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Irregularity Notions for Digraphs

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Abstract

Locally irregular graphs are those graphs in which any two adjacent vertices have different degrees, while locally irregular decompositions are edge-partitions of graphs where each part induces a locally irregular graph. These notions were introduced in a seminal work of Baudon *et al.*, in connection, in particular, to the so-called 1-2-3 Conjecture. Since then, several of their aspects of interest have been investigated in literature, including the existence of such decompositions with few parts, and the complexity of finding such ones.

In this work, we pursue investigations on generalisations of these notions to digraphs. In particular, we mainly investigate one variant in which the notion of irregularity for a digraph requires, for every arc from a vertex u to a vertex v , the outdegree of u be different from the indegree of v . We establish several results on this variant, covering upper bounds on the number of needed parts in decompositions, complexity aspects, and the impact of being able to choose the graph orientation, which results we compare to both the undirected setting and previous investigations on the directed one.

Keywords: graph irregularity; graph decomposition; edge-colouring; digraph.

1. Introduction

In this work, we explore ways of generalising **locally irregular graphs** and **locally irregular decompositions** from graphs¹ to digraphs. Before entering into the details of our actual contribution, we start by recalling what all these notions are about.

A common notion in graph theory is that of *regular graphs*, which can be defined as those graphs in which all vertices have the same degree. It is a well known fact, however, that, the one-vertex graph apart, no simple graph can be fully irregular, in the sense that there must always be a pair of vertices with the same degree. This led several researchers, such as Chartrand *et al.* [11], to investigate notions of graph irregularity that could stand as plausible antonyms to that of graph regularity. The notion of local irregularity is one instance of such. Formally, a graph G is *locally irregular* if we have $d(u) \neq d(v)$ for every edge uv of G , or, in other words, if no two adjacent vertices of G have the same degree.

Locally irregular graphs, as just defined above, arose in the study of several problems, and have been more particularly studied following investigations by Baudon, Bensmail, Przybyło, and Woźniak in [3] on locally irregular decompositions. Recall that a *k-decomposition* of a graph G is a partition of the edge set $E(G)$ into k parts, and that such a decomposition is *locally irregular* if each part indeed induces a locally irregular graph. Obviously, a graph admits a locally irregular 1-decomposition if and only if it is locally irregular itself. However, if a graph G is not locally irregular, then Baudon *et al.* are

¹Throughout this work, the term *graph* is used to refer to a simple undirected graph.

interested in measuring how far from locally irregular G is, which they do through locally irregular decompositions. More precisely, G is considered closer to locally irregular when it admits locally irregular decompositions with fewer parts. This leads to introducing the parameter $\chi_{\text{irr}}(G)$, which is the smallest $k \geq 1$ (if any) such that G admits locally irregular k -decompositions.

Without entering too much into the details, let us mention that one source of motivation for investigating locally irregular decompositions in [3] is their interplay with other notions of graph theory, namely proper labellings and the so-called 1-2-3 Conjecture posed by Karoński, Łuczak, and Thomason [13]. Since then, these decompositions have been receiving increasing attention, resulting, in particular, in the following works and results.

A first result established by Baudon *et al.*, in [3], is the exhaustive list of graphs G for which $\chi_{\text{irr}}(G)$ is not defined, in the sense that G does not admit any locally irregular decomposition (no matter the number of parts) at all. For convenience, from now on, let us say that a graph G is *decomposable* if it does admit locally irregular decompositions, and that it is an *exception* otherwise. It is easy to see that exceptions comprise graphs of \mathcal{P} and \mathcal{C} , the sets of all odd-length paths and odd-length cycles, respectively. Baudon *et al.* came up with a third set of exceptions, \mathcal{T} , whose members can be built as follows. Start from C_3 , the cycle of length 3, and repeatedly apply (at least once) the following: choose any degree-2 vertex u belonging to some triangle, and attach to u either:

- one end of an even-length path, or
- one end of an odd-length path at the other end of which is attached a triangle.

In brief, the members of \mathcal{T} can be seen as trees with maximum degree 3 in which some degree-3 vertices are actually triangles, and such that pendant paths and paths joining triangles satisfy some properties. Baudon *et al.* proved there are no other exceptions.

Theorem 1.1 (Baudon, Bensmail, Przybyło, Woźniak [3]). *Exceptions are exactly the graphs in $\mathcal{P} \cup \mathcal{C} \cup \mathcal{T}$.*

Based on the fact that they managed to prove this for a few classes of decomposable graphs (such as trees, complete graphs, some Cartesian products, and regular graphs with large degree), Baudon *et al.* suspected that, perhaps, all decomposable graphs G should have $\chi_{\text{irr}}(G) \leq 3$. This was later supported by further results (the conjecture was proved for *e.g.* split graphs [14] and graphs of large degree [19]). However, Sedlar and Škrekovski recently exhibited one 10-vertex graph G , called the bow-tie graph (obtained from an edge uv by attaching two triangles at u and v), with $\chi_{\text{irr}}(G) = 4$ [21]. Since it is still unclear whether other such graphs exist, it might be that the following refined conjecture holds:

Conjecture 1.2. *Decomposable graphs G other than the bow-tie graph satisfy $\chi_{\text{irr}}(G) \leq 3$.*

Towards Conjecture 1.2, a first constant upper bound on $\chi_{\text{irr}}(G)$ for decomposable graphs G was established by Bensmail, Merker, and Thomassen [10], whom proved that $\chi_{\text{irr}}(G) \leq 328$ holds. Later on [17], Lužar, Przybyło, and Soták improved upon their proof, establishing that even $\chi_{\text{irr}}(G) \leq 220$ holds. Among other notable works, let us also mention [6], in which Baudon, Bensmail, and Sopena investigated complexity aspects, proving that determining $\chi_{\text{irr}}(G)$ for a given graph G is NP-complete in general, but can be solved in polynomial time when G is a tree. A consequence of the former result is that Conjecture 1.2, if true, would be best possible in general.

For the sake of keeping the current introduction short, note that facts of lesser importance on locally irregular decompositions will be recalled later on, as they are needed.

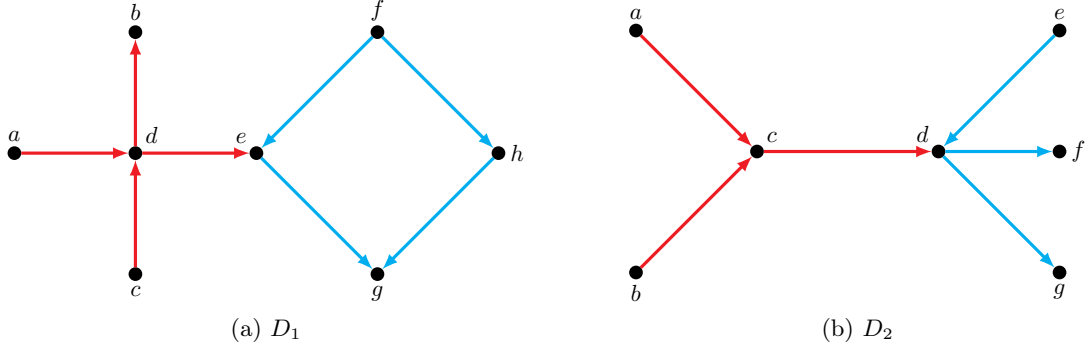


Figure 1: Two digraphs D_1 (a) and D_2 (b). Note that D_1 is not $(+, -)$ -irregular (because $d^+(d) = 2 = d^-(e)$), while it can be checked that D_1 is $(-, +)$ -irregular. Also, in (a), colours form a $(+, -)$ -irregular 2-decomposition. Thus, $\chi_{\text{irr}}^{(+,-)}(D_1) = 2$ and $\chi_{\text{irr}}^{(-,+)}(D_1) = 1$. On the other hand, D_2 is not $(-, +)$ -irregular (because $d^-(c) = 2 = d^+(d)$), while it can be checked that D_2 is $(+, -)$ -irregular. Also, in (b), colours form a $(-, +)$ -irregular 2-decomposition. Thus, $\chi_{\text{irr}}^{(+,-)}(D_2) = 1$ and $\chi_{\text{irr}}^{(-,+)}(D_2) = 2$.

In this work, we explore possible ways of generalising locally irregular decompositions from graphs to digraphs. Note that there are of course lots of directions to explore, as there are many ways to consider that a digraph is locally irregular, given that, in this context, to each vertex v are associated two degree notions, namely its *indegree* $d^-(v)$ and its *outdegree* $d^+(v)$. Drawing inspiration from the series of works [1, 5, 7, 12], we here consider irregularity notions defined over the following terminology. We associate the symbol $-$ to the indegree parameter, and the symbol $+$ to the outdegree parameter. Now, for any two symbols $\alpha, \beta \in \{-, +\}$, a digraph D is said (α, β) -irregular if, for every arc \vec{uv} (i.e., from u to v) of D , the degree parameter of u associated to α is different from the degree parameter of v associated to β . In turn, we generalise to digraphs all previous notions related to locally irregular decompositions: an (α, β) -irregular k -decomposition of a digraph D is a partition of the arc set $A(D)$ of D into k parts each inducing an (α, β) -irregular digraph, and, w.r.t. this notion, D either is an (α, β) -exception, meaning it admits no (α, β) -irregular decompositions at all, or is (α, β) -decomposable and there is a smallest $k \geq 1$, denoted $\chi_{\text{irr}}^{(\alpha, \beta)}(D)$, such that (α, β) -irregular k -decompositions of D exist. To make sure most of these notions are clear, we illustrate some of these in Figure 1.

The previous formalism leads to four sets of notions (which we call *variants* throughout), which are each more or less close to the original notions. In particular, in a graph G , every edge uv contributes to both $d(u)$ and $d(v)$, which two parameters come into play when establishing whether G is locally irregular. Now, in a digraph D , every arc \vec{uv} contributes to both $d^+(u)$ and $d^-(v)$, which, w.r.t. \vec{uv} , are exactly the two parameters of interest in the $(+, -)$ variant, contain only one of the two parameters of interest in the $(-, -)$ and $(+, +)$ variants, and contain none of them in the $(-, +)$ variant. From this point of view, the $(+, -)$ variant can thus be regarded as the variant that is the closest to the original one, while the $(-, +)$ variant can be regarded as the farthest, and the other two, the $(-, -)$ and $(+, +)$ variants, can be considered as intermediate in terms of closeness.

For each of the four variants, we have concerns that are reminiscent of those in the undirected case. Namely, for any two $\alpha, \beta \in \{-, +\}$, we wonder:

- Are there (α, β) -exceptions? If yes, can they be described easily?
- How large can $\chi_{\text{irr}}^{(\alpha, \beta)}(D)$ be, for an (α, β) -decomposable digraph D ?
- Can we bound $\chi_{\text{irr}}^{(\alpha, \beta)}(D)$ above for any (α, β) -decomposable digraph D ?

- Is there a “nice” classification of (α, β) -decomposable digraphs D w.r.t. $\chi_{\text{irr}}^{(\alpha, \beta)}(D)$?
- For a graph G , what can be said about its orientations D , w.r.t. $\chi_{\text{irr}}^{(\alpha, \beta)}(D)$? In particular, are there good orientations of G ? Bad ones?

To date, only the $(-, -)$ and $(+, +)$ variants have received some attention in literature, namely in [4, 8]. Note that these two variants, w.r.t. the concerns above, can actually be regarded as a single one, since, for a digraph D , a $(-, -)$ -irregular decomposition of D yields a $(+, +)$ -irregular decomposition of $R(D)$, the digraph obtained from D by reversing the orientation of every arc, and *vice versa*. In [8], the authors, among other things, observed that there are neither $(-, -)$ -exceptions nor $(+, +)$ -exceptions, conjectured that $\chi_{\text{irr}}^{(-, -)}(D)$ and $\chi_{\text{irr}}^{(+, +)}(D)$ cannot exceed 3 for a digraph D , proved that deciding whether $\chi_{\text{irr}}^{(-, -)}(D) \leq 2$ or $\chi_{\text{irr}}^{(+, +)}(D) \leq 2$ holds for a given digraph D is NP-complete (thereby showing that the conjectured upper bounds, if true, would be tight), and established that $\chi_{\text{irr}}^{(-, -)}(D) \leq 6$ and $\chi_{\text{irr}}^{(+, +)}(D) \leq 6$ hold for every digraph D . This was later improved in [4], as the authors proved that $\chi_{\text{irr}}^{(-, -)}(D) \leq 5$ and $\chi_{\text{irr}}^{(+, +)}(D) \leq 5$ hold for every digraph D .

In this work, we initiate the study of the $(+, -)$ and $(-, +)$ variants, focusing mainly on the former, motivated, as mentioned earlier, by the fact that it is the closest to the undirected setting (while, on the other hand, the latter is the farthest). We start in Section 2 by introducing a transformation which, given a digraph, produces a graph which is, depending on the variant considered, more or less close to the input digraph in terms of decompositions, from which we deduce several results (in terms of decomposability and of bounds on the minimum number of parts needed in a decomposition). Building upon these, we then focus on the $(+, -)$ variant in Section 3. In particular, we establish constant upper bounds on $\chi_{\text{irr}}^{(+, -)}(D)$ for all $(+, -)$ -decomposable digraphs D and investigate the impact of being allowed to choose the orientation of a given graph. For the latter concern, we characterise graphs that can be oriented to a $(+, -)$ -decomposable digraph, show how to produce good orientations for some classes of graphs, and establish that orienting a given graph to get a $(+, -)$ -irregular digraph is NP-complete. In Section 4, we provide one more result for the $(-, +)$ variant, showing that determining $\chi_{\text{irr}}^{(-, +)}(D)$ for a given digraph D is NP-complete, a result that has no equivalent in the $(+, -)$ variant yet.

As going along, we disseminate questions and problems which we think are of interest and could be subject to further work. Some of these are discussed further in concluding Section 5, in which we also introduce more questions of interest. We also take this occasion to give Table 1, which summarises all works to date on the four aforementioned variants, so that the reader gets a good grasp of the current knowledge on this line of research.

2. A connection with the undirected setting, and some consequences

In this section, we introduce a graph transformation which, from a digraph D , produces a graph $B(D)$ such that $(+, -)$ -irregular decompositions and $(-, +)$ -irregular decompositions of D relate to locally irregular decompositions of $B(D)$. From this connection, we get to a full understanding of the $(+, -)$ variant, and get some results on the $(-, +)$ variant.

We start by introducing the aforementioned graph transformation. Let G be a balanced bipartite graph with bipartition U, V , *i.e.*, with $|U| = |V|$. Assuming we have an ordering (u_1, \dots, u_n) over the vertices of U and an ordering (v_1, \dots, v_n) over the vertices of V , we say that U and V are *aligned* if $u_i v_i$ is not an edge for every $i \in \{1, \dots, n\}$, or, in other words, if $\{u_1 v_1, \dots, u_n v_n\}$ is a perfect matching in the complement of G . Now, for a digraph

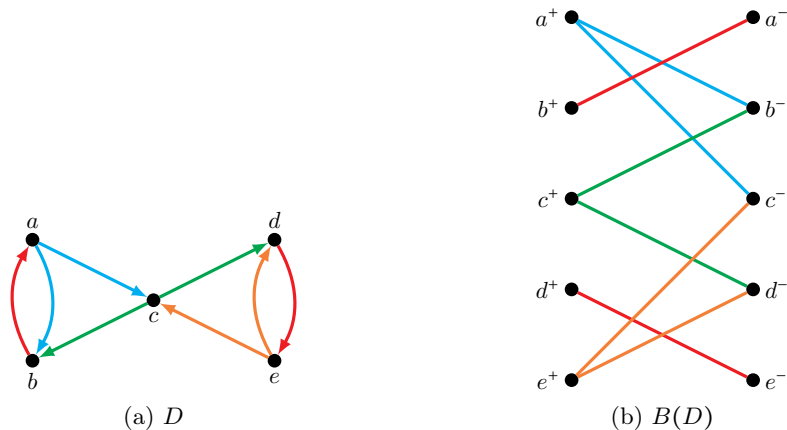


Figure 2: A digraph D (a) and the corresponding graph $B(D)$ (b). Colours on the arcs and edges show how a decomposition of a digraph yields one of the corresponding graph, and *vice versa*. Here, $B(D)$ is not decomposable because of the edges b^+a^- and d^+e^- , implying that D is not $(+, -)$ -decomposable.

D , its *corresponding graph* $B(D)$ is the bipartite graph with bipartition U^+, U^- obtained as follows (see Figure 2 for an illustration):

- for every vertex v of D , we have, in $B(D)$, a vertex v^+ in U^+ and a vertex v^- in U^- ;
- for every arc \vec{uv} of D , we have, in $B(D)$, the edge u^+v^- .

For this transformation to be clear, note that it is important that the orderings over the vertices of U^+ and U^- are clear, which we will always make sure to guarantee, throughout later uses. In particular, these orderings must guarantee that the bijection between the vertices of U^+ and those of U^- is clear (to make sure of this, most of the time we will use $+$ and $-$ symbols as superscripts, as in the previous definitions). Note also that there is a bijection between the arcs of any digraph D and the edges of the corresponding graph $B(D)$; for this reason, given a decomposition of D , one directly deduces a *corresponding decomposition* of $B(D)$, and *vice versa*. Stronger deductions can even be made for the $(+, -)$ variant (and, to some extent, for the $(-, +)$ variant), as highlighted below.

2.1. Consequences for the $(+, -)$ variant

In the context of the $(+, -)$ variant, the crucial point is that there is a straight equivalence between decomposing a digraph and its corresponding graph. Indeed:

Observation 2.1. *Let D be a digraph, ϕ' be a decomposition of $B(D)$, and ϕ be the corresponding one of D . Then, ϕ' is locally irregular if and only if ϕ is $(+, -)$ -irregular.*

Proof. Assuming ϕ and ϕ' are k -decompositions, the claim follows from the fact that, for any vertex v of D , and any $i \in \{1, \dots, k\}$, we have² $d_i^+(v) = d_i(v^+)$ and $d_i^-(v) = d_i(v^-)$. \square

In other words, decomposing a digraph in a $(+, -)$ -irregular way, which, in general, is a tedious task to perform by hand, can actually be done through decomposing its corresponding graph into locally irregular graphs, which is far easier to comprehend in general.

²Throughout this work, given a k -decomposition of a digraph, for any vertex v and some $i \in \{1, \dots, k\}$, we denote by $d_i^-(v)$ the number of arcs in the i th part incoming to v , and by $d_i^+(v)$ the number of arcs in the i th part outgoing from v . Similarly, in the context of a graph, we denote by $d_i(v)$ the number of edges in the i th part incident to any vertex v , for any $i \in \{1, \dots, k\}$.

Observation 2.1 therefore has several consequences, the first one of which deals with exceptions. Throughout what follows, a *sink-source path* in a digraph D is an oriented path (v_1, \dots, v_p) satisfying the following properties:

- the v_i 's are sinks and sources;
- thus, no two consecutive of the v_i 's are both sinks or both sources;
- every sink in $\{v_2, \dots, v_{p-1}\}$ has indegree 2;
- every source in $\{v_2, \dots, v_{p-1}\}$ has outdegree 2;
- each of v_1 and v_p is either a sink with indegree 1 or a source with outdegree 1.

We can now characterise $(+, -)$ -exceptions.

Theorem 2.2. *A digraph D is a $(+, -)$ -exception if and only if $B(D)$ is an exception, that is, if and only if D contains a sink-source path.*

Proof. By Observation 2.1, if $B(D)$ is decomposable, then D is $(+, -)$ -decomposable. Thus, for D to be a $(+, -)$ -exception, $B(D)$ must be an exception. Now, since $B(D)$ is bipartite, and the only connected bipartite exceptions are odd-length paths (recall Theorem 1.1), $B(D)$ must contain an odd-length path as a connected component, which, back in D , actually forms a sink-source path. \square

Observation 2.1 means that the $(+, -)$ variant is actually equivalent to the original decomposition problem in bipartite graphs. Namely, the following should now be clear.

Theorem 2.3. *If D is a $(+, -)$ -decomposable digraph, then $\chi_{\text{irr}}^{(+,-)}(D) = \chi_{\text{irr}}(B(D))$.*

Regarding locally irregular decompositions of bipartite graphs, a first constant upper bound was provided in [10], in which Bensmail, Merker, and Thomassen proved that $\chi_{\text{irr}}(G) \leq 10$ holds for every decomposable bipartite graph G . Later on, in [17], Lužar, Przybyło, and Soták proved that we even have $\chi_{\text{irr}}(G) \leq 7$ for every such G . Together with Theorem 2.3, the latter result yields the following upper bound:

Corollary 2.4. *If D is a $(+, -)$ -decomposable digraph, then $\chi_{\text{irr}}^{(+,-)}(D) \leq 7$.*

Regarding the other direction, that is, $(+, -)$ -decomposable digraphs that would be somewhat bad for $(+, -)$ -irregular decompositions, one way to investigate is “reversing” corresponding graphs. Formally, this leads to *e.g.* the following:

Theorem 2.5. *For every decomposable bipartite graph G , we can construct a $(+, -)$ -decomposable digraph D such that $B(D)$ contains G as a connected component.*

Proof. Let U, V denote the bipartition of G . Note that we may have $|U| \neq |V|$. Regardless, assuming $|U| = n$ and $|V| = m$, we set $U = \{v_1^+, \dots, v_n^+\}$ and $V = \{v_{n+1}^-, \dots, v_{n+m}^-\}$. Now add to U m new isolated vertices $v_{n+1}^+, \dots, v_{n+m}^+$ and similarly add to V n new isolated vertices v_1^-, \dots, v_n^- . Denoting by U^+ and U^- the resulting sets (where U^+ originates from U and U^- originates from V), and by G' the resulting bipartite graph, note that we have $|U^+| = |U^-| = n + m$ and that G' contains G as a connected component. So G' is balanced, and the way we have denoted the vertices of U^+ and U^- shows they are aligned (through the orderings $(v_1^+, \dots, v_{n+m}^+)$ and $(v_1^-, \dots, v_{n+m}^-)$).

We can now construct a digraph D such that $B(D) = G'$ by just adding, to D , a new vertex v_i for every $i \in \{1, \dots, n + m\}$, and an arc $\overline{v_i v_j}$ for every i and j such that $v_i^+ v_j^-$ is an edge of G' . In particular, we always have $i \neq j$ since G' has no edge of the form $v_i^+ v_i^-$. \square

Corollary 2.6. *There exist infinitely many digraphs D with $\chi_{\text{irr}}^{(+,-)}(D) = 3$.*

Proof. This follows from Theorem 2.5 and the fact that there are infinitely many bipartite graphs G with $\chi_{\text{irr}}(G) = 3$. For instance, all cycles with length congruent to 2 modulo 4 have this property [3]. There actually are infinitely many trees T with $\chi_{\text{irr}}(T) = 3$, see [6]. \square

In particular, it is worth mentioning that these ideas cannot be used to construct a digraph D with $\chi_{\text{irr}}^{(+,-)}(D) = 4$. Indeed, recall that there is only one known graph G with $\chi_{\text{irr}}(G) = 4$ (exhibited in [21]), and that this graph G is not bipartite. So the previous tools (Theorem 2.5 in particular) cannot be used to construct a digraph D with $B(D) = G$.

There are also downsides behind the equivalence between the $(+, -)$ variant and the original decomposition problem. In particular, some of the questions we raised earlier on the $(+, -)$ variant actually correspond to questions and problems on locally irregular decompositions of bipartite graphs, some of which remain open to date. As a particular example, recall that determining whether $\chi_{\text{irr}}(G) \leq 2$ holds for a given graph G is NP-complete in general [6], while the complexity of the same problem is still unknown when G is assumed to be bipartite. Due to the connections we have exhibited earlier, likewise we are not able to establish what is the complexity of the problem of determining whether $\chi_{\text{irr}}^{(+,-)}(D) \leq 2$ holds for a given digraph D .

2.2. Consequences for the $(-, +)$ variant

Regarding the $(-, +)$ variant, the use of corresponding graphs is far less straightforward. Note indeed that constructing a $(-, +)$ -irregular decomposition of a digraph D can, again, sort of be modelled as constructing a particular decomposition of $B(D)$. This time, however, constructing a locally irregular decomposition of $B(D)$ is not what we are looking for. Indeed, note that the edges in $B(D)$ are not representative of the constraints we have when decomposing D in a $(-, +)$ -irregular way. In particular, D might have an arc \vec{uv} yielding the edge u^+v^- in $B(D)$, while, in a $(-, +)$ -irregular decomposition of D , if \vec{uv} belongs to some i th part, then it might not be required to have $d_i^+(u) \neq d_i^-(v)$. Indeed, this is only required if \vec{vu} is also an arc of D . In short, for the $(-, +)$ variant $B(D)$ models D decomposition-wise, but not constraint-wise. What we mean by this is that, when decomposing a digraph D in a $(-, +)$ -irregular way, we can still work in the corresponding graph $B(D)$, which, most of the times, makes it easier to compute degrees in each part of the decomposition. However, in general a locally irregular decomposition of $B(D)$ does not have to yield a $(-, +)$ -irregular decomposition of D , and *vice versa*.

Still, there are ways to make this approach work, in particular in order to deduce upper bounds on $\chi_{\text{irr}}^{(-,+)}(D)$ for particular digraphs D . One way to do so is through decomposing $B(D)$ in such a way that the corresponding decomposition in D is actually $(-, +)$ -irregular, regardless of the actual constraints defining $(-, +)$ -irregular decompositions of D .

Observation 2.7. *Let D be a $(-, +)$ -decomposable digraph, ϕ' be a k -decomposition of $B(D)$, and $\mathcal{P}, \mathcal{M} \subset \mathbb{N}^*$ be sets such that, in $B(D)$, for every $i \in \{1, \dots, k\}$, vertices $v^+ \in U^+$ with $d_i(v^+) > 0$ satisfy $d_i(v^+) \in \mathcal{P}$ and vertices $v^- \in U^-$ with $d_i(v^-) > 0$ satisfy $d_i(v^-) \in \mathcal{M}$. If $\mathcal{P} \cap \mathcal{M} = \emptyset$, then the k -decomposition ϕ of D corresponding to ϕ' is $(-, +)$ -irregular.*

Proof. This is because, by ϕ' and ϕ , for every vertex v of D and every $i \in \{1, \dots, k\}$, we get $d_i^+(v) = d_i(v^+)$ and $d_i^-(v) = d_i(v^-)$ (recall Observation 2.1). By the assumptions of the statement, this means all non-zero $d_i^+(v)$'s lie in \mathcal{P} while all non-zero $d_i^-(v)$'s lie in \mathcal{M} . Now, in D , if some arc \vec{uv} belongs to the i th part, then, by assumption, we have $d_i^+(v) \in \mathcal{P}$ and $d_i^-(u) \in \mathcal{M}$, and $d_i^-(u) \neq d_i^+(v)$ since $\mathcal{P} \cap \mathcal{M} = \emptyset$. In particular, the fact that D is

$(-, +)$ -decomposable implies we do not have $d^-(u) = d^+(v) = 0$ for any arc \vec{uv} of D (or, equivalently, $d(v^+) = d(u^-) = 0$ in $B(D)$). Thus, ϕ is $(-, +)$ -irregular. \square

Observation 2.7 yields a sufficient condition for a decomposition of a digraph to be $(-, +)$ -irregular, but we would like to insist on the fact that this condition is far from being necessary in general. In particular, one can easily come up with examples of $(-, +)$ -irregular k -decompositions by which two (distant) vertices u and v satisfy $d_i^-(u) = d_i^+(v)$ for some i , which property disagrees with the conditions in the statement of Observation 2.7.

Looking at the work of Lužar, Przybyło, and Soták in [17], the results they establish on decomposable bipartite graphs rely, in particular, on the fact that, under certain circumstances, bipartite graphs with bipartition U, V admit k -decompositions where, for every $i \in \{1, \dots, k\}$, every vertex $u \in U$ with $d_i(u) > 0$ has $d_i(u)$ odd, while every vertex $v \in V$ with $d_i(v) > 0$ has $d_i(v)$ even. Note that, under such properties, we actually fall into the conditions in the statement of Observation 2.7, and can thus get partial results.

In particular, in [17], the authors established several results of this nature, but these require to consider bipartite graphs in which the vertices in some part of the bipartition all have even degree. In particular, for such bipartite graphs, they were able to prove that such 4-decompositions exist, and provided some improvements in some cases for the existence of such 3-decompositions and even 2-decompositions. All these results will be covered more in details in later Section 3. For now, let us just mention that, regarding the $(-, +)$ variant, these results lead to upper bounds in certain contexts, namely when, for a $(-, +)$ -decomposable digraph D , the corresponding graph $B(D)$ has all its vertices in one part of the bipartition being of even degree. Looking at the equivalence between a digraph D and the corresponding graph $B(D)$, in our case this mainly makes sense for digraphs in which all indegrees are even, or all outdegrees are even.

3. Results on the $(+, -)$ variant

We here provide more results on the $(+, -)$ variant, focusing on two directions. We first consider the general problem where a digraph D is given and we wonder about its $(+, -)$ -irregular decompositions. Then we move to the different problem where a graph G is given, and we wonder about best ways to orient it regarding $(+, -)$ -irregular decompositions.

3.1. When the orientation is fixed

As explained throughout Subsection 2.1, the $(+, -)$ variant is essentially equivalent to the initial decomposition problem in bipartite graphs. Due to the current knowledge we have on locally irregular decompositions in bipartite graphs, at this point it thus makes sense to raise the following:

Conjecture 3.1. *If D is a $(+, -)$ -decomposable digraph, then $\chi_{\text{irr}}^{(+,-)}(D) \leq 3$.*

Conjecture 3.1, if true, would be best possible, recall Corollary 2.6. Also, at this point, towards Conjecture 3.1 the best result we currently have is Corollary 2.4.

Other results from literature also provide further support to Conjecture 3.1. For instance, we can easily prove the conjecture holds for $(+, -)$ -decomposable oriented trees and $(+, -)$ -decomposable oriented cycles.

Theorem 3.2. *If D is a $(+, -)$ -decomposable oriented tree, then $\chi_{\text{irr}}^{(+,-)}(D) \leq 3$.*

Proof. Given that D is an oriented tree, we have that $B(D)$ is a forest (as the existence of cycles in $B(D)$ would imply the existence of cycles in the graph underlying D). Also, the fact that D is $(+, -)$ -decomposable implies $B(D)$ has no odd-length path and is thus decomposable (recall Theorem 2.2). Hence, $B(D)$ is a decomposable forest, and we have $\chi_{\text{irr}}(B(D)) \leq 3$ by a result from [6]. The result then follows from Theorem 2.3. \square

Note that the arguments in the proof of Theorem 3.2 actually imply that, for a given oriented tree D , we can determine $\chi_{\text{irr}}^{(+,-)}(D)$ in polynomial time. Indeed, constructing $B(D)$ can clearly be done in polynomial time, and, from $B(D)$, we can, in polynomial time, determine whether D is $(+, -)$ -decomposable (by checking whether a connected component of $B(D)$ is an odd-length path, recall Theorem 2.2) and, if yes, determine $\chi_{\text{irr}}^{(+,-)}(D)$ (by computing $\chi_{\text{irr}}(B(D))$) as explained in [6], which is what is desired, by Theorem 2.3).

As a side note, let us also mention that Theorem 3.2 is best possible, as there exist decomposable trees T with $\chi_{\text{irr}}(T) = 3$, see [5]. Then, indeed, due to Theorem 2.5, we can come up with $(+, -)$ -decomposable oriented trees D with $\chi_{\text{irr}}^{(+,-)}(D) = 3$.

Theorem 3.3. *If D is a $(+, -)$ -decomposable oriented cycle, then $\chi_{\text{irr}}^{(+,-)}(D) \leq 3$.*

Proof. When D is an oriented cycle, note that $B(D)$ is either a cycle (if D is a directed cycle) or a forest of paths (otherwise). Furthermore, if $B(D)$ is a cycle, then it is of even length since $B(D)$ is bipartite. Also, by Theorem 2.2, since D is $(+, -)$ -decomposable it must be that no connected component of $B(D)$ is an odd-length path. The result now follows from [3], wherein it was proved that we have $\chi_{\text{irr}}(G) \leq 3$ for every even-length path or even-length cycle G . \square

Again, the arguments in the proof of Theorem 3.3 easily yield a polynomial-time algorithm to determine $\chi_{\text{irr}}^{(+,-)}(D)$ when D is an oriented cycle. In particular, see [3], it is known that for an even-length cycle G we have $\chi_{\text{irr}}(G) = 3$ if G has length congruent to 2 modulo 4, and $\chi_{\text{irr}}(G) = 2$ otherwise, if its length is a multiple of 4.

The proofs of previous Theorems 3.2 and 3.3 could actually be caught by the following more general result, which requires a bit of extra care to be proved.

Theorem 3.4. *If D is a $(+, -)$ -decomposable oriented cactus, then $\chi_{\text{irr}}^{(+,-)}(D) \leq 3$.*

Proof. The main point is proving, under the assumption that D is an oriented cactus, that $B(D)$ is itself a (bipartite) cactus, so that the result follows from [22] (in which it was proved that we have $\chi_{\text{irr}}(G) \leq 3$ for every decomposable bipartite cactus G). It is not too complicated to see that if $B(D)$ had two cycles sharing at least two vertices, then we would deduce that D also does (recall indeed that there is a bijection between the arcs of D and the edges of $B(D)$). Thus, the claim follows. \square

In the very same way, other existing results can provide weaker upper bounds for some classes of digraphs, provided their corresponding graphs have certain properties. For instance, we clearly have $\Delta(B(D)) \leq \Delta(D)$ for every digraph D (where $\Delta(D)$ refers to the maximum degree of the graph underlying D); since decomposable subcubic graphs G satisfy $\chi_{\text{irr}}(G) \leq 4$ (see [17]), we thus have $\chi_{\text{irr}}^{(+,-)}(D) \leq 4$ for every $(+, -)$ -decomposable digraph D with maximum degree at most 3. Similar upper bounds can also be deduced for $(+, -)$ -decomposable digraphs with larger maximum degree (due *e.g.* to upper bounds from [2]). Likewise, degeneracy properties are preserved by corresponding graphs, so we would also get similar results for $(+, -)$ -decomposable degenerate digraphs (due to upper bounds *e.g.* from [9, 10]), thus including outerplanar digraphs, planar digraphs, etc.

As mentioned earlier, improving the upper bound in Corollary 2.4 (towards what is conjectured in Conjecture 3.1) can be done in certain contexts, but doing this in general does not seem obvious. Recall that, strictly speaking, the $(+, -)$ variant is actually equivalent to the original decomposition problem in a very particular subclass of bipartite graphs, being those bipartite graphs being balanced and whose complements admit perfect matchings. Unfortunately, we are not sure to perceive how such properties could be useful for showing *e.g.* that decomposable corresponding graphs admit locally irregular 6-decompositions.

3.2. When the orientation can be chosen

To get further understanding over the $(+, -)$ variant and Conjecture 3.1, for the rest of this section we turn to a different problem, being to focus on graph orientations. That is, we are now considering graphs, and we investigate ways to orient them so that we can then construct $(+, -)$ -irregular decompositions with few parts. In light of Conjecture 3.1 and Corollary 2.4, we are more particularly interested in orientations admitting $(+, -)$ -irregular decompositions with at most six parts.

We mainly prove a series of four results. An important first point is wondering whether all graphs can be oriented in the manner above; we prove this is not the case by establishing the precise list of graphs that cannot be oriented this way. Second, we then focus on orientable graphs, and prove they all admit an orientation admitting $(+, -)$ -irregular decompositions with at most five parts. Then, thirdly, we investigate how to improve this result in certain settings, by looking at particular contexts in which we can get three parts to suffice. Last, we establish that determining whether a graph admits a $(+, -)$ -irregular orientation is NP-complete. As a side result, we thus get that, for many graphs, all $(+, -)$ -decomposable orientations need at least two parts in their $(+, -)$ -irregular decompositions.

3.2.1. Characterising orientable graphs

Throughout the rest of Section 3, we say a graph G is *orientable* if it admits a $(+, -)$ -decomposable orientation. Assuming G is orientable, we denote by $\text{ori}^{(+,-)}(G)$ the smallest $k \geq 1$ such that G admits an orientation D with $\chi_{\text{irr}}^{(+,-)}(D) = k$.

In this section, we give a full characterisation of orientable graphs, which we do by exploiting tools from [3, 10] that were introduced to prove Theorem 1.1.

Theorem 3.5 (Baudon, Bensmail, Przybyło, Woźniak [3]). *Every decomposable graph admits a decomposition into paths of length 2, claws, and claws in which two edges have been subdivided exactly once.*

From Theorem 3.5, we get the following:

Corollary 3.6. *Every decomposable graph admits a $(+, -)$ -decomposable orientation.*

Proof. Let G be a decomposable graph, and E_1, \dots, E_k be a decomposition of G into paths of length 2, claws, and claws in which two edges have been subdivided exactly once (such a decomposition exists by Theorem 3.5). Consider now D , the orientation of G obtained by considering every $i \in \{1, \dots, k\}$ and orienting the edges in E_i as follows:

- if $G[E_i]$ is a path uvw of length 2, then add \vec{vu} and \vec{vw} to D ;
- if $G[E_i]$ is a claw with center u and leaves v_1, v_2, v_3 , then add $\vec{uv_1}, \vec{uv_2}, \vec{uv_3}$ to D ;
- if $G[E_i]$ is a claw with center u in which two edges have been subdivided once (resulting in edges $uv_1, uv_2, uv_3, v_1w_1, v_2w_2$), then add $\vec{uv_1}, \vec{uv_2}, \vec{uv_3}, \vec{w_1v_1}, \vec{w_2v_2}$ to D .

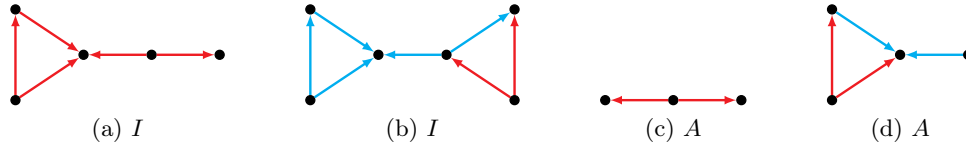


Figure 3: How pieces constituting exceptions are oriented and 2-decomposed in the proof of Theorem 3.7. In every picture, the two parts of the decomposition are depicted as red and blue arcs.

Note that D is indeed an orientation of G . For every $i \in \{1, \dots, k\}$, let A_i be the set of arcs of D containing exactly the arcs corresponding to the edges in E_i . Due to how we oriented edges, note that $D[A_i]$ is $(+, -)$ -irregular for every $i \in \{1, \dots, k\}$. Thus, A_1, \dots, A_k is a $(+, -)$ -irregular decomposition of D , and G is orientable. \square

Thus, non-orientable graphs, by Theorem 1.1, must belong to $\mathcal{P} \cup \mathcal{C} \cup \mathcal{T}$, *i.e.*, must be exceptions. Actually, we can prove graphs in \mathcal{T} are orientable.

Theorem 3.7. *Every graph of \mathcal{T} admits a $(+, -)$ -decomposable orientation.*

Proof. According to the definition, every graph G in \mathcal{T} is obtained from a triangle by repeatedly (at least once) choosing a degree-2 vertex u belonging to some triangle, and attaching, at u , an even-length path or an odd-length path at the other end of which is attached a new triangle. Since one of these two operations must be applied at least once, we could similarly consider that G was obtained starting from an initial piece I being either a triangle with an even-length path attached, or a triangle to which was attached an odd-length path with another triangle attached at the other end, and then repeatedly attaching (possibly zero times) another piece A , being either a path of even length, or a path of odd length at the end of which another triangle is attached.

To prove the claim, it then suffices to show that all possible initial pieces I can be oriented so that they admit a $(+, -)$ -irregular decomposition, and similarly for all possible added pieces A . We actually prove that all these pieces can be oriented so that $(+, -)$ -irregular 2-decompositions exist. We describe how to obtain such orientations in what follows, and how to decompose them into only two parts (see Figure 3 for illustrations).

- For I being a triangle $uvwu$ with an even-length path $vv_1 \dots v_{2k}$ attached, we can *e.g.* orient edges so that $\overrightarrow{uv}, \overrightarrow{uw}, \overrightarrow{wv}$ are arcs, and the remaining edges are oriented so that every v_i with odd i becomes a source. For the resulting orientation, a $(+, -)$ -irregular 2-decomposition A_1, A_2 is obtained when adding all of $\overrightarrow{uv}, \overrightarrow{uw}, \overrightarrow{wv}$ to A_1 , and then, for every $i \equiv 1 \pmod{4}$ adding the two (outgoing) arcs incident to v_i to A_1 , and for every $i \equiv 3 \pmod{4}$ adding the two (outgoing) arcs incident to v_i to A_2 .
- For I being two triangles $uvwu$ and $u'v'w'u'$ being joined by a path $vv_1 \dots v_kv'$ of odd length (thus $k \geq 0$ is even), we can *e.g.* orient edges so that $\overrightarrow{uv}, \overrightarrow{uw}, \overrightarrow{wv}$ are arcs, every v_i with odd i becomes a source, and $\overrightarrow{v'v_k}, \overrightarrow{u'v'}, \overrightarrow{u'w'}, \overrightarrow{v'w'}$ are arcs. A $(+, -)$ -irregular 2-decomposition A_1, A_2 of this orientation is then obtained as follows. We first add all of $\overrightarrow{uv}, \overrightarrow{uw}, \overrightarrow{wv}$ to A_1 . Then, for every $i \equiv 1 \pmod{4}$ we add the two (outgoing) arcs incident to v_i to A_1 , and for every $i \equiv 3 \pmod{4}$ we add the two (outgoing) arcs incident to v_i to A_2 . Last, assuming $k-1 \equiv 1 \pmod{4}$, we add $\overrightarrow{v'v_k}$ and $\overrightarrow{v'w'}$ to A_2 , and $\overrightarrow{u'v'}$ and $\overrightarrow{u'w'}$ to A_1 . Otherwise, if $k-1 \equiv 3 \pmod{4}$, then we add $\overrightarrow{v'v_k}$ and $\overrightarrow{v'w'}$ to A_1 , and $\overrightarrow{u'v'}$ and $\overrightarrow{u'w'}$ to A_2 .

- For A being a path $v_1 \dots v_k$ of even length (thus $k \geq 3$ is odd), we can *e.g.* consider the orientation where v_i is a source for every even i . Then a $(+, -)$ -irregular 2-decomposition A_1, A_2 of this orientation can be obtained by adding, for every $i \equiv 2 \pmod{4}$, the two (outgoing) arcs incident to v_i to A_1 , and by adding, for every $i \equiv 0 \pmod{4}$, the two (outgoing) arcs incident to v_i to A_2 .
- For A being a triangle $uvwu$ with an odd-length path $vv_1 \dots v_k$ (thus with $k \geq 1$ odd) attached, we can *e.g.* consider the following orientation. Orient edges so that $\overrightarrow{uv}, \overrightarrow{uw}, \overrightarrow{vw}, \overrightarrow{v_1v}$ are arcs, and, from here, make sure every v_i with i even is a source. For the resulting orientation, we can get a $(+, -)$ -irregular 2-decomposition A_1, A_2 as follows. Add \overrightarrow{uv} and \overrightarrow{uw} to A_1 , and \overrightarrow{vw} and $\overrightarrow{v_1v}$ to A_2 . Then, for every $i \equiv 2 \pmod{4}$ add the two (outgoing) arcs incident to v_i to A_1 , and for every $i \equiv 0 \pmod{4}$ add the two (outgoing) arcs incident to v_i to A_2 .

This concludes the proof. □

We are now ready to characterise non-orientable graphs.

Theorem 3.8. *A graph is not orientable if and only if it is an odd-length path or cycle.*

Proof. By Corollary 3.6, every decomposable graph is orientable. Furthermore, every member of \mathcal{T} is orientable by Theorem 3.7. Thus, by Theorem 1.1, it remains to consider odd-length paths and odd-length cycles. More precisely, we need to show they are not orientable. Assume this is wrong, and let G be an odd-length path or an odd-length cycle admitting a $(+, -)$ -decomposable orientation D . Consider the corresponding graph $B(D)$. Since G has odd size, D also has odd size. Also, since G is a path or a cycle, note that $B(D)$ is either a cycle or a forest of paths. The former case is not possible, since $B(D)$ is bipartite, and thus it cannot be an odd-length cycle. In the latter case, since $B(D)$ has odd size, it must be that $B(D)$ contains a path of odd length, which is not decomposable. Thus, in any case we get a contradiction to the existence of G . □

3.2.2. Orientations requiring at most five parts in their $(+, -)$ -irregular decompositions

As mentioned earlier, the proof of Corollary 2.4 relies on results of Lužar, Przybyło, and Soták from [17], which allowed them to prove that $\chi_{\text{irr}}(G) \leq 7$ holds for every decomposable bipartite graph G . An ingredient behind their proof of this upper bound is the following colouring tool, which deals with *even bipartite graphs*, being those bipartite graphs such that one of the two partition classes contains vertices of even degree only.

Theorem 3.9 (Lužar, Petruševski, Škrekovski [16]). *Let G be an even bipartite graph with bipartition U, V such that all vertices in U have even degree. Then, G admits a 4-decomposition such that, for every $i \in \{1, 2, 3, 4\}$, every vertex $u \in U$ with $d_i(u) > 0$ satisfies $d_i(u)$ odd, while every vertex $v \in V$ with $d_i(v) > 0$ satisfies $d_i(v)$ even.*

We will use the previous result together with a few other tools.

Lemma 3.10 (Baudon, Bensmail, Przybyło, Woźniak [3]). *Every connected graph with even size admits a decomposition into even-length paths.*

The following one is the exact statement of previous Theorem 3.5 in [3].

Lemma 3.11 (Baudon, Bensmail, Przybyło, Woźniak [3]). *Every connected graph G with odd size contains a claw or a claw with two edges subdivided once, H , such that all connected components of $G - E(H)$ have even size.*

From the previous tools and results, we can prove the following:

Theorem 3.12. *If G is a connected orientable graph, then $\text{ori}^{(+,-)}(G) \leq 5$.*

Proof. If G has even size, then let H be the empty graph. Otherwise, by Lemma 3.11, G contains H , a claw possibly having two of its edges subdivided once, such that $G' = G - E(H)$ has all its connected components being of even size. Note that the same conclusion holds if H is empty. Now, by Lemma 3.10, G' admits a decomposition $E_1, \dots, E_{\lfloor \frac{|E(G')|}{2} \rfloor}$ into paths of length 2, and by considering each uvw of these paths, and orienting its edges so that \overrightarrow{vu} and \overrightarrow{vw} are arcs, we get an orientation D' of G' such that, denoting by U^+, U^- the bipartition of the corresponding graph $B(D')$, all vertices in U^+ have even degree (regarding any path uvw , note that, in $B(D')$, we increased $d(v^+)$ by exactly 2, while $d(u^+)$ and $d(w^+)$ were not altered). Thus, in other words, $B(D')$ is an even bipartite graph, and $B(D')$ admits a 4-decomposition as described in the proof of Theorem 3.9, which is a locally irregular decomposition, and is thus $(+, -)$ -irregular when considered back in D' , by Observation 2.1.

If H is empty, then D' actually forms an orientation of G admitting a $(+, -)$ -irregular 4-decomposition. Now, if H is not empty, then, by arguments from the proof of Corollary 3.6, there is a $(+, -)$ -irregular orientation D'' of H . This D'' , together with the previous D' , forms an orientation D of G , which orientation admits a $(+, -)$ -irregular 5-decomposition. \square

Note that we can go a bit beyond Theorem 3.12 in particular contexts. For instance, as highlighted in the proof, we actually have $\text{ori}^{(+,-)}(G) \leq 4$ under the additional assumption that G has even size (case where H is empty). In [16], the authors also go beyond Theorem 3.9 when considering graphs with certain properties. For instance, it restricts to 2-decompositions for even bipartite graphs having vertices of even degree only. For such graphs G , our proof scheme in the proof of Theorem 3.12 yields that $\text{ori}^{(+,-)}(G) \leq 3$ holds.

3.2.3. Orientations complying with Conjecture 3.1

In this section, we essentially show that Conjecture 3.1 holds for three graph classes, provided we are allowed to choose the orientation. Namely, we consider complete graphs, trees, and sparse graphs. For the former two classes, we actually prove that there even exists an orientation such that only two parts suffice to design a $(+, -)$ -irregular decomposition.

We start off by considering complete graphs. This is done by focusing on acyclic tournaments, *i.e.*, tournaments with no directed cycles.

Theorem 3.13. *Every acyclic tournament \overrightarrow{T}_n of order $n \geq 4$ satisfies $\chi_{\text{irr}}^{(+,-)}(\overrightarrow{T}_n) \leq 2$.*

Proof. We prove the claim by induction on n , through a stronger hypothesis. Namely, we prove that, for every $n \geq 4$, every acyclic tournament \overrightarrow{T}_n of order n admits a $(+, -)$ -irregular 2-decomposition A_1, A_2 such that, for some $i \in \{1, 2\}$, the unique source of \overrightarrow{T}_n has all its incident (outgoing) arcs belonging to A_i , while the unique sink has incident (incoming) arcs in both A_1 and A_2 (*i.e.*, all its incident arcs do not belong to a single A_i).

As a base case, we remark that the claim holds when $n \in \{4, 5\}$, see Figure 4 for examples of $(+, -)$ -irregular 2-decompositions. In particular, in each case note that the extra requirements are indeed met. We now turn to proving the general case. That is, consider some $x \geq 6$ such that the claim holds for every acyclic tournament \overrightarrow{T}_x with $n < x$. Let us denote by v_1, \dots, v_x the vertices of \overrightarrow{T}_x so that $\overrightarrow{v_i v_j}$ is an arc whenever $i < j$. By the induction hypothesis, the acyclic tournament induced by vertices v_2, \dots, v_{x-1} admits a $(+, -)$ -irregular 2-decomposition ϕ' with parts A'_1, A'_2 such that, w.l.o.g., all (outgoing) arcs incident to v_2 belong to A'_1 , while v_{x-1} does not have all its incident arcs belonging to

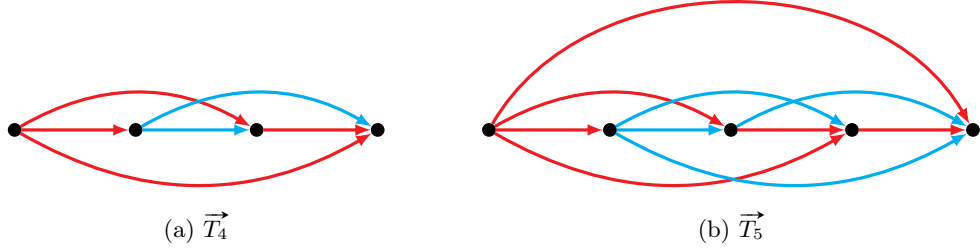


Figure 4: $(+, -)$ -irregular 2-decompositions of \vec{T}_4 and \vec{T}_5 , the acyclic tournaments of order 4 and 5. In each picture, the two parts of the decomposition are depicted as red and blue arcs.

some single A'_i . We extend this decomposition to \vec{T}_x by adding all (outgoing) arcs incident to v_1 to A'_2 , and adding all (incoming) arcs incident to v_x to A'_2 , with the exception of $\vec{v_2v_x}$ which we add to A'_1 . Let us denote by ϕ the resulting 2-decomposition of \vec{T}_x , and by A_1, A_2 its parts (where each A_i originates from A'_i).

We claim ϕ is $(+, -)$ -irregular. To see this is true, note first that $d_2^+(v_1) = x - 1$, while only v_x has indegree $x - 1$ but $d_2^-(v_x) = x - 2$. Thus, v_1 cannot be involved in conflicts. Likewise, note that $d_1^+(v_2) = x - 2$, while only v_{x-1} and v_x have indegree at least $x - 2$, but $d_1^-(v_x) = 1$ and $d_1^-(v_{x-1}) < x - 2$ (since $\vec{v_1v_{x-1}} \in A_2$). Thus, also v_2 cannot be involved in conflicts. Now, for every $i, j \in \{3, \dots, x - 1\}$ with $i < j$, note that, as going from ϕ' to ϕ , we have that $d_1^-(v_i), d_1^-(v_j), d_1^+(v_i)$, and $d_1^+(v_j)$ did not change, while all of $d_2^-(v_i), d_2^-(v_j), d_2^+(v_i)$, and $d_2^+(v_j)$ have been increased by 1. Thus, since v_i and v_j were not in conflict by ϕ' , they are still not in conflict by ϕ . Last, regarding v_x , recall that $d_1^-(v_x) = 1$ and $d_2^-(v_x) = x - 2$. As mentioned earlier, v_x can be in conflict with neither v_1 nor v_2 . Now, for every $i \in \{3, \dots, x - 1\}$, recall that $\vec{v_iv_x} \in A_2$, while $d_2^-(v_i) < x - 2$. Thus, v_i and v_x cannot be in conflict, and ϕ is $(+, -)$ -irregular. Note further that it fulfils all required extra requirements: in particular, all (outgoing) arcs incident to v_1 belong to A_2 , while v_x has incident (incoming) arcs belonging both to A_1 and A_2 . \square

Corollary 3.14. *Every complete graph K_n of order $n \geq 4$ satisfies $\text{ori}^{(+,-)}(K_n) \leq 2$.*

We now consider orientable trees, going beyond Theorem 3.2.

Theorem 3.15. *If T is a tree different from an odd-length path, then $\text{ori}^{(+,-)}(T) \leq 2$.*

Proof. We prove the result through a double induction on the number x of vertices of degree at least 3 in T and on the order n of T . We consider small values of x first.

- If $x = 0$, then T is an even-length path; in this case, denoting by $v_1 \dots v_{2k+1}$ the consecutive vertices of T from one end to the other, we consider (regardless of n) the orientation of T where all v_i with i even are sources, which admits a $(+, -)$ -irregular 2-decomposition A_1, A_2 (obtained *e.g.* by adding the two (outgoing) arcs incident to any v_i with $i \equiv 2 \pmod 4$ to A_1 , and adding the two (outgoing) arcs incident to any v_i with $i \equiv 0 \pmod 4$ to A_2).
- If $x = 1$, then T is a subdivided star with center u , to which are attached at least three branches (pendant paths). We consider (regardless of n) the following orientation \vec{T} of T , and some 2-decomposition A_1, A_2 of it:

- for every neighbour v of u , we have the arc \vec{uv} in \vec{T} ;
- for every branch $uv_1 \dots v_{2k}$ with even length attached at u , we have the arc $\vec{v_2v_1}$;

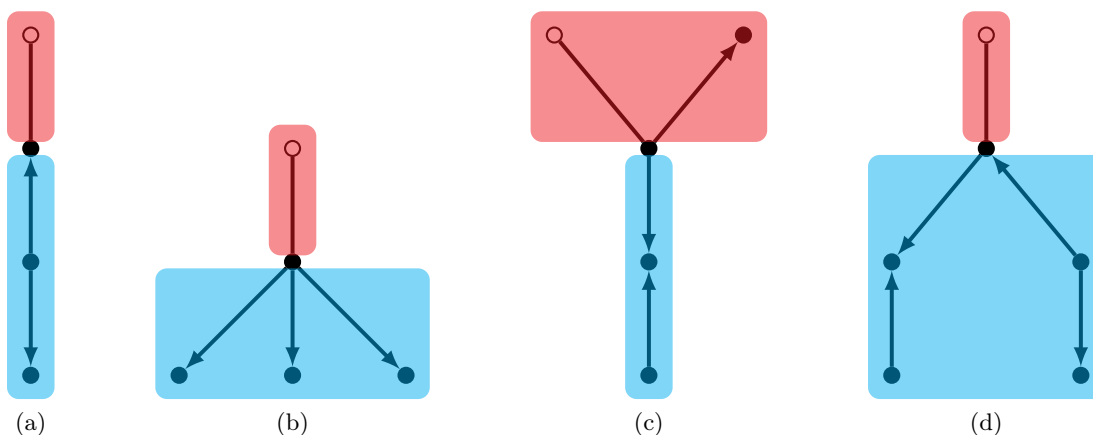


Figure 5: Cases covered in the proof of Theorem 3.15. Areas in red and blue depict the two parts of a $(+, -)$ -irregular 2-decomposition of some orientation, the area in blue containing the subgraph removed before the induction is invoked. In each case, the white vertex is the parent.

- note that all edges we have oriented at this point are at distance at most 2 from u , and the way we oriented them guarantees the corresponding arcs of \vec{T} induce a $(+, -)$ -irregular digraph. We add all these arcs to A_1 .
- The edges of T that remain to be oriented induce even-length paths. For every such path $v_1 \dots v_{2k+1}$ (where v_1 is the end that is the closest to u), we orient its edges in \vec{T} so that every v_i with even i becomes a source. Then, for every i with $i \equiv 2 \pmod{4}$, we add the two (outgoing) arcs incident to v_i to A_2 , while, for every i with $i \equiv 0 \pmod{4}$, we add the two (outgoing) arcs incident to v_i to A_1 .

Altogether, it can be checked that A_1 and A_2 form a $(+, -)$ -irregular 2-decomposition of \vec{T} . Thus, we have $\text{ori}^{(+,-)}(T) \leq 2$ as claimed.

Assume now the claim holds for all values of x up to some $i \geq 1$; we prove it also holds for $x = i + 1$. Let T be a tree with x vertices of degree at least 3. We can assume further the claim holds for all trees with x vertices of degree at least 3 but less vertices than T . Set as r any vertex of T , defined as the root of T . Given two adjacent vertices u and v of T , we say u is the *parent* of v if u is closer to r than v is. Otherwise, u is a *child* of v . For a vertex v of T , its *descendants* are all vertices having v lying on their unique path to r .

Consider now v , a vertex of degree at least 3 being at largest distance from r , and let u denote the parent of v . Then, all descendants of v are of degree at most 2. In other words, to v are attached $d \geq 2$ pendant paths, whose vertices are precisely the descendants of v .

If one of the pendant paths $v = v_1 \dots v_k$ attached to v has length at least 3, then we can be done with T in the following way (see Figure 5 (a)). Let $T' = T - \{v_{k-1}, v_k\}$. Note that T' has x vertices of degree at least 3 (implying, in particular, that T' cannot be an odd-length path) but is smaller than T . Thus, by the induction hypothesis, there is an orientation of T' admitting a $(+, -)$ -irregular 2-decomposition A'_1, A'_2 . We extend these to T by orienting $v_{k-2}v_{k-1}$ and $v_{k-1}v_k$ so that $\overrightarrow{v_{k-1}v_{k-2}}$ and $\overrightarrow{v_{k-1}v_k}$ are arcs, and, assuming the arc defined over $v_{k-3}v_{k-2}$ belongs, say, to A'_1 , we add these two new arcs to A'_2 . It can be checked we get an orientation of T and a $(+, -)$ -irregular 2-decomposition.

From now on, we may thus assume the d pendant paths attached at v have length at most 2. If $d \geq 3$, then we can proceed as follows (Figure 5 (b) shows a particular case). Let T' be the tree obtained from T by removing all vertices other than v from the d pendant paths attached at v . That is, if vv_1 is any pendant path of length 1 attached at v then

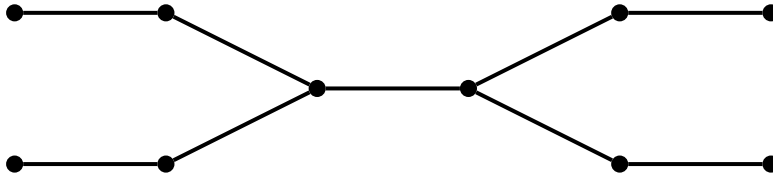


Figure 6: A tree T with $\text{ori}^{(+,-)}(T) > 1$.

we remove v_1 , while, if vv_1v_2 is any pendant path of length 2 then we remove v_1 and v_2 . Note that T' has $x - 1 \geq 1$ vertices of degree 3 (implying, again, that it cannot be that T' is an odd-length path); thus, by induction, there is an orientation of T' admitting a $(+, -)$ -irregular 2-decomposition A'_1, A'_2 . We extend these to T as follows. For every pendant path vv_1 of length 1 attached to v we removed, we add the arc $\overrightarrow{vv_1}$ to the orientation. For every pendant path vv_1v_2 of length 2 attached to v we removed, we add the arcs $\overrightarrow{vv_1}$ and $\overrightarrow{v_2v_1}$. Note that all these new arcs form a $(+, -)$ -irregular digraph. Thus, assuming the arc defined over uv belongs to, say, A'_1 , we obtain a $(+, -)$ -irregular 2-decomposition of the resulting orientation of T when adding all these new arcs to A'_2 .

It remains to consider the cases where $d = 2$. If both children of v are leaves (thus if the two paths attached at v have length 1), then we can be done through the same arguments as in the previous case. If the two pendant paths vv_1v_2 and vw_1w_2 attached at v have length 2, then we can be done similarly as in the previous case, by orienting the four removed edges so that $\overrightarrow{vv_1}$, $\overrightarrow{v_2v_1}$, $\overrightarrow{w_1v}$, and $\overrightarrow{w_1w_2}$ are arcs (see Figure 5 (d)). It remains to consider the case where the two pendant paths vv_1 and vw_1w_2 attached at v have length 1 and 2, respectively. Here, we instead consider $T' = T - \{w_1, w_2\}$, which has $x - 1 \geq 1$ vertices of degree at least 3 and is thus not an odd-length path. Again, by the induction hypothesis, T' has an orientation admitting a $(+, -)$ -irregular 2-decomposition A'_1, A'_2 . By that decomposition, note that the arcs defined over the edges uv and vv_1 must necessarily belong to the same A'_i , say A'_1 (since a single arc is not a $(+, -)$ -irregular digraph). Then we reach the same conclusions as previously, when orienting vw_1 and w_1w_2 so that $\overrightarrow{w_1v}$ and $\overrightarrow{w_1w_2}$ are arcs, and we add both arcs to A'_2 (Figure 5 (c)). This concludes the proof. \square

We note that Theorem 3.15 is best possible, in that there exist orientable trees that do not admit $(+, -)$ -irregular orientations. An example of such is depicted in Figure 6.

We end up this section by investigating a last family of sparse graphs. Recall that the *average degree* of a graph G is defined as the quantity $\frac{2|E(G)|}{|V(G)|}$, and that the *maximum average degree* of G , denoted $\text{mad}(G)$, is defined as the maximum average degree over all subgraphs of G . Our upcoming result deals with graphs having maximum average degree less than $\frac{5}{2}$. A point of interest behind our result is that we establish it by exploiting the notion of corresponding graph; that is, we prove that orientable graphs G with $\text{mad}(G) < \frac{5}{2}$ can be oriented to an orientation in which the following ideas apply:

Observation 3.16. *Let G be a graph, and \vec{G} be an orientation of G . If $B(\vec{G})$ is a forest with no odd-length path as a connected component, then $\chi_{\text{irr}}(B(\vec{G})) \leq 3$ according to [6], and, thus, by Theorem 2.3, we have $\text{ori}^{(+,-)}(G) \leq 3$.*

Recall that establishing 3 as an upper bound on $\text{ori}^{(+,-)}(G)$ for some graph G makes particular sense in light of Conjecture 3.1. In some sense, this establishes that G satisfies the conjecture, provided we are allowed to choose the orientation freely.

Let us mention that, following the exact same lines as in the proof of Theorem 3.8, it is easy to sort of establish that a graph cannot be oriented as described in the statement

of Observation 3.16 if and only if it is an odd-length path or an odd-length cycle. Indeed, to be more precise, we can more generally wonder about graph orientations admitting k -decompositions where each part, in the corresponding graph, translates into a forest with no odd-length paths. From ideas from the proof of Theorem 3.8, we can establish that every graph different from an odd-length path and an odd-length cycle admits such an orientation admitting such a k -decomposition for some k . Besides, by Observation 3.16, if we could prove that this is even true for $k = 1$, then we would establish that $\text{ori}^{(+,-)}(G) \leq 3$ holds for every graph G different from an odd-length path and an odd-length cycle. Below, we prove this to be true in the class of graphs with maximum average degree less than $\frac{5}{2}$.

So, our main result in this section reads as follows:

Theorem 3.17. *Every graph G with $\text{mad}(G) < \frac{5}{2}$ different from an odd-length path and an odd-length cycle admits an orientation \vec{G} such that $B(\vec{G})$ is a forest with no odd-length path as a connected component.*

Proof. We prove the claim through the so-called *discharging method*, which, recall, goes as follows. We assume the claim is wrong, that is, that there exist graphs G with $\text{mad}(G) < \frac{5}{2}$, that are neither an odd-length path nor an odd-length cycle, that do not admit an orientation as in the statement. Among all such counterexamples, we focus on G , a smallest one in terms of $|V(G)| + |E(G)|$. Clearly, G is connected. Our ultimate goal is to contradict the existence of G , which is done through two main steps.

- As a first step, we start by proving that G being a smallest counterexample to the claim, it cannot contain certain configurations, called *reducible configurations*. Formally, a configuration is a certain sparse subgraph H defined in terms of adjacent vertices having certain degrees. The main idea is that when removing (part of) H from G , we end up with a graph G' that satisfies $\text{mad}(G') < \frac{5}{2}$, is smaller than G , and, assuming it is neither an odd-length path nor an odd-length cycle, thus admits an orientation as in the statement, which, due to how permissive the structure of H is, can be extended to one of G , a contradiction.
- Once we have identified a certain set of reducible configurations, we then prove, given that G cannot contain any of these sparse configurations, that it is impossible for G to satisfy $\text{mad}(G) < \frac{5}{2}$, yet another contradiction. This is done through a certain *discharging process*, which will be described later on.

We thus start by proving that G , due to its minimality, cannot contain certain reducible configurations. To describe these, we employ the following terminology. For some $k \geq 1$, a k -vertex in a graph is a vertex of degree k , while a k^+ -vertex is a vertex with degree at least k . Similarly, a k -neighbour of some vertex is an adjacent k -vertex, and a k^+ -neighbour is an adjacent k^+ -vertex.

Claim 3.18. *G contains neither of the following configurations:*

- (C1) *a 1-vertex with a 2-neighbour;*
- (C2) *a 1-vertex with a 3-neighbour;*
- (C3) *a 1-vertex with a 4⁺-neighbour;*
- (C4) *a 2-vertex with two 2-neighbours;*
- (C5) *a 2-vertex with a 2-neighbour and a 3-neighbour;*

(C6) a 3-vertex with three 2-neighbours;

(C7) a 4-vertex with four 2-neighbours.

Proof of the claim. We consider each of the configurations separately (where, when being done with any Configuration (Ci), it is then assumed, from there, when dealing with later configurations, that G cannot contain Configuration (Ci)). For simplicity, below we say a graph is *orientable* if it contains neither an odd-length path nor an odd-length cycle as a connected component. Also, whenever dealing with corresponding graphs, we do so employing the terminology from Section 2 (in particular, every bipartition is assumed to have parts U^+, U^- , vertices of U^+ are denoted with a + superscript, and vertices of U^- are denoted with a - superscript).

- Configuration (C1).

Assume G contains a 1-vertex u with a 2-neighbour v . Let w be the second neighbour of v . We may assume $d(w) \geq 2$, as otherwise G would be just a path of length 2, contradicting clearly that G is a counterexample to the claim. We here consider $H = G - \{u, v\}$. If H is orientable, then it admits an orientation \vec{H} that is as desired. To extend it to one of G , we proceed as follows. Assuming w.l.o.g. that w has outdegree at least 1 in \vec{H} , we orient the edges uv and vw so that \vec{wv} and \vec{uv} are arcs. Note that, in the corresponding graph, this attaches a path of length 2 to the tree containing w^+ ; thus, the corresponding graph still has the desired properties.

If H is not orientable, then it is either an odd-length path or an odd-length cycle. In the former case of an odd-length path, either an end of H is w , in which case we deduce G is an odd-length path itself (a contradiction), or w is an inner vertex of H , in which case G is actually a subdivided claw, and a desired orientation of G can be obtained *e.g.* following the arguments in the proof of Theorem 3.15. Now, in the latter case where H is an odd-length cycle, G is actually an odd-length cycle with a path of length 2 attached, in which case it can be checked that the claim holds.

- Configuration (C2).

Assume G contains a 1-vertex u with a 3-neighbour v . Denote by w_1, w_2 the other two neighbours of v . Note that if $d(w_1) = d(w_2) = 1$, then G is actually a claw, contradicting clearly that G is a counterexample to the claim.

- If, say, $d(w_1) = 1$ and $d(w_2) \geq 2$, then we set $H = G - \{u, w_1\}$. If H is orientable, then it admits an orientation \vec{H} as desired. To extend it to one of G , we proceed as follows. Assuming w.l.o.g. that $\vec{vw_2}$ is an arc of \vec{H} , we just orient vu and vw_1 so that \vec{vu} and $\vec{vw_1}$ are arcs. In the corresponding graph, note that this attaches two pendant edges at v^+ . Thus, the tree containing v^+ has maximum degree at least 3, and thereby it cannot be an odd-length path.

Now, if H is not orientable, then, because v has degree 1 in H , and H is connected, it must be that H is an odd-length path whose one end is v . Thus, G is a subdivided claw, and we can come up with a desired orientation of G using *e.g.* the ideas from the proof of Theorem 3.15.

- If $d(w_1), d(w_2) \geq 2$, then consider $H = G - \{uv, vw_2\}$. If H is orientable, then it admits a desired orientation \vec{H} . We extend it to G as follows. Assume w.l.o.g. that $\vec{w_1v}$ is an arc of \vec{H} . Then we orient vu and vw_2 so that \vec{vu} and $\vec{vw_2}$ are arcs. In the corresponding graph, note that this attaches a path of length 2 at w_2^- ,

thereby either adding a path of length 2 to the forest, or attaching a pendant path of length 2 to one of the trees. In particular, note that, in $B(\vec{H})$, vertex v^+ has degree 0, *i.e.*, belongs to none of the trees.

If H is not orientable, then several situations can occur. Recall that, in H , we have $d(v) = 1$, $d(w_1) \geq 2$, and $d(w_2) \geq 1$. If H has only one connected component, then it must be an odd-length path with end v . If w_2 is the second end of that path, then G is a cycle with an edge attached, a case in which it is not too hard to deduce a desired orientation of G . Otherwise, if w_2 is an inner vertex of that path, then, because G does not contain Configuration (C1), we must have that a neighbour of w_2 is a 1-vertex (the second end of the path). Then G is essentially a cycle in which two adjacent vertices have a pendant vertex attached each. Again, it is not too complicated to obtain a desired orientation of G in such a case. So assume last H has two connected components, one C_1 containing vw_1 , and one C_2 containing w_2 . If C_1 is not orientable, then it is an odd-length path with end v , and, because $d(w_1) \geq 2$, that path has length at least 3, implying that G contains Configuration (C1). So, we can assume C_1 is orientable, and thus C_2 is not orientable. If C_2 is an odd-length path, then it can be noticed that, regardless of whether that path contains w_2 as an end or as an inner vertex, necessarily G must contain Configuration (C1). So, it must be that C_2 is an odd-length cycle. In that case, when removing all edges of C_2 from G , we obtain an orientable graph (since v has degree 3 in that graph), which can be oriented as desired by the induction hypothesis, and, from here, it is not too hard to extend the orientation to a desired one of G .

- Configuration (C3).

Assume G contains a 1-vertex u with a 4^+ -neighbour v . Set $H = G - u$. Note that H is connected with maximum degree at least 3; thus, H is orientable, and admits an orientation \vec{H} with the desired properties. We extend it to one of G as follows. Since v has degree at least 3 in H , we may assume w.l.o.g. that the outdegree of v in \vec{H} is at least 2. We orient uv so that \vec{vu} is an arc. Then, in the corresponding graph, the tree containing v^+ , which already had at least two edges incident to v^+ , gets attached the edge v^+u^- , making that tree being of maximum degree at least 3. Thus, that tree is not an odd-length path, and we have our conclusion.

- Configuration (C4).

Assume G contains a path $uvwxy$ where $d(v) = d(w) = d(x) = 2$. We here set $H = G - w$. If H is orientable, then it admits an orientation \vec{H} such that the corresponding graph is as desired. If both \vec{uv} and \vec{yx} are arcs of \vec{H} , then note that we get a desired orientation of G when orienting the edges incident to w so that \vec{vw} and \vec{xw} are arcs. Indeed, this adds to the corresponding graph a new connected component being a path of length 2. Similar arguments also hold when both \vec{vu} and \vec{xy} are arcs of \vec{H} . So assume now that, w.l.o.g., we have the arcs \vec{uv} and \vec{xy} . Here, we are done when orienting so that \vec{vw} and \vec{xw} are arcs, since, in the corresponding graph, this attaches a pendant path of length 2 to the tree containing x^+ .

Assume now H is not orientable. Since v and x have degree 1 in H , note that no connected component of H can be an odd-length cycle. So, it must be that H contains an odd-length path P . If v and x are the ends of P (then $H = P$), then note that G would be an even-length cycle, in which case a desired orientation can easily

be designed. So, P must have only one of v and x as an end, and we deduce that G is a graph with a pendant path of length 2. Then G contains Configuration (C1), which we dealt with earlier.

- Configuration (C5).

Assume G contains a 2-vertex u with a 2-neighbour v and a 3-neighbour w . We here consider $H = G - u$. If H is orientable, then there is an orientation \vec{H} of H as desired. We extend it to G as follows. Assuming w.l.o.g. that, in \vec{H} , the unique arc incident to v is oriented towards v , we orient uv and uw so that \vec{vu} and \vec{wu} are arcs. In the corresponding graph, note that this attaches a pendant path of length 2 at w^+ , creating a new tree of size 2 or attaching a pendant path of length 2 to some tree. Meanwhile, note that, in \vec{H} , vertex v^+ has degree 0. Thus, we get our conclusion.

If H is not orientable, then several cases can occur. Note that, in H , we have $d(v) = 1$ and $d(w) = 2$. If H is connected, then it must be an odd-length path, having v as an end and w as an inner vertex. Then G is a cycle with a pendant path attached, a case in which it is not too hard to come up with a desired orientation of G . Now, assume H has two connected components C_1 and C_2 , where, say, C_1 contains v and C_2 contains w . If H is not orientable because of C_1 , then, because v has degree 1 in H , it must be that C_1 is an odd-length path, in which case we would deduce G necessarily contains Configuration (C1). So, we can assume C_1 is orientable, and thus that C_2 is not. If C_2 is an odd-length path, thus with w as an inner vertex, then, since C_2 has odd length, necessarily we deduce that G must contain Configuration (C1). So, it remains to consider when C_2 is an odd-length cycle. In this case, we can instead remove the vertices in $V(C_2)$ or $V(C_2) \cup \{u\}$ from G so that what remains is not an odd-length path, apply induction, and, from, here, it is not too hard to see that the resulting orientation can be extended to the whole of G .

- Configuration (C6).

Assume G contains a 3-vertex u with three 2-neighbours v_1, v_2, v_3 . Set $H = G - u$. If H is orientable, then there is an orientation \vec{H} of H as desired, which we extend to G as follows. In \vec{H} , note that we must have either $d^-(v_1) + d^-(v_2) + d^-(v_3) \leq 1$ or $d^+(v_1) + d^+(v_2) + d^+(v_3) \leq 1$. Assume the former situation holds, *i.e.*, at most one of the v_i 's has its unique incident arc incoming. Then we are done when orienting uv_1, uv_2, uv_3 so that $\vec{uv_1}, \vec{uv_2},$ and $\vec{uv_3}$ are arcs. In the corresponding graph, note that this adds a claw with center u^+ , possibly attached, through one leaf, to at most one tree (to one v_i^-). Then we have the desired properties.

If H is not orientable, then we treat G as follows. Note that all v_i 's have degree 1 in H . Thus, all connected components of H have minimum degree 1. If one of these connected components is an odd-length path containing only one of the v_i 's as an end, then note that we would deduce that G contains Configuration (C1). So, if H is not orientable, it must be because of a connected component being an odd-length path containing two of the v_i 's as its ends. Because G does not contain Configuration (C4), note that this path must be of length exactly 1. Thus, we may assume v_1 and v_2 are adjacent. Here, we can orient G as follows. Consider the graph $G - v_1$. If $G - v_1$ is an odd-length path, then G is basically an even-length path with a triangle attached at the end, in which case a desired orientation of G can be obtained following *e.g.* arguments in the proof of Theorem 3.7. Now, assume $G - v_1$ admits a desired orientation. By that orientation, suppose w.l.o.g. $\vec{uv_2}$ is an arc. Then we get a desired orientation of G by orienting v_1v_2 and uv_1 so that $\vec{v_1v_2}$ and $\vec{uv_1}$ are arcs.

Indeed, note that, in the correspond graph, this attaches two pendant edges (one at u^+ and one at v_2^-) to the tree containing u^+ .

- Configuration (C7).

Assume G contains a 4-vertex u with four 2-neighbours v_1, v_2, v_3, v_4 . Let v'_1, v'_2, v'_3, v'_4 denote the other neighbours of v_1, v_2, v_3, v_4 , respectively. Set $H = G - u$. If H is orientable, then there is an orientation \vec{H} of H as desired. We extend it to G in the following manner. In \vec{H} , note that we must have either $d^-(v_1) + d^-(v_2) + d^-(v_3) + d^-(v_4) \leq 2$ or $d^+(v_1) + d^+(v_2) + d^+(v_3) + d^+(v_4) \leq 2$. If we have either $d^-(v_1) + d^-(v_2) + d^-(v_3) + d^-(v_4) \leq 1$ or $d^+(v_1) + d^+(v_2) + d^+(v_3) + d^+(v_4) \leq 1$, then we can proceed similarly as how we treated Configuration (C6), orienting all of uv_1, uv_2, uv_3, uv_4 either towards or away from u . So the last case to consider is when $d^-(v_1) + d^-(v_2) + d^-(v_3) + d^-(v_4) = 2$ and $d^+(v_1) + d^+(v_2) + d^+(v_3) + d^+(v_4) = 2$. Assume w.l.o.g. that, in \vec{H} , we have the arcs $\vec{v_1v'_1}, \vec{v_2v'_2}, \vec{v_3v'_3}$, and $\vec{v_4v'_4}$. We here orient the remaining four edges so that $\vec{uv_1}, \vec{uv_2}, \vec{v_3u}$, and $\vec{v_4u}$ are arcs. As a result, it can be checked that this adds to the corresponding graph two new trees, being paths of length 2. Then we have what is desired.

If H is not orientable, then we can apply the exact same arguments as in Configuration (C6), when we fell into the situation where H is not orientable.

This concludes the proof of the claim. ◇

We now turn to proving that G cannot satisfy $\text{mad}(G) < \frac{5}{2}$, due to Claim 3.18. For that, we assign an initial charge $\omega(v)$ to every vertex v of G , where $\omega(v) = d(v) - \frac{5}{2}$. Then, 1-vertices have initial charge $-\frac{3}{2}$, 2-vertices have charge $-\frac{1}{2}$, 3-vertices have charge $\frac{1}{2}$, and so on. Now, we note that

$$\sum_{v \in V(G)} \omega(v) = \sum_{v \in V(G)} \left(d(v) - \frac{5}{2} \cdot |V(G)| \right)$$

and

$$\sum_{v \in V(G)} d(v) \leq |V(G)| \cdot \text{mad}(G) < \frac{5}{2} \cdot |V(G)|,$$

which implies that the sum of the initial charges is strictly negative. Through the upcoming discharging process, we will move these charges locally, from neighbours to neighbours, according to some rules. The point is to show that, due to Claim 3.18, eventually the sum of charges becomes positive, which is a final contradiction.

The discharging rules we apply are the following:

(R1) Every 3-vertex sends $\frac{1}{4}$ to each of its 2-neighbours.

(R2) Every 4⁺-vertex sends $\frac{1}{2}$ to each of its 2-neighbours.

For every vertex v of G , let us denote by $\omega^*(v)$ the resulting charge of v once the rules above have been applied. To be done, it suffices to show that $\omega^*(v) \geq 0$ holds for every vertex v of G . In the analysis, we consider separately the possible degree values for v .

- $d(v) = 1$.

By Configurations (C1), (C2), and (C3), it cannot be that G contains a 1-vertex. Thus, this case does not have to be treated.

- $d(v) = 2$.

Recall that $\omega(v) = -\frac{1}{2}$. Note that v did not send any charge through Rules (R1) and (R2). Since G does not contain Configurations (C1) and (C4), the two neighbours u_1 and u_2 of v cannot be 1-vertices, and one of these, say u_1 , must be a 3^+ -vertex.

- If u_2 is a 2-vertex, then, because G does not contain Configuration (C5), u_1 must be a 4^+ -vertex. Then u_1 sent $\frac{1}{2}$ to v by Rule (R2), and, hence, we deduce $\omega^*(v) = \omega(v) + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0$.
- If u_2 is a 3^+ -vertex, then both u_1 and u_2 sent at least $\frac{1}{4}$ to v by either Rule (R1) or (R2). Then, $\omega^*(v) \geq \omega(v) + 2 \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{2} = 0$.

- $d(v) = 3$.

Recall that $\omega(v) = \frac{1}{2}$. Note that v cannot receive any charge through Rules (R1) and (R2). By Rule (R1), v sent $\frac{1}{4}$ to each of its 2-neighbours, while v has at most two 2-neighbours since G does not contain Configuration (C6). Then, we here have $\omega^*(v) \geq \omega(v) - 2 \times \frac{1}{4} = \frac{1}{2} - \frac{1}{2} = 0$.

- $d(v) = 4$.

Recall that $\omega(v) = \frac{3}{2}$, and that v did not receive any charge through Rules (R1) and (R2). Through Rule (R2), v sent $\frac{1}{2}$ to each of its 2-neighbours. However, since Configuration (C7) cannot occur in G , the number of 2-neighbours of v is at most 3. Then, $\omega^*(v) \geq \omega(v) - 3 \times \frac{1}{2} = \frac{3}{2} - \frac{3}{2} = 0$.

- $d(v) \geq 5$.

Recall that $\omega(v) = d(v) - \frac{5}{2} = \frac{2d(v)-5}{2}$, and that v did not receive any charge through Rules (R1) and (R2). Note that v sent $\frac{1}{2}$ to each of its 2-neighbours, by Rule (R2). Thus,

$$\omega^*(v) \geq \omega(v) - d(v) \times \frac{1}{2} = \frac{2d(v)-5}{2} - \frac{d(v)}{2} = \frac{d(v)-5}{2} \geq 0.$$

The total sum of charge, since every $\omega^*(v)$ is positive, is thus positive, which is a contradiction. Thus, G cannot exist, and the claim holds. \square

As mentioned earlier, a consequence of Observation 3.16 and Theorem 3.19 we get is:

Theorem 3.19. *If G is an orientable graph with $\text{mad}(G) < \frac{5}{2}$, then $\text{ori}^{(+,-)}(G) \leq 3$.*

3.2.4. Getting a $(+, -)$ -irregular orientation is NP-complete

Before we start proving the result we need some preparation. Be aware that some of the constructions to be introduced are more general than what we actually need, but we feel these more general versions might be useful later on, for later investigations on the $(+, -)$ variant. We refer the reader to Figure 7 for illustrations.

Given a graph G , by *attaching a pentagon* at some vertex v of G , we mean adding a new cycle $vabcdv$ of length 5, where a, b, c, d are new vertices (thus of degree 2). This construction will be used in conjunction with the following properties:

Observation 3.20. *Let H be the graph obtained from a graph G by attaching a pentagon $vabcdv$ at some vertex v . Then, in every $(+, -)$ -irregular orientation D of H , the edges va and vd must be oriented in different directions w.r.t. v . Furthermore, if, say, \vec{va} and \vec{dv} are arcs of D , then either $d^-(a) = 1$ and $d^+(d) = 2$, or $d^-(a) = 2$ and $d^+(d) = 1$. Also, omitting any conflict involving vertices of G , there are $(+, -)$ -irregular orientations of H that realise any of these two configurations of indegrees and outdegrees for a and d .*

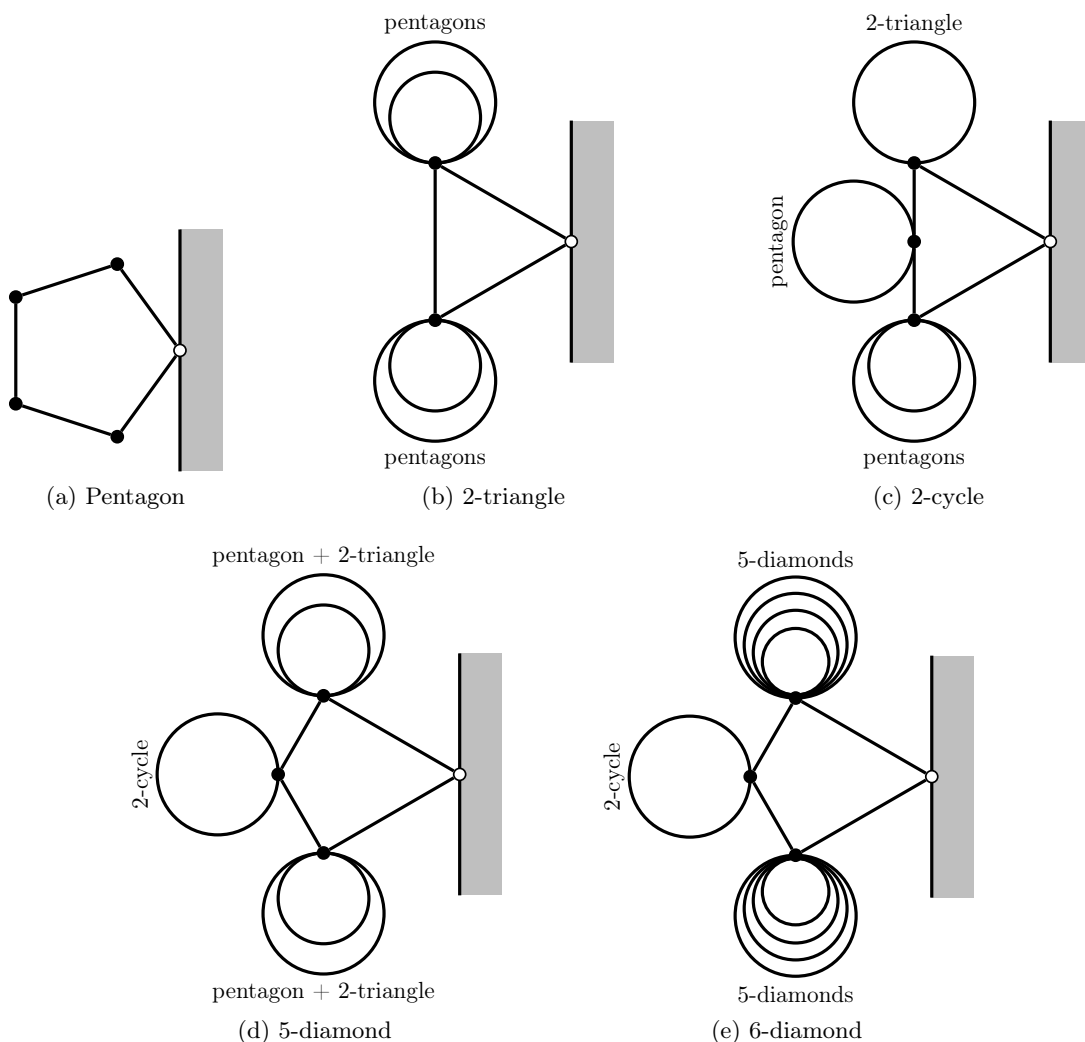


Figure 7: Constructions used in the proof of Theorem 3.25. In each case, the structure is being attached at the white vertex, the original graph being represented by the gray area.

Proof. Focusing on the pentagon, we note first that, in any $(+, -)$ -irregular orientation of H , it must be that the edges va and bc are oriented in opposite directions (that is, \vec{va} and \vec{cb} are arcs, or \vec{av} and \vec{bc} are arcs), and similarly for the edges bc and dv , and the edges ab and cd . Indeed, having e.g. \vec{va} and \vec{bc} as arcs of an orientation D would necessarily imply $d^+(a) = d^-(b) = 1$ while \vec{ab} is an arc (if indeed \vec{ab} is an arc), or $d^+(b) = d^-(a) = 2$ while \vec{ba} is an arc (otherwise), making it impossible for D to be $(+, -)$ -irregular. Obviously, these arguments apply for the other two pairs of edges as well.

From this, we directly get that, in any $(+, -)$ -irregular orientation D of H , either \vec{va} , \vec{cb} , and \vec{dv} are arcs, or \vec{av} , \vec{bc} , and \vec{vd} are arcs. Assuming now that \vec{va} and \vec{dv} are arcs of D , we have that either \vec{ab} and \vec{dc} are arcs in which case $d^-(a) = 1$ and $d^+(d) = 2$, or \vec{ba} and \vec{cd} are arcs in which case $d^-(a) = 2$ and $d^+(d) = 1$. In the former situation, we actually get $d^+(a) = 1$, $d^-(b) = 2$, $d^-(c) = d^+(c) = 1$, and $d^+(d) = 2$ from which we deduce that, locally, the orientation of the pentagon is $(+, -)$ -irregular. In the latter situation, we get $d^-(a) = 2$, $d^-(b) = d^+(b) = 1$, $d^+(c) = 2$, and $d^-(d) = d^+(d) = 1$ from which we get the same conclusion. Thus, the last part of the statement also holds. \square

We now introduce a second construction. Given a graph G and some $k \geq 1$, by *attaching*

a k -triangle uvw at some vertex u of G , we mean adding three edges uv , vw , and wu to the graph (where v and w are new vertices), attaching k pentagons at v , and attaching k pentagons at w . The properties of interest are the following:

Observation 3.21. *Let H be the graph obtained from a graph G by attaching, for some $k \geq 2$, a k -triangle uvw at some vertex u . Then, in every $(+, -)$ -irregular orientation D of H , either \overrightarrow{uv} and \overrightarrow{uw} are arcs, or \overrightarrow{vu} and \overrightarrow{wu} are arcs. Furthermore, assuming \overrightarrow{vw} is an arc, in the former case we have $d^-(v) = k + 1$ and $d^-(w) = k + 2$, while in the latter case we have $d^+(v) = k + 2$ and $d^+(w) = k + 1$. Also, omitting any conflict involving vertices of G , there are $(+, -)$ -irregular orientations of H that realise any way to orient uv and uw .*

Proof. By Observation 3.20, recall that, in any $(+, -)$ -irregular orientation D of H , the k pentagons attached at v bring k to both $d^-(v)$ and $d^+(v)$, and similarly the k pentagons attached at w bring k to both $d^-(w)$ and $d^+(w)$. Assuming, now, that e.g. \overrightarrow{vw} is an arc of D , note that if \overrightarrow{uv} and \overrightarrow{uw} were arcs then we would have $d^+(v) = k + 1 = d^-(w)$ (while \overrightarrow{vw} is an arc), contradicting that D is $(+, -)$ -irregular. Likewise, if \overrightarrow{vu} and \overrightarrow{wu} were arcs then we would have $d^+(v) = k + 2 = d^-(w)$, another contradiction since, again, \overrightarrow{vw} is an arc. Thus, either \overrightarrow{uv} and \overrightarrow{uw} are arcs, or \overrightarrow{vu} and \overrightarrow{wu} are arcs, and the indegrees and outdegrees of v and w must be as stated.

To see now that the last part of the statement holds, consider the following partial orientation D of H . Orient edges so that \overrightarrow{vw} is an arc, and similarly for \overrightarrow{uv} and \overrightarrow{uw} . Next orient the k pentagons attached at v in any correct way described in Observation 3.20. Note that this guarantees v cannot be in conflict with some vertex in a pentagon attached at v , since, because $k \geq 2$, we have $d^-(v) = d^+(v) = k + 1 \geq 3$ (while every vertex in a pentagon has indegree and outdegree at most 2). Now regarding w , orient its attached pentagons so that, for any resulting arc \overrightarrow{wa} where a belongs to some pentagon, we have $d^-(a) \neq k$ (which is possible by Observation 3.20). This guarantees w cannot be in conflict, since, because $k \geq 2$, we get $d^-(w) = k + 2 \geq 4$ and $d^+(w) = k \geq 2$. Thus, locally the resulting orientation is $(+, -)$ -irregular. Similar arguments apply the same way to get a similar conclusion when \overrightarrow{vw} , \overrightarrow{vu} , and \overrightarrow{wu} are arcs, concluding the proof. \square

We introduce another construction. For any $k \geq 1$, given a graph G with a vertex u , by attaching a k -cycle $uv_1 \dots v_{k+1}u$ at u we mean adding a new cycle $uv_1 \dots v_{k+1}u$ to G where the v_i 's are new vertices, attaching a 2-triangle at v_1 , and finally, for every $i \in \{2, \dots, k+1\}$, attaching $i - 1$ pentagons at v_i . Thus, v_2 gets attached one pentagon, v_3 gets attached two pentagons, and so on, up to v_{k+1} which gets attached k pentagons. This construction has the following properties.

Observation 3.22. *Let H be the graph obtained from a graph G by attaching, for some $k \geq 1$, a k -cycle $uv_1 \dots v_{k+1}u$ at some vertex u . Then, in every $(+, -)$ -irregular orientation D of H :*

- *if k is even, then either $\overrightarrow{uv_1}$ and $\overrightarrow{uv_{k+1}}$ are arcs, or $\overrightarrow{v_1u}$ and $\overrightarrow{v_{k+1}u}$ are arcs. Furthermore, in the former case we have $d^-(v_1) = 2$ and $d^-(v_{k+1}) = k + 2$, while in the latter case we have $d^+(v_1) = 2$ and $d^+(v_{k+1}) = k + 2$;*
- *if k is odd, then the edges uv_1 and uv_{k+1} must be oriented in different directions w.r.t. u . Furthermore, if $\overrightarrow{uv_1}$ and $\overrightarrow{v_{k+1}u}$ are arcs of D , then $d^-(v_1) = 2$ and $d^+(v_{k+1}) = k + 2$, while, if $\overrightarrow{v_1u}$ and $\overrightarrow{uv_{k+1}}$ are arcs, then $d^+(v_1) = 2$ and $d^-(v_{k+1}) = k + 2$.*

Also, omitting any conflict involving vertices of G , there are $(+, -)$ -irregular orientations of H that realise any of these ways to orient uv_1 and uv_{k+1} and any of these configurations of indegrees and outdegrees for v_1 and v_{k+1} .

Proof. By Observation 3.21, in any $(+, -)$ -irregular orientation D of H , the 2-triangle attached at v_1 must be oriented so that the two arcs incident to v_1 bring 2 to either $d^-(v_1)$ or $d^+(v_1)$. Also, if these two arcs are oriented away from v_1 then v_1 has out-neighbours with indegree 3 and 4, while it is adjacent to in-neighbours with outdegree 3 and 4 otherwise. Besides, by Observation 3.20, for every $i \in \{2, \dots, k+1\}$ the arcs of the $i-1$ pentagons attached at v_i must be oriented so that the $2i-2$ arcs incident to v_i bring $i-1$ to both $d^-(v_i)$ and $d^+(v_i)$.

We can assume $\overrightarrow{uv_1}$ is an arc of D (as all arguments below can be symmetrised). If the two arcs incident to v_1 of the attached 2-triangle are directed towards v_1 , then note that, regardless of the orientation of v_1v_2 , we must have $d^-(v_1) \in \{3, 4\}$, contradicting that D is $(+, -)$ -irregular (since v_1 has in-neighbours with indegree 3 and 4). Thus, it must be that the two arcs incident to v_1 from the 2-triangle are directed away from v_1 . Note that this forces $\overrightarrow{v_2v_1}$ to be an arc of D , as otherwise we would have $d^+(v_1) = 3$, while v_1 , in the 2-triangle, has an out-neighbour with indegree 3. Then, $d^-(v_1) = d^+(v_1) = 2$. Now, recall that one pentagon was attached at v_2 , and that its edges must be oriented so that they bring 1 to both $d^-(v_2)$ and $d^+(v_2)$. Then, so that $d^+(v_2) \neq d^-(v_1)$, it must be that the last arc incident to v_2 (going to v_3 or u) is oriented away from v_2 , so that $d^+(v_2) = 3$. Note also that $d^-(v_2) = 1$. Next, if v_3 exists, then, due to the two attached pentagons, we have $d^-(v_3) \in \{3, 4\}$, and, so that we do not have a conflict between v_2 and v_3 (recall $\overrightarrow{v_2v_3}$ is an arc), it must be that $d^-(v_3) = 4$, which requires that the last edge incident to v_3 (going to v_4 or u) is oriented towards v_3 . Also, $d^+(v_3) = 2$. These arguments generalise along the v_i 's. In particular, looking only how the edges of the cycle $uv_1 \dots v_{k+1}u$ are oriented, the v_i 's with $i > 1$ must alternate between sinks and sources. For every such v_i that is a sink, we get $d^-(v_i) = i+1$ and $d^+(v_i) = i-1$, while, for every such v_i that is a source, we get $d^-(v_i) = i-1$ and $d^+(v_i) = i+1$. Hence, if $k+1$ is odd, then $\overrightarrow{uv_{k+1}}$ must be an arc of D , while, otherwise, $\overrightarrow{v_{k+1}u}$ must be an arc, as claimed.

Regarding the last part of the statement, recall that, by the orientation D described above, we get $d^-(v_1) = d^+(v_1) = 2$. By Observation 3.21, recall that the 2-triangle attached at v_1 can indeed be oriented with the two arcs incident to v_1 going away from v_1 , and so that the two neighbours of v_1 inside this 2-triangle have indegree 3 and 4 (and thus are not in conflict with v_1). Likewise, recall that v_2 and every later even v_i has outdegree at least 3 and thus cannot be in conflict with any neighbour in an attached pentagon (since it has indegree and outdegree at most 2), while every such pentagon does admit a $(+, -)$ -irregular orientation by Observation 3.20. The same goes for v_3 and every later odd v_i , since its indegree is at least 4. Actually, every v_i with $i \geq 4$ gets both indegree and outdegree at least 3 (due to the attached pentagons), so no conflict can occur within a pentagon for such vertices. The last possible issues to consider, then, are the fact that v_2 has indegree 1 and v_3 has outdegree 2. By Observation 3.20, recall that we can actually orient the pentagon attached at v_2 so that if $\overrightarrow{wv_2}$ is an arc of D with w being part of the pentagon, then $d^+(w) \neq 1$. Likewise, we can orient the two edges attached at v_3 so that, if $\overrightarrow{v_3w}$ is an arc with w in some pentagon, then $d^-(w) \neq 2$. From all these arguments, it then follows that $(+, -)$ -irregular orientations of H with the claimed properties do exist. \square

We now introduce our last construction, starting with an initial piece. Given a graph G with a vertex v , by *attaching a 5-diamond $vabcv$ at v* we mean adding a new cycle $vabcv$ of length 4 to G , attaching both a pentagon and a 2-triangle at a , attaching a 2-cycle at b , and attaching both a pentagon and a 2-triangle at c . Here are our properties of interest:

Observation 3.23. *Let H be the graph obtained from a graph G by attaching a 5-diamond $vabcv$ at some vertex v . Then, in every $(+, -)$ -irregular orientation D of H , the edges va*

and vc must be oriented in different directions w.r.t. v . Furthermore, if, say, \vec{va} and \vec{cv} are arcs of D , then $d^-(a) = 5$ and $d^+(c) = 5$. Also, omitting any conflict involving vertices of G , there are $(+, -)$ -irregular orientations of H that realise this configuration of indegree and outdegree for a and c .

Proof. In every $(+, -)$ -irregular orientation D of H , recall that the pentagons attached at a and c bring 1 to $d^-(a)$, $d^+(a)$, $d^-(c)$, and $d^+(c)$ (by Observation 3.20), the 2-triangles attached at a and c bring 2 to either $d^-(a)$ or $d^+(a)$, and 2 to either $d^-(c)$ or $d^+(c)$ (by Observation 3.21), and that the 2-cycle attached at b brings 2 to either $d^-(b)$ or $d^+(b)$ (by Observation 3.22). Furthermore, if the 2-triangle attached at a is oriented so that the two arcs incident to a are oriented away from a , then a has out-neighbours with indegree 3 and 4, while, otherwise, it has in-neighbours with outdegree 3 and 4. The same goes for c . Last, b is adjacent, in the attached 2-cycle, to either an out-neighbour with indegree 2 and an out-neighbour with indegree 4 (if both incident arcs are oriented away from b) or to an in-neighbour with outdegree 2 and an in-neighbour with outdegree 4 (otherwise).

Assume that, in D , we have \vec{va} . Note that the two edges incident to a being part of the attached 2-triangle cannot be oriented away from a , as otherwise we would have $d^+(a) \in \{3, 4\}$ regardless of how the edge ab is oriented, which would be a contradiction since a would have out-neighbours with indegree 3 and 4 in the 2-triangle. So these two edges must be oriented towards a , and it must be that \vec{ba} is an arc, as otherwise we would have $d^-(a) = 4$ while a has, in the 2-triangle, an in-neighbour with outdegree 4. Now, if \vec{vc} was an arc of D , then, by the same arguments, we would deduce that \vec{bc} is also an arc. As a result, either $d^+(b) = 2$ (if the two incident edges in the attached 2-cycle are oriented away from b) while b has, in the 2-cycle, an out-neighbour with indegree 2, or $d^-(b) = 4$ (otherwise) while b has an in-neighbour with outdegree 4. In both cases, we thus get a contradiction to the fact that D is a $(+, -)$ -irregular orientation of H .

Thus, if \vec{va} is an arc of D , then it must be that \vec{cv} is an arc, which, altogether, implies \vec{ba} and \vec{cb} are arcs, and thus $d^-(a) = 5$ and $d^+(c) = 5$. To see now that the last part of the statement holds, note that this implies also $d^+(a) = 1$, while, by Observation 3.20, the pentagon attached at a can be oriented so that for the arc \vec{aw} where w belongs to the pentagon, we have $d^-(w) \neq 1$. A similar thing holds for c . Now, regarding b , by Observation 3.22 the attached 2-cycle can be oriented so that the two edges incident to b are directed away from b , which yields $d^-(b) = 1$ and $d^+(b) = 3$, while the two neighbours of b in the 2-cycle get indegree 2 and 4. Meanwhile, recall that $d^+(c) = d^-(a) = 5$. Thus, there cannot be any conflict, and the claimed orientations of H do exist. \square

5-diamonds can be generalised in the following way. For any successive $k \geq 6$, by attaching a k -diamond $vabcv$ at some vertex v in a graph G , we mean adding a cycle $vabcv$ to G , attaching $k - 2$ $(k - 1)$ -diamonds at a , attaching a 2-cycle at b , and attaching $k - 2$ $(k - 1)$ -diamonds at c . Then we have the following:

Observation 3.24. *For some $k \geq 6$, let H be the graph obtained from a graph G by attaching a k -diamond $vabcv$ at some vertex v . Then, in every $(+, -)$ -irregular orientation D of H , the edges va and vc must be oriented in different directions w.r.t. v . Furthermore, if, say, \vec{va} and \vec{cv} are arcs of D , then $d^-(a) = k$ and $d^+(c) = k$. Also, omitting any conflict involving vertices of G , there are $(+, -)$ -irregular orientations of H that realise this configuration of indegree and outdegree for a and c .*

Proof. We show the claim for $k = 6$, since all arguments then generalise to consecutive larger values of k . By Observation 3.23, in every $(+, -)$ -irregular orientation D of H , the four 5-diamonds attached at a bring 4 to $d^-(a)$ and $d^+(a)$, and a has in-neighbours with

outdegree 5 and out-neighbours with indegree 5. The same holds for c . Regarding b , we have the same properties as those described in the proof of Observation 3.23.

Assume now that we have *e.g.* \vec{va} in D . Then we must have $d^-(a) \in \{5, 6\}$, and, so that a is not involved in a conflict, we must actually have \vec{ba} in D so that $d^-(a) = 6$. Conversely, if we had \vec{av} in D , then we would also have \vec{ab} so that $d^+(a) = 6$. The same arguments apply to c as well: either \vec{vc} and \vec{bc} are arcs, or \vec{cv} and \vec{cb} are arcs. Now, for similar reasons as in the proof of Observation 3.23, it cannot be that either \vec{ab} and \vec{cb} are arcs, or \vec{ba} and \vec{bc} are arcs, as otherwise we would get a conflict between b and one of its neighbours in the 2-cycle attached at b . Thus, either \vec{va} and \vec{cv} are arcs of D , or \vec{av} and \vec{vc} are arcs.

By arguments above and arguments used in the proof of Observation 3.23, we can also deduce that, for each arc configuration, there is indeed a $(+, -)$ -irregular orientation of H that realises it, as claimed in the last part of the statement. \square

We are now ready to prove our next result:

Theorem 3.25. *Deciding if a graph has a $(+, -)$ -irregular orientation is NP-complete.*

Proof. Since the problem is clearly in NP, we focus on proving it is NP-hard, which we do by reduction from the MONOTONE NOT-ALL-EQUAL 3-SAT problem, which is known to be NP-hard [20]. An instance of MONOTONE NOT-ALL-EQUAL 3-SAT is a 3CNF formula F over clauses C_1, \dots, C_m and variables x_1, \dots, x_n such that the clauses contain exactly three (positive) variables each, and the problem is to decide whether F can be satisfied in a *not-all-equal way*, that is, whether we can set the variables to *true* or *false* so that each clause contains at least one true variable and at least one false variable. Free to consider the formula $F \wedge F \wedge F$, which is clearly not-all-equal satisfiable if and only if F is, note that we can assume each variable appears in at least three clauses. Also, we can assume every clause contains exactly three distinct variables (since the 2-COLOURABILITY OF 3-UNIFORM HYPERGRAPH problem is NP-complete [15]). From such an instance F of MONOTONE NOT-ALL-EQUAL 3-SAT, we construct, in polynomial time, a graph G such that F is not-all-equal satisfiable if and only if G admits a $(+, -)$ -irregular orientation.

The construction of G goes as follows. We start from the graph modelling the structure of F . That is, for every clause C_i of F we have a *clause vertex* c_i in G , for every variable x_i of F we have a *variable vertex* x_i in G , and, whenever a variable x_i appears in some clause C_j of F , we have the *formula edge* $x_i c_j$ in G . We then modify G further as follows:

- to every clause vertex c_i , we attach both a 5-diamond and a pentagon;
- to every variable vertex x_i , assuming variable x_i appears in $k \geq 3$ clauses of F , we attach $3k$ diamonds at x_i , including at least one $(3k + 1)$ -diamond, at least one $(3k + 2)$ -diamond, and so on, up to at least one $(4k - 1)$ -diamond. The other diamonds can be *e.g.* $(4k - 1)$ -diamonds.

Clearly, the construction of G is achieved in polynomial time.

Let us now describe how a $(+, -)$ -irregular orientation D of G should behave, given all the structures we have added to G .

- Regarding clause vertices c_i , recall that the pentagon attached at c_i brings 1 to both $d^-(c_i)$ and $d^+(c_i)$ (Observation 3.20), and similarly for the 5-diamond attached at c_i (Observation 3.23). Also, through an outgoing arc, c_i is adjacent to a vertex with indegree 5, and, through an incoming arc, to a vertex with outdegree 5. This implies that all three formula edges incident to c_i cannot all be oriented towards or away

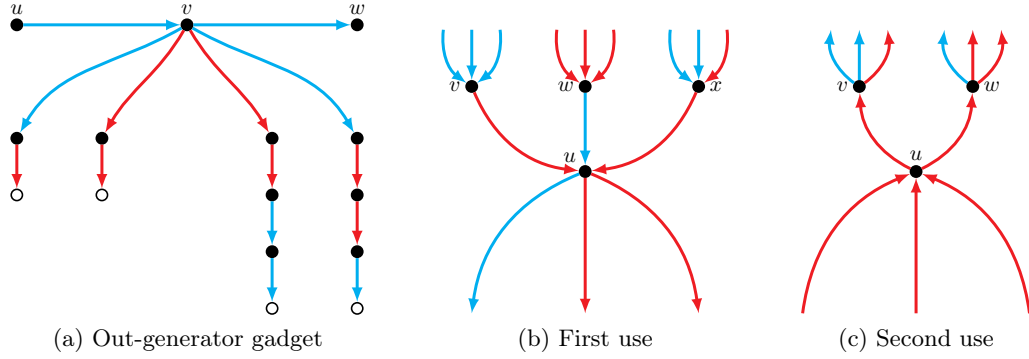


Figure 8: The generator gadget (a), given together with a $(-, +)$ -irregular 2-decomposition (omitting the output ends, which are the white vertices). (b) and (c) showcase the two situations described in the statements of Observations 4.2 and 4.3, as well as valid $(-, +)$ -irregular 2-decompositions. In each picture, the two parts of the decomposition are depicted as red and blue arcs.

from c_i , as otherwise we would have $d^-(c_i) = 5$ or $d^+(c_i) = 5$, a contradiction to D being $(+, -)$ -irregular. Meanwhile, if exactly two formula edges are oriented away from c_i then $d^-(c_i) = 3$ and $d^+(c_i) = 4$, while, if only one formula edge is oriented away from c_i , then $d^-(c_i) = 4$ and $d^+(c_i) = 3$.

- Regarding variable vertices x_i incident to $k \geq 3$ formula edges, recall that the $3k$ diamonds attached at x_i bring $3k$ to both $d^-(x_i)$ and $d^+(x_i)$ (Observations 3.23 and 3.24). Furthermore, due to the diamonds we have attached, x_i is adjacent, through outgoing arcs, to vertices with all indegree values in $\{3k+1, \dots, 4k-1\}$, and, through incoming arcs, to vertices with all outdegree values in $\{3k+1, \dots, 4k-1\}$. Thus, the k formula edges incident to x_i must all be oriented away from x_i (so that $d^-(x_i) = 3k$ and $d^+(x_i) = 4k$) or towards x_i (so that $d^-(x_i) = 4k$ and $d^+(x_i) = 3k$). Note furthermore that, since $k \geq 3$, we have $3k \geq 9$, so, for any clause vertex c_j , we must have $d^-(c_j), d^+(c_j) < d^-(x_i), d^+(x_i)$; thus, variable vertices cannot conflict with clause vertices.

To see now that the claimed equivalence between F and G holds, suppose that, in D , having an arc $\overrightarrow{c_j x_i}$ models that x_i brings, to C_j , truth value *true* by a truth assignment to the variables of F , and *false* otherwise, if $\overrightarrow{x_i c_j}$ is an arc. The property above for clause vertices then depicts the fact that, by a truth assignment to the variables of F , a clause is considered satisfied in a not-all-equal way if and only if it contains at least one true variable and at least one false variable. The property above for variable vertices then depicts the fact that, by a truth assignment, any variable brings the same truth value to all clauses that contain it. From this, we can deduce a not-all-equal truth assignment to the variables of F from a $(+, -)$ -irregular orientation of G , and *vice versa*. In particular, one should keep in mind that, by Observations 3.20, 3.23, and 3.24, when orienting a pentagon or a diamond, there is always a bit of freedom to guarantee there is no conflict involving the vertex to which it is attached. This concludes the proof. \square

4. Result on the $(-, +)$ variant

Our goal in this section is to establish that determining $\chi_{\text{irr}}^{(-,+)}(D)$ for a given $(-, +)$ -decomposable digraph D is NP-complete in general.

We start by introducing some gadget and showing how we will use it later on. The *generator gadget* D is the digraph obtained as follows (see Figure 8 (a)). We start from

D containing two arcs \overrightarrow{uv} and \overrightarrow{vw} . We can next grow this structure by attaching some *outputs* at v , which can be of two kinds:

- by adding a *short output* to the generator, we mean adding the arcs \overrightarrow{va} and \overrightarrow{ab} to D , where a and b are new vertices;
- by adding a *long output* to the generator, we mean adding the arcs \overrightarrow{va} , \overrightarrow{ab} , \overrightarrow{bc} , and \overrightarrow{cd} to D , where a , b , c , and d are new vertices.

In each of these two cases, we call the resulting pendant vertex (b in the former case, d in the latter one) the *end* of the output, and its unique incident (incoming) arc its *end-arc*. This gadget, as is, already has some properties of interest:

Observation 4.1. *Let D be a generator gadget with any number of short and long outputs. Then, in every $(-, +)$ -irregular 2-decomposition A_1, A_2 of D , the end-arcs of all short outputs belong to A_1 while those of all long outputs belong to A_2 , or vice versa. Furthermore, assuming some output end-arc incident to some end x belongs to A_i , we have $d_i^-(x) = 0$ or $d_i^-(x) = 1$. Also, omitting any conflict involving output ends, there are $(-, +)$ -irregular 2-decompositions of D that realise any of these indegrees for every output end.*

Proof. As in the definition of the generator gadget above, let us denote by \overrightarrow{uv} and \overrightarrow{vw} the two arcs of D such that the outputs are attached at v . Let A_1, A_2 be a $(-, +)$ -irregular 2-decomposition of D . Assuming \overrightarrow{vw} belongs *e.g.* to A_1 , note that we necessarily have $d_1^+(w) = 0$. Since \overrightarrow{uv} is the only arc incoming to v , it must be that \overrightarrow{uv} belongs to A_1 too, so that $d_1^-(v) \neq 0$ and we do not have a conflict between v and w .

Note that these arguments remain invariant under the assumption that there are paths attached to v oriented away from v . So assume now there is an arbitrarily long directed path (v, v_1, v_2, \dots) attached at v . Note that if $\overrightarrow{vv_1} \in A_1$, then we must have $\overrightarrow{v_1v_2} \in A_2$, as otherwise we would have $d_1^-(v) = d_1^+(v_1) = 1$, a contradiction. Then, we must have $\overrightarrow{v_2v_3} \in A_2$, as otherwise we would have $d_2^-(v_1) = d_2^+(v_2) = 0$. Next, we must have $\overrightarrow{v_3v_4} \in A_1$ since otherwise we would get $d_2^-(v_1) = d_2^+(v_2) = 1$. From here, the arguments generalise along the directed path: that is, pair of consecutive arcs must alternate between belonging to A_1 and A_2 . From this, we deduce that, if we have $\overrightarrow{vv_1} \in A_2$, then we must also have $\overrightarrow{v_1v_2} \in A_2$, then $\overrightarrow{v_2v_3}, \overrightarrow{v_3v_4} \in A_1$, and so on. In particular, assuming a vertex incident to some arc is at distance 1 from it, every arc of the directed path at distance 2 modulo 4 from v must lie in A_2 , while every arc at distance 0 modulo 4 must lie in A_1 .

Among the consequences of these remarks, we deduce (still assuming $\overrightarrow{uv}, \overrightarrow{vw} \in A_1$) that all short outputs of D must have their end-arcs in A_2 , while all long outputs must have their end-arcs in A_1 . Also, every end x incident to an end-arc in some A_i can satisfy either $d_i^-(x) = 0$ or $d_i^-(x) = 1$ depending on whether the arc of the corresponding output incident to v belongs to A_1 or A_2 . The arguments above show that the last part of the statement thus holds, regardless of the number of outputs that D has. \square

Note that by reversing all arcs of a generator gadget, all properties of Observation 4.1 translate (since $(-, +)$ -irregularity is preserved upon reversing the direction of all arcs), the main difference being that we get a digraph in which all ends are incident to a unique outgoing arc. In the sequel, we will call the original gadget an *out-generator gadget*, while its reversed version we will call an *in-generator gadget*.

The next two observations provide situations (depicted in Figures 8 (b) and (c)) and ways to use out-generator gadgets and in-generator gadgets, which will be more apparent in the proof of our main result in this section, upcoming Theorem 4.4.

Observation 4.2. Let D be a digraph given with a $(-, +)$ -irregular 2-decomposition A_1, A_2 , u be a vertex of D with $d^+(u) = 3$, and v, w, x be three other vertices such that:

- $\vec{vu} \in A(D)$, $d_1^-(v) = 3$, and $d_2^-(v) = 0$;
- $\vec{wu} \in A(D)$, $d_1^-(w) = 0$, and $d_2^-(w) = 3$; and
- $\vec{xu} \in A(D)$, $d_1^-(x) = 2$, and $d_2^-(x) = 1$.

Then, we must have $d_1^+(u) = 1$ and $d_2^+(u) = 2$.

Proof. We note first that we cannot have $d_1^+(u) = 3$. Indeed:

- if $\vec{vu} \in A_1$, then, since $d_1^-(v) = 3$, all three arcs outgoing from u cannot belong to A_1 ;
- if $\vec{vu} \in A_2$, then, since $d_2^-(v) = 0$, it must be that at least one arc outgoing from u belongs to A_2 ; thus, the three arcs outgoing from u cannot belong to A_1 .

Thus, we must have $d_1^+(u) \in \{1, 2\}$. To be done, it suffices to show that we cannot have $d_1^+(u) = 2$. Assume otherwise that two arcs outgoing from u belong to A_1 , while the remaining one belongs to A_2 .

- If $\vec{xu} \in A_1$, then we get a contradiction, since $d_1^-(x) = 2 = d_1^+(u)$.
- If $\vec{xu} \in A_2$, then we get another contradiction, since $d_2^-(x) = 1 = d_2^+(u)$.

This concludes the proof. □

Observation 4.3. Let D be a digraph given with a $(-, +)$ -irregular 2-decomposition A_1, A_2 , u be a vertex of D with $d^-(u) = 3$, and v, w be two other vertices such that:

- $\vec{uv} \in A(D)$, $d_1^+(v) = 2$, and $d_2^+(v) = 1$; and
- $\vec{uw} \in A(D)$, $d_1^+(w) = 1$, and $d_2^+(w) = 2$.

Then, we must have either $d_1^-(u) = 3$ and $d_2^-(u) = 0$, or $d_1^-(u) = 0$ and $d_2^-(u) = 3$.

Proof. If we have $d_1^-(u) = 2$ and $d_2^-(u) = 1$, then we can have neither $\vec{uv} \in A_1$ (since $d_1^-(u) = 2 = d_1^+(v)$) nor $\vec{uv} \in A_2$ (since $d_2^-(u) = 1 = d_2^+(v)$). Similar conclusions can be reached regarding \vec{uw} if we have $d_1^-(u) = 1$ and $d_2^-(u) = 2$. The claim thus follows. □

So that the previous observations can be invoked, we will use the following operation. Given a generator gadget and any number k of its distinct outputs, by *joining* these outputs we mean identifying their k ends to a single vertex.

We are now ready to prove our main result in this section.

Theorem 4.4. Deciding whether $\chi_{\text{irr}}^{(-,+)}(D) \leq 2$ holds for a given digraph D is NP-complete.

Proof. Since the problem is clearly in NP, we focus on proving its NP-hardness. This is done from the CUBIC MONOTONE 1-IN-3 3SAT problem, which is known to be NP-hard [18]. An instance of this problem is a 3CNF formula F defined over clauses C_1, \dots, C_m and variables x_1, \dots, x_n such that every clause contains three (positive) variables, every variable appears in exactly three distinct clauses, and the question is whether F can be satisfied in a 1-in-3 way, that is, whether there is a truth assignment for the variables so that every clause contains exactly one true variable. Note that we can also assume that F does not contain a clause of the form $(x_i \vee x_i \vee x_i)$ or $(x_i \vee x_i \vee x_j)$ (where $i \neq j$), as

otherwise either F is a negative instance or F can be simplified. In other words, we can assume the clauses contain three distinct positive variables each. From F , we construct, in polynomial time, a digraph D such that F can be satisfied in a 1-in-3 way if and only if D admits $(-, +)$ -irregular 2-decompositions.

The construction of D goes as follows. We start from the bipartite graph modelling the structure of F , similarly as in the proof of Theorem 4.4. This time, note that this graph is cubic. So we again have clause vertices c_1, \dots, c_m , variable vertices x_1, \dots, x_n , which are joined through formula edges. We orient these edges to *formula arcs*, going from the clause vertices to the variable vertices (that is, if cx would be a formula edge, then \overrightarrow{cx} is a formula arc of D). Next we add an out-generator gadget G^+ (which, for now, can be assumed to have arbitrarily many short and long outputs; the exact numbers will be clarified later on), which we connect to the rest as follows:

- for every clause C_i of F , we pick three new long outputs of G^+ and join them to a vertex v_{c_i} , and add the arc $\overrightarrow{v_{c_i}c_i}$;
- for every clause C_i of F , we pick three new short outputs of G^+ and join them to a vertex w_{c_i} , and add the arc $\overrightarrow{w_{c_i}c_i}$;
- for every clause C_i of F , we pick two long outputs and one new short output of G^+ and join them to a vertex x_{c_i} , and add the arc $\overrightarrow{x_{c_i}c_i}$.

We also add an in-generator gadget G^- to D (for which, again, the number of short and long outputs will be discussed later on), connected to the rest as follows:

- for every variable x_i of F , we pick two new long outputs and one new short output of G^- and join them to a vertex v_{x_i} , and add the arc $\overrightarrow{x_i v_{x_i}}$;
- for every variable x_i of F , we pick one new long output and two new short outputs of G^- and join them to a vertex w_{x_i} , and add the arc $\overrightarrow{x_i w_{x_i}}$.

To finish off the construction of D , we consider every variable vertex x_i and attach four pendant arcs $\overrightarrow{x_i p_i}$, $\overrightarrow{x_i p'_i}$, $\overrightarrow{x_i q_i}$, and $\overrightarrow{x_i q'_i}$, where p_i, p'_i, q_i , and q'_i are four new vertices.

Note that the number of short and long outputs we need G^+ and G^- to have is a linear function of the number of clauses and variables of F . Then the whole construction of D is achieved in polynomial time from F , as desired.

Let us now describe how a $(-, +)$ -irregular 2-decomposition A_1, A_2 of D should behave. By Observation 4.1, we can assume that *e.g.* the end-arcs of all long outputs of G^+ belong to A_1 while the end-arcs of all short outputs of G^+ belong to A_2 . Then, for every clause vertex c_i , we must have $d_1^-(v_{c_i}) = 3$ and $d_2^-(v_{c_i}) = 0$, $d_1^-(w_{c_i}) = 0$ and $d_2^-(w_{c_i}) = 3$, and $d_1^-(x_{c_i}) = 2$ and $d_2^-(x_{c_i}) = 1$. Note that we are then in the context described in Observation 4.2, and we deduce that exactly one of the three (outgoing) formula arcs incident to c_i must belong to A_1 while the other two must belong to A_2 . This models the fact that, in a truth assignment to the variables of F , a clause is satisfied if and only if it contains exactly one true variable.

Likewise, the end-arcs of all short outputs of G^- must belong to some A_α , while the end-arcs of all long outputs must belong to A_β , where $\alpha, \beta \in \{1, 2\}$ and $\beta \neq \alpha$. Regardless, we then deduce that, for every variable vertex x_i , one of the two vertices v_{x_i} and w_{x_i} has two outgoing arcs in A_1 and the last one in A_2 , while the other has one outgoing arc in A_1 and the other two in A_2 . Then we get the configuration described in Observation 4.3, from which we deduce that the three (incoming) formula arcs incident to x_i must all belong to

	Exceptions	Max. value	Complexity	Min. orientation
(-, -) variant (+, +) variant	none [8]	$3 \leq \max \leq 5$ [4, 8]	$\leq 1?$ in P $\leq 2?$ NPC [8]	min = 1 [5]
(+, -) variant	sink-source path Thm. 2.2	$3 \leq \max \leq 7$ Cor. 2.4, 2.6	$\leq 1?$ in P $\leq 2?$ unknown [6], Obs. 2.1	$2 \leq \min \leq 5$ Thm. 3.12, 3.25
(-, +) variant	unknown	$3 \leq \max \leq ?$ Thm. 4.4	$\leq 1?$ in P $\leq 2?$ NPC Thm. 4.4	$? \leq \min \leq ?$

Table 1: Summary of some of the knowledge we currently have on the (α, β) variants. Cells in green contain problems and questions that are now mostly understood, cells in orange contain open problems and questions towards which some significant steps have already been made, and cells in red contain problems and questions that, to date, have not been considered as such yet.

A_1 or all belong to A_2 . This models the fact that, in a truth assignment to the variables of F , any variable brings the same truth value to all clauses that contain it.

Note also that if the three (incoming) formula arcs incident to x_i belong, say, to A_1 , then, to avoid any conflict, all of $\overrightarrow{x_i p_i}, \overrightarrow{x_i p'_i}, \overrightarrow{x_i q_i}, \overrightarrow{x_i q'_i}$ must belong to A_1 (as otherwise all of $d_2^-(x_i), d_2^+(p_i), d_2^+(p'_i), d_2^+(q_i),$ and $d_2^+(q'_i)$ would be 0, with $\overrightarrow{x_i p_i}, \overrightarrow{x_i p'_i}, \overrightarrow{x_i q_i}, \overrightarrow{x_i q'_i} \in A_2$). This implies we have $d_1^+(x_i) \in \{4, 6\}$, while, for every clause vertex c_j (recall $\overrightarrow{c_j x_i} \in A_1$), we have $d_1^-(c_j) \leq 3$ since $d^-(c_j) = 3$. Thus, clause vertices and variable vertices cannot conflict. Likewise, as described in the proof of Observation 4.1, recall that there are always two ways to decompose every output of G^+ and G^- , w.r.t. whether the end-arc belongs to A_1 or A_2 . In particular, this implies we can always make sure the v_{c_i} 's, w_{c_i} 's, x_{c_i} 's, v_{x_i} 's, and w_{x_i} 's are not involved in any conflict with any vertex from a joined output.

From these arguments, we thus get that we can deduce a 1-in-3 truth assignment to the variables of F from a $(-, +)$ -irregular 2-decomposition of D , and *vice versa*. \square

5. Conclusion

In this work, we have pursued the investigations on the (α, β) variants initiated in [4, 8]. More precisely, we mainly focused on the $(+, -)$ variant, and gave some look at the $(-, +)$ variant on the way. We first introduced the notion of corresponding graph in Section 2, which allowed to derive some results on these variants. In particular, we were able to characterise all $(+, -)$ -exceptions (Theorem 2.2), and to derive some constant upper bounds on $\chi_{\text{irr}}^{(+, -)}(D)$ and $\chi_{\text{irr}}^{(-, +)}(D)$ for some decomposable digraphs D , sometimes in particular contexts (recall, in particular, Corollary 2.4). We then focused more deeply on the $(+, -)$ variant in Section 3, raising Conjecture 3.1 as our main guideline. We noticed that the conjecture holds for certain classes of digraphs, recall Theorems 3.2 and 3.3. Realising that it would not be easy to go beyond Corollary 2.4 in general, we then focused on the different problem where a graph is given, and the task is to find an orientation that complies with Conjecture 3.1. In that setting, we first characterised orientable graphs (Theorem 3.8), before proving a better upper bound (Theorem 3.12), and then showing that Conjecture 3.1, sometimes in an even stronger form, holds for more classes of graphs (Corollary 3.14, Theorems 3.15 and 3.19). We also proved in Theorem 3.25 that not all graphs can be oriented to a $(+, -)$ -irregular digraph, since deciding this is an NP-complete problem. Last, in Section 4, we gave further attention to the $(-, +)$ variant, showing in Theorem 4.4 that determining $\chi_{\text{irr}}^{(-, +)}(D)$ for a digraph D is NP-complete, the main point being that the same result is not known to hold yet for the $(+, -)$ variant, which complies with our feeling that the $(-, +)$ variant is harder than the $(+, -)$ variant to comprehend.

We have gathered in Table 1 some of the current knowledge we have on all the (α, β) variants. Among the aspects of interest we chose to summarise, are, for some (α, β) variant, the characterisation of (α, β) -exceptions, the maximum number of parts required in an (α, β) -irregular decomposition, the complexity of determining whether a given number of parts in such a decomposition suffices, and the problem of finding a best orientation in terms of parts of an (α, β) -irregular decomposition.

- Regarding exceptions, only $(-, +)$ -exceptions remain to be characterised. Note that some of our results already provide a partial answer, and show such exist. In particular, as mentioned in Section 2, when given a digraph D , the corresponding graph $B(D)$ is equivalent, for the $(-, +)$ variant, in terms of decomposition but not in terms of constraints. Still, we can sometimes establish that D is not $(-, +)$ -decomposable when $B(D)$ has some properties. For instance, if D has an arc \overrightarrow{uv} such that, in $B(D)$, we have $d(u^-) = d(v^+) = 0$, then D is not $(-, +)$ -decomposable.
- Regarding the maximum value that $\chi_{\text{irr}}^{(\alpha, \beta)}(D)$ can take for some (α, β) -decomposable digraph D , we already know it is at least 3 for all (α, β) . The $(-, +)$ variant is the only variant, to date, for which a constant upper bound is still missing. It might be that, with some efforts, the ideas behind the proof of Corollary 2.4 could be adapted. However, since the equivalence between a digraph D and the corresponding graph $B(D)$ is less obvious in the context of the $(-, +)$ variant, this is not quite clear.
- Regarding complexity aspects, note that deciding whether $\chi_{\text{irr}}^{(\alpha, \beta)}(D) \leq 2$ holds for some digraph D is known to be NP-complete for all variants but the $(+, -)$ one. As mentioned in Section 2, this problem actually relates to the problem of determining whether $\chi_{\text{irr}}(G) \leq 2$ holds for a given bipartite graph G , a problem (mentioned *e.g.* in [6]) whose complexity is still unknown to date.
- Finally, regarding the problem of finding a best orientation for some (α, β) variant, note that, in the current work, we mainly investigated it for the $(+, -)$ variant, thereby confirming Conjecture 3.1 for more classes of graphs in this context. Actually, given the results we provided, we believe that the following conjecture could hold:

Conjecture 5.1. *If G is an orientable graph, then $\text{ori}^{(+, -)}(G) \leq 2$.*

Note that Conjecture 5.1, if true, would be best possible due to Theorem 3.25. Actually, even for trees T we sometimes have $\text{ori}^{(+, -)}(T) > 1$, recall the tree from Figure 6. Finally, regarding the other variants, let us recall, for the $(-, -)$ and $(+, +)$ variants, that every graph admits an orientation where every two adjacent vertices have different outdegrees (or similarly different indegrees; see *e.g.* [5]); meanwhile, the question still has to be investigated for the $(-, +)$ variant.

Many other questions and aspects of interest could be subject to further work. For instance, it could be interesting to generalise Theorem 3.19 to graphs with larger maximum average degree. Regarding Theorem 3.13, it could be interesting to consider general tournaments, and, regarding Corollary 3.14, to investigate $(+, -)$ -irregular tournaments. Regarding the tree T in Figure 6, we were able to observe that combinations of T lead to other trees that have not $(+, -)$ -irregular orientations, and one could wonder whether there is a characterisation or whether Theorem 3.25 holds when restricted to trees. More generally speaking, we still do not know much about the $(-, +)$ variant, and it could deserve a dedicated study, in particular regarding constant upper bounds. Finally, regarding the $(-, -)$ and $(+, +)$ variants, it would be interesting to prove whether every digraph admits a corresponding decomposition with at most four parts.

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