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THE USE OF FUZZY NUMBERS IN DECISION ANALYSIS

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A survey of decision-analysis-oriented methods based on the concept of a fuzzy number, is proposed, together with new results likely to improve the reviewed material. Fuzzy numbers are useful to perform sensitivity analysis on utility-based models or scoring methods, when probability or utility values, weights of attributes... cannot be precisely estimated but are obtained through verbal statements. Algorithms for computing fuzzy expectations of utility or fuzzy global ratings are provided. Lastly, new possibilistic scalar comparison indices are suggested for the purpose of ranking fuzzy numbers which represent the overall worth of alternative decisions.

KEYWORDS : Fuzzy numbers - Decision Analysis - Utility theory - Scoring methods - Comparison indices - Ranking procedures.

INTRODUCTION

Fuzzy set theory becomes more and more widely acknowledged as being relevant to decision analysis and decision-making situations. The survey book written by Kickert /22/ clearly indicates that numerous existing frameworks such as utility theory, linear programming models, multiple objective and multi-stage decision making, social choice theories, ect.. may have their descriptive power enhanced by letting irreducible imprecision and fuzziness be taken into account.

Among the numerous mathematical concepts existing in fuzzy set theory, that of a fuzzy number seems to be quite important, because much of the data provided is linguistic rather than precisely numerical. Situations in which precise worth estimations, subjective probability values, grades of importance of objectives can be obtained are very scarce in practice. On the other hand, linguistic terms denoting imprecise quantities are very naturally modelled by fuzzy numbers. Methods for processing and combining fuzzy numbers in the scope of decision analysis have been recently put forward by many researchers. The purpose of this paper is to provide a survey of results and algorithms dealing with the use of fuzzy numbers in decision analysis models. Not only does it try to unify scattered past works, but it also suggests new approaches whenever possible and needed.

In order to implement any methodology based on fuzzy numbers the following problems must be solved

P1) how to estimate the membership functions ? Several estimation procedures exist in the literature, a survey of which can be found in /10, chap.IV.1/. As far as linguistic terms are concerned, the most straightforward approach is to use parametered shapes of membership functions where parameters have a clear interpretation, and determine the parameter values through direct query. See Kuz'min /23/ for a very recent published

work on this topic, a topic which is not primarily ours in this paper.

P2) how to carry out calculations with fuzzy quantities ? Fuzzy arithmetic (see /10/) is by now starting to be developed. Specific problems of decision analysis involve cases where linear interaction between variables must be dealt with.
P3) How to compare fuzzy numbers so as to perform rankings ? This problem occurs when-ever fuzzily-rated alternatives or potential decisions have to be compared.

Part 1, of the paper is devoted to the extension of unidimensional utility theory, to cope with linguistic probabilities. The question of the invariance of a fuzzy utility function up to a scaling effect is addressed. Issues regarding the normalization of a set of linguistic probabilities, the calculation of fuzzy expectations of utility are investigated. Part 2 surveys fuzzy scoring methods for multi-attribute decision-making and suggests a new approach which overcomes numerical complexity when many attributes are involved. Both parts 1 and 2 contain significant material to deal with problem P2. Part 3 is specific of problem P3. It is indicated that most of the proposed comparison indices are optimistic, which results in a poor discrimination power. New pessimistic indices are suggested, in the framework of possibility theory, to be used conjointly with their already known optimistic counterpart.

1 - DECISION MAKING UNDER UNCERTAINTY

1.1 - Motivating fuzzy numbers

In the classical unidimensional utility theory - see Keeney and Raiffa /21/ for an extensive account - utility values are assessed by collecting the decider's preferences among expected consequences of some decision. Uncertain consequences are described in terms of lotteries involving precise values of probabilities regarding the occurrence of events. Such probabilities are called "subjective probabilities", because they are -

directly or implicitly - provided by the decider as a numerical assessment of his belief in the occurrence of consequences. Usually utility assessment procedures involve consistency checks in order to verify that the decider's answers are non-contradictory (e.g. transitivity of preferences) and fit the axiomatic framework of probability theory. In such a normative approach, the decider is often asked to modify his preference pattern so as to meet the requirements of the theory. If such an updating stage may help him to "straighten out his head", as Keeney and Raiffa stress it, it may also introduce a biasing effect in the sense that the decider may give up some significant part of his preference pattern for the sake of fulfilling too rigid requirements. In other words, if it is clear that any theory of utility should be somewhat normative, too restrictive a definition of rationality may prevent from modeling some significant decision making attitudes.

Among the requirements of unidimensional theory which can be questioned is the necessity for assessing utilities in terms of precise numbers. Recently several researchers (Jain /17/, /19/ ; Watson et al. /32/ ; Dubois and Prade /9/, /11/ ; Freeling /15/, /16/ ; Adamo /1/) have pointed out that utility assessment could more realistically be carried out by using fuzzy numbers. Freeling /16/ provides convincing arguments to support this approach :

1/ - deciders never use the whole scale $[0,1]$ to assess their judgments but only integral multiples of 0.1, where say .3 actually means "about .3".

2/ - the calculation of fuzzy expected utility by means of extended sum and products (e.g. Dubois and Prade /10/) comes down to a multi-level sensitivity analysis, and is rather simple to perform.

Some comments are in order. First, none of the mentioned authors has investigated whether a fuzzy utility did preserve essential features. This point is investigated in the following section. On the other hand, no simple algorithm for handling n-ary fuzzy decision trees is available yet. This is because, as first pointed out by Zadeh /35/ the calculation of a fuzzy expectation with fuzzy probability values involve linearly interactive variables, i.e. the usual sup-min extension principle underlying non interactive variables does not apply. Let us review cited past works. Jain /17/, /19/ uses fuzzy numbers. However he performs fuzzy operations with a non standard extension principle where the supremum is changed into the probabilistic sum $(a+b - ab)$. His approach is discussed by Dubois & Prade /8/, and falls outside the scope of this paper. Watson & al. /32/ only consider binary decision nodes. As clearly indicated by Freeling /15/ a n-ary fuzzy decision node is not equivalent to its expanded version as a sequence of binary decision subtrees, contrary to the classical case. This fact motivates the development of some procedure for tackling n-ary probability nodes in fuzzy decision trees. Dubois & Prade /9/ solve ternary

decision trees with precise utility values by providing an exact formula calculated on a general parametered representation of the fuzzy probabilities ; this result is extended in /11/ and is recalled in section 1.3. Adamo /1/ proposes to solve an approximate formulation of the general n-ary case through linear programming. Based on /11/ and level-cutting we formulate in section 1.3 a simple algorithm which is likely to perform better.

1.2 - Strategically equivalent fuzzy utilities

Classical unidimensional utility theory is usually described as follows : the problem is of choosing between actions denoted $a \in A$, which have consequences denoted $x_1 \dots x_n$. These consequences are not necessarily indifferent to the decider, who is able to rank them according to his preference :

$$x_1 \leq x_2 \leq \dots \leq x_n$$

where x_1 (resp. x_n) is the least (resp. most) preferred consequence. The consequences of some action a_i are not precisely known, but only a probability distribution is available.

Utility values u_j , $j=1, n$ for consequences are evaluated through a query regarding the risk behaviour of the decider. Namely the worst consequence is given a utility u_1 , the best one a utility u_n . The decider is then asked to provide for each x_j , $j=1, n$, a probability π_j such that he declares to be indifferent between the two following options: (see Keeney & Raiffa /21/).

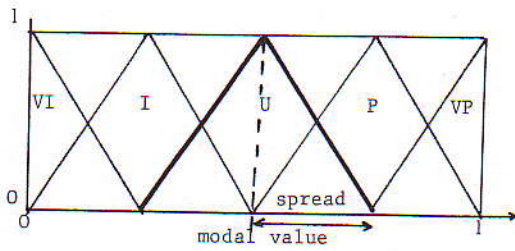
- certainty option : receive x_j
 - risky option : receive the best consequence with probability π_j and the worst one with probability $1-\pi_j$.
 (x_n, π_j, x_1) is called a lottery. The expected utility of this lottery is :

$$u_j = u_n \cdot \pi_j + u_1 \cdot (1-\pi_j) \quad (1)$$

$$= \pi_j \text{ if the scale } [u_1, u_n] \text{ is } [0, 1]$$

x_j is called a certainty equivalent of the lottery (x_1, π_j, x_n) . Two utility functions u_j , $j=1, n$ and u'_j , $j=1, n$ are called strategically equivalent if and only if they imply the same preference ranking for any two lotteries. The basic theorem of utility theory claims that two utility functions u and u' are strategically equivalent if and only if there are two positive constants h and k such that $u' = hu+k$, i.e. they are the same up to a scaling operation.

It is patent that the situation in which a decider does provide precise values of π_j , $j=1, n$ while assessing the utility function is idealistic. In practice, a decider is only able to distinguish between a few confidence levels such as "very probable", "probable", "uncertain", "improbable", "very improbable", which he eventually translate into numbers in $[0,1]$ in a somewhat arbitrary manner. In other words the π_j 's being numerical translations of linguistic terms, are far better represented as fuzzy numbers. Fig.1 suggests a term set for the π_j 's viewed as linguistic variables (Zadeh /35/)



VI = very improbable
 I = improbable
 U = uncertain
 P = probable
 VP = very probable

Figure 1

The spreads of the fuzzy numbers may be difficult to assess, but can be viewed as parameters to be used in a sensitivity analysis of the final ranking of alternatives. Estimates of modal values can be obtained directly from the decider.

Definition 1 : A linguistic lottery is an uncertain situation in which consequences x_1, \dots, x_n of a decision are supposed to occur with (linguistic) probabilities $\tilde{p}_1, \dots, \tilde{p}_n$, where the \tilde{p}_i 's are linguistic values represented by fuzzy numbers on $[0, 1]$.

Fuzzy numbers of $[0, 1]$ are convex normalized fuzzy sets of $[0, 1]$ such as those pictured in fig.1. Specifically there is a modal value \bar{p}_i such that $\forall i, \mu_{\tilde{p}_i}(\bar{p}_i) = 1$, where $\mu_{\tilde{p}_i}$ is the membership function of \tilde{p}_i .

In the deterministic case the p_i 's are such that $\sum_{i=1}^n p_i = 1$. In the fuzzy case, this condition becomes : $\sum_{i=1}^n \bar{p}_i = 1$ (2)

as will be seen later on. For instance when $n=2$, $\tilde{p}_1 = \tilde{p}$, and $\tilde{p}_2 = 1 \ominus \tilde{p}$ satisfy (2). Here \ominus is the extended subtraction (see /10/) and

$$\mu_{1 \ominus \tilde{p}}(x) = \mu_{\tilde{p}}(1-x) \quad \forall x \in [0, 1] \quad (3)$$

Linguistic lotteries naturally yield fuzzy utility functions. A fuzzy utility function can be assessed as follows :

1/ - let $[x_0, x^*]$ be a (continuous) range of consequences where x_0 is the worst possible consequence and x^* the best possible consequence

2/ - let u_0 and u^* be the respective utilities of x_0 and x^* so that $[u_0, u^*]$ is the utility scale. u_0 and u^* are not supposed to be fuzzy because (x_0, x^*) are ideal (worst and best) consequences in the sense that finding better (resp : worse) than x^* (resp. : x_0) is not possible. Moreover this is consistent with the framework of fuzzy sets theory in which universes (here $[u_0, u^*]$) are not fuzzy

3/ - the utility of a consequence $x_j \in [x_0, x^*]$ is defined by $\tilde{u}(x_j)$ such that :

$$\begin{aligned} \mu_{\tilde{u}(x_j)}(u) &= \sup_{p: u=(1-p)u_0+pu^*} \mu_{\tilde{p}_j}(p) \\ &= \mu_{\tilde{p}_j} \left(\frac{u-u_0}{u^*-u_0} \right) \end{aligned} \quad (4)$$

where \tilde{p}_j is a linguistic probability (fuzzy number on $[0, 1]$)

x_j is a certainty equivalent of the linguistic lottery (x_0, \tilde{p}_j, x^*)

It is clear that $\tilde{u}(x_j)$ is a fuzzy set whose support

$$S(\tilde{u}(x_j)) \triangleq \{u | \mu_{\tilde{u}(x_j)}(u) > 0\} \quad (5)$$

is included in $[u_0, u^*]$

Definition 2 : Let \tilde{u} and \tilde{u}' be two fuzzy utility functions built on the same ordered set of consequences $x_1 \leq x_2 \dots \leq x_n$. \tilde{u} and \tilde{u}' are said to be strategically equivalent if and only if they derive from the same set of linguistic lotteries (x_0, \tilde{p}_j, x^*) . Strategic equivalence is denoted $\tilde{u} \sim \tilde{u}'$.

Remark : In the non-fuzzy case, the concept of strategical equivalence relies on the possibility of ranking lotteries in terms of their expected utility. However in the fuzzy case the expected utility is fuzzy and, since fuzzy numbers are no longer linearly ordered, the classical definition of strategical equivalence is not relevant. This is why definition 2 is based on the lotteries encountered in the assessment procedure.

Now we generalize the basic result on the unicity of the utility function up to a scaling transformation.

Proposition 1 : $\tilde{u} \sim \tilde{u}'$ if and only if there are two positive scalar constants h and k such that $\forall j = 1, n$

$$\tilde{u}'(x_j) = h \oplus k \cdot \tilde{u}(x_j) \quad (6)$$

where \oplus is the extended addition /10/.

Proof : (6) means

$$\forall u', \mu_{\tilde{u}'}(x_j)(u') = \mu_{h \oplus k \tilde{u}(x_j)}(u') = \mu_{\tilde{u}(x_j)} \left(\frac{u'-h}{k} \right)$$

according to Zadeh's extension principle.

From (4) and strategical equivalence of u and u' , $\exists u_0, u_*, u'_0, u'_*$ such that $\forall j = 1, n$

$$\begin{aligned} \mu_{\tilde{u}(x_j)}(u) &= \mu_{\tilde{p}_j} \left(\frac{u-u_0}{u_*-u_0} \right) \quad \forall u \in [u_0, u_*] \\ \mu_{\tilde{u}'(x_j)}(u') &= \mu_{\tilde{p}_j} \left(\frac{u'-u'_0}{u'_*-u'_0} \right) \quad \forall u' \in [u'_0, u'_*] \end{aligned} \quad (7)$$

Now

$$\mu_{k \oplus h \tilde{u}(x_j)}(u') = \mu_{\tilde{p}_j} \left(\frac{u'-h-ku_0}{k(u_*-u_0)} \right) \quad (8)$$

Hence obviously from (7) and (8), (6) holds if and only if

$$\begin{aligned}
 u'_0 &= h + k u_0 \\
 u'_x - u'_0 &= k(u_x - u_0) \\
 \text{i.e. } k &= \frac{u'_x - u'_0}{u_x - u_0} ; \quad h = \frac{u_x u'_0 - u'_x u_0}{u_x - u_0} \quad \text{Q.E.D.}
 \end{aligned}$$

This result indicates that a fuzzy utility function is still a utility function in the usual sense.

1.3 - Calculation of fuzzy expected utility

In this paragraph are presented methods for actually evaluating the expected utility of alternative decisions, from the knowledge of the (possibly fuzzy) utility of consequences and (possibly linguistic) probabilities of consequences of decisions.

1.3.1 - Statement of the problem

Formally let
 $X = \{x_1 \dots x_n\}$ be the set of consequences
 $A = \{a_1 \dots a_m\}$ be the set of possible actions or decisions.
 $\tilde{u}_j, j=1, n$, denotes the fuzzy utility of x_j , fuzzy number on $[0, 1]$
 \tilde{p}_{ij} denotes the linguistic probability that x_j occurs given that a_i is chosen $\forall i=1, m, \forall j=1, n$.
 \vec{p} denotes $(p_1 \dots p_n)$, \vec{u} denotes $(u_1 \dots u_n)$;
 $\vec{p} \cdot \vec{u} \triangleq \sum_{j=1}^n p_j u_j$

The fuzzy expected utility of action a_i is the fuzzy set U_i of the real line defined by $\forall u$:

$$\mu_{U_i}(u) = \sup_{\vec{u}, \vec{p}} \min_{j=1, n} (\min(\mu_{\tilde{u}_j}(u_j), \mu_{\tilde{p}_{ij}}(p_j))) \quad (9)$$

under the constraints :

$$u = \vec{u} \cdot \vec{p} \quad (10)$$

$$1 = \sum_{j=1}^n p_j \quad (11)$$

and $\mu_{U_i}(u) = 0$ if (10)-(11) define an empty domain.

Clearly if \tilde{u}_j and \tilde{p}_{ij} are real numbers (9), (10), (11) reduces to the usual expectation

$$U_i = \vec{u} \cdot \vec{p}_i$$

(11) is a constraint expressing a linear interaction between the variables fuzzily restricted by the \tilde{p}_{ij} 's. As a consequence, as soon as the \tilde{p}_{ij} 's are not mere numbers, U_i cannot be calculated via extended sums (⊕) and products (⊗). Namely the quantity

$$U'_i = (\tilde{u}_1 \otimes \tilde{p}_{i1}) \otimes \dots \otimes (\tilde{u}_n \otimes \tilde{p}_{in}) \quad (12)$$

is not equal to U_i , but corresponds to the problem (9)-(10). Clearly we have the inclusion

$$U_i \subseteq U'_i \quad (\text{i.e. } \mu_{U_i} \leq \mu_{U'_i}) \quad (13)$$

which means that U_i is more precise than U'_i . In (13) the equality holds as soon as the \tilde{p}_{ij} 's are real numbers. In this case (11) is always met, and the utility of a_j can be viewed as a random

variable taking fuzzy values. This type of fuzzy random variable has been investigated first by Van Eeden /31/. Its mathematical properties are studied at length by Kwakernaak /24/ /25/.

1.3.2 - α -Cuts of the fuzzy expected utility

The α -cut of any fuzzy set A on a universe U is the crisp set

$$A_\alpha = \{u \in U \mid \mu_A(u) \geq \alpha\}, \quad \forall \alpha \in]0, 1] \quad (14)$$

A general result on the commutativity of the α -cutting and the extension principle has been established by Nguyen /26/. This result is quite useful for making explicit calculations of fuzzy utilities level by level.

In order to use this commutativity property on the system (9) (10) (11) Nguyen's result must be generalized as follows.

Proposition 2 : Let D be a subset of the cartesian product $U_1 \times \dots \times U_n \times W$, $A_1 \dots A_n$ be fuzzy sets on $U_1 \dots U_n$ respectively. Denote $D(w)$ the subset of $U_1 \times \dots \times U_n$ obtained by fixing $w \in W$, i.e.

$$D(w) = \{(u_1 \dots u_n) \mid (u_1 \dots u_n, w) \in D\}$$

and

$$D(A_1 \dots A_n) \text{ a fuzzy set of } W \text{ defined by}$$

$$\mu_{D(A_1 \dots A_n)}(w) = \sup_{\vec{u} \in D(w)} \min_{i=1, n} \mu_{A_i}(u_i) \quad (15)$$

Then

$$D(A_{1\alpha} \dots A_{n\alpha}) = D(A_1 \dots A_n)_\alpha, \quad \forall \alpha \in]0, 1]$$

if and only if $\forall w$, the supremum in (15) is attained for values $(u_1^* \dots u_n^*) \in D(w)$.

Proof : for simplicity we assume $n=2$ $A_1=A$, $U_1=U$, $U_2=V$, $A_2=B$.

i) necessity let $t \triangleq \mu_{D(A,B)}(w)$; then

$$w \in D(A,B)_\alpha = D(A_\alpha, B_\alpha), \quad \forall \alpha \leq t$$

Hence $\exists (u^*, v^*) \in A_\alpha \times B_\alpha$ with $(u^*, v^*) \in D(w)$. The supremum in (15) is attained by (u^*, v^*) since

$$\begin{aligned}
 t &= \sup_{(u,v) \in D(w)} \min(\mu_A(u), \mu_B(v)) \\
 &\geq \min(\mu_A(u^*), \mu_B(v^*)) \geq t
 \end{aligned}$$

ii) sufficiency first note that $D(A_\alpha, B_\alpha) \subseteq D(A, B)_\alpha$ trivially holds without any extra condition. Let $w \in D(A, B)_\alpha$ and $(u^*, v^*) \in D(w)$ such that

$$\mu_{D(A,B)}(w) = \min(\mu_A(u^*), \mu_B(v^*)) \geq \alpha$$

Thus $u^* \in A_\alpha, v^* \in B_\alpha$
Hence $w \in D(\{u^*\}, \{v^*\}) \subseteq D(A_\alpha, B_\alpha)$ Q.E.D.

Corollary 1 : Denoting the solution of (9)-(10)-(11) by

$$U_i \triangleq \text{Ext}(\vec{u}, \vec{p} \mid \tilde{p}_{ij}, \tilde{u}_j, \sum p_j = 1),$$

when \tilde{p}_{ij} and \tilde{u}_j are fuzzy numbers with upper-semi-continuous membership functions then $\forall \alpha \in [0, 1]$,

$$U_{i\alpha} = \text{Ext}(\vec{u} \cdot \vec{p} | \tilde{p}_{ij\alpha}, \tilde{u}_{j\alpha}, \sum p_j = 1)$$

Proof : Here $D(u)$ is defined by (10)-(11) which is clearly a closed domain. Moreover the closures of the supports $S(\tilde{p}_{ij})$, $S(\tilde{u}_j)$ are compact since \tilde{p}_{ij} and \tilde{u}_j are fuzzy numbers. Hence, using the same proof as that in /26 theorem 5.1/, the supremum is always attained in (9). The corollary follows from prop. 2 Q.E.D.

Now it is established that (9) can be solved working with α -cuts. Note that the case when \tilde{p}_{ij} and \tilde{u}_j are closed intervals is encompassed by corollary 1.

1.3.3 - Height of the fuzzy expected utility

Let \tilde{x} be a fuzzy number. Its being normalized means that we can assert that a value of the variable x restricted by \tilde{x} does belong to $S(\tilde{x})$. If we happen to know that x is also restricted by the fuzzy number \tilde{b} , then we may check whether $\tilde{x} \wedge \tilde{b}$ is normalized or not. If :

$$\forall u, \mu_{\tilde{x} \wedge \tilde{b}}(u) \stackrel{\Delta}{=} \min(\mu_{\tilde{x}}(u), \mu_{\tilde{b}}(u)) < 1$$

then we may question the consistency of our information. Zadeh defines the consistency degree of \tilde{a} and \tilde{b} as restricting the same variable :

$$C(\tilde{a}, \tilde{b}) = \Pi(u=v | \tilde{a}, \tilde{b}) = \sup_t \min(\mu_{\tilde{a}}(t), \mu_{\tilde{b}}(t)) \quad (16)$$

Where $\Pi(u=v | \tilde{a}, \tilde{b})$ denotes the grade of possibility (Zadeh /36/) of the event $u=v$, when u is restricted by \tilde{a} , v by \tilde{b} , and both u and v are restricted to have the same value.

If the data consists of a set of fuzzy probabilities \tilde{p}_{ij} , $j=1, n$, the reliability of this information can be evaluated by checking to what extent the variables v_j restricted by \tilde{p}_{ij} satisfy the usual condition $\sum v_j = 1$. A possibilistic approach

to this problem is to calculate the index $C(\tilde{p}_{ij}, j=1, n)$ defined by

$$C(\tilde{p}_{ij}, j=1, n) \stackrel{\Delta}{=} \Pi(\sum v_j = 1 | \tilde{p}_{ij}, j=1, n) \quad (17)$$

where the right hand side means "possibility of the event $\sum v_j = 1$ knowing that \tilde{p}_{ij} restricts $v_j \forall j$ ". Following Zadeh, we have

$$C(\tilde{p}_{ij}, j=1, n) = \sup_t \min_{j=1, n} \mu_{\tilde{p}_{ij}}(t_j) \quad (18)$$

$$\begin{aligned} \sum_j t_j &= 1 \\ &= \mu_{\tilde{p}_{i1}} \oplus \dots \oplus \mu_{\tilde{p}_{in}} \quad (19) \end{aligned}$$

where \oplus is the usual extended addition

$C(\tilde{p}_{ij}, j=1, n) = 1$ is clearly equivalent to :

$$\exists \bar{p}_1, \dots, \bar{p}_n \text{ such that } \begin{cases} \forall j=1, n, \mu_{\tilde{p}_{ij}}(\bar{p}_j) = 1 \\ \sum_j \bar{p}_j = 1 \end{cases} \quad (20)$$

when the supremum is reached in (18) ; (20) is nothing but (2) as suggested in a previous section. The condition $C(\tilde{p}_{ij}, j=1, n) = 1$ can be viewed as the normalization condition of a set of linguistic probabilities.

If a fuzzy expected utility U_i is to be calculated according to (9) (10) (11) then we have the following result regarding its height.

Proposition 3 : Provided involved fuzzy sets are fuzzy numbers with upper semi-continuous membership functions, hence normalized fuzzy sets :

i) $\sup U_i = C(\tilde{p}_{ij}, j=1, n) \stackrel{\Delta}{=} c_i$

ii) Moreover if we change $\{\tilde{u}_j, \tilde{p}_{ij}, j=1, n\}$ into $\{\bar{u}_j, \bar{p}_{ij}, j=1, n\}$ such that :

$$\mu_{\bar{u}_j} = \min(c_i, \mu_{\tilde{u}_j}) \quad \text{and} \quad \mu_{\bar{p}_{ij}} = \min(c_i, \mu_{\tilde{p}_{ij}})$$

$$\text{Then } U_j = \text{Ext}(\vec{p} \cdot \vec{u} | \bar{p}_{ij}, \bar{u}_j, \sum p_j = 1) \quad (21)$$

Proof : If $c_i = 1$, then (20) holds. Choosing \bar{u}_j such that $\forall j, \mu_{\bar{u}_j}(\bar{u}_j) = 1$, clearly from (9)-(10)-(11)

$$\mu_{U_i}(\sum \bar{p}_j \cdot \bar{u}_j) = 1$$

Now assume $c_i < 1$; from (18) $\forall p_1 \dots p_n$ such that $\sum p_j = 1$, we have $\min_j \mu_{\tilde{p}_{ij}}(p_j) < c_i$

Hence $\forall u_1 \dots u_n, p_1 \dots p_n$ satisfying (10) and (11) we have

$$\min_j \min(\mu_{\tilde{p}_{ij}}(p_j), \mu_{\tilde{u}_j}(u_j)) < c_i \quad (22)$$

(22) holds whatever the value $u = \sum u_j p_j$ be. Hence $\forall u, \mu_{U_i}(u) < c_i$, i.e. U_i is not normalized and its height is $\sup \mu_{U_i} < c_i$. Actually $\sup \mu_{U_i} = c_i$ since the supremum in (18) is attained.

(22) implies :

$$\begin{aligned} \min_j \min(\mu_{\tilde{p}_{ij}}(p_j), \mu_{\tilde{u}_j}(u_j)) &= \\ \min_j \min(c_i, \mu_{\tilde{p}_{ij}}(p_j), \mu_{\tilde{u}_j}(u_j)) & \end{aligned}$$

and this equality justifies the truncation leading to (21) Q.E.D.

Thus U_i is normalized if and only if $c_i = 1$. As a consequence if the set $\{\tilde{p}_{ij} | j=1, n\}$ is not normalized we have a simple method to evaluate the height of U_j by means of an extended sum. Extended sums are very efficiently performed as described in /10/. Moreover Proposition 3 enables to spare computational effort by restricting it to the interval $[0, c_i]$.

In the following we assume $c_i = 1$ which, owing to prop. 3, is not restrictive.

1.3.4 - Algorithms for the calculation of fuzzy expected utilities

1.3.4.1 - Crisp probabilities, fuzzy utilities
This situation occurs when the utilities of con-

sequences have been assessed by means of a linguistic lottery and the expected consequences of decisions can be objectively evaluated out of available histograms on past decisions. Computationally this case is the easiest since we have :

$$\forall i \quad U_i = p_{i1} \cdot \tilde{u}_1 \oplus \dots \oplus p_{in} \cdot \tilde{u}_n \quad (23)$$

When the \tilde{u}_j 's are any upper semi-continuous fuzzy numbers, then getting α -cuts, the algorithm described in /10/ can be used, noting that $\forall \alpha \in]0, 1[$, \tilde{u}_j^α is a closed interval $[a_j(\alpha), b_j(\alpha)]$ and

$$\forall \alpha \in]0, 1[\quad U_{i\alpha} = \left[\sum_{j=1}^n p_{ij} \cdot a_j(\alpha), \sum_{j=1}^n p_{ij} \cdot b_j(\alpha) \right] \quad (24)$$

Hence by discretizing $[0, 1]$, as many points of μ_{U_i} as possible can be calculated.

Using a L-R representation /10/ for the fuzzy numbers is very efficient. A fuzzy number A is said to be L-R represented if

$$\begin{aligned} \mu_A(x) &= L\left(\frac{\bar{a}-x}{\hat{a}}\right) & x \leq \bar{a} & , \hat{a} \geq 0 \\ &= R\left(\frac{x-\bar{a}}{\check{a}}\right) & x \geq \bar{a} & , \check{a} \geq 0 \end{aligned}$$

where L and R are non increasing upper-semi-continuous mappings from $[0, +\infty)$ to $[0, 1]$ with $L(0) = R(0) = 1$. When A is of finite support, $L(1) = R(1) = 0$. We write $A = (\bar{a}, \hat{a}, \check{a})_{LR}$ for short. More details appear in /10/. \bar{a} is the modal value ; \hat{a}, \check{a} are spreads. If $\tilde{u}_j = (\bar{u}_j, \psi_j, \psi_j)_{LR}$, $\forall j$, then we have (see /11/) :

$$U_i = (\bar{u}_i, \psi_i, \psi_i)_{LR} \quad (25)$$

1.3.4.2 - Fuzzy probabilities, crisp utilities

In this case the following interactive sum must be calculated :

$$U_i = \text{Ext}(\bar{p}, \bar{u} | \bar{p}_{ij}, \sum p_j = 1) \quad (26)$$

This problem has been completely solved in a previous paper /11/, where proofs can be found. Here we just summarize the results.

Proposition 4 : Let the α -cut \tilde{p}_{ij}^α be denoted $[m_{ij}(\alpha), M_{ij}(\alpha)]$ and assume the u_j 's are ranked in an increasing order. The α -cut of U_i is the closed interval $U_{i\alpha}$ whose bounds are :

$$\begin{aligned} \inf U_{i\alpha} &= \max \left(\sum_{j=1}^{k-1} M_{ij}(\alpha) u_j \right. \\ &\quad \left. + (1 - \sum_{j=1}^{k-1} M_{ij}(\alpha) - \sum_{j=k+1}^n m_{ij}(\alpha)) u_k \right. \\ &\quad \left. + \sum_{j=k+1}^n m_{ij}(\alpha) u_j \right) \quad (27) \end{aligned}$$

$$\begin{aligned} \sup U_{i\alpha} &= \min \left(\sum_{j=1}^{k-1} m_{ij}(\alpha) u_j \right. \\ &\quad \left. + (1 - \sum_{j=1}^{k-1} m_{ij}(\alpha) - \sum_{j=k+1}^n M_{ij}(\alpha)) u_k \right. \\ &\quad \left. + \sum_{j=k+1}^n M_{ij}(\alpha) u_j \right) \quad (28) \end{aligned}$$

Proof : See the appendix of /11/.

It is established in /11/ that there is only one k^- such that the quantity

$$(1 - \sum_{j=1}^{k^-} M_{ij}(\alpha) - \sum_{j=k^-+1}^n m_{ij}(\alpha)) \in \tilde{p}_{ik^-}$$

in (27) i.e. is feasible, and this k^- is the maximizing value in (27). A similar remark holds for (28). Note that :

$$U_{i\alpha} = \{u | u = \bar{p} \cdot \bar{u}, p_j \in \tilde{p}_{ij\alpha} \forall j, \sum p_j = 1\}$$

Proposition 4 underlies $c_i = 1$; if $c_i < 1$, $U_{i\alpha} = \emptyset \forall \alpha > c_i$ and $U_{i\alpha}$ is given by (27)-(28) for $\alpha \leq c_i$.

The above remarks motivate the following algorithm for computing U_i when the membership functions of the \tilde{p}_{ij} 's are general.

A.1 - Step 1 rank the u_j 's in the increasing order.

Step 2 choose a new $\alpha \in]0, c_i]$

Step 3 find k^- and k^+ such that

$$1 - \sum_{j=1}^{k^-} M_{ij}(\alpha) - \sum_{j=k^-+1}^n m_{ij}(\alpha) \in [m_{ik^-}(\alpha), M_{ik^-}(\alpha)]$$

$$1 - \sum_{j=1}^{k^+} m_{ij}(\alpha) - \sum_{j=k^++1}^n M_{ij}(\alpha) \in [m_{ik^+}(\alpha), M_{ik^+}(\alpha)]$$

Step 4 calculate $\inf U_i$, using (27) with $k=k^-$
calculate $\sup U_i$, using (28) with $k=k^+$
Go to step 2 if other points are needed

A further simplification takes place if the \tilde{p}_{ij} 's have a L-L representation, say

$$\forall j, \tilde{p}_{ij} = (\bar{p}_{ij}, \epsilon_{ij}, \eta_{ij})_{LL} \quad (29)$$

and

$$\sum \bar{p}_{ij} = 1$$

Then it is proved in /11/ that U_i is still LL-represented i.e.

$$\begin{aligned} U_i &= (\bar{U}_i, \hat{U}_i, \check{U}_i)_{LL} \quad \text{with} \quad \bar{U}_i = \sum_{j=1}^n u_j \bar{p}_{ij} \\ \hat{U}_i &= \min_{k=1, n} \left(\sum_{j=1}^{k-1} (u_k - u_j) \eta_{ij} + \sum_{j=k+1}^n (u_j - u_k) \epsilon_{ij} \right) \\ \check{U}_i &= \min_{k=1, n} \left(\sum_{j=1}^{k-1} (u_k - u_j) \epsilon_{ij} + \sum_{j=k+1}^n (u_j - u_k) \eta_{ij} \right) \quad (30) \end{aligned}$$

1.3.4.3 - Fuzzy probabilities and fuzzy utilities

In this section we give an algorithm for solving the general problem (9) (10) (11) using α -cuts. First the following result is established.

Proposition 5 : $U_{i\alpha}$ is a closed interval, $\forall \alpha \in]0, c_i]$

Proof : $U_{i\alpha} = \{u = \bar{u} \cdot \bar{p} | u_j \in u_{j\alpha}, p_j \in \tilde{p}_{ij\alpha}, \sum p_j = 1\}$

From corollary 1, $v = \inf U_{i\alpha}$ and $V = \sup U_{i\alpha}$ are attained for values (u_j^*, p_j^*) and (u_j^*, p_j^*) respectively. If $x \in [v, V]$ then for some $\epsilon \in]0, 1]$:

$$\begin{aligned}
 x &= \epsilon v + (1-\epsilon)V \\
 &= \sum_j \epsilon u_j p_j^* + (1-\epsilon)u_j^{**} p_j^* \\
 &= \sum_j (\epsilon p_j^* + (1-\epsilon)p_j^{**}) y_j \text{ for some } y_j^* \in [u_j^*, u_j^{**}] \subseteq \tilde{u}_j \alpha \\
 &= \sum_j p_j' y_j \text{ for } p_j' \in [p_j^*, p_j^{**}] \subseteq \tilde{p}_j \alpha
 \end{aligned}$$

Hence $x \in U_{i\alpha}$ Q.E.D.

Now let $\tilde{u}_{j\alpha} = [a_j(\alpha), b_j(\alpha)]$, $\tilde{p}_{ij\alpha} = [m_{ij}(\alpha), M_{ij}(\alpha)]$
 $\inf U_{i\alpha}$ (resp. $\sup U_{i\alpha}$) can be obtained by the following algorithm (where for simplicity α is dropped), for $\alpha \leq c_i$

A.2 - Step 1 rank the a_j 's (resp b_j 's) so that

$$a_{11} \leq a_{12} \leq \dots \leq a_{1n}$$

Step 2 find k^- (resp. k^+) as in step 3 of algorithm A1

Step 3 use (27) (resp.(28)) with $k=k^-$ (resp. k^+) and $u_j = a_{1j}$ (resp. b_{1j}) to calculate $\inf U_{i\alpha}$ (resp. $\sup U_{i\alpha}$)

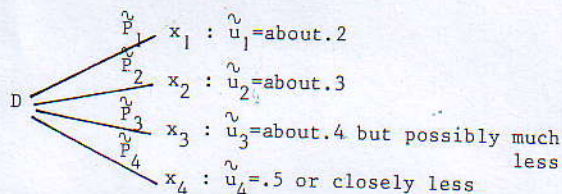
Proof of A2

$$\begin{aligned}
 \inf U_{i\alpha} &= \inf_{\vec{u}} \inf_{\vec{p}} \{ \sum_j p_j u_j \mid u_j \in [a_j, b_j], p_j \in [m_{ij}, M_{ij}], \sum p_j = 1 \} \\
 &= \inf_{\vec{p}} \{ \sum_j p_j a_{1j} \mid p_j \in [m_{ij}, M_{ij}], \sum p_j = 1 \}
 \end{aligned}$$

Hence proposition 4 applies with $u_j = a_{1j}$, $j=1, n$, lower bounds of $\tilde{u}_{j\alpha}$ Q.E.D.

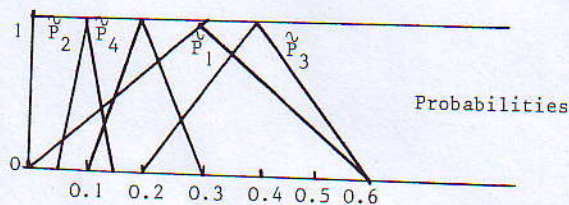
Since the extended product of fuzzy numbers does not preserve the L-R representation there is no hope of using this representation to simplify this algorithm.

Example : Consider the following decision node having 4 possible consequences



$\tilde{p}_1 = \text{about } .3$, with much irregularity
 $\tilde{p}_2 = \text{very close to } .1$
 $\tilde{p}_3 = \text{about } .4$
 $\tilde{p}_4 = \text{close to } .2$

They are modelled by linear fuzzy numbers as pictured below



To see how A2 works let us calculate the .75-cut of the expected utility U. The following table gives the .75-cuts of the data

i	1	2	3	4
u_i	[.175, .225]	[.275, .325]	[.325, .425]	[.475, .5]
p_i	[.225, .375]	[.0875, .1125]	[.35, .45]	[.175, .225]

Lower bound : it is attained for :
 (.175, .275, .325, .475) = \vec{a} (.75)
 Step 1 = already ranked
 Step 2 = searching k^-

The feasible \vec{p} is one of the four 4-tuples obtained by assigning smallest (resp. greatest) probability values to greatest (resp. smallest) utility values :

- a) $p_1 = .375$; $p_2 = .1125$, $p_3 = .45$ + $p_4 = .0625$: not feasible
- b) $p_1 = .375$ $p_2 = .1125$ $p_4 = .175$ + $p_3 = .3375$: not feasible
- c) $p_1 = .375$ $p_3 = .35$ $p_4 = .175$ + $p_2 = 0.1$ feasible

We know the feasible 4-tuple is unique (see /11/) so we may end the search :

$$k^- = 2$$

Step 3 =
 $\inf U_{.75} = .375x_{.175} + .1x_{.275} + .35x_{.325} + .175x_{.475} = 0.29$

NB : One may check that using the 4-tuples a) and b) would yield estimates of $\inf U_{.75}$ smaller than .29 but of course not feasible.

Upper bound

Step 1 : $\vec{b}(.75) = (.225, .325, .425, .5)$
 Step 2 : $p_1 = .225$ $p_3 = .45$ $p_4 = .225$ + $p_2 = .1$ feasible i.e. $k^+ = 2$

Step 3 :
 $\sup U_{.75} = .225x_{.225} + .1x_{.325} + .45x_{.425} + .225x_{.5} = .386875 \approx .39$

The result of the overall calculation, for the support ($\alpha=0$) and the α -cuts, $\alpha \in \{0.25, .5, .75, 1\}$ is pictured on figure 3; U is nearly linear. In dashed line is the fuzzy set obtained by dropping the interaction constraint on the probability variables i.e.

$$U' = (\tilde{p}_1 \otimes \tilde{u}_1) \oplus (\tilde{p}_2 \otimes \tilde{u}_2) \oplus (\tilde{p}_3 \otimes \tilde{u}_3) \oplus (\tilde{p}_4 \otimes \tilde{u}_4)$$

using the extended product \otimes and sum \oplus . It is patent that the interaction constraint enables to significantly improve the precision of the result.

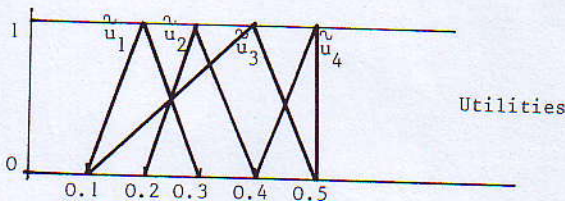
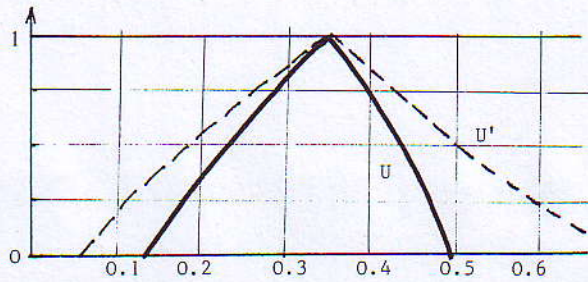


Figure 2



α	0	0.25	0.5	0.75	1
U_α	[.14, .49]	[.18, .46]	[.24, .43]	[.29, .39]	{0.35}

Figure 3

1.4 - Perspectives

We are now able to compute with a very simple method the expected utility of any n-ary decision node in a fuzzy decision tree. Further work along the same research line may deal with the case of dependent consequences of decisions. Solving this case presupposes a fuzzy extension of Bayes formula to linguistic probabilities (which contrasts with the extension of this formula to probabilities of fuzzy events, as done by Okuda et al.27// Adamo /1//s approach to this problem involves algorithms which may become intractable for large decision trees, and thus simple methods should be developed.

A radically different framework for the modelling of decision trees has been suggested by Yager /33/. His paper is concerned with situations where only possibility levels for consequences of decisions are available and utility values are grades of possibility too. In other words Yager considers possibilistic lotteries, in which a certain consequence is balanced with the occurrence of other consequences with various levels of possibility. If u_j is the possibilistic utility of x_j and Π_{ij} is the possibility of occurrence of x_j if decision i is chosen, then the possibilistic expectation of the utility of decision i is defined by

$$U_i = \max_{j=1,n} \min(u_j, \Pi_{ij}) \tag{31}$$

instead of the classical $U_i = \sum p_j u_j$

Freeling /16/ points out that the choice of operators max and min is rather arbitrary ; but the idea of taking utility theory out of the usual probabilistic frame-work is quite interesting. Freeling /16/ suggests to generalize max and min into more general types of fuzzy set unions and intersections. This implies that in a given decision situation, it is possible to identify which operators are relevant, i.e. requires the knowledge of an axiomatic system for set-functions, which encompass both probability and possibility theory, and explicitly uses generalized fuzzy set-theoretic operators. Such a framework is proposed in Dubois and Prade /13/ and could be useful for a non probabilistic utility theory. Be-

sides it is clear that Yager /33//s approach could be reconducted with linguistic levels of possibility.

NB : When consequences can no longer be linearly ordered but only a fuzzy preference relation is available, the classical utility theory is no longer relevant. For a discussion on this topic see Sagaama /29/.

2 - AGGREGATION OF FUZZILY WEIGHTED CRITERIA

2.1 - Previous approaches

The considered situation is here that of choosing between objects or decisions which are statically evaluated according to several points of view. Let $A=\{a_1, \dots, a_n\}$ be the set of attributes and $O=\{O_1, \dots, O_m\}$ be the set of objects to be ranked. A classical approach to this choice problem is the scoring method in which attributes are assigned weights which may depend upon the considered object i , and each object is rated according to each attribute. In other words we have a weight matrix $W=[w_{ij}]$ and a rate matrix $R=[r_{ij}]$. The overall rate of object i is then defined by the weighted sum

$$\rho_i = \frac{\sum_j w_{ij} r_{ij}}{\sum_j w_{ij}} \tag{32}$$

$$\Delta = \frac{\vec{w}_i \cdot \vec{r}_i}{|\vec{w}_i|}$$

Although this method is quite convenient, one of its drawbacks stems from the fact that the final ranking of the objects according to the ρ_i 's strongly depends upon the numerical values of the weights. These weights are usually directly obtained from the decider who is some-times puzzled by the necessity of providing precise numbers, where he would better say "important", "rather unimportant", etc...

Consequently several authors have tried to use (32) without precisely setting the values of the weights, and, in the same spirit, the ratings. Kahne /20/ modelled weights and ratings by probability distributions and applied probabilistic rules to derive the distribution of ρ_i . Jain /18/ modelled w_{ij} and r_{ij} as fuzzy numbers but without normalizing the weighted sum as in (32); moreover the membership function of ρ_i was obtained by means of an already mentioned exotic extension principle. These features slightly invalidate his approach (see /8/).

More recently, Baas and Kwakernaak /2/, modeling rates and weights as Jain did, applied the extension principle to (32) in order to derive the membership function of ρ_i , i.e. performed a multilevel sensitivity analysis on (32).

They criticize Kahne's approach which is computationally burdensome, involves biasing effects, and where the use of probability theory is questionable.

However using the extension principle on (32) is

not obvious and may become very difficult if the set of alternatives is large. Namely the usual methods of fuzzy arithmetic (see /10/) do not apply here because the domains in which the mapping

$$\vec{w}_i \rightarrow (\vec{w}_i, \vec{r}_i) / |\vec{w}_i|$$

is monotonic are difficult to find for more than 3 attributes. Baas and Kwakernaak /2/ only deal with a 2-attribute example which is simple to solve directly. In the general case they only suggest to find the monotonicity domains by trial and error methods. On the same model, McInnis and Cheng /7/ use α -cuts to reduce the fuzzy expression to several crisp ones, but the same problem is met : the bounds of the range of $(\vec{w}_i, \vec{r}_i) / |\vec{w}_i|$ when \vec{w}_{ij} 's belong to prescribed closed intervals are still hard to reach.

2.2 - A new fuzzy scoring method

An alternative approach consists of applying algorithm A2 with normalized set of weights in the sense of section 1.3.3.

We start from term sets of linguistic weights such as

{very important, important, rather unimportant...} Each weight is modelled by a fuzzy number of $[0,1]$ say $\vec{w}_{ij} \in [0,1]$ is taken as the weight scale. Then we normalize each n-tuple of fuzzy weights \vec{w}_i , i.e. consistently modify them into a n-tuple \vec{p}_i such that

$$\mu_{\vec{p}_{i1} \otimes \dots \otimes \vec{p}_{in}}(1) = 1 \tag{33}$$

2.2.1 - Fuzzy normalization

A first hint is to choose

$$\forall j, \vec{p}_{ij} = \text{Ext} \left[\vec{w}_{ij} / |\vec{w}_i| \mid \vec{w}_{ij}, \forall j \right] \tag{34}$$

If the α -cut of \vec{w}_{ij} is $\vec{w}_{ij\alpha} = [\underline{w}_{ij}(\alpha), \bar{w}_{ij}(\alpha)]$

then the α -cut $\vec{p}_{ij\alpha}$ is clearly a closed interval and

$$\sup \vec{p}_{ij\alpha} = \bar{w}_{ij}(\alpha) / (\bar{w}_{ij}(\alpha) + \sum_{k \neq j} \bar{w}_{ik}(\alpha)) \tag{35}$$

$$\inf \vec{p}_{ij\alpha} = \underline{w}_{ij}(\alpha) / (\underline{w}_{ij}(\alpha) + \sum_{k \neq j} \bar{w}_{ik}(\alpha)) \tag{36}$$

(33) is then satisfied because if m_{ij} denotes an element of \vec{w}_{ij} (the 1-cut of \vec{w}_{ij}) then

$$\forall j, m_{ij} / |m_i| \in \vec{p}_{ij}$$

$$\text{Hence } 1 = \sum_j m_{ij} / |m_i| \in (\vec{p}_{i1} \otimes \dots \otimes \vec{p}_{in})$$

Note that the \vec{p}_{ij} 's are more imprecise than the original weights.

2.2.2 - Crisp normalization

Another simpler normalization consists in choosing:

$$\forall j, \vec{p}_{ij} = \vec{w}_{ij} / a_i \tag{37}$$

with $a_i \in \{x \mid \mu_{\vec{w}_{i1} \otimes \dots \otimes \vec{w}_{in}}(x) = 1\}$

The choice of a_i is rather arbitrary. Note that choosing a_i as large as possible leads to keep the 1-cut of $\vec{p}_{i1} \otimes \dots \otimes \vec{p}_{in}$ inside the unit interval (see figure 4) : Consequently, there is only one

\vec{p} such that $p_j \in \vec{p}_{ij}$ and $|\vec{p}| = 1$ (see figure 4).

A similar remark holds when choosing a_i as small as possible.

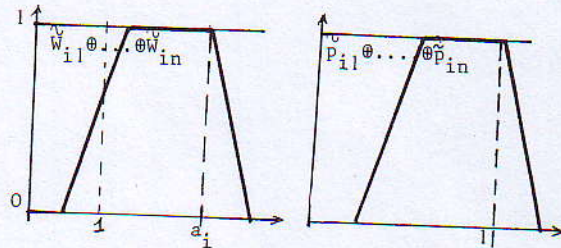


Figure 4

Now \vec{p}_i is defined by

$$\vec{p}_i = \text{Ext} \left[\sum p_i r_i \mid \vec{p}_{ij}, \vec{r}_{ij}, \sum p_i = 1 \right] \tag{38}$$

similarly to U_i in (9) (10) (11).

The calculation of \vec{p}_i is made easy by algorithm A2 even for a large number of attributes, whatever the normalization (34 or 37) be.

2.3 - Comparison of methods

Since the commutativity of α -cutting and the extension principle holds as well for Baas & Kwakernaak's method as in the proposed one, they can be compared with weights and ratings being crisp intervals. Moreover in (32) ρ_i is an increasing function of the r_{ij} 's so that

$$\inf \{ \vec{w}_i, \vec{r}_i / |\vec{w}_i| \mid r_{ij} \in [\underline{r}_{ij}, \bar{r}_{ij}], w_{ij} \in \vec{w}_{ij\alpha} \}$$

is reached for $r_{ij} = \underline{r}_{ij}$ and the supremum for $r_{ij} = \bar{r}_{ij}$

Hence for the purpose of comparing the methods, r_{ij} can be taken as a real number.

2.3.1 - Baas-Kwakernaak method and fuzzy normalization

To simplify the notations let

$$A(\alpha, \beta) = \{ \vec{w}_i, \vec{r}_i \mid \forall i=1, n, w_i \in [\alpha_i, \beta_i] \} \text{ (B. \& K.)}$$

$$U(\gamma, \delta) = \{ \vec{p}_i, \vec{r}_i \mid \forall i=1, n, p_i \in [\gamma_i, \delta_i], |\vec{p}| = 1 \} \text{ (Algor. A1)}$$

The two methods are equivalent if and only if

$$\forall \vec{\alpha}, \vec{\beta} \in \mathbb{R}_+^n, \vec{\gamma}, \vec{\delta} \in [0, 1]^n : A(\alpha, \beta) = U(\gamma, \delta)$$

Since $\forall i, \{w_i / |\vec{w}| \mid w_j \in [\alpha_j, \beta_j], \forall j\}$ is the closed

$$\text{interval } \left[\frac{\alpha_i}{\alpha_i + \sum_{j \neq i} \beta_j}, \frac{\beta_i}{\beta_i + \sum_{j \neq i} \alpha_j} \right]$$

The only possible values for γ_i and δ_i are the bounds of this interval, which is now taken for granted, and motivates (34).

Proposition 6 : $A(\alpha, \beta) \subseteq U(\gamma, \delta)$
The equality holds for $n=2$ (2 attributes).

Proof : Let $x \in A(\alpha, \beta)$; then $\forall i$,
 $\exists \bar{w}_i \in [\alpha_i, \beta_i], \bar{w}_i / |\bar{w}| = x$
Hence $\forall i, \bar{w}_i / |\bar{w}| \in [\gamma_i, \delta_i]$; hence $x \in U(\gamma, \delta)$
When $n=2$, assume $r_1 \leq r_2$

if $r_1 = r_2$ then $A(\alpha, \beta) = U(\gamma, \delta) = \{r_1\}$
if $r_1 < r_2$ the mapping $(w_1, w_2) \rightarrow (w_1 r_1 + w_2 r_2) / (w_1 + w_2)$
is decreasing in w_1 and increasing with w_2
Hence :

$$A(\alpha, \beta) = \left[\frac{\beta_1 r_1 + \alpha_2 r_2}{\beta_1 + \alpha_2}, \frac{\alpha_1 r_1 + \beta_2 r_2}{\alpha_1 + \beta_2} \right] \quad (39)$$

$$= [\delta_1 r_1 + \gamma_2 r_2, \gamma_1 r_1 + \delta_2 r_2]$$

Clearly $\delta_1 + \gamma_2 = \gamma_1 + \delta_2 = 1$

From (27) :

$$\inf U(\gamma, \delta) = \max(r_1 \delta_1 + r_2 (1 - \delta_1), r_2 \gamma_2 + r_1 (1 - \gamma_2))$$

$$= \delta_1 r_1 + \gamma_2 r_2 \text{ due to the above remark.}$$

Similarly, $\sup U(\gamma, \delta) = \gamma_1 r_1 + \delta_2 r_2$ Q.E.D.

Consequently when only 2 attributes are concerned, Baas & Kwakernaak method is equivalent to algorithm A2 with normalization (34). But in such a trivial case, the α -cuts of $\tilde{\beta}_i$ are readily obtained using (39). Moreover if

$$\tilde{w}_i = (\bar{w}_i, a_i, b_i)_{LL} \text{ (see sect. 1.3.4.1)}$$

Then the membership function of $\tilde{\beta} =$

$$\text{Ext} \left(\frac{w_1 r_1 + w_2 r_2}{w_1 + w_2} \mid \tilde{w}_1, \tilde{w}_2 \right) \text{ is, for } r_1 \leq r_2$$

$$\forall z \leq \frac{\bar{w}_1 r_1 + \bar{w}_2 r_2}{\bar{w}_1 + \bar{w}_2}, \mu_{\tilde{\beta}}^v(z) = L \left(\frac{\bar{w}_1 (r_1 - z) + \bar{w}_2 (r_2 - z)}{a_2 (r_2 - z) - b_1 (r_1 - z)} \right)$$

$$\forall z \geq \frac{\bar{w}_1 r_1 + \bar{w}_2 r_2}{\bar{w}_1 + \bar{w}_2}, \mu_{\tilde{\beta}}^v(z) = L \left(\frac{\bar{w}_1 (r_1 - z) + \bar{w}_2 (r_2 - z)}{a_1 (r_1 - z) - b_2 (r_2 - z)} \right)$$

When more than 3 attributes are concerned, the B-K method becomes more and more tricky, while our method remains easy to carry out. Proposition 6 indicates that using normalization (34) the obtained results are consistent with the B-K method, although fuzzier, i.e. $\tilde{\beta}_i$ (B-K) $\subseteq \tilde{\beta}_i$ (eqn (34) + (38)).

The following counter-example shows that for $n > 2$, $A(\alpha, \beta) \neq U(\gamma, \delta)$

Example

Attributes	α	β	r	γ	δ
1	1	2	1	1/18	1/7
2	4	7	2	1/4	7/16
3	8	10	3	8/17	2/3

In the above table, γ and δ are calculated from

α and β .

Using algorithm A1 yields

$$U(\gamma, \delta) = \left[\frac{277}{119}, \frac{47}{18} \right] \approx [2.32, 2.61]$$

Besides

$$\sup_{\vec{w}} \frac{r_1 w_1 + r_2 w_2 + r_3 w_3}{w_1 + w_2 + w_3} = \sup_{w_2} \frac{\alpha_1 r_1 + r_2 w_2 + \beta_3 r_3}{\alpha_1 + w_2 + \beta_3}$$

it is attained for

$$\vec{w} = (\alpha_1, \beta_2, \beta_3) \text{ if } \alpha_1 (r_2 - r_1) > \beta_3 (r_3 - r_2)$$

$$= (\alpha_1, \alpha_2, \beta_3) \text{ otherwise}$$

This remark and the dual one at the lower bound lead to find in the example

$$A(\alpha, \beta) = \left[\frac{40}{17}, \frac{39}{15} \right] \approx [2.35, 2.60]$$

2.3.2 - Baas-Kwakernaak method and crisp normalization

Now, the normalization (37) corresponds to the set

$$V_a(\varphi, \psi) = \{ \vec{p}, \vec{r} \mid p_i \in [\varphi_i, \psi_i], |\vec{p}| = 1 \}$$

With

$$\varphi_i = \frac{\alpha_i}{a}, \psi_i = \frac{\beta_i}{a}, a \in [\sum_i \alpha_i, \sum_i \beta_i]$$

Now letting $y_i = \bar{w}_i / a \forall i$, $A(\alpha, \beta)$ can be written as

$$A(\alpha, \beta) = \left\{ \frac{y \cdot r}{|\vec{y}|}, \forall i, y_i \in [\varphi_i, \psi_i] \right\}$$

$$= \bigcup_k \left\{ \frac{y \cdot r}{k}, \forall i, y_i \in [\varphi_i, \psi_i], |\vec{y}| = k \right\} \quad (40)$$

Consequently $V_a(\varphi, \psi) \subseteq A(\alpha, \beta)$, $\forall a$ it is easy to figure out that for

$$a = \sum_i \alpha_i : V_a(\varphi, \psi) = \left\{ \frac{\alpha \cdot r}{|\alpha|} \right\}$$

$$a = \sum_i \beta_i : V_a(\varphi, \psi) = \left\{ \frac{\beta \cdot r}{|\beta|} \right\}$$

i.e. in both cases the result is a precise number.

Thus, the result of algorithm A2, with normalization (37) gives consistent but more precise results than the B-K method. However the choice of the normalization coefficient "a" may bias the final ranking of the fuzzily-rated objects. In other words several values of a_i should be tried to check the worth of the ratings obtained using (37). This fact tends to favour (34) rather than (37) although the former produces ratings slightly less discriminating than the B-K method.

N.B. : If the weights \tilde{w}_{ij} do not depend upon the considered object i , then it is useless to normalize the ratings in (32). Hence ρ_i is obtained by simply extending $\sum_j w_j$ according to the extension principle with the linear interactivity constraint $\sum w_j = 1$ when $[0, 1]$ is the weighting scale, i.e. algorithm A2 is relevant and sufficient.

Remark

Another framework for criteria aggregation has

been suggested by Bellman and Zadeh /5/ ten years ago. Namely the set of objects sharing some attribute j is a fuzzy set G_j over the set O . the set of globally preferred objects D can thus be obtained through a fuzzy set theoretic combination of the G_i 's. Bellman and Zadeh have suggested

$$D = \bigwedge_j G_j \quad (41)$$

where \bigwedge is expressed via the minimum operator.

If the ratings are fuzzy the membership grades in G_i are fuzzy too, i.e. G_i is a type two-fuzzy set; (41) thus becomes (see Dubois and Prade /12/).

$$\forall O_i, \mu_D(O_i) = \overset{\sim}{\min}_j \mu_{G_j}(O_i) \quad (42)$$

where $\overset{\sim}{\min}$ is the extended minimum operator.

Moreover, it is patent that other aggregation operators than "min" are possible in (41). An axiomatic system is proposed in /12/ for such operators, together with its extension to fuzzy ratings. Empirical investigations on this topic are reported by Zimmermann and Zysno /37/. Starting with fuzzy partial ratings in Bellman and Zadeh's framework, ultimately yields fuzzy global ratings, once more.

3 - RANKING OF FUZZILY RATED ALTERNATIVES

In the preceding sections, several decision making situations have been described in which human judgments have been assessed by fuzzy numbers and fuzzy global ratings of decisions have been obtained. We are now faced to the problem of ranking these fuzzy ratings, in order to perform a choice of decision.

3.1 - Past work

Numerous approaches for ranking fuzzy numbers have been proposed in the litterature. Jain /18/ proposes to match each fuzzy number N_i $i=1,n$ with a fuzzy goal G , fuzzy set of the considered rating scale whose meaning is "as great as possible". The numbers are then ranked according to their consistency (Zadeh /36/) with G , i.e. in terms of $\{C_i = \sup \mu_{N_i \wedge G} | i=1,n\}$

Such a method has been criticized by Dubois and Prade /8/ and Baldwin and Guild /4/ as likely to provide counter-intuitive rankings. Dubois and Prade /8/ and Freeling /16/ have suggested the use of the extended maximum and minimum operators (/10/). However, as Baldwin and Guild /3/ point out, $\max(N_1, N_2)$ may be neither N_1 nor N_2 , so that in some instances, no discrimination is achieved where intuition would discriminate. Besides N_1 and N_2 may be very close to each other while $\max(N_1, N_2) = N_2$ and intuition would not be so definite.

Baas & Kwakernaak /2/ have proposed to extend the inequality $u_1 \geq u_2$ in a canonical way to fuzzy numbers :

$$d(N_1 \geq N_2) = \sup_{u_1 \geq u_2} \min(\mu_{N_1}(u_1), \mu_{N_2}(u_2)) \quad (43)$$

It is clear that

$$\begin{aligned} \max(d(N_1 \geq N_2), d(N_2 \geq N_1)) &= 1 \\ \min(d(N_1 \geq N_2), d(N_2 \geq N_1)) &= \text{height of } N_1 \wedge N_2 \end{aligned}$$

These authors readily extend (43) to n fuzzy numbers as follows : the grade of membership of N_i to the set I of greatest fuzzy numbers among $N_1 \dots N_m$ is

$$\forall i, \mu_I(i) = \sup_{u_i \geq \max_j u_j} \min_j \mu_{N_j}(u_j) \quad (44)$$

Although this approach is very natural in the framework of fuzzy set theory, Baldwin and Guild /4/ have stressed its lack of discrimination power, when the N_i 's significantly overlap.

Watson et al. /32/ proposed a different point of view on comparing two fuzzy numbers, but ultimately they come with a scalar index which is nothing but $1-d(N_1 \geq N_2)$ (see /10, chap.IV.3/), i.e. is redundant with (43).

Baas and Kwakernaak /2/ also define a fuzzy preference index P_i for object O_i rated by N_i : however, ultimately, the P_i 's have still to be ranked and the difficulty remains unsolved.

Lastly, Yager /34/ bases his ranking procedure on the computation of the Hamming distance between fuzzy sets. Yager's purpose is to rank fuzzy truth values τ_i viewed as fuzzy sets of the unit interval, in terms of their proximity to the standard fuzzy truth value "True" defined by $\mu_{\text{true}}(u) = u, \forall u \in [0,1]$. This proximity is expressed by means of the Hamming distance.

$$\Delta(\tau_i, \text{true}) = \int_0^1 |\mu_{\tau_i}(u) - u| du \quad (45)$$

Clearly Yager's procedure is very specific and is not relevant to the framework of this paper, although (45) may be viewed as a probabilistic similarity index between fuzzy numbers.

3.2 - New pessimistic scalar ranking indices

It is patent that Baas and Kwakernaak's ranking index, despite the mentioned drawback, is the most natural one in accordance with the basic principles of possibility theory. Its lack of discrimination power does not make it irrelevant, but indicates that some other indices should be used simultaneously. Namely it is clearly an optimistic index.

Baldwin and Guild /4/ modify (43) replacing $u_1 \geq u_2$ by a fuzzy ordering relation R so as to improve the discrimination power of the index, i.e.

$$d_R(N_1 \geq N_2) = \sup_{u_1, u_2} \min(\mu_{N_1}(u_1), \mu_{N_2}(u_2), \mu_R(u_1, u_2)) \quad (46)$$

Although this idea is very much worth-while, d_R is still an optimistic index.

A complete set of indices including (43) has been recently proposed by Dubois and Prade /14/, which takes advantage of possibility theory. This ap-

proach is only sketched here for the sake of brevity.

If N is a given fuzzy number then the set of numbers greater or equal to N is denoted $[N, +\infty)$ such that

$$\forall x, \mu_{[N, +\infty)}(x) = \sup_{x > u} \mu_N(u) \quad (47)$$

Similarly the set of numbers smaller or equal to N is $(-\infty, N]$

$$\forall x, \mu_{(-\infty, N]}(x) = \sup_{x < u} \mu_N(u) \quad (48)$$

Now the set of numbers strictly greater (resp. smaller) than N is $]N, +\infty)$ ($(-\infty, N[$ = $[N, +\infty)$) where the complement \bar{A} of the fuzzy set A has a membership function $1 - \mu_A$.

The question of a fuzzy number N_1 being greater than a fuzzy number N_2 can now be addressed in four distinct ways.

(1) Is N_1 possibly greater than or equal to N_2 ?

We are now concerned with the intersection (or consistency) of N_1 and $[N_2, +\infty)$. Hence index $\text{Pos}(N_1 \geq N_2)$ defined by :

$$\begin{aligned} \text{Pos}(N_1 \geq N_2) &= \sup_{u_1} \min(\mu_{N_1}(u_1), \mu_{[N_2, +\infty)}(u_1)) \\ &= \sup_{u_1 \geq u_2} \min(\mu_{N_1}(u_1), \mu_{N_2}(u_2)) \end{aligned}$$

by an obvious reduction. This is Baas and Kwakernaak index. (43).

(2) Is N_1 necessarily greater or equal to N_2 ?

We are now concerned with the inclusion $N_1 \subseteq [N_2, +\infty)$. In other words we must check whether $\bar{N}_1 \cap]N_2, +\infty)$ is the real line. Index $\text{Nes}(N_1 \geq N_2)$ is defined by

$$\begin{aligned} \text{Nes}(N_1 \geq N_2) &= \inf_{u_1} \max(1 - \mu_{N_1}(u_1), \mu_{[N_2, +\infty)}(u_1)) \\ &= \inf_{u_1} \sup_{u_2 \leq u_1} \max(1 - \mu_{N_1}(u_1), \mu_{N_2}(u_2)) \end{aligned}$$

(3) Is N_1 possibly greater than N_2 ?

Now we deal with $N_1 \cap]N_2, +\infty)$ i.e.

$$\begin{aligned} \text{Pos}(N_1 > N_2) &= \sup_{u_1} \min(\mu_{N_1}(u_1), 1 - \mu_{(-\infty, N_2]}(u_1)) \\ &= \sup_{u_1} \inf_{u_2 \geq u_1} \min(\mu_{N_1}(u_1), 1 - \mu_{N_2}(u_2)) \end{aligned}$$

(4) Is N_1 necessarily greater than N_2 ?

We must check $N_1 \subseteq]N_2, +\infty)$ i.e.

$$\begin{aligned} \text{Nes}(N_1 > N_2) &= \inf_{u_1} \max(1 - \mu_{N_1}(u_1), 1 - \mu_{(-\infty, N_2]}(u_1)) \\ &= \inf_{u_1 \leq u_2} (1 - \min(\mu_{N_1}(u_1), \mu_{N_2}(u_2))) \end{aligned}$$

Actually $\text{Nes}(N_1 > N_2)$ is Watson et al. /32/'s index. The following properties hold for normalized fuzzy sets of the real line : (see/14/)

$$\text{Pos}(N_1 \geq N_2) \geq \max(\text{Pos}(N_1 > N_2), \text{Nes}(N_1 \geq N_2)) \quad (49)$$

$$\min(\text{Nes}(N_1 \geq N_2), \text{Pos}(N_1 > N_2)) \geq \text{Nes}(N_1 > N_2) \quad (50)$$

$$\text{Pos}(N_1 \geq N_2) = 1 - \text{Nes}(N_2 > N_1) \quad (51)$$

Figure 5 visualizes the values of the indices on an example

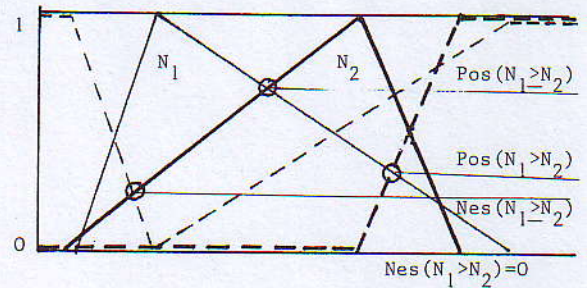


Figure 5

(49) and (50) are natural inequalities which fit the intuition.

(51) stems from the definition of necessity of some event A as the grade of impossibility of the opposite event (see /10/)

$$\text{Nes}(A) = 1 - \text{Pos}(\bar{A}) \quad (52)$$

Indices $\text{Nes}(N_1 > N_2)$ and $\text{Pos}(N_1 > N_2)$ are the pessimistic evaluations we were looking for, and are provided by none of the previously mentioned approaches. Natural n-ary extensions of $\text{Nes}(N_1 > N_2)$ and $\text{Pos}(N_1 > N_2)$ are defined in /14/, consistently with (44).

An alternative approach to the ranking of the set $\{N_1, \dots, N_m\}$ of fuzzy numbers is the one by Tong and Bonissone /30/ which build a fuzzy relation from pairwise evaluations $\text{Pos}(N_i > N_j)$, $\forall i, j, i \neq j$. They give a method for turning this fuzzy outranking relation into a total ordering. This fuzzy relation clearly expresses a possibility of preference or indifference. Other fuzzy relations are obtained from $\text{Nes}(N_i > N_j)$ (necessity of preference or indifference) and $\text{Pos}(N_i > N_j)$ (possibility of strict preference), are described in /14/ where specific methods for their processing are discussed at length. The papers by Blin /6/ or Roy /28/ are also quite relevant for deriving consistent orderings out of fuzzy relations.

CONCLUSION

This paper has proposed an overview of results and algorithms which demonstrate the attractiveness of the fuzzy number concept for decision analysis under uncertainty or several objectives. The use of linguistic probabilities enables to soften the drastic normative framework of utility theory, and efficient algorithms make possible a fast processing of fuzzy data. Scoring methods in multicriteria decision making can also benefit from the use of fuzzy numbers and of these algorithms. Lastly a strongly motivated approach to fuzzy number ranking, based on pessimistic and optimistic indices, seems to emerge as canonical

in the framework of possibility theory.

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