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Takaya Uchida, COAPS ([tuchida@fsu.edu](mailto:tuchida@fsu.edu))

Dhruv Balwada, LDEO ([dhruvbalwada@gmail.com](mailto:dhruvbalwada@gmail.com))

Quentin Jamet, INRIA ([quentin.jamet@inria.fr](mailto:quentin.jamet@inria.fr))

William K. Dewar, FSU ([wdewar@fsu.edu](mailto:wdewar@fsu.edu))

Bruno Deremble, CNRS ([bruno.deremble@univ-grenoble-alpes.fr](mailto:bruno.deremble@univ-grenoble-alpes.fr))

Thierry Penduff, CNRS ([thierry.PENDUFF@cnrs.fr](mailto:thierry.PENDUFF@cnrs.fr))

Julien Le Sommer, CNRS ([julien.Lesommer@univ-grenoble-alpes.fr](mailto:julien.Lesommer@univ-grenoble-alpes.fr))

# Cautionary tales from the mesoscale eddy transport tensor

Takaya Uchida<sup>a,b,\*</sup>, Dhruv Balwada<sup>c</sup>, Quentin Jamet<sup>b,d</sup>, William K. Dewar<sup>b,e</sup>, Bruno Deremble<sup>b</sup>, Thierry Penduff<sup>b</sup>, Julien Le Sommer<sup>b</sup>

<sup>a</sup>*Center for Ocean-Atmospheric Prediction Studies, Florida State University, Tallahassee, Florida, USA*

<sup>b</sup>*Université Grenoble Alpes, CNRS, IRD, Grenoble-INP, Institut des Géosciences de l'Environnement, Grenoble, France*

<sup>c</sup>*Lamont Doherty Earth Observatory, Columbia University in the City of New York, Palisades, New York, USA*

<sup>d</sup>*INRIA, ODYSSEY group, Ifremer, Plouzané, France*

<sup>e</sup>*Department of Earth, Ocean and Atmospheric Science, Florida State University, Tallahassee, Florida, USA*

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## Abstract

The anisotropic mesoscale eddy transport tensor is diagnosed using passive tracers advected online in both an idealized 101-member mesoscale-resolving quasi-geostrophic (QG) double-gyre ensemble, and a realistic 24-member eddy-resolving (1/12°) ensemble of the North Atlantic. We assert that the Reynolds decomposition along the ensemble dimension, rather than the spatial or temporal dimension, allows us to capture the intrinsic spatiotemporal variability of the mean flow and eddies. The tensor exhibits good performance in reconstructing the eddy fluxes of passive tracers, here defined as fluctuations about the ensemble thickness-weighted averaged (TWA) mean. However, the inability of the tensor to reconstruct eddy fluxes of QG potential vorticity, which encapsulates the eddy-mean flow interaction, and other active tracers raises the question: To what extent can the diagnosed tensor be applied to inform the parametrization of mesoscale dynamics?

*Keywords:* Ensemble simulation, mesoscale eddy, eddy diffusivity, thickness-weighted average

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\*Email: tuchida@fsu.edu



## 1. Introduction

In the field of computational oceanography, there is never enough computational power to resolve all the spatiotemporal scales of interest, spanning from on the order of  $O(1\text{ m})$  to  $O(1000\text{ km})$  and seconds to centuries respectively. Nevertheless, with the continuous increase in computational power, global fully-coupled ocean simulations with spatial resolution of  $O(10\text{ km})$  have emerged (e.g. Small et al., 2014; Uchida et al., 2017; Chang et al., 2020). This new generation of ocean simulations have improved the representation of oceanic processes, e.g. the oceanic jets in the separated Gulf Stream and Atlantic Circumpolar Current, and the global oceanic heat transport estimates match better with observations (Griffies et al., 2015; Chassignet et al., 2020). Nevertheless, forced ocean simulations with higher model resolution have shown that even at  $O(10\text{ km})$  mesoscale eddies and their feedback onto the jets are not sufficiently resolved (Chassignet and Xu, 2021; Uchida et al., 2019, 2022c; Hewitt et al., 2022).

In order to overcome the insufficient representation of the eddies and their variability, there has been a growing field of research on how to parametrize the effects of eddies in simulations at the grey zone resolution, when the eddy variability is partially resolved. The missing dynamical effects, due to limited resolution, include processes such as energy-backscattering and dissipation of energy and enstrophy (e.g. Bachman et al., 2017; Jansen et al., 2019; Bachman, 2019; Guillaumin and Zanna, 2021; Uchida et al., 2022a, and references therein). In this study, we examine the possibility of capturing the effects of these missing dynamical processes within the framework of eddy transport coefficients, where we represent the sub-grid fluxes of tracers via the gradient flux of the resolved tracer fields. In doing so, we utilized two sets of ensemble simulations: an idealized 101-member quasi-geostrophic (QG) double-gyre ensemble at mesoscale-resolving resolution and a realistic eddying ( $1/12^\circ$ ) 24-member North Atlantic (NA) ensemble.

In the parametrization literature, it is common to frame the eddies as the sub-grid variability and mean flow as the resolved flow under limited model resolution, where the eddy and mean are defined using a Reynold's decomposition (Bachman et al., 2015). The assumption, and hope, is that the reduced variability in the mean flow, diagnosed by filtering a high-resolution or observed flow field, would mimic the partially resolved variability at coarser model resolution. In a seminal paper, Young (2012) demonstrated that when the equations of oceanic motions are Reynold's decomposed via a thickness-

38 weighted averaging, the net effect of the eddies onto the mean flow can be  
39 represented in terms of eddy Ertel’s potential vorticity (PV) fluxes on the  
40 thickness-weighted averaged (TWA) mean momentum equations. With this  
41 interpretation, the goal of the parametrization problem is to represent the  
42 eddy QG or Ertel’s PV fluxes in terms of the mean fields (Young, 2012; Mar-  
43 shall et al., 2012; Vallis, 2017). In this study we address the question: Can  
44 the eddy fluxes be expressed in terms of a tensor transport model, where  
45 the eddy flux is related to the local gradient through a transport tensor?  
46 Also, is the transport tensor for active tracers (e.g. PV) similar to a trans-  
47 port tensor for passive tracers? Our approach is to diagnose the transport  
48 coefficients using passive tracers outputs and then to examine whether this  
49 can be applied to reconstruct eddy fluxes of active tracers. The potential  
50 interchangeability between passive (in the sense that they do not affect the  
51 dynamics) and active tracers such as PV has its history in that the governing  
52 equations for the two take a similar form of an advective-diffusive equation  
53 (with the caveat that PV is directly linked to the momentum and buoyancy  
54 while passive tracers are not; Killworth, 1997; Wilson and Williams, 2006;  
55 Eden and Greatbatch, 2008).

56 For practical reasons, a time or spatial mean has often been employed to  
57 Reynold’s decompose the flow into its eddy and mean in diagnostic studies.  
58 However, this comes with its own issues regarding interpretation. For a  
59 temporally varying system, such as the real ocean, defining the mean flow  
60 via a time mean conflates intrinsic variability of the ocean resulting from the  
61 non-linearities in governing equations with the variability in the atmospheric  
62 forcing (Aiki and Richards, 2008; Fedele et al., 2021; Uchida et al., 2022b).  
63 There are also issues surrounding how to choose the filtering scale, and what  
64 model resolution, if any, the scale corresponds to (Bachman et al., 2015). An  
65 alternative approach is to define the mean flow via an ensemble mean, where  
66 the intrinsic variability is expressed as eddies and the oceanic response to  
67 atmospheric forcing as the mean (Sérazin et al., 2018; Leroux et al., 2018;  
68 Uchida et al., 2022b). Taking the ensemble outputs, we, therefore, define the  
69 eddy-mean flow decomposition along the ensemble dimension and attempt  
70 to reconstruct the eddy flux of tracers defined as fluctuations about the  
71 ensemble TWA mean. Unlike the spatiotemporal filtering approaches, there  
72 is less ambiguity about how the eddy and mean are defined (Jamet et al.,  
73 2022), but the connection to parametrization in coarse-resolution models  
74 still remains tenuous. Nonetheless, in this study we examine the eddy-mean  
75 flow problem under this framework, which is relatively novel in the ocean

76 turbulence literature. We also note that because the ensemble dimension is  
77 orthogonal to the spatiotemporal dimensions, our Reynold’s decomposition  
78 is exact. Our approach can be rephrased as us having the lofty long-term  
79 goal to parametrize the oceanic intrinsic variability otherwise resolved under  
80 sufficient model resolution.

81 The paper is organized as follows: We describe the ensemble model config-  
82 urations and framework of the anisotropic eddy transport tensor in Section 2.  
83 The results are given in Section 3 with a discussion on the structural simi-  
84 larity in eddy transport coefficients that emerge between the two ensembles.  
85 We conclude and provide some cautionary notes in Section 4.

## 86 2. Methods

87 We describe the quasi-geostrophic (QG) and realistic North Atlantic (NA)  
88 ensembles (sections 2.1 and 2.2), and then provide a brief description of the  
89 eddy transport tensor (section 2.3).

### 90 2.1. Quasi-geostrophic ensemble

91 The model outputs used here are those of Uchida et al. (2021b). For  
92 completeness, we provide a brief overview of the configuration here. We use  
93 the QG configuration of the Multiple Scale Ocean Model (MSOM; Deremble  
94 and Martinez, 2020, hereon referred to as MSQG), based on the Basilisk  
95 language (Popinet, 2015), to simulate a three-layer double-gyre flow with  
96 a rigid lid and flat bottom. The characteristic length scale of the Rossby  
97 radius is prescribed as 50 km and horizontal resolution is  $\sim 4$  km, so we have  
98 roughly 12 grid points per radius; our simulation can be considered mesoscale  
99 resolving (Hallberg, 2013).

100 The model was forced with a stationary wind stress curl without any  
101 buoyancy forcing at the surface. A seasonally varying background stratifica-  
102 tion was prescribed at the first interface but kept stationary at the second  
103 interface, which is consistent with the seasonal variability of stratification  
104 being confined in the upper few hundred meters in the real ocean (Chelton  
105 et al., 1998). The 101 ensemble members are initialized with stream func-  
106 tions slightly perturbed at a single grid point randomly selected in the first  
107 layer per member and the surface wind stress and temporally varying back-  
108 ground stratification are kept identical amongst the members. We refer the  
109 interested reader to Uchida et al. (2021b) for further details on the model  
110 configuration and ensemble generation.

111 In addition, we added four passive tracers to each ensemble member with  
 112 the governing equation:

$$113 \quad C_{it} + J(\psi, C_i) = -\mathcal{T}^{-1}(C_i - \dot{C}_i), \quad (1)$$

114 where  $C_i, i = 0, 1, 2, 3$  are the four passive tracers and  $\psi$  the stream function.  
 115 They are relaxed towards a profile orthogonal to each other

$$116 \quad \dot{C}_0 = \frac{x - L/2}{L/2}, \quad \dot{C}_1 = \frac{y - L/2}{L/2}, \quad \dot{C}_2 = \sin \frac{2\pi x}{L}, \quad \dot{C}_3 = \sin \frac{2\pi y}{L}, \quad (2)$$

117 with a relaxation time scale of  $\mathcal{T} \sim 360$  days for all three vertical layers.  
 118  $L (= 4000 \text{ km})$  is the zonal and meridional domain extent,  $x = [0, L], y =$   
 119  $[0, L]$ , resulting in the tracers taking values between  $[-1, 1]$ . The time scale  
 120 chosen is similar to previous studies (e.g. Bachman et al., 2020), and meant  
 121 to be longer than that the typical eddy turn over time scale. In other words,  
 122 the tracers are passively stirred by the flow realized by the intrinsic variability  
 123 of each ensemble member but are relaxed towards identical profiles amongst  
 124 members. The relaxation avoids the tracers from homogenizing (i.e.  $\nabla_{\text{h}} C_i \neq$   
 125  $\mathbf{0}$  where  $\nabla_{\text{h}}$  is the horizontal gradient operator). The tracer profiles were  
 126 initialized with their respective relaxation profiles at the beginning of the  
 127 ensemble generation and advected prognostically online.

128 The QG eddy flux is  $\mathbf{J}^{C_i} = J^{C_i 1} \mathbf{i} + J^{C_i 2} \mathbf{j} \stackrel{\text{def}}{=} \overline{u' C_i'} \mathbf{i} + \overline{v' C_i'} \mathbf{j}$  where  $\overline{(\cdot)}$  is the  
 129 ensemble mean operator and  $(\cdot)' \stackrel{\text{def}}{=} (\cdot) - \overline{(\cdot)}$ .  $\mathbf{i}, \mathbf{j}$  are the horizontal unit vectors  
 130 in geopotential coordinates. Snapshots of the eddy kinetic energy (EKE) and  
 131 tracer variance are shown in the Supplementary Material Fig. S1.

## 132 2.2. Realistic North Atlantic ensemble

133 We use the model outputs from the realistic simulations described in  
 134 Jamet et al. (2019, 2020), and Uchida et al. (2022b), which are 48 air-sea  
 135 partially coupled ensemble members of the NA ocean at mesoscale-permitting  
 136 resolution ( $1/12^\circ$ ) using the hydrostatic configuration of the Massachusetts  
 137 Institute of Technology general circulation model (MITgcm; Marshall et al.,  
 138 1997). The modelled domain was configured to wrap around zonally in order  
 139 to reduce memory allocation in running the simulation. Similar to the QG  
 140 ensemble, in the latter subset of 24 members, four passive tracers per member  
 141 were added online using the Relaxation Boundary Conditions (RBCS) package  
 142 with the relaxation profiles of

$$143 \quad \dot{C}_0 = \frac{y - L^y/2}{L^y/2}; \quad \dot{C}_1 = \sin \frac{2\pi x}{L^x}; \quad \dot{C}_2 = \frac{z - H/2}{H/2}; \quad \dot{C}_3 = \sin \frac{2\pi y}{L^y}, \quad (3)$$

144 and the relaxation time scale of  $\mathcal{T} = 365$  days.  $L^x, L^y$  are the zonal and  
 145 meridional domain extent respectively and  $H$  is the deepest depth in the  
 146 domain bathymetry. The tracer profiles were again initialized with their  
 147 respective relaxation profiles at the beginning of the ensemble generation  
 148 and advected prognostically online.

149 The connection between primitive equations viewed in the thickness-  
 150 weighted averaged (TWA) framework and quasi geostrophy is that in the  
 151 latter, the layer thickness  $\sigma \stackrel{\text{def}}{=} \zeta_{\bar{b}}$  only fluctuates on the order of Rossby  
 152 number ( $\zeta$  is the depth of the neutral surface and the subscript  $\bar{(\cdot)}$  denotes  
 153 derivatives in density coordinates). Hence, it can be argued that QG vari-  
 154 ables are implicitly TWA and that for a fair comparison, the primitive equa-  
 155 tions should also be TWA (cf. Marshall et al., 2012). The three-dimensional  
 156 oceanic motions become quasi two dimensional upon thickness-weighted av-  
 157 eraging (Aoki, 2014; Loose et al., 2022), further elucidating the similarity  
 158 to quasi geostrophy. We proceed in defining our eddy tracer fluxes within  
 159 the TWA framework, which gives  $\mathbf{J}^{C_i} = J^{C_i1} \bar{\mathbf{e}}_1 + J^{C_i2} \bar{\mathbf{e}}_2 \stackrel{\text{def}}{=} \widehat{u'' C_i''} \bar{\mathbf{e}}_1 + \widehat{v'' C_i''} \bar{\mathbf{e}}_2$   
 160 where  $\widehat{(\cdot)} \stackrel{\text{def}}{=} \bar{\sigma}^{-1} \overline{\sigma(\cdot)}$  and  $(\cdot)'' \stackrel{\text{def}}{=} (\cdot) - \widehat{(\cdot)}$  (Young, 2012).  $\bar{\mathbf{e}}_1$  and  $\bar{\mathbf{e}}_2$   
 161 are the horizontal unit vectors along the neutral surface, which reduce to  $\mathbf{i}$  and  
 162  $\mathbf{j}$  respectively when the neutral surfaces are flat ( $\widetilde{\nabla}_h \zeta = 0$  where  $\widetilde{\nabla}_h$  is the  
 163 horizontal gradient operator in density coordinates). We note that due to the  
 164 3<sup>rd</sup> order direct-space-time (DST) flux-limiter advective scheme used, there  
 165 are two possible ways to define the eddy flux, i.e.  $\widehat{\mathbf{u}'' C_i''} \approx \bar{\sigma}^{-1} (\overline{\sigma \mathbf{F}^{C_i}}) - \widehat{\mathbf{u}} C_i$   
 166 where  $\mathbf{F}^{C_i}$  is the DST advective flux computed by MITgcm. The advective  
 167 scheme was chosen to be consistent with the schemes used for temperature  
 168 and salinity. Snapshots of the EKE and tracer variance are given in the  
 169 Supplementary Material Fig. S2.

170 In transforming the three-dimensional model outputs in geopotential co-  
 171 ordinates to TWA variables, we: i) compute the approximately neutral den-  
 172 sity surfaces for each member by removing the effect of compressibility, ii)  
 173 remap the momentum, isopycnal depth and tracers onto the neutral sur-  
 174 faces, iii) thickness weight the variables and take the ensemble mean, iv)  
 175 apply Reynold's decomposition about the ensemble TWA mean to obtain  
 176 the eddy terms  $(\cdot)''$ , and v) thickness-weighted average the eddy covariance  
 177 to get the eddy flux. Further details regarding the coordinate remapping  
 178 from geopotential to approximately neutral surfaces using the `xgcm` Python  
 179 package (Abernathey et al., 2021), and the averaging are given in Uchida  
 180 et al. (2022b).

181 *2.3. Eddy transport tensor*

182 As the eddy flux of tracers is generally poorly resolved in global forced and  
 183 coupled ocean simulations, there has been an effort to parametrize this flux  
 184 in the quasi-adiabatic interior via a local-gradient based model along neutral  
 185 surfaces (Redi, 1982; Griffies, 2004; Wilson and Williams, 2006; Holmes et al.,  
 186 2022)

$$187 \quad \mathbf{J}^{C_i} = -\kappa \nabla_{\text{h}} \overline{C_i}, \quad (4)$$

188 where  $\kappa$  is the scalar eddy diffusivity, hereon referred to as the transport  
 189 coefficient. The terminology of ‘diffusivity’ has its origin in the works by  
 190 Redi (1982); Gent and McWilliams (1990) and Gent et al. (1995), but eddies  
 191 not only diffuse but advect tracers (Smith and Gent, 2004; Griffies, 2004;  
 192 Haigh et al., 2021b), so we have opted for a more general term.

193 While it is tempting to directly infer a scalar eddy transport coeffi-  
 194 cient from equation (4), assuming an isotropic transport coefficient for an  
 195 anisotropic flow in realistic simulations is a poor approximation (Smith and  
 196 Gent, 2004; Ferrari and Nikurashin, 2010; Rypina et al., 2012; Fox-Kemper  
 197 et al., 2013; Kamenkovich et al., 2015, 2020). A more general or appropri-  
 198 ate model might be to relate the eddy fluxes to the mean gradients via a  
 199 transport tensor with four parameters (Plumb and Mahlman, 1987), which  
 200 has some justification in linear wave theories and mixing-length based mod-  
 201 els (Bachman and Fox-Kemper, 2013). In both these models, the inherent  
 202 assumption is that the scalar transport coefficient or transport tensor is only  
 203 a function of the flow, and is tracer independent.

204 We, therefore, take the approach of estimating the eddy transport tensor  
 205 ( $\mathbf{K}$ ) from a least-squares best fit to (Plumb and Mahlman, 1987; Abernathey  
 206 et al., 2013; Bachman and Fox-Kemper, 2013)

$$207 \quad \underbrace{\begin{pmatrix} J^{C_01} & J^{C_02} \\ J^{C_11} & J^{C_12} \\ J^{C_21} & J^{C_22} \\ J^{C_31} & J^{C_32} \end{pmatrix}}_{\mathbf{J}} = \mathbf{G} \cdot \underbrace{\begin{pmatrix} \kappa^{uu} & \kappa^{vu} \\ \kappa^{uv} & \kappa^{vv} \end{pmatrix}}_{\mathbf{K}}, \quad (5)$$

208 where for the QG ensemble

$$209 \quad \mathbf{G} = - \begin{pmatrix} \overline{C_{0x}} & \overline{C_{0y}} \\ \overline{C_{1x}} & \overline{C_{1y}} \\ \overline{C_{2x}} & \overline{C_{2y}} \\ \overline{C_{3x}} & \overline{C_{3y}} \end{pmatrix}, \quad (6)$$

210 and for the realistic NA ensemble

$$211 \quad \mathbf{G} = - \begin{pmatrix} \widehat{C}_{0\bar{x}} & \widehat{C}_{0\bar{y}} \\ \widehat{C}_{1\bar{x}} & \widehat{C}_{1\bar{y}} \\ \widehat{C}_{2\bar{x}} & \widehat{C}_{2\bar{y}} \\ \widehat{C}_{3\bar{x}} & \widehat{C}_{3\bar{y}} \end{pmatrix}, \quad (7)$$

212 respectively. The least-squares fit can be estimated as  $\mathbf{K} = \mathbf{G}^+ \mathbf{J}$  where  $\mathbf{G}^+$  is  
 213 the Moore-Penrose pseudo inverse of  $\mathbf{G}$  for each data point (Bachman et al.,  
 214 2015). The rotational components in  $\mathbf{J}^{C_i}$  are not removed prior to inversion.  
 215 It is possible to invert equation (5) with just two tracers whose gradients are  
 216 not aligned with each other, and this will return a unique solution specific to  
 217 the pair of tracers chosen for the inversion (e.g. Haigh and Berloff, 2021; Sun  
 218 et al., 2021). Here, we instead focus on estimating a single tensor that works  
 219 for all possible tracer orientations, even if it does not perfectly reconstruct  
 220 the eddy flux for any single tracer. We have, thus, kept the system over-  
 221 determined by using four tracers. The assumption is that by keeping it over-  
 222 determined,  $\mathbf{K}$  would extract the universal component in the relation between  
 223 eddy fluxes and gradient flux of the mean tracer fields. From a practical point  
 224 of view, all tracers should be associated with the same transport coefficients  
 225 in order to reduce the number of model parameters.

226 The eddy flux and mean tracer gradient fields for the QG ensemble were  
 227 coarse grained by  $4 \times 4$  grid-point boxcar filter prior to inverting (5) in order  
 228 to reduce the computational cost. For the NA ensemble, we first spatially  
 229 smoothed the eddy flux and mean tracer fields by applying a Gaussian kernel  
 230 with the standard deviation of 50 km using the `gcm-filters` Python package  
 231 (Grooms et al., 2021) as we do not expect the linear model (5) to capture  
 232 grid-scale features. The eddy flux and mean tracer gradient fields were then  
 233 further coarse grained by  $10 \times 10$  grid-point boxcar filter in order to reduce  
 234 the computational cost of inversion. (We demonstrate that other than the  
 235 significant reduction in computation, the coarse graining has no qualitative  
 236 impact on our results; Fig. S3. We also note that a boxcar coarse graining  
 237 commutes with the spatial derivatives and does not result in extra Clark  
 238 terms. cf. Bachman et al., 2015) Each row in  $\mathbf{J}$  and  $\mathbf{G}$  was then normalized  
 239 by horizontal median of the magnitude of each mean tracer gradient flux  
 240 (i.e.  $\frac{(\mathbf{J}^{C_i}, \mathbf{G}^{C_i})}{\text{median}[|\mathbf{G}^{C_i}|]}$  where  $\mathbf{J}^{C_i}$  and  $\mathbf{G}^{C_i}$  are the smoothed and coarse-grained eddy  
 241 flux and mean gradient flux of an arbitrary tracer  $C_i$ ) prior to the inversion  
 242 so that each tracer had roughly equal weighting in inverting equation (5).

243 **3. Results**

244 We present results using the first time step of the fifth year, a time when  
 245 the ensemble spread has converged for both ensembles (Uchida et al., 2021b;  
 246 Jamet et al., 2019). The QG outputs are instantaneous snapshots while the  
 247 North Atlantic (NA) outputs are five-day averaged.

248 *3.1. Quasi-geostrophic ensemble*

249 The four components in the eddy transport tensor are provided in the top  
 250 two rows of Fig. 1 with values reaching up to  $O(10^4 \text{ m}^2 \text{ s}^{-1})$  in the first and  
 251 second layer. The bottom layer is quiescent with transport coefficients on the  
 252 order of  $O(10^2 \text{ m}^2 \text{ s}^{-1})$ . The diagonal components of the tensor tend to take  
 253 positive values over the entire domain while the antidiagonal components  
 254 tend to change signs across the jet centered around  $y = 2000 \text{ km}$ . This  
 255 suggests that the eddies are broadly working to dissipate small-scale variance,  
 256 as one might expect. Focusing on the first layer,  $\kappa^{uv}$  tends to be coherently  
 257 negative south of the jet and positive north of the jet. In the lower two layers,  
 258  $\kappa^{uv}$  and  $\kappa^{vu}$  tend to mirror each other where there is a sign change across  
 259 the jet but also within each idealized subtropical and subpolar gyre. These  
 260 cross diagonal terms of the tensor are largely associated with eddy-induced  
 261 advection, and often act to oppose the mean flow (Marshall, 2011).

262 Taking the diagnosed tensor, we examine the reconstruction of the eddy  
 263 flux of  $C_2$ , i.e.  $\mathbf{J}_{\text{reconstructed}}^{C_2} \stackrel{\text{def}}{=} -\nabla_{\text{h}} \overline{C_2} \cdot \mathbf{K}$ . The middle two rows of Fig. 1 show  
 264 that the performance of reconstruction is good across all three layers. The  
 265 spatial correlations between the actual eddy flux and its reconstruction are  
 266 higher than 0.9 for all four tracers (Table 1). The spatial correlation was  
 267 computed as

$$268 \quad r^{C_{ij}} \stackrel{\text{def}}{=} \frac{\sum \left[ (J^{C_{ij}} - \langle J^{C_{ij}} \rangle) \left( J_{\text{reconstructed}}^{C_{ij}} - \langle J_{\text{reconstructed}}^{C_{ij}} \rangle \right) \right]}{\sqrt{\sum (J^{C_{ij}} - \langle J^{C_{ij}} \rangle)^2} \sqrt{\sum \left( J_{\text{reconstructed}}^{C_{ij}} - \langle J_{\text{reconstructed}}^{C_{ij}} \rangle \right)^2}}, \quad (8)$$

269 where  $\langle \cdot \rangle$  is the horizontal spatial mean,  $j = 1, 2$  correspond to the zonal and  
 270 meridional orientation, and the summation is taken over the entire domain  
 271 of interest.

272 We also exhibit the spatial median of normed relative errors  $|\varepsilon^{C_{ij}}|$  com-  
 273 puted as

$$274 \quad \varepsilon^{C_{ij}} \stackrel{\text{def}}{=} \frac{J^{C_{ij}} - J_{\text{reconstructed}}^{C_{ij}}}{J^{C_{ij}}}, \quad (9)$$



Table 1: Spatial correlation for the four passive tracers and QGPV between the eddy flux and its reconstruction for the zonal and meridional orientation and all three layers. The PV coefficients are not shown for the middle and bottom layer as it is evident from Fig. 1 that the flux and reconstruction are completely decorrelated.

Tracer	Top	Middle	Bottom
$r^{C_01}$	0.952	0.989	0.999
$r^{C_02}$	0.957	0.990	0.996
$r^{C_11}$	0.980	0.966	0.981
$r^{C_12}$	0.999	0.999	0.997
$r^{C_21}$	0.939	0.978	0.993
$r^{C_22}$	0.949	0.966	0.973
$r^{C_31}$	0.992	0.969	0.980
$r^{C_32}$	0.999	0.999	0.997
$r^{q1}$	0.844	–	–
$r^{q2}$	0.906	–	–

275 in Table 2. This allows us to quantify the normalized error in addition to the  
 276 sign;  $\varepsilon^{C_{ij}}$  would return positive values if the reconstruction had the correct  
 277 sign as the actual flux even if it had an offset. When the reconstruction has  
 278 the wrong sign, the errors would become negative. The spatial median of  
 279 the normed errors are all positive with values smaller than unity indicating  
 280 that the reconstruction overall predicts the correct sign of the eddy flux. The  
 281 spatial fields given in the Supplementary Material Fig. S4 also support the  
 282 good performance.

283 As the inversion was done as a least squares fit to (5), the good recon-  
 284 struction of  $\mathbf{J}^{C_i}$  may not come as a surprise and is consistent with previous  
 285 studies (e.g. Abernathey et al., 2013; Bachman et al., 2015, 2020). However,  
 286 it is important to note that such a good fit, although not perfect (Figs. S3  
 287 and S4), suggests that the transport tensor model with four free parameters  
 288 is a good model to represent how the eddies flux different tracers along neu-  
 289 tral surfaces. This model is able to separate the contribution of the flow from  
 290 any dependence on the orientation of tracer gradients well, and this model  
 291 of reduced complexity can be a target for parametrizations that are able to  
 292 represent the eddy fluxes of any arbitrary passive tracer.

293 One may now ask the question: Can the tensor diagnosed from passive  
 294 tracers be applied to active tracers, here chosen as QG potential vorticity  
 295 (PV), which is the sole active tracer in quasi geostrophy? Looking at the

Table 2: Spatial median of the normed relative error for the four passive tracers and QGPV between the eddy flux and its reconstruction for the zonal and meridional orientation and all three layers. The PV errors are not shown for the middle and bottom layer for the same reasons as correlation.

Tracer	Top	Middle	Bottom
$ \varepsilon^{C_0^1} $	0.0440	0.0190	0.0122
$ \varepsilon^{C_0^2} $	0.0622	0.0517	0.0169
$ \varepsilon^{C_1^1} $	0.141	0.0746	0.0252
$ \varepsilon^{C_1^2} $	0.0596	0.0153	0.00721
$ \varepsilon^{C_2^1} $	0.0770	0.0326	0.0189
$ \varepsilon^{C_2^2} $	0.112	0.0838	0.0280
$ \varepsilon^{C_3^1} $	0.135	0.0994	0.0373
$ \varepsilon^{C_3^2} $	0.0596	0.0210	0.0112
$ \varepsilon^{q^1} $	1.161	–	–
$ \varepsilon^{q^2} $	0.945	–	–

296 bottom two rows of Fig. 1, we see that while the reconstruction  $-\nabla_{\text{h}}\bar{q} \cdot \mathbf{K}$   
 297 captures some features in the top layer (cf. Table 1) and in the regions away  
 298 from the jet in the middle layer, the reconstruction performs poorly for the  
 299 bottom layers and in the jet in the middle layer, with the spatial correlations  
 300 being smaller than 0.1. The normed relative errors are also quite large being  
 301 on the order of  $O(1)$  (Table 4; Fig. S5).

### 302 3.2. Realistic North Atlantic ensemble

303 Following the QG analyses, we now present the four components of the  
 304 tensor  $\mathbf{K}$  from the NA ensemble. The magnitude of the tensor components  
 305 are similar to the QG ensemble, reaching up to  $O(10^4 \text{ m}^2 \text{ s}^{-1})$  (top left four  
 306 panels in Fig. 2). Also similar to the QG ensemble, the diagonal compo-  
 307 nents tend to take positive values while the antidiagonal components tend  
 308 to change signs across the separated Gulf Stream (GS). The separated GS  
 309 can be identified in the top right panel in Fig. 2 where the ensemble-mean  
 310 depth associated with the neutral surface shoals around  $35^\circ\text{N}$ . We also show  
 311 the vertical transects along  $300^\circ\text{E}$ . The patterns in sign persist over depth  
 312 with the order of magnitude decreasing towards  $O(10^2 \text{ m}^2 \text{ s}^{-1})$  in the abyssal  
 313 ocean (bottom four panels in Fig. 2).

314 We remind the reader that we define the eddy flux as  $\mathbf{J}^{C_i} \stackrel{\text{def}}{=} \widehat{\mathbf{u}''C_i''}$ , and  
 315 following the convention of the TWA framework, the reconstructed eddy flux

Table 3: Spatial correlation for the four passive tracers and temperature and salinity between the eddy flux and its reconstruction for the zonal and meridional orientation along the neutral surface shown in Fig. 2. The correlation coefficients are shown for when they are diagnosed over the entire horizontal domain and only between 10°N-30°N both excluding the hatched regions in Figs. 3 and 4.

Tracer	Entire domain	10°N-30°N
$r^{C_01}$	0.918	0.956
$r^{C_02}$	0.917	0.971
$r^{C_11}$	0.935	0.959
$r^{C_12}$	0.917	0.936
$r^{C_21}$	0.975	0.948
$r^{C_22}$	0.983	0.969
$r^{C_31}$	0.924	0.967
$r^{C_32}$	0.931	0.977
$r^{\Theta 1}$	0.344	0.303
$r^{\Theta 2}$	0.494	0.353
$r^{S1}$	0.181	0.390
$r^{S2}$	0.308	0.128

316 of  $C_1$  becomes  $\mathbf{J}_{\text{reconstructed}}^{C_1} \stackrel{\text{def}}{=} -\tilde{\nabla}_h \widehat{C}_1 \cdot \mathbf{K}$ . In examining the reconstruction,  
 317 we limit it to regions where the neutral surface is deeper than 150 m and  
 318 to grids upon the  $10 \times 10$  coarse graining included no land cells in order to  
 319 minimize the effect of diabatic mixing (here parametrized by the K-Profile  
 320 Parametrization; Large et al., 1994). As we see from Fig. 3, the reconstruc-  
 321 tion of eddy passive tracer fluxes is good generally across the entire three  
 322 dimensional domain with the spatial correlation again higher than 0.9 (Ta-  
 323 ble 3). The largest disagreements between  $\mathbf{J}^{C_i}$  and  $\mathbf{J}_{\text{reconstructed}}^{C_i}$  can be seen  
 324 in the separated GS region where eddy activity and vertical fluctuations of  
 325 the neutral surface are vigorous (Uchida et al., 2022b). The horizontal spa-  
 326 tial correlation improves for the quiescent gyre interior slightly for all four  
 327 passive tracers.

328 Similar to the QG ensemble, we also show the normalized relative error  
 329 from the NA ensemble in Table 4. The magnitude of errors are smaller than  
 330 unity over most of the domain (Fig. S6 in Supplementary Material). As  
 331 expected from the improvement seen in the spatial correlations, the normed  
 332 relative errors for passive tracers generally decrease when they are taken over  
 333 the subdomain of 10°N-30°N.

Table 4: Spatial median of the normed relative errors for the four passive tracers and temperature and salinity between the eddy flux and its reconstruction for the zonal and meridional orientation along the neutral surface shown in Fig. 2. The errors are shown for when they are diagnosed over the entire horizontal domain and only between 10°N-30°N both excluding the hatched regions in Figs. 3 and 4.

Tracer	Entire domain	10°N-30°N
$ \varepsilon^{C_01} $	0.0257	0.0141
$ \varepsilon^{C_02} $	0.0208	0.00593
$ \varepsilon^{C_11} $	0.0118	0.0257
$ \varepsilon^{C_12} $	0.0108	0.00847
$ \varepsilon^{C_21} $	0.0372	0.0324
$ \varepsilon^{C_22} $	0.0726	0.0582
$ \varepsilon^{C_31} $	0.0334	0.00343
$ \varepsilon^{C_32} $	0.0261	0.00153
$ \varepsilon^{\Theta 1} $	0.343	0.286
$ \varepsilon^{\Theta 2} $	0.428	0.331
$ \varepsilon^{S1} $	0.234	0.183
$ \varepsilon^{S2} $	0.314	0.222

334 We may now also examine if we can use the tensor to reconstruct the  
335 eddy temperature and salinity fluxes, variables which were not included in  
336 the inversion of (5) and are active tracers. Overall the level of reconstruc-  
337 tion is poorer than that of passive tracers particularly in the separated GS  
338 region; the discrepancy north of 30°N extends vertically down to  $\sim 1000$  m  
339 (bottom two rows in Fig. 4). It is interesting, however, that there seems to  
340 be some utility of the tensor in the quiescent gyre interior (top three rows  
341 of Fig. 4; Table 3);  $\mathbf{J}_{\text{reconstructed}}^{\Theta}$  and  $\mathbf{J}_{\text{reconstructed}}^S$  capture the sign structure  
342 south of 30°N in  $\mathbf{J}^{\Theta}$  and  $\mathbf{J}^S$  respectively. The normed errors are also on the  
343 order of  $O(0.1)$  indicating some level of reconstructing the eddy flux partic-  
344 ularly in the quiescent eastern part of the gyre (cf. Fig. S7 in Supplementary  
345 Material). Given the QGPV results from the previous section, we did not  
346 attempt the reconstruction for Ertel’s PV.

#### 347 4. Conclusion and discussion

348 In this study, we have diagnosed the anisotropic mesoscale eddy transport  
349 tensor  $\mathbf{K}$  using passive tracer outputs from two sets of ensemble simulations:  
350 an idealized 101-member quasi-geostrophic (QG) double-gyre ensemble and

351 a realistic 24-member ensemble of the North Atlantic (NA). In decomposing  
 352 the eddies and mean flow, we have chosen to take the averaging operator over  
 353 the ensemble dimension rather than the often employed spatial or temporal  
 354 dimension (e.g. Balwada et al., 2020; Bachman et al., 2020; Kamenkovich  
 355 et al., 2020; Haigh and Berloff, 2021) with the aim of capturing the intrinsic  
 356 variability of eddy transport. We have also investigated the tensor in  
 357 the thickness-weighted averaged (TWA) context, which to our knowledge,  
 358 is the first study to do so. While we have only utilized one time slice of  
 359 output, we do not expect the performance of inverting for  $\mathbf{K}$  to qualitatively  
 360 vary over time in the quasi-adiabatic interior of the ocean; in Fig. S8 in the  
 361 Supplementary Material, we exhibit the time series of the spatial median of  
 362 the major and minor eigenvalues of the symmetric part of the QG tensor  
 363 ( $\mathbf{S} = (\mathbf{K} + \mathbf{K}^T)/2$  where  $\mathbf{K}^T$  is the transpose). The physical significance of the  
 364 eigenvalues is that the eddies tend to diffuse the tracers in the direction of  
 365 the eigenvectors with an associated transport coefficient of each respective  
 366 eigenvalue (Abernathy et al., 2013; Haigh et al., 2021a). We find a robust  
 367 seasonal cycle repeating itself (larger values during late summer and smaller  
 368 values during late winter) with a lag to the seasonality in the prescribed  
 369 background stratification. When examining the climatological summer average  
 370 of transport coefficients diagnosed at each time step and averaged over  
 371 five years, they exhibit less spatial structure but with the same tendencies  
 372 as Fig. 1, i.e. the diagonal components are largely positive and antidiagonal  
 373 components change sign across the separated jet (Fig. S9).

374 The diagnosed tensor shows good performance in reconstructing the eddy  
 375 fluxes of passive tracers from the gradient flux of mean passive tracer fields,  
 376 which were weakly restored with a one-year relaxation timescale. The agree-  
 377 ment between the spatial patterns in the transport coefficients emerging  
 378 from the QG and NA ensemble in respect to the position of the jet is also  
 379 comforting; they have similar orders of magnitude and the diagonal compo-  
 380 nents ( $\kappa_{uu}, \kappa_{vv}$ ) tend to be positive while ( $\kappa_{uv}, \kappa_{vu}$ ) tend to have opposite  
 381 signs across the jet and change signs within each gyre. This partially jus-  
 382 tifies our assumption that there is a universal eddy transport tensor, which  
 383 is able to represent the eddy flux across passive tracers (at least for pas-  
 384 sive tracers with similar source/sink terms; cf. Fig. S10). As noted in Sec-  
 385 tion 2.2, there are two possible ways to define the eddy tracer fluxes as soon  
 386 as the advective scheme becomes more complicated than a 2<sup>nd</sup>-order centered  
 387 scheme. We also diagnosed the tensor with the eddy fluxes defined as

388  $\mathbf{J}^{C_i} \stackrel{\text{def}}{=} \overline{\sigma^{-1}(\sigma \mathbf{F}^{C_i})} - \hat{\mathbf{u}} \hat{C}_i$  but the performance of reconstruction deteriorated  
 389 compared to when  $\mathbf{J}^{C_i} \stackrel{\text{def}}{=} \widehat{\mathbf{u}'' C_i''}$  (not shown). We speculate that the linear  
 390 gradient flux model (5) is unable to capture the non-linearities in the flux-  
 391 limiter advective scheme used in our simulations for all passive and active  
 392 tracers.

393 For studies using ensemble simulations, whether the ensemble size is large  
 394 enough to achieve statistical convergence is always in question. We exhibit  
 395 the QG transport coefficients in the top layer diagnosed from a subset of  
 396 12, 25 and 50 members in Fig. S11. While noisier, the transport coeffi-  
 397 cients diagnosed from 25 members already demonstrate similar values and  
 398 spatial patterns to the top left panels in Fig. 1. The histograms also align  
 399 more closely to the distribution from 101 members and the tails of histogram  
 400 shorten as the size of subset increases (bottom four panels in Fig. S11).

401 Given the diagnosed transport tensor from passive tracers, we have fur-  
 402 ther examined whether it can be carried over to inform the parametrization  
 403 of active tracer fluxes. However, when applying the tensor to reconstruct  
 404 the eddy fluxes of active tracers, here QG potential vorticity (PV), our re-  
 405 sults suggest that passive and active tracers have significantly different eddy  
 406 transport coefficients. In other words, passive and active tracers have differ-  
 407 ent relations between the eddy fluxes and mean fields and this likely stems  
 408 from the fact that for QGPV, there is no restoring force, similar to what we  
 409 prescribed for the passive tracers, particularly below the first layer; the gradi-  
 410 ent of mean QGPV had very little spatial structure in the separated jet region  
 411 as opposed to the passive tracers (Figs. S12 and S13). Without any restoring  
 412 for passive tracers, their concentrations would completely homogenize over  
 413 time ( $\nabla_{\mathbf{h}} C_i = \mathbf{0}$ ) rendering the gradient flux model (5) useless. Our empha-  
 414 sis on the eddy PV flux is because it encapsulates the energy backscattering  
 415 onto the mean flow (Young, 2012; Marshall et al., 2012; Uchida et al., 2021a).  
 416 Unlike previous studies (e.g. Mak et al., 2016; Haigh et al., 2021a), we have  
 417 not split the eddy flux of passive tracers into its divergent and rotational  
 418 component in diagnosing  $\mathbf{K}$  as our interest has been in whether this tensor  
 419 can be used to approximate the total eddy PV flux itself ( $\mathbf{F}^\#$  in Young, 2012),  
 420 and not its divergence, that forces the mean momentum. This being said,  
 421 our least-square approach does fairly well at reconstructing also the eddy flux  
 422 divergence of passive tracers in the quiescent gyre interior where the relative  
 423 errors are small (Figs. S13 and S14). Machine learning methods or further  
 424 generalizations to the framework may provide a pathway forward in finding

425 the relation for active tracers (e.g. Zanna and Bolton, 2020; Guillaumin and  
426 Zanna, 2021; Frezat et al., 2021; Lu et al., 2022).

427 Nonetheless, the development of a prognostic and physically consistent **K**  
428 will likely benefit biogeochemical modelling since biogeochemical tracers are  
429 passive (cf. Jones and Abernathy, 2019; Uchida et al., 2020). We end on  
430 the note that for the realistic NA ensemble, the tensor shows some skill in  
431 reconstructing the eddy temperature and salinity fluxes in the gyre interior.  
432 While temperature and salinity are not purely passive as they affect the  
433 dynamics via the hydrostatic pressure, the partial reconstruction may suggest  
434 that they have less of a direct role on the dynamics compared to PV.

#### 435 **Data availability statement**

436 The Jupyter notebooks used for the analyses of the QG and NA en-  
437 sembles are available via Github ([https://github.com/roxyboy/qg\\_eddy\\_](https://github.com/roxyboy/qg_eddy_diffusivity)  
438 [diffusivity](https://github.com/roxyboy/TWA-eddy-diffusivity) and <https://github.com/roxyboy/TWA-eddy-diffusivity>  
439 respectively; DOIs will be added upon acceptance of the manuscript).

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441 The authors declare no conflict of interest.

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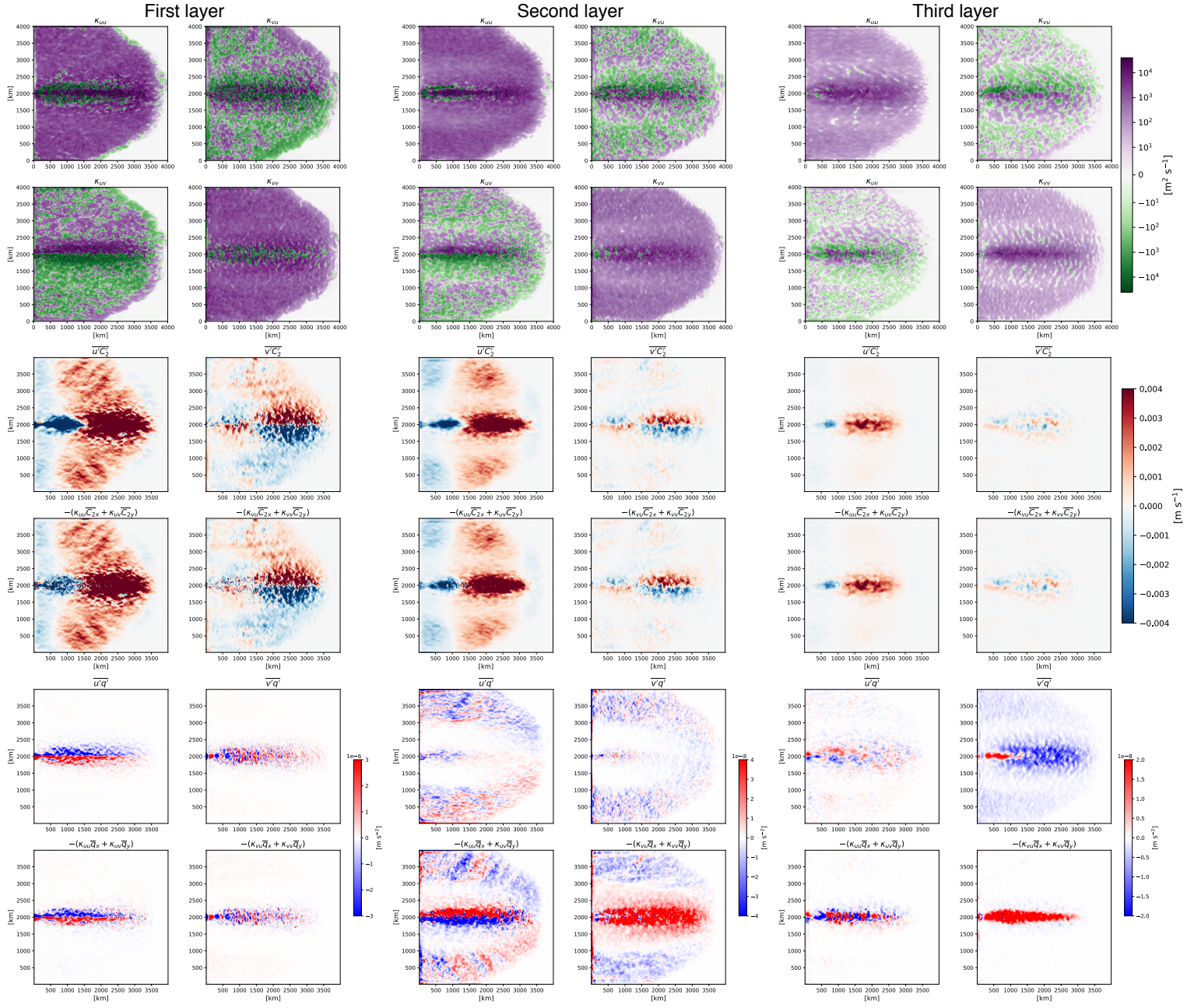


Figure 1: The eddy transport tensor  $\mathbf{K}$  diagnosed by inverting (5), eddy flux of  $C_2$  and PV and their reconstruction from the QG ensemble for all three layers. The four components of the tensor are exhibited in the top two rows, the eddy flux  $\mathbf{J}^{C_2}$  and its reconstruction  $-\nabla_{\mathbf{h}} \overline{C_2} \cdot \mathbf{K}$  in the middle two rows, and the tensor applied to PV ( $q$ ) in the bottom two rows. The eddy PV fluxes in the bottom two layers are two orders of magnitude smaller than the top layer.



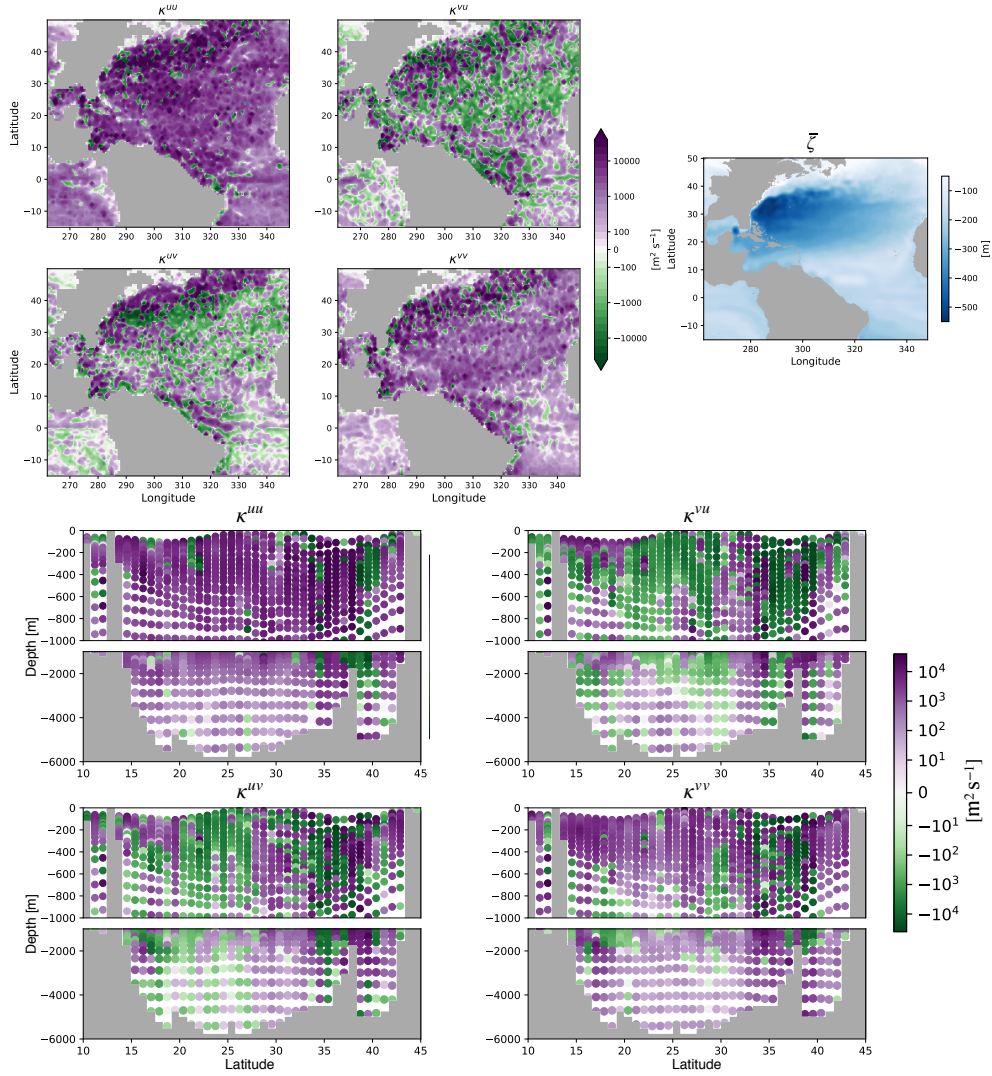


Figure 2: The eddy transport tensor  $\mathbf{K}$  diagnosed by inverting (5) from the realistic North Atlantic ensemble on January 3, 1967. The top left four panels show the four components along the neutral surface associated with the ensemble-mean depth  $\bar{z}$  shown in the right top panel. The bottom four panels exhibit the vertical transect of the four components along  $300^\circ\text{E}$ .

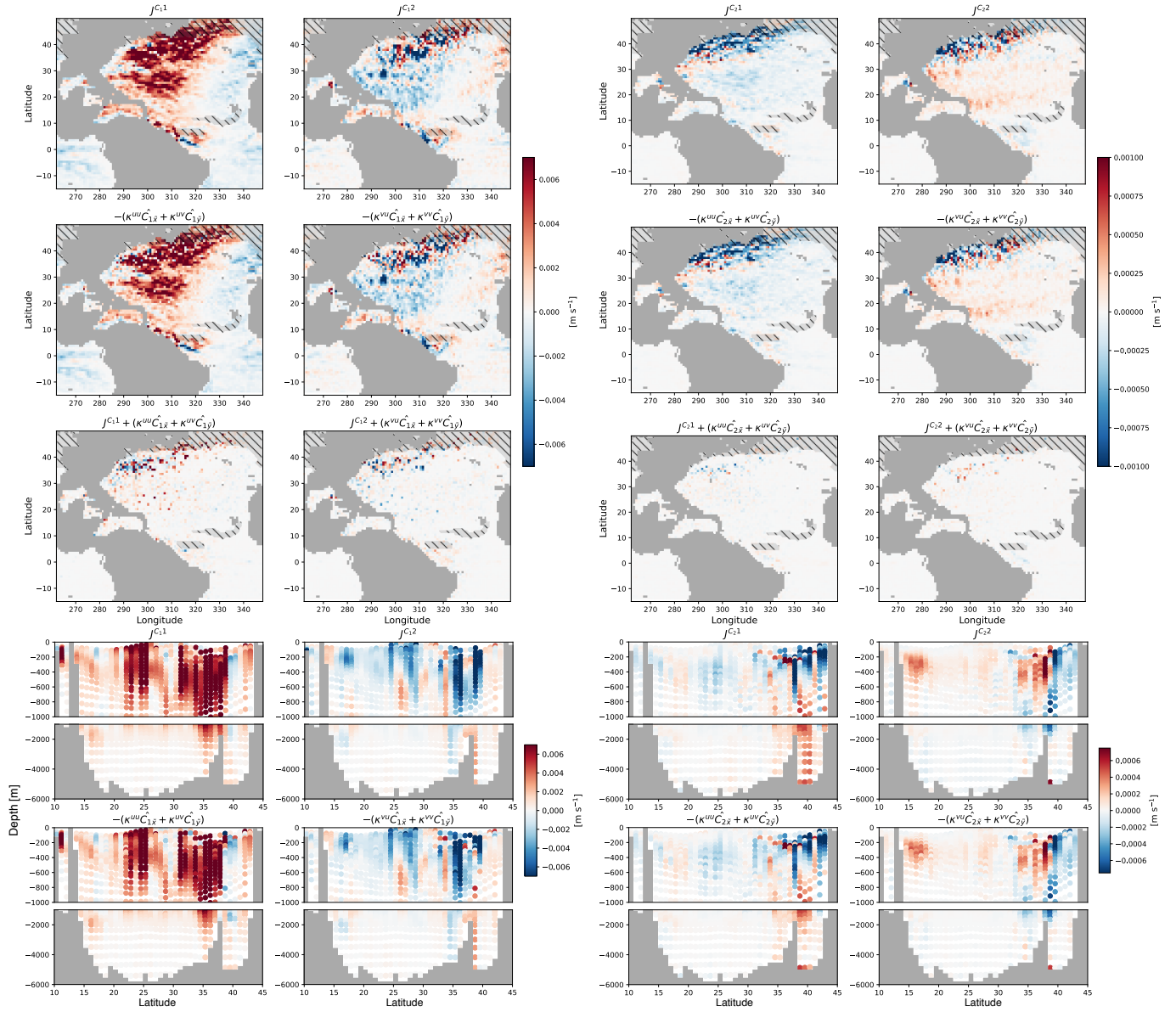


Figure 3: The eddy flux of tracers  $C_1$  and  $C_2$  and their reconstruction from the realistic North Atlantic ensemble on January 3, 1967. The top three rows present the eddy flux  $\mathbf{J}^{C_i}$ , reconstruction  $-\tilde{\nabla}_h \widehat{C}_i \cdot \mathbf{K}$ , and the difference between the two respectively. The hatches indicate regions where the neutral surface is shallower than 150 m. The bottom two rows exhibit the vertical transect of the flux and reconstruction along 300°E.

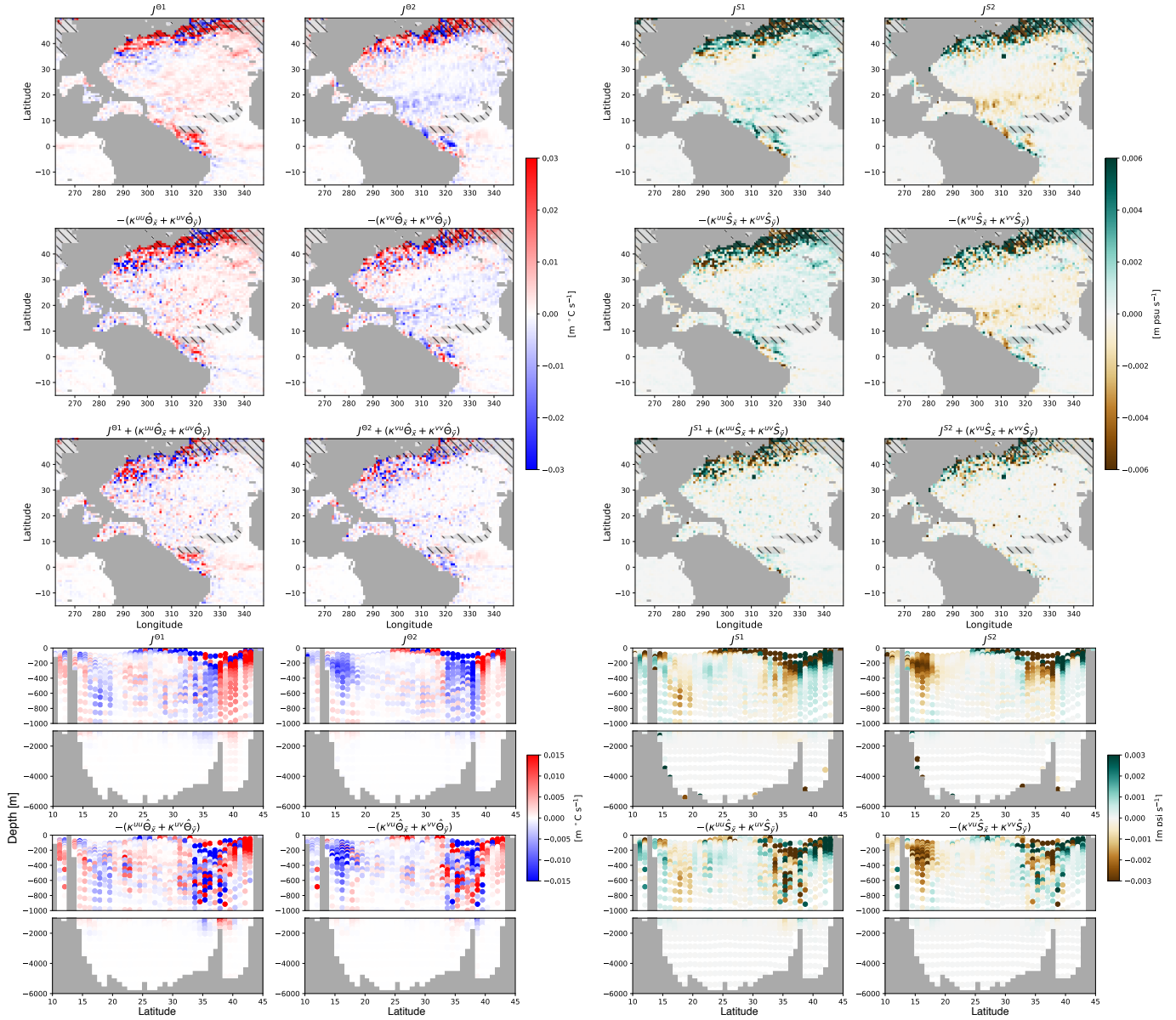


Figure 4: The eddy flux of potential temperature and practical salinity, and their reconstruction from the realistic North Atlantic ensemble on January 3, 1967. The top three rows present the eddy flux  $\mathbf{J}^{\theta}$ ,  $\mathbf{J}^S$ , reconstruction  $-\nabla_{\text{h}}(\hat{\Theta}, \hat{S}) \cdot \mathbf{K}$ , and the difference between the two respectively. The hatches indicate regions where the neutral surface is shallower than 150 m. The bottom two rows exhibit the vertical transect of the flux and reconstruction along 300°E.