r) = \int_0^r s(\mu(\varepsilon) - m_{0,d}(\varepsilon))a^\varepsilon b^\varepsilon (1 - f_d^2) ds.$$

Since $a^\varepsilon > 0$, this proves that $(\frac{a^\varepsilon}{b^\varepsilon})'(r) > 0$ as soon as $\frac{a^\varepsilon}{b^\varepsilon} > 0$ in $[0, r[$ and consequently $b^\varepsilon(r) > 0$ for all $r \geq 0$. But this is in contradiction with $\int_0^{\frac{1}{\varepsilon}} (1 - f_d^2) a^\varepsilon b^\varepsilon s ds = 0$. Thus, if $\mu(\varepsilon) > m_{0,d}(\varepsilon)$ is an eigenvalue for $n = 0$, we have $|\mu(\varepsilon) - 1| \geq C$, with C independent of ε .

Now let us prove that $\tilde{m}_{0,d}(\varepsilon) \geq 1 + C$, with $C > 0$ independent of ε , where $\tilde{m}_{0,d}$ is defined in (1.24). If \tilde{a}_0^ε realizes $\tilde{m}_{0,d}$, we can estimate

$$\tilde{m}_{0,d}(\varepsilon) \geq m_{0,d}(\varepsilon) + \varepsilon^2 \frac{\int_0^{\frac{1}{\varepsilon}} 2r f_d (a_0^\varepsilon)^2 dr}{\int_0^{\frac{1}{\varepsilon}} r (1 - f_d^2) (a_0^\varepsilon)^2 dr}.$$

But, with the notations of Proposition 2.5, we can write $a_0^\varepsilon = A_1^\varepsilon b_1^\varepsilon + A_2^\varepsilon b_2^\varepsilon$, and, as above $A_1^\varepsilon \rightarrow A_1$, $A_2^\varepsilon \rightarrow A_2$, for some real numbers A_1 and A_2 . And since $a_0^\varepsilon \rightarrow a_0$, that can be supposed to be equal to a_1 , we have $A_1 \neq 0$. Since $1 - f_d^2 = \frac{d^2}{r^2} + O(\frac{1}{r^4})$, a calculus gives some $M > 0$ independent of ε such that

$$\tilde{m}_{0,d}(\varepsilon) \geq m_{0,d}(\varepsilon) + M.$$

4 The proof of Lemma 1.1.

To begin with, let us recall some notation for the eigenvalue problem (1.18). We consider the operator $\mathcal{T}_{n,d} : \mathcal{H}_{n,d} \rightarrow \mathcal{H}'_{n,d}$ defined by

$$\langle -\mathcal{T}_{n,d}(a, b), (u, v) \rangle_{\mathcal{H}'_{n,d}, \mathcal{H}_{n,d}} := \int_0^1 (r a' u' + r b' v' + \frac{(n-d)^2}{r} a u + \frac{(n+d)^2}{r} b v + \frac{r}{\varepsilon^2} f^2 (a+b)(u+v)) dr.$$

We remark that

$$((a, b), (u, v)) \mapsto \langle \mathcal{T}_{n,d}(a, b), (u, v) \rangle_{\mathcal{H}'_{n,d}, \mathcal{H}_{n,d}}$$

is a scalar product on $\mathcal{H}_{n,d}$. So, $\mathcal{T}_{n,d}$ is an isomorphism, by the Riesz Theorem.

Last, let us define the embedding

$$I : \mathcal{H}_{n,d} \rightarrow \mathcal{H}'_{n,d} \\ (a, b) \mapsto ((u, v) \mapsto \int_0^1 r (a u + b v) dr)$$

Since the embedding $H_0^1(B(0, 1)) \times H_0^1(B(0, 1)) \subset L^2(B(0, 1)) \times L^2(B(0, 1))$ is compact, then I is compact.

Let us define $\mathcal{C} = \frac{1}{\varepsilon^2} (1 - f^2) I$. Since \mathcal{C} is a compact operator and thanks to the continuity of $\mathcal{T}_{n,d}^{-1}$, then $\mathcal{T}_{n,d}^{-1} \mathcal{C}$ is a compact operator from $\mathcal{H}_{n,d}$ into itself. By the standard theory of self adjoint compact operators, there exists a Hilbertian base of $\mathcal{H}_{n,d}$ formed of eigenvectors of $\mathcal{T}_{n,d}^{-1} \mathcal{C}$. We will consider $(\zeta_i)_{i \in J}$ such a base. We can write the eigenvalue problem as $\mathcal{T}_{n,d}(a, b) = \mu(\varepsilon) \mathcal{C}(a, b)$. We can normalized the base (ζ_j) in order to have

$$\langle \mathcal{C} \zeta_i, \zeta_j \rangle_{(L^2 \times L^2)(B(0,1))} = 0 \quad \text{for } i \neq j \quad \text{and} \quad \langle \mathcal{C} \zeta_j, \zeta_j \rangle_{(L^2 \times L^2)(B(0,1))} = 1.$$

Let us suppose that (a, b) is a bounded solution of (1.15). Let $\frac{1}{2} < N < 1$ be given, let us define (a^{cut}, b^{cut}) by

$$(a^{cut}, b^{cut})(r) = \begin{cases} (a, b)(r) & \text{for } 0 \leq r \leq \frac{N}{\varepsilon} \\ ((a, b)(r)(1 - h(r))) & \text{for } \frac{N}{\varepsilon} \leq r \leq \frac{1}{\varepsilon} \end{cases}$$

where

$$h(r) = \frac{(r - \frac{N}{\varepsilon})^3}{(\frac{1}{\varepsilon} - \frac{N}{\varepsilon})^3}.$$

We have $a^{cut} e^{id\theta} \in (H^2 \cap H_0^1)(B(0, \frac{1}{\varepsilon}))$ and $b^{cut} e^{id\theta} \in (H^2 \cap H_0^1)(B(0, \frac{1}{\varepsilon}))$, since this is true for a and b and since moreover the first two derivatives of $a^{cut} e^{id\theta}$ and $b^{cut} e^{id\theta}$ are continuous.

In view of the possible behaviors at $+\infty$ given in Proposition 2.1, we have, for ε small enough

$$|a(r)| \leq Cr^{-n} \text{ for } \frac{N}{\varepsilon} < r < \frac{1}{\varepsilon}, \quad |a^{cut}| \leq |a| \quad \text{and} \quad r(1 - f_d^2) = O(\frac{1}{r}) \quad \text{at } +\infty$$

and we verify that

$$\begin{aligned} & \langle (a^{cut} - a, b^{cut} - b), (1 - f_d^2)(a^{cut} - a, b^{cut} - b) \rangle_{(L^2 \times L^2)(B(0, \frac{1}{\varepsilon}))} \\ &= \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(1 - f_d^2)((a - a^{cut})^2 + (b - b^{cut})^2) dr = O(\varepsilon^{2n}) \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (4.65)$$

Then, let us define

$$(\tilde{a}^{cut}, \tilde{b}^{cut})(r) = (a^{cut}, b^{cut})(\frac{r}{\varepsilon}) \quad 0 < r < 1.$$

We write

$$(\tilde{a}^{cut}, \tilde{b}^{cut}) = \sum_{i \in J} \alpha_i(\varepsilon) \zeta_i$$

and

$$\begin{aligned} & \langle (1 - f_d^2)(a^{cut}, b^{cut}), (a^{cut}, b^{cut}) \rangle_{(L^2 \times L^2)(B(0, \frac{1}{\varepsilon}))} \\ &= \langle \mathcal{C} \sum_{j \in J} \alpha_j \zeta_j, \sum_{j \in J} \alpha_j(\varepsilon) \zeta_j \rangle_{(L^2 \times L^2)(B(0, 1))} = \sum_{i \in J} \alpha_i^2(\varepsilon). \end{aligned}$$

By (4.65),

$$\langle (1 - f_d^2)(a^{cut}, b^{cut}), (a^{cut}, b^{cut}) \rangle_{L^2(B(0, \frac{1}{\varepsilon}))} \longrightarrow \int_0^{+\infty} r(1 - f_d^2)(a^2 + b^2) dr \text{ as } \varepsilon \rightarrow +\infty.$$

Consequently there exists $I \subset J$, such that

$$I \neq \emptyset \text{ and for all } i \in I, \quad \alpha_i^2(\varepsilon) \not\rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \quad (4.66)$$

Now we write

$$(\mathcal{T} - \mathcal{C})(\tilde{a}^{cut}, \tilde{b}^{cut}) = \sum_{i \in J} \alpha_i(\varepsilon)(\mu_i(\varepsilon) - 1) \mathcal{C} \zeta_i$$

that gives

$$\langle (\mathcal{T} - \mathcal{C})(\tilde{a}^{cut}, \tilde{b}^{cut}), \sum_{i \in J} \alpha_i(\varepsilon)(\mu_i(\varepsilon) - 1) \zeta_i \rangle_{\mathcal{H}'_{n,d}, \mathcal{H}_{n,d}} = \sum_{i \in J} \alpha_i^2(\varepsilon)(\mu_i(\varepsilon) - 1)^2. \quad (4.67)$$

But $(\mathcal{T} - \mathcal{C})(\tilde{a}^{cut}, \tilde{b}^{cut})$ is represented by a function of $L^2(B(0, 1)) \times L^2(B(0, 1))$. Using this identification, we can estimate the rhs of (4.67) as follows,

$$\begin{aligned}
& \langle (\mathcal{T} - \mathcal{C})(\tilde{a}^{cut}, \tilde{b}^{cut}), \sum_{i \in J} \alpha_i (\mu_i(\varepsilon) - 1) \zeta_i \rangle_{\mathcal{H}'_{n,d}, \mathcal{H}_{n,d}} \\
&= \langle (\mathcal{T} - \mathcal{C})(\tilde{a}^{cut}, \tilde{b}^{cut}), \frac{\varepsilon^2}{1 - f^2} (\mathcal{T} - \mathcal{C})(\tilde{a}^{cut}, \tilde{b}^{cut}) \rangle_{(L^2 \times L^2)(B(0,1))} \\
&= \int_N^1 \frac{r \varepsilon^2}{1 - f^2} \left[\left((\tilde{a}^{cut})'' + \frac{(\tilde{a}^{cut})'}{r} - \frac{(n-d)^2}{r^2} \tilde{a}^{cut} - \frac{1}{\varepsilon^2} f^2 (\tilde{a}^{cut} + \tilde{b}^{cut}) - \frac{1}{\varepsilon^2} (1 - f^2) \tilde{a}^{cut} \right)^2 \right. \\
&\quad \left. + \left((\tilde{b}^{cut})'' + \frac{(\tilde{b}^{cut})'}{r} - \frac{(n+d)^2}{r^2} \tilde{b}^{cut} - \frac{1}{\varepsilon^2} f^2 (\tilde{a}^{cut} + \tilde{b}^{cut}) - \frac{1}{\varepsilon^2} (1 - f^2) \tilde{b}^{cut} \right)^2 \right] dr. \\
&= \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1 - f_d^2} \left[\left((a^{cut})'' + \frac{(a^{cut})'}{r} - \frac{(n-d)^2}{r^2} a^{cut} - f_d^2 (a^{cut} + b^{cut}) - (1 - f_d^2) a^{cut} \right)^2 \right. \\
&\quad \left. + \left((b^{cut})'' + \frac{(b^{cut})'}{r} - \frac{(n+d)^2}{r^2} b^{cut} - f_d^2 (a^{cut} + b^{cut}) - (1 - f_d^2) b^{cut} \right)^2 \right] dr.
\end{aligned}$$

Let us estimate each term, as $\varepsilon \rightarrow 0$.

We use

$$\frac{r}{1 - f_d^2} = O(r^3) \quad \text{at } +\infty$$

to get

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1 - f^2} \frac{(a^{cut})^2}{r^4} dr = O(\varepsilon^{2n}).$$

Taking advantage that $a + b = O(r^{-n-2})$ at $+\infty$, a similar estimate for $a^{cut} + b^{cut}$ gives

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1 - f_d^2} f_d^2 (a^{cut} + b^{cut})^2 dr = O(\varepsilon^{2n}).$$

Now

$$a^{cut'} = a'(1 - h) + ah' \quad \text{and} \quad |a'| \leq Cr^{-n-1} \quad \text{and} \quad \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} h'^2 dr = O(\varepsilon).$$

We deduce that

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1 - f_d^2} \frac{(a^{cut'})^2}{r^2} dr = O(\varepsilon^{2n}).$$

Now, since

$$|a''| \leq Cr^{-n-2} \quad \text{and} \quad \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} h''^2 dr = O(\varepsilon^3)$$

we get

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1 - f_d^2} (a^{cut''})^2 dr = O(\varepsilon^{2n}).$$

We have proved that

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} (a^{cut''} + \frac{a^{cut'}}{r} - \frac{(n-d)^2}{r^2} a^{cut} - f_d^2(a^{cut} + b^{cut}) - (1-f_d^2)a^{cut})^2 dr = O(\varepsilon^{2n})$$

and with the same proof we have

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} (b^{cut''} + \frac{b^{cut'}}{r} - \frac{(n+d)^2}{r^2} b^{cut} - f_d^2(a^{cut} + b^{cut}) - (1-f_d^2)b^{cut})^2 dr = O(\varepsilon^{2n})$$

and eventually

$$\langle (\mathcal{T} - \mathcal{C})(\tilde{a}^{cut}, \tilde{b}^{cut}), \sum_{i \in J} \alpha_i(\varepsilon)(\mu_i(\varepsilon) - 1)\zeta_i \rangle_{\mathcal{H}'_{n,d}, \mathcal{H}_{n,d}} = O(\varepsilon^{2n}) \quad (4.68)$$

But (4.68) and (4.67) give

$$\sum_{i \in J} \alpha_i^2(\varepsilon)(\mu_i(\varepsilon) - 1)^2 = O(\varepsilon^{2n}).$$

So, for all $i \in J$ we have

$$|\alpha_i(\varepsilon)(\mu_i(\varepsilon) - 1)| = O(\varepsilon^n).$$

Since $n \geq 1$, we are led to

$$\mu_i(\varepsilon) - 1 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \text{ for all } i \in I,$$

where the set I is defined in (4.66).

We have proved Lemma 1.1, for $n \geq 1$.

5 The proof of Theorem 1.2 completed.

First, let $\omega = \sum_{n \geq 1} (a_n e^{-in\theta} + b_n e^{in\theta}) + a_0$, where a_n and b_n are real valued functions. We are going to prove that

$$| \langle \mathcal{L}_\varepsilon(\omega), \omega \rangle_{L^2(B(0,1))} | \geq C \langle \mathcal{C}\omega, \omega \rangle_{L^2(B(0,1))}. \quad (5.69)$$

The second inequality of (1.4) will follow, since we remark that if we define $\eta := \sum_{n \geq 1} (a_n e^{-in\theta} - b_n e^{in\theta}) + a_0$ and if α_n, β_n are the real valued functions defined by

$$\mathcal{L}_\varepsilon(\eta) = \sum_{n \geq 1} (\alpha_n e^{-in\theta} + \beta_n e^{in\theta}) + \alpha_0,$$

we have

$$\mathcal{L}_\varepsilon(i\omega) = \sum_{n \geq 1} i(\alpha_n e^{-in\theta} - \beta_n e^{in\theta}) + i\alpha_0 + i \frac{f^2}{\varepsilon^2} 2a_0.$$

Consequently, we have

$$\langle \mathcal{C}(i\omega), i\omega \rangle_{L^2(B(0,1))} = \langle \mathcal{C}\eta, \eta \rangle_{L^2(B(0,1))}$$

and

$$\langle \mathcal{L}_\varepsilon(i\omega), i\omega \rangle_{L^2(B(0,1))} = \langle \mathcal{L}_\varepsilon(\eta), \eta \rangle_{L^2(B(0,1))} + 2\frac{f^2}{\varepsilon^2}|a_0|^2.$$

Thus, the inequality

$$\langle \mathcal{L}_\varepsilon(i\omega), i\omega \rangle_{L^2(B(0,1))} \geq C \langle \mathcal{C}(i\omega), i\omega \rangle_{L^2(B(0,1))}$$

follows from the inequality

$$\langle \mathcal{L}_\varepsilon(\eta), \eta \rangle_{L^2(B(0,1))} \geq C \langle \mathcal{C}\eta, \eta \rangle_{L^2(B(0,1))}.$$

We define the operator \mathcal{T} by

$$\mathcal{L}_\varepsilon = -\mathcal{T} + \mathcal{C}$$

where $\mathcal{C}\omega = \frac{\omega}{\varepsilon^2}(1 - f^2)$.

We consider the Hilbert space

$$\oplus_{n \geq 1} \mathcal{H}_{n,d},$$

that is

$$\oplus_{n \geq 1} \mathcal{H}_{n,d} := \left\{ \omega \in \mathcal{H}, \omega = \sum_{n \geq 1} (a_n e^{-in\theta} + b_n e^{in\theta}), a_n \text{ and } b_n \text{ real valued, } a_n(1) = b_n(1) = 0 \right\},$$

We may endow $\oplus_{n \geq 1} \mathcal{H}_{n,d}$ with that scalar product

$$\langle \omega, \tilde{\omega} \rangle := \langle \mathcal{T}\omega, \tilde{\omega} \rangle_{L^2(B(0,1))}$$

and this scalar product let $\oplus_{n \geq 1} \mathcal{H}_{n,d}$ be a Hilbert space.

We let $(\zeta_j)_{j \in J}$ be a Hilbertian base, each ζ_j being an eigenvector for the operator $\mathcal{T}^{-1}\mathcal{C}$.

We let the eigenvalue $\mu_i(\varepsilon)$ be defined by $\mathcal{T}\zeta_j = \mu_j(\varepsilon)\mathcal{C}\zeta_j$. This implies that

$$\langle \mathcal{C}\zeta_i, \zeta_j \rangle_{L^2(B(0,1))} = 0 \quad \text{for } i \neq j.$$

We define $(W^\perp)_1$ the subspace of W_1^\perp corresponding to the eigenvalues $\mu_j(\varepsilon) > 1$ and $(W^\perp)_2$ corresponding to the eigenvalues $\mu_j(\varepsilon) < 1$. And we define $J = J_1 \cup J_2$. By Theorem 1.6 there exists some $C > 0$ independent of ε such that

$$1 - \mu_j(\varepsilon) > C \quad \text{for } j \in J_1 \quad \text{and} \quad 1 - \mu_j(\varepsilon) < -C \quad \text{for } j \in J_2.$$

We write

$$\omega = \sum_{i \in J} \alpha_i \zeta_i$$

From the definition of $(\zeta_j)_{j \in J}$, we infer that

$$\mathcal{T}\omega = \sum_{j \in J} \alpha_j \mu_j \mathcal{C}\zeta_j$$

and consequently

$$\mathcal{L}_\varepsilon \omega = \sum_{j \in J} \alpha_j (1 - \mu_j) \mathcal{C}\zeta_j,$$

that gives

$$\langle \mathcal{L}_\varepsilon \omega, \omega \rangle_{L^2(B(0,1))} = \sum_{j \in J} \alpha_j^2 (1 - \mu_j) \langle \mathcal{C} \zeta_j, \zeta_j \rangle.$$

If $\omega \in (W^\perp)_1$, $\alpha_j = 0$ unless $j \in J_1$. We deduce that

$$\langle \mathcal{L}_\varepsilon \omega, \omega \rangle_{L^2(B(0,1))} \leq -C \langle \mathcal{C} \omega, \omega \rangle_{L^2(B(0,1))}.$$

If $\omega \in (W^\perp)_2$, $\alpha_j = 0$ unless $j \in J_1$. We deduce that

$$\langle \mathcal{L}_\varepsilon \omega, \omega \rangle_{L^2(B(0,1))} \geq C \langle \mathcal{C} \omega, \omega \rangle_{L^2(B(0,1))}.$$

In any case, we have (5.69) and this proves (1.4), for $\omega \in \oplus_{n \geq 1} \mathcal{H}_{n,d}$.

As remarked in the beginning of this section, we obtain (1.4) for $\omega_{\mathcal{R}} \in \oplus_{n \geq 1} \mathcal{H}_{n,d}$ and for $\omega_{\mathcal{I}} \in \oplus_{n \geq 1} \mathcal{H}_{n,d}$.

Now, we know from Theorem 1.6 (ii) that if $a_0 \in \mathcal{H}_{0,d}$ is real valued, then

$$\langle \mathcal{L}_\varepsilon a_0, a_0 \rangle_{L^2(B(0, \frac{1}{\varepsilon}))} \geq C \langle \mathcal{C} a_0, a_0 \rangle_{L^2(B(0, \frac{1}{\varepsilon}))}$$

and if $ia_0 \in \langle i\Phi_0 \rangle^\perp$, then

$$\langle \mathcal{L}_\varepsilon ia_0, ia_0 \rangle_{L^2(B(0, \frac{1}{\varepsilon}))} \geq C \langle \mathcal{C} a_0, a_0 \rangle_{L^2(B(0, \frac{1}{\varepsilon}))}$$

where, in the both cases, C is independent of ε . That terminates the proof of (1.4).

Now, if $\omega \in W^\perp \cap \mathcal{H}$ is a solution of (1.3), as claimed in (1.11), if we denote $h = h_{\mathcal{R}} + ih_{\mathcal{I}}$, we have

$$(\mathcal{L}_\varepsilon \omega = h) \Leftrightarrow (\mathcal{L}_\varepsilon \omega_{\mathcal{R}} = h_{\mathcal{R}}, \mathcal{L}_\varepsilon i\omega_{\mathcal{I}} = ih_{\mathcal{I}}).$$

We consider first $\omega_{\mathcal{R}}$ and $h_{\mathcal{R}}$.

The equality

$$\langle \mathcal{L}_\varepsilon \omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} = \langle h_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))}$$

together with (1.4) gives

$$|\langle h_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))}| \geq M \langle \mathcal{C} \omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))}.$$

We deduce that

$$\langle \mathcal{C} \omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} \leq M \langle \mathcal{C} \omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))}^{\frac{1}{2}} \left(\int_0^1 r \frac{\varepsilon^2}{1-f^2} |h_{\mathcal{R}}|^2 dr \right)^{\frac{1}{2}}$$

and consequently

$$\langle \mathcal{C} \omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} \leq M \int_0^1 r \frac{\varepsilon^2}{1-f^2} |h_{\mathcal{R}}|^2 dr. \quad (5.70)$$

But since $\mathcal{L} = -\mathcal{T} + \mathcal{C}$, we are led to

$$\langle \mathcal{T} \omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} = - \langle h_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} + \langle \mathcal{C} \omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))}.$$

That gives

$$\langle \mathcal{T}\omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} \leq \langle \mathcal{C}\omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))}^{\frac{1}{2}} \left(\int_0^1 r \frac{\varepsilon^2}{1-f^2} |h_{\mathcal{R}}|^2 dr \right)^{\frac{1}{2}} + \langle \mathcal{C}\omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))}$$

and using (5.70), we obtain

$$\langle \mathcal{T}\omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} \leq M \int_0^1 r \frac{\varepsilon^2}{1-f^2} |h_{\mathcal{R}}|^2 dr.$$

Now we use

$$1 - f^2 \geq C \frac{\varepsilon^2}{\varepsilon^2 + r^2} \quad \text{for all } r \geq 0 \quad (5.71)$$

with C independent of ε to obtain

$$\langle \mathcal{T}\omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} \leq M \int_0^1 r(\varepsilon^2 + r^2) |h_{\mathcal{R}}|^2 dr. \quad (5.72)$$

that gives (1.6). Moreover, we use (5.71) in (5.70) to obtain

$$\int_0^1 \frac{r}{\varepsilon^2 + r^2} |\omega_{\mathcal{R}}|^2 dr \leq M \int_0^1 r(\varepsilon^2 + r^2) |h_{\mathcal{R}}|^2 dr. \quad (5.73)$$

Now (5.72) and (5.73) are valid for $i\omega_{\mathcal{I}}$ instead of $\omega_{\mathcal{R}}$ and $ih_{\mathcal{I}}$ instead of $h_{\mathcal{R}}$.

Thus (5.73) gives

$$\int_0^1 \frac{r}{\varepsilon^2 + r^2} (|\omega_{\mathcal{R}}|^2 + |\omega_{\mathcal{I}}|^2) dr \leq M \int_0^1 r(\varepsilon^2 + r^2) (|h_{\mathcal{R}}|^2 + |h_{\mathcal{I}}|^2) dr$$

that gives (1.5) and (1.7).

Now, let us recall that

$$\begin{aligned} \langle \mathcal{T}\omega_{\mathcal{R}}, \omega_{\mathcal{R}} \rangle_{L^2(B(0,1))} &= \sum_{n \geq 1} \int_0^1 \left(r|a'_n|^2 + r|b'_n|^2 + \frac{(n-d)^2}{r} |a_n|^2 + \frac{(n+d)^2}{r} |b_n|^2 + \frac{|a_n + b_n|^2}{r} f^2 \right) dr + \\ &\quad + \int_0^1 (r|a'_0|^2 + \frac{d^2}{r} |a_0|^2) dr \end{aligned}$$

and that

$$\begin{aligned} \langle \mathcal{T}i\omega_{\mathcal{I}}, i\omega_{\mathcal{I}} \rangle_{L^2(B(0,1))} &= \sum_{n \geq 1} \int_0^1 \left(r|a'_n|^2 + r|b'_n|^2 + \frac{(n-d)^2}{r} |a_n|^2 + \frac{(n+d)^2}{r} |b_n|^2 + \frac{|a_n - b_n|^2}{r} f^2 \right) dr + \\ &\quad + \int_0^1 (r|a'_0|^2 + \frac{d^2}{r} |a_0|^2 + 2rf_d |a_0|^2) dr \end{aligned}$$

We conclude that (1.6) and (1.8) are valid when a_n and b_n are complex valued.

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