Wetting on a wall and wetting in a well: Overview of equilibrium properties

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WETTING ON A WALL AND WETTING IN A WELL: OVERVIEW OF EQUILIBRIUM PROPERTIES

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Abstract. We study the wetting model, which considers a random walk constrained to remain above a hard wall, but with additional pinning potential for each contact with the wall. This model is known to exhibit a *wetting* phase transition, from a localized phase (with trajectories pinned to the wall) to a delocalized phase (with unpinned trajectories). As a preamble, we take the opportunity to present an overview of the model, collecting and complementing well-known and other folklore results. Then, we investigate a version with elevated boundary conditions, which has been studied in various contexts both in the physics and the mathematics literature; it can alternatively be seen as a wetting model in a square well. We complement here existing results, focusing on the equilibrium properties of the model, for a general underlying random walk (in the domain of attraction of a stable law). First, we compute the free energy and give some properties of the phase diagram; interestingly, we find that, in addition to the wetting transition, a so-called *saturation* phase transition may occur. Then, in the so-called Cramér's region, we find an exact asymptotic equivalent of the partition function, together with a (local) central limit theorem for the fluctuations of the left-most and right-most pinned points, jointly with the number of contacts at the bottom of the well.

Keywords: wetting, pinning, polymers, random walk, large deviations, central limit theorem.

MSC2020 AMS classification: Primary: 82B41, 60K35 ; Secondary: 82D60, 60F05.

Figure 1. Wetting of a square well by a droplet. On the left, the droplet does not reach the bottom of the well (dry phase); on the right the droplet reaches the bottom of the well (wet phase).

Contents

1. Introduction

The goal of the present article is to study the random walk wetting model, which has been introduced following the work of [Abr80, Abr81] as an effective model for interfaces in the 2D Ising model, and quickly drew a lot of interest from the physics community, see for instance the seminal paper [Fis84]. The model is based on a random walk trajectory which is constrained to remain above a hard wall, with a reward (or penalty) for every point of contact with the wall. It can be used as an effective model for a one-dimensional interface interacting with a substrate, or a model for the adsorption of polymers on a substrate. Over the last decades, the wetting model has been widely studied, both in the physics and in the mathematical literature, either in its homogeneous or disordered version. We refer to [Gia07, Gia10] for a general overview of the model and its relation to the pinning model.

Here, we study a version of the (homogeneous) wetting model with elevated boundary conditions, in analogy with [dCD87, Pat98] (for the Solid-On-Solid model), [AdCD89, PV99] (for interfaces in the 2D Ising model) or [BFO09] (for d-dimensional Gaussian random walks pinned to a subspace). This can also be seen as a model for the wetting of a (square) well or cavity, see Figure 1. One motivation of the latter interpretation is the description of a wetting transition for droplets on a grooved surface, see [dCDH11]. In [dCDH11, LT15], the authors study this model from a dynamical point of view: in particular, they show that for some region of parameters in the phase diagram, the model exhibits a metastable transition (the dry phase might be stable whereas the equilibrium measure is in the wet phase, or vice versa). In the present paper, we investigate further the properties of the equilibrium measure.

We consider a general setting where the underlying random walk is in the domain of attraction of some α-stable law, with $\alpha \in (0, 2]$. To our knowledge, the literature mostly focuses on specific cases, such as simple or Gaussian random walks in [LT15], resp. [BFO09], integer or real valued Solid-On-Solid (SOS) models in [Pat98], resp. [dCD87]. This also seems to generalize the setting usually considered for the standard wetting model, see e.g. [Gia07, §1.3] and references therein (authors consider random walks in the domain of attraction of the normal law, *i.e.* $\alpha = 2$). The wetting model is known to undergo a phase transition between a delocalized (or unpinned) phase where trajectories wander away from the wall, and a localized (or pinned) phase where trajectories stick the wall. The remarkable fact noticed by Fisher [Fis84] (for the simple random walk) is that the critical point and the critical behavior of this model can be understood precisely. For convenience (and because our setting is a bit more general than in the literature), we provide in Section 2 a complete introduction of the model.

Overview of the results. Our first main result consists in providing an expression for the free energy of the wetting model with elevated boundary condition: it is based on an optimization between the free energy of the wetting model and the large deviation rate functions of the underlying random walk (in analogy with [LT15, Prop. 2.1]). As a byproduct of the proof, we also obtain the location of the left-most and right-most point of contact of the interface with the bottom of the well. We also provide some properties of the phase diagram. We show in particular that if the rate functions are non-trivial (which occurs only in the case where the random walk admits a finite variance, *i.e.* $\alpha = 2$), then the wetting phase transition is always of first order, in contrast with what happens in the standard wetting model where the phase transition is of second order. Interestingly, depending on the underlying random walk, another (or two other) phase transition may occur: it corresponds to a *saturation* transition, when the bottom corners of the well become wet.

Our second set of results deals with the so-called Cramér regime of the model, when the optimization problem that defines the free energy attains its maximum at points where the rate functions are strictly convex. In that case, we are able to obtain an exact asymptotic for the partition function, together with sharp trajectory estimates. In particular, we are able to prove a (local) central limit theorem for the left and right-most points of contact, jointly with the number of contacts of the walk with the bottom of the well.

Overview of the article. Let us now briefly present the organization of the paper.

In Section 2, we introduce the wetting model in a general setting; we consider either discrete or continuous random walks, in the domain of attraction of a stable law. Since this setting is not usually the one presented in the literature (as far as we know), we give a complete overview of the model: we explain its relation to the pinning model, give an implicit expression for the free energy and provide the critical behavior of the model. This section mostly collects well-known facts on the model and could serve as a rather complete overview of the homogeneous model; to complete the overview, we also include some (folklore) *integrable* models where the free energy admits an explicit expression, in Appendix A. Readers familiar with the subject may skip this section entirely.

In Section 3, we turn our attention to the wetting model with elevated boundary conditions or *wetting in a (square) well*. We present our main results on the free energy (see Theorem 3.1), together with some properties of the phase diagram, in particular regarding the critical curve. We also make the maximizer(s) of the variational problem that defines the free energy explicit (see Lemma 3.5, there are cases where the maximizer is not unique!). We then give some consequences on the behavior of path trajectories. Finally, we state our sharper results in Cramér's region (see Theorem 3.10) and we also give some properties of the phase transition.

In Section 4, we discuss several natural questions one is led to consider. The rest of the paper is devoted to the proofs of the different results.

2. A review of the (standard) wetting model

2.1. **Wetting of a random walk on a hard wall.** Let $(X_i)_{i\geq 1}$ be i.i.d. real valued random variables and let $(S_n)_{n\geq 0}$ be the associated random walk, that is $S_n = \sum_{i=1}^n X_i$ for $n \geq 0$ (with $S_0 = 0$ by convention); we denote by $\mathbf{P}$ its law. For $N \in \mathbb{N}$, the directed random walk $(n, S_n)_{0\leq n\leq N}$ is used to describe an effective interface between two solvents or alternatively the trajectory of a polymer of length N (with (n, S_n) representing the position of the n-th monomer), see Figure 2.

We focus on two specific cases: either the random walk is $\mathbb{Z}$-valued and aperiodic (discrete case) or the X_i's are have a density with respect to the Lebesgue measure (continuous case). We denote by $f(\cdot)$ the density of the law of X_i with respect to μ, where μ is the counting measure on $\mathbb{Z}$ in the discrete case and μ is the Lebesgue measure in the continuous case: $\mathbf{P}(X_1 \in \mathrm{d}x) = f(x)\mu(\mathrm{d}x)$.

For $\lambda > 0$, we introduce the following family of Gibbs measures $\mathbf{P}_{N,\lambda}$ on the space of trajectories of $(S_0, \dots, S_N)$, known as the (δ-pinning) wetting model:

$$\mathrm{d}\mathbf{P}_{N,\lambda}(s_1, \dots, s_n) = \frac{1}{Z_{N,\lambda}} \prod_{i=1}^{N-1} \left(\mathbb{1}_{\{s_i>0\}}\mu(\mathrm{d}s_i) + \lambda\delta_0(\mathrm{d}s_i)\right) \prod_{i=1}^{N} f(s_i - s_{i-1})\, \lambda\delta_0(\mathrm{d}s_N)\,, \tag{2.1}$$

with δ_0 denoting the Dirac mass at 0 and $s_0 = 0$ by convention. The quantity $Z_{N,\lambda}$ i the partition function of the model, that normalizes $\mathbf{P}_{N,\lambda}$ to a probability. Let us stress that the measure $\mathbf{P}_{N,\lambda}$ puts weights only on trajectories that stay above the wall, and gives a weight λ to each contact with the wall. We refer to Figure 2 for an illustration.

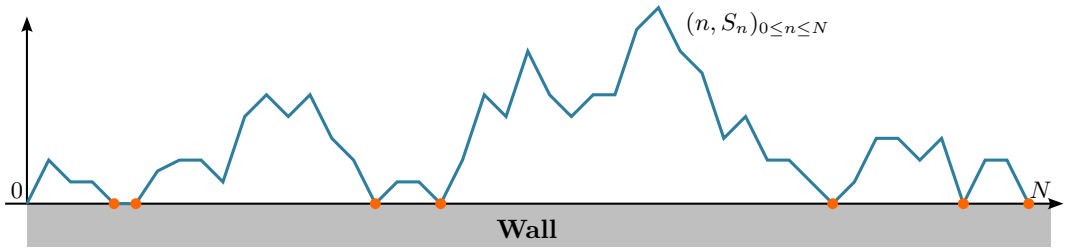


FIGURE 2. Illustration of the wetting of a random walk on a hard wall — the random walk trajectory $(n, S_n)_{0\le n\le N}$ may represent a directed polymer or some effective interface. The trajectory is subject to a hard wall constraint (*i.e.* one cannot have $S_i < 0$) and the measure $\mathbf{P}_{N,\lambda}$ additionally gives weight λ to each contact with the hard wall (represented by dots in the figure).

We will see in Section 2.2 below that the wetting model (2.1) reduces to a renewal pinning model, somehow forgetting the underlying random walk path and considering only the instants of return to 0. However, it is important in our setting to keep the random walk interpretation, in particular if we want to consider a model with elevated boundary conditions.

Let us observe that, because of the term $\delta_0(\mathrm{d}s_n)$ in (2.1), the endpoint of the walk is pinned to the wall: this is the *constrained* version of the model. It is also possible to consider the *free* version of the model, removing the last constraint $\delta_0(\mathrm{d}s_n)$ in (2.1); results would then be very similar and we refer to [Gia07, Ch. 1&2] for more discussion on the free case (that we do not discuss further).

Remark 2.1. *In the discrete setting, which is the one considered originally in* [Fis84] *(and in most of the literature), the Gibbs measure* $\mathbf{P}_{N,\lambda}$ *can be written as*

$$\mathrm{d}\mathbf{P}_{N,\lambda}(S) = \frac{1}{Z_{N,\lambda}}\lambda^{H_N(S)}\mathbb{1}_{\Omega_N^+(S)}\mathbb{1}_{\{S_N=0\}}\mathrm{d}\mathbf{P}(s_1,\dots,s_N)\,, \qquad \text{with } H_N(S) := \sum_{n=1}^{N}\mathbb{1}_{\{S_n=0\}}\,, \tag{2.2}$$

where we have set $\Omega_N^+(S) := \{S_1 \ge 0, \dots, S_N \ge 0\}$*; this corresponds to the Gibbs measure considered in* [LT15]*. The assumption that* $(S_n)_{n\ge 0}$ *is aperiodic is needed here so that the constraint* $\{S_N = 0\}$ *can be verified, at least for* N *large; if the random walk were periodic of period* d*, then one would simply need to restrict to lengths* $N \in d\mathbb{N}$*.*

Our setting is similar to that of [DGZ05, CGZ06], where the density $f(\cdot)$ of the X_i's is put in the form $f(x) = e^{-V(x)}$ for some potential V; the authors assume there that the increments X_i are centered and in the domain of attraction of the normal law. Our main assumption is the following (it is similar for instance to [CC13, Hyp. 2.1]).

Assumption 1. *The random walk* $(S_n)_{n\ge 0}$ *is in the domain of attraction of a strictly stable law with index* $\alpha \in (0, 2]$ *and positivity parameter* $\varrho \in (0, 1)$*. More precisely, there exists a sequence* $(a_n)_{n\ge 1}$ *which is regularly varying with index* $1/\alpha$ *such that* $a_n^{-1}S_n$ *converges weakly to some (strictly)* α*-stable random variable* $\mathcal{Z}$ *with positivity parameter* $\varrho \in (0, 1)$*.*

In the discrete case, we assume that $(S_n)_{n\ge 0}$ *is aperiodic; in the continuous case, we assume that the density* f_n *of* S_n *verifies* $f_n(0) < +\infty$ *for all* $n \ge 1$ *and is essentially bounded for some* n*.*

We denote by f_α the density of the limiting α-stable random variable $\mathcal{Z}$. We stress that $f_\alpha(0) > 0$ thanks to the fact that $\varrho \in (0, 1)$.

2.2. **Renewal representation of the model.** As noticed in [CGZ06], the wetting model (2.1) can be rewritten using a renewal process. Let us define, for $n \geq 1$,

$$f_n^+(0) := \int_{s_1>0,\dots,s_{n-1}>0} \prod_{i=1}^{n} f(s_i - s_{i-1}) \prod_{i=1}^{n-1} \mu(\mathrm{d}s_i)\,, \qquad \text{with} \quad s_0 := s_n := 0\,. \tag{2.3}$$

This corresponds to the density $f_n^+(x) = \frac{1}{\mu(\mathrm{d}x)}\mathbf{P}(S_1 > 0, \dots, S_{n-1} > 0, S_n \in \mathrm{d}x)$ at $x = 0$. In particular, in the discrete case, we have $f_n^+(0) = \mathbf{P}(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0)$.

Let us denote $\kappa := \sum_{n=1}^{\infty} f_n^+(0)$ and observe that $\kappa \in (0, +\infty)$ holds under Assumption 1, thanks to Lemma 2.3 below (and formula (B.1) to see that $f_n^+(0) < +\infty$ for all n in the continuous case). Then we define the following probability density on $\mathbb{N}$:

$$K(n) := \kappa^{-1} f_n^+(0) \qquad \text{for all} \quad n \geq 1\,. \tag{2.4}$$

Now, if we write explicitly the partition function

$$Z_{N,\lambda} = \int_{(\mathbb{R}_+)^N} \prod_{i=1}^{N-1} \left(\mathbb{1}_{\{s_i>0\}}\mu(\mathrm{d}s_i) + \lambda\delta_0(\mathrm{d}s_i)\right) \prod_{i=1}^{N} f(s_i - s_{i-1})\,\lambda\delta_0(\mathrm{d}s_N)\,,$$

then decomposing it with respect to the number and positions of contact with the wall, *i.e.* decomposing the integral over the possible sets $I = \{i \in \{1, \dots, N\}, s_i = 0\}$, we obtain

$$Z_{N,\lambda} = \sum_{k=1}^{N} \sum_{0=:i_0<i_1<\cdots<i_k=N} \lambda^k \prod_{j=1}^{k} f^+_{i_j - i_{j-1}}(0) = \sum_{k=1}^{N} \sum_{0=:i_0<i_1<\cdots<i_k=N} (\kappa\lambda)^k \prod_{j=1}^{k} K(i_j - i_{j-1})\,.$$

Introducing a (recurrent) renewal process $\tau = (\tau_i)_{i\geq 0}$ with inter-arrival distribution $K(\cdot)$, *i.e.* letting $\tau_0 := 0$ and $(\tau_i - \tau_{i-1})_{i\geq 1}$ be i.i.d. $\mathbb{N}$-valued random variables with law $\mathbf{P}(\tau_1 = n) = K(n)$, the above partition function can be rewritten as

$$Z_{N,\lambda} = \mathbf{E}\Big[(\kappa\lambda)^{H_N(\tau)}\mathbb{1}_{\{N\in\tau\}}\Big]\,, \qquad \text{with } H_N(\tau) := \sum_{n=1}^{N} \mathbb{1}_{\{n\in\tau\}}\,.$$

(With some abuse of notation, we also interpret $\tau = \{\tau_i\}_{i\geq 0}$ as a subset of $\mathbb{N}\cup\{0\}$.) In this context we can rewrite the Gibbs measure (2.1) as

$$\frac{\mathrm{d}\mathbf{P}_{N,\lambda}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z_{N,\lambda}}(\kappa\lambda)^{H_N(\tau)}\mathbb{1}_{\{N\in\tau\}}\,, \qquad \text{with } H_N(\tau) = \sum_{n=1}^{N} \mathbb{1}_{\{n\in\tau\}}\,. \tag{2.5}$$

This is the usual formulation of the homogeneous pinning model, see [Gia07, Ch. 2] (write $e^\beta = \kappa\lambda$).

Notice here that the wall constraint has been absorbed in the definition of τ (see the definition (2.3) of $f_n^+(0)$) and we have reduced to a recurrent renewal τ at the cost of a change of parameter $\lambda \rightsquigarrow \kappa\lambda$. This way, one can interpret τ as the return times of the random walk $(S_n)_{0\leq n\leq N}$ to 0 conditioned on staying non-negative (due to the definition of $f_n^+(0)$) and having finite excursions away from 0 (due to the normalization by $\frac{1}{\kappa}$ in (2.4)). Let us also stress that the Gibbs law (2.5) describes only the distribution of the renewal process τ, but this is enough to describe completely the measure (2.1) on random walk trajectories: indeed, conditionally on the return times to 0, the law of the excursions of $(S_n)_{n\geq 0}$ between τ_{k-1} and τ_k is unchanged by (2.1).

Analogously to what is observed in [CGZ06, App. A], under Assumption 1 we are able to obtain information on the law of the renewal process τ. In particular, we have the following results. Lemma 2.2 is a direct consequence of [AD99]. Lemma 2.3 is given in [CC13, Prop. 4.1-(4.5)] for

the discrete case and [CC13, Thm. 5.1-(5.2)] for the continuous case. For the sake of completeness, we provide (simple) proofs of these results in Appendix B.1.

Lemma 2.2. *Let $\bar{T}_1 := \min\{n \geq 1, S_n \leq 0\}$ and $\bar{H}_1 = -S_{\bar{T}_1}$ be the first (weak) ladder epoch and ladder height of the random walk. Then, we have*

$$\kappa := \sum_{n=1}^{\infty} f_n^+(0) = \begin{cases} \mathbf{P}(\bar{H}_1 = 0) & \text{in the discrete case,} \\ f_{\bar{H}_1}(0) & \text{in the continuous case,} \end{cases}$$

where $f_{\bar{H}_1}$ is the density of $\bar{H}_1$ w.r.t. the Lebesgue measure in the continuous case.

Lemma 2.3. *Suppose that Assumption 1 holds and let $f_n^+(0)$ be defined as in (2.3). Then,*

$$f_n^+(0) \sim \frac{c_0}{n a_n} \qquad \text{as} \quad n \to \infty\,,$$

with $c_0 := f_\alpha(0)\mathbf{P}(\bar{H}_1 > 0)$, where we recall that f_α is the density of the limiting α-stable random variable $\mathcal{Z}$; observe also that $\mathbf{P}(\bar{H}_1 > 0) = 1$ in the continuous case.
Note that if $\sigma^2 := \mathbf{E}[X_1^2] < +\infty$ ($\mathbf{E}[X_1] = 0$), then $f_n^+(0) \sim c_1 n^{-3/2}$ with $c_1 := \frac{1}{\sigma\sqrt{2\pi}}\mathbf{P}(\bar{H}_1 > 0)$.

Let us now highlight one consequence of Lemma 2.3 on the inter-arrival probability distribution of τ. Since the normalizing sequence a_n in Assumption 1 is regularly varying with index $1/\alpha$, we find that there exists a function $L(\cdot)$ slowly varying at infinity[1] such that

$$K(n) = \mathbf{P}(\tau_1 = n) = L(n) n^{-(1+\frac{1}{\alpha})}\,. \tag{2.6}$$

To summarize, we have rewritten the Gibbs measure (2.1) in terms of the standard homogeneous pinning model, see (2.5), with underlying (recurrent) renewal τ whose inter-arrival distribution $K(\cdot)$. The relation (2.6) is often the underlying assumption when studying the pinning model (with a wider range for the parameter $\frac{1}{\alpha}$, which is here restricted to $[\frac{1}{2}, +\infty)$). We refer to [Gia07, Ch. 2] for a complete overview of the (homogeneous) pinning model, but we collect (and complement) below some of the results.

2.3. **Free energy and phase transition.** An important physical quantity of the wetting (or pinning) model is the *free energy*, defined by

$$\mathrm{F}(\lambda) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\lambda}\,.$$

The fact that the limit exist follows easily once one realizes that the sequence $(\log Z_{N,\lambda})_{N\geq 1}$ is super-additive. It is standard to see that the free energy $\mathrm{F}(\lambda)$ verifies:

(i) $\mathrm{F}(\lambda) \geq 0$, since we have $Z_{N,\lambda} \geq \kappa\lambda \mathbf{P}(\tau_1 = N)$ and (2.6), and $\mathrm{F}(\lambda) = 0$ for all $\kappa\lambda \leq 1$;
(ii) $\lambda \mapsto \mathrm{F}(\lambda)$ is non-decreasing, since $\lambda \mapsto \log Z_{N,\lambda}$ is non-decreasing for any $N \geq 1$.
(iii) $\beta \mapsto \mathrm{F}(e^\beta)$ is convex, since $\beta \mapsto \log Z_{N,e^\beta}$ is convex for any $N \geq 1$.

Hence, we can define a critical point

$$\lambda_c = \sup\{\lambda, \mathrm{F}(\lambda) = 0\} = \inf\{\lambda, \mathrm{F}(\lambda) > 0\}\,.$$

Let us stress that from convexity arguments, we obtain that whenever $\mathrm{F}'(\lambda)$ exists (from Theorem 2.4 below, this is the case for all $\lambda > 0$ except possibly at $\lambda = \lambda_c$), we have

$$\mathrm{F}'(\lambda) = \lim_{N\to\infty} \frac{\partial}{\partial\lambda}\frac{1}{N}\log Z_{N,\lambda} = \lambda^{-1} \lim_{N\to\infty} \mathbf{E}_{N,\lambda}\Big[\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{n\in\tau\}}\Big]\,.$$

[1] A function $L(\cdot)$ is said to be slowly varying at infinity if for any $a > 0$, $\lim_{x\to\infty} L(ax)/L(x) = 1$.

Hence, $\mathrm{F}'(\lambda)$ is (λ^{-1} times) the limiting density of contacts under the Gibbs measure (2.5). This shows that λ_c marks the transition between a *delocalized* phase for $\lambda < \lambda_c$ (with $\mathrm{F}'(\lambda) = 0$, zero density of contact) and a *localized* phase (with $\mathrm{F}'(\lambda) > 0$, positive density of contacts)

We now collect a number of results on the free energy: they show that the wetting model (or the pinning model when considering the definition (2.5)) is solvable. We state the results with our notation, but we refer to [Gia07, Ch. 2] for the general context of the pinning model. Define the Laplace transform of τ_1 (recall that $\sum_{n=1}^{\infty} K(n) = 1$): for any $\vartheta \geq 0$,

$$\mathcal{K}(\vartheta) := \sum_{n=1}^{\infty} K(n) e^{-\vartheta n} = \mathbf{E}\big[e^{-\vartheta \tau_1}\big]\,. \tag{2.7}$$

Let us note that $\mathcal{K} : (0, +\infty) \to (0, 1)$ is decreasing and analytic.

Theorem 2.4 ([Gia07], Thm. 2.1)**.** *The free energy is characterized by the following relation:*

$$\mathrm{F}(\lambda) \text{ is the solution of } \mathcal{K}(\vartheta) = (\kappa\lambda)^{-1} \text{ if a solution exists, and } \mathrm{F}(\lambda) = 0 \text{ otherwise.} \tag{2.8}$$

In particular, the critical point is given by $\lambda_c = 1/\kappa$, and the implicit function theorem shows that $\lambda \mapsto \mathrm{F}(\lambda)$ is analytic on $(\lambda_c, +\infty)$. Additionally, we have

$$\mathrm{F}(\lambda_c + u) \sim \widetilde{L}(u) u^{\min(1,\alpha)} \qquad \text{as } u \downarrow 0\,, \tag{2.9}$$

where $\widetilde{L}$ is some slowly varying function (which depends explicitly on L, α in (2.6)).

In the context of the wetting model, where $K(n) := \frac{1}{\kappa} f_n^+(0)$ with some explicit expressions for κ and $f_n^+(0)$ from Lemma 2.2-2.3, one is able to describe the critical behavior of F more explicitly than in (2.9). The proof is identical to that of [Gia07, Thm. 2.1], making some inversion formulas explicit; we refer to Appendix B.2 for details.

Proposition 2.5. *Suppose Assumption 1 holds and choose the normalization $(a_n)_{n\geq 1}$ as follows:*
— *if $\alpha \in (0, 2)$, let $(a_n)_{n\geq 1}$ be such that $\mathbf{P}(|X_1| > a_n) \sim \frac{1}{n}$ as $n \to \infty$;*
— *if $\alpha = 2$, let $(a_n)_{n\geq 1}$ be such that $\sigma^2(a_n) a_n^{-2} \sim \frac{1}{n}$, where $\sigma^2(x) := \mathbf{E}[(X_1)^2 \mathbb{1}_{\{|X_1| \leq x\}}]$.*
Then, we have the following asymptotic behaviors, as $u \downarrow 0$:

- *If $\sum_{n=1}^{\infty} \frac{1}{a_n} < +\infty$, i.e. if the random walk $(S_n)_{n\geq 0}$ is transient (in particular if $\alpha < 1$), then*

$$\mathrm{F}(\lambda_c + u) \sim c_1 u \qquad \text{with } c_1 := \kappa^2 \Big/ \sum_{n=1}^{\infty} n f_n^+(0)\,. \tag{2.10}$$

- *If $\alpha = 1$ and $\sum_{n=1}^{\infty} \frac{1}{a_n} = +\infty$, letting $c_0 := f_\alpha(0)\mathbf{P}(\bar{H}_1 > 0)$ as in Lemma 2.3, we have*

$$\mathrm{F}(\lambda_c + u) \sim \frac{\kappa^2}{c_0} \frac{u}{\mu(1/u)} \qquad \text{with } \mu(x) := \int_0^x \frac{\mathrm{d}s}{s^2 \mathbf{P}(|X_1| > s)}\,. \tag{2.11}$$

- *If $\alpha \in (1, 2)$, then*

$$\mathrm{F}(\lambda_c + u) \sim \mathbf{P}\big(|X_1| > c_2/u\big) \qquad \text{with } c_2 := \alpha c_0 \Gamma\big(\tfrac{\alpha-1}{\alpha}\big)/\kappa^2\,. \tag{2.12}$$

- *If $\alpha = 2$, then*

$$\mathrm{F}(\lambda_c + u) \sim c_3 u^2 \sigma^2(1/u) \qquad \text{with } c_3 := \tfrac{1}{2}\kappa^4/\mathbf{P}(\bar{H}_1 > 0)^2\,. \tag{2.13}$$

2.4. **Sharp asymptotic of the partition function and some path properties.** Let us stress that the sharp asymptotic of the partition function are known, in the delocalized ($\lambda < \lambda_c$), critical ($\lambda = \lambda_c$) and localized ($\lambda > \lambda_c$) regimes, as can be found in Theorem 2.2 of [Gia07]. All these asymptotic behaviors are derived from the following representation:

$$Z_{N,\lambda} = e^{N\mathrm{F}(\lambda)} \widetilde{\mathbf{P}}_\lambda\big(N \in \widetilde{\tau}\big)\,, \tag{2.14}$$

where $\widetilde{\mathbf{P}}_\lambda$ is the law of a renewal $\widetilde{\tau}$ with inter-arrival distribution given by

$$\widetilde{\mathbf{P}}_\lambda(\widetilde{\tau}_1 = n) = \kappa\lambda K(n) e^{-\mathrm{F}(\lambda)n} \qquad \text{for } n \geq 1\,. \tag{2.15}$$

Indeed, writing $Z_{N,\lambda}(A) := \mathbf{E}\big[(\kappa\lambda)^{H_N(\tau)} \mathbb{1}_{\{N\in\tau\}} \mathbb{1}_A\big]$ for any event A, we have for $0 =: i_0 < i_1 < \cdots < i_k = N$,

$$\begin{aligned} Z_{N,\lambda}\big(\tau \cap [0,N] = \{i_1,\dots,i_k\}\big) &= (\kappa\lambda)^k \prod_{j=1}^k K(i_j - i_{j-1}) = e^{N\mathrm{F}(\lambda)} \prod_{j=1}^k \widetilde{\mathbf{P}}_\lambda(\widetilde{\tau}_1 = i_j - i_{j-1}) \\ &= e^{\mathrm{F}(\lambda)N} \widetilde{\mathbf{P}}_\lambda\big(\tau \cap [0,N] = \{i_1,\dots,i_k\}\big)\,. \end{aligned}$$

Summing over $k \geq 1$ and $i_1,\dots,i_{k-1}$, we obtain (2.14). Let us stress that in general, since we have $\mathbf{P}_{N,\lambda}(A) = \frac{1}{Z_{N,\lambda}} Z_{N,\lambda}(A)$, we obtain the following representation for the pinning (or wetting) measure $\mathbf{P}_{N,\lambda}$ (see also [Gia07, Rem. 2.8]):

$$\mathbf{P}_{N,\lambda}\big(\tau \cap [0,N] \in \cdot\big) = \widetilde{\mathbf{P}}_\lambda\big(\widetilde{\tau} \cap [0,N] \in \cdot \,\big|\, N \in \widetilde{\tau}\big)\,. \tag{2.16}$$

Note that, in view of (2.8), $\widetilde{\tau}$ defined by (2.15) is recurrent if $\lambda \geq \lambda_c$ and transient if $\lambda < \lambda_c$. In the super-critical case $\lambda > \lambda_c$, we have that $\widetilde{\mathbf{P}}_\lambda(\widetilde{\tau}_1 = j)$ decays exponentially fast (with exponential decay rate $\mathrm{F}(\lambda) > 0$), so $\widetilde{\tau}_1$ is positive recurrent. Therefore, using the renewal theorem, we get from (2.14) that

$$Z_{N,\lambda} \sim \frac{1}{m_\lambda} e^{\mathrm{F}(\lambda)N} \quad \text{as } N \to \infty\,, \qquad \text{with } m_\lambda := \widetilde{\mathbf{E}}_\lambda[\widetilde{\tau}_1] = \kappa\lambda \sum_{n=1}^\infty n K(n) e^{-n\mathrm{F}(\lambda)}\,. \tag{2.17}$$

We refer to [Gia07, Thm. 2.2] for the corresponding results in the critical ($\lambda = \lambda_c$) and sub-critical ($\lambda < \lambda_c$) cases: we have

$$\begin{aligned} &\text{if } \lambda < \lambda_c \qquad Z_{N,\lambda} \sim \frac{\kappa\lambda}{(1-\kappa\lambda)^2} K(N)\,, \\ &\text{if } \lambda = \lambda_c \qquad Z_{N,\lambda} \sim \begin{cases} \mathbf{E}[\min(\tau_1, N)]^{-1} & \text{if } \alpha \in (0,1]\,, \\ \frac{\alpha}{\pi}\sin(\pi\alpha) N^{\alpha-1}/L(N) & \text{if } \alpha \in (1,2]\,. \end{cases} \end{aligned}$$

To complement the above results, let us also give an application of (2.16) in the super-critical case. Indeed, it gives that for any $k \in \{0,\dots,N\}$,

$$\mathbf{P}_{N,\lambda}\big(H_N(\tau) = k\big) = \widetilde{\mathbf{P}}_\lambda\big(\big|\widetilde{\tau} \cap [0,n]\big| = k \,\big|\, n \in \widetilde{\tau}\big) = \frac{1}{\widetilde{\mathbf{P}}_\lambda(n \in \tau)} \widetilde{\mathbf{P}}_\lambda\big(\widetilde{\tau}_k = n\big)\,. \tag{2.18}$$

Indeed, using the local central limit theorem for $\widetilde{\tau}_k$, see e.g. [GK54, Ch. 9, §50], and the renewal theorem, one directly obtains the following result.

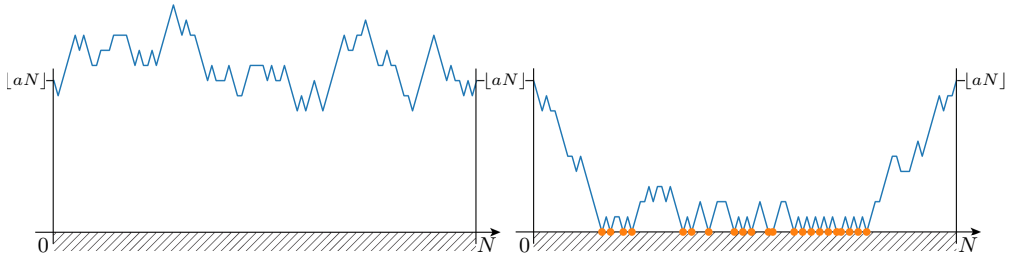


FIGURE 3. Wetting of a lazy random walk in a rectangular well of depth $\lfloor aN \rfloor$. On the left, the system is in the delocalized regime; on the right it is in the localized regime.

Proposition 2.6. *For any $\lambda > \lambda_c$, we have the following local limit theorem for $H_N = H_N(\tau)$ under $\mathbf{P}_{N,\lambda}$:*

$$\sup_{k\geq 0} \Big| \sqrt{N}\, \frac{1}{m_\lambda} \mathbf{P}_{N,\lambda}\big(H_N = k\big) - \phi_{\sigma_\lambda}\Big(\frac{N - k m_\lambda}{\sqrt{N}}\Big)\Big| \xrightarrow{N\to\infty} 0\,.$$

where $\phi_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}$ is the density of $\mathcal{N}(0,\sigma^2)$, and $m_\lambda := \widetilde{\mathbf{E}}_\lambda[\widetilde{\tau}_1]$, $\sigma_\lambda = \widetilde{\mathbf{V}}\mathbf{ar}_\lambda(\widetilde{\tau}_1)$.

As a direct consequence of a Riemann sum approximation, we obtain that under $\mathbf{P}_{N,\lambda}$, the rescaled number of contacts $\frac{1}{\sqrt{N}}(H_N - \frac{1}{m_\lambda}N)$ converges in distribution to $\mathcal{N}(0, \sigma_\lambda^2/m_\lambda^2)$.

3. WETTING WITH ELEVATED BOUNDARY CONDITIONS

3.1. **Wetting in a square well.** We now turn to the wetting model with elevated boundary condition, also seen as a model of wetting of a square well. We will focus here on the discrete case, so from now on $(S_n)_{n\geq 0}$ is an aperiodic centered random walk on $\mathbb{Z}$; the continuous case can be treated analogously. The model consists in lowering the hard wall to a depth $-\lfloor aN \rfloor$, where $a > 0$ is a parameter tuning the depth of the well, see Figure 3.

$$\frac{\mathrm{d}\mathbf{P}^a_{N,\lambda}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z^a_{N,\lambda}} \lambda^{H_N(S)} \mathbb{1}_{\Omega^+_{N,a}(S)} \mathbb{1}_{\{S_N = 0\}}\,, \qquad \text{with} \quad H^a_N(S) := \sum_{n=1}^{N} \mathbb{1}_{\{S_n = -\lfloor aN \rfloor\}}\,, \tag{3.1}$$

with $Z^a_{N,\lambda}$ the partition function of the model and $\Omega^+_{N,a}(S) = \{S_1 \geq -\lfloor aN \rfloor, ..., S_N \geq -\lfloor aN \rfloor\}$.

The definition (3.1) is equivalent to considering the wetting model (2.2) with elevated boundary conditions $S_0 = S_N = \lfloor aN \rfloor$. One could also consider non-symmetric boundary conditions $S_0 = \lfloor aN \rfloor$, $S_N = \lfloor bN \rfloor$ with $a, b > 0$, as done for instance in [Pat98, PV99, BFO09]. We chose for simplicity to focus on the symmetric case $a = b$; we comment below how some of our results would be modified.

3.2. **First results: free energy, left-most and right-most points of contact.** Here again, one may define the free energy of the wetting model with elevated boundary condition:

$$\mathrm{F}(\lambda, a) = \lim_{N\to\infty} \frac{1}{N} \log Z^a_{N,\lambda}\,. \tag{3.2}$$

For simplicity, we keep the notation $\mathrm{F}(\lambda) = \mathrm{F}(\lambda, 0)$ for the free energy of the original wetting model. Theorem 3.1 below shows that the free energy exists and makes it explicit in terms of $\mathrm{F}(\lambda)$ and of the left and right large deviation rate functions for S_n.

3.2.1. *Free energy and large deviations.* First, let us define the (left and right) log-moment generating functions of X_1: for $t \geq 0$, let

$$\Lambda_+(t) := \log \mathbf{E}[e^{tX_1}], \qquad \Lambda_-(t) := \log \mathbf{E}[e^{-tX_1}]. \tag{3.3}$$

Let us also define the Fenchel–Legendre transforms of $\Lambda_\pm$: for $x \geq 0$,

$$\mathrm{I}_+(x) := \sup_{t\geq 0}\{tx - \Lambda_+(t)\}, \qquad \mathrm{I}_-(x) := \sup_{t\geq 0}\{tx - \Lambda_-(t)\}. \tag{3.4}$$

Let also $\Lambda(t) := \Lambda_+(t)\mathbb{1}_{\{t\geq 0\}} + \Lambda_-(-t)\mathbb{1}_{\{t<0\}}$ and $\mathrm{I}(x) := \mathrm{I}_+(x)\mathbb{1}_{\{x\geq 0\}} + \mathrm{I}_-(-x)\mathbb{1}_{\{x<0\}}$. By Cramér's theorem, $\mathrm{I}_+, \mathrm{I}_-$ are the (upward and downward) large deviation rate functions for the random walk $(S_n)_{n\geq 0}$, see e.g. [DZ09, Thm. 2.2.3] or Section 5 below, (5.1). Our first result is to obtain the value of the free energy in terms of $\mathrm{I}_+, \mathrm{I}_-$, in analogy with [LT15, Prop. 2.1].

Theorem 3.1. *The free energy* $\mathrm{F}(\lambda, a)$ *defined in* (3.2) *exists and* $\mathrm{F}(\lambda, a) = \max\{\psi(\lambda, a), 0\}$, *where*

$$\begin{aligned} &\psi(\lambda, a) := \sup_{0\leq u\leq v\leq 1} g_{\lambda,a}(u, v), \\ &\textit{with} \quad g_{\lambda,a}(u, v) := (v-u)\mathrm{F}(\lambda) - u\,\mathrm{I}_-\Big(\frac{a}{u}\Big) - (1-v)\,\mathrm{I}_+\Big(\frac{a}{1-v}\Big). \end{aligned} \tag{3.5}$$

Remark 3.2. *For the wetting model* (2.2) *with non-symmetric elevated boundary conditions* $S_0 = \lfloor aN \rfloor$, $S_N = \lfloor bN \rfloor$, *Theorem 3.1 is easily adapted and one obtains that the free energy is* $\max\{\psi(\lambda, a, b), 0\}$ *with* $\psi(\lambda, a, b) := \sup_{0\leq u\leq v\leq 1}\big\{(v-u)\mathrm{F}(\lambda) - u\,\mathrm{I}_-(\frac{a}{u}) - (1-v)\,\mathrm{I}_+(\frac{b}{1-v})\big\}$.

Then, for any $a \geq 0$, we can define the critical point

$$\lambda_c(a) := \inf\{\lambda, \mathrm{F}(\lambda, a) > 0\} = \inf\{\lambda, \psi(\lambda, a) > 0\},$$

and one easily sees that $a \mapsto \lambda_c(a)$ is non-decreasing. The point $\lambda_c(a)$ marks a *localization* or *wetting* phase transition, from a delocalized phase $\mathrm{F}(\lambda, a) = 0$ (with zero density of contact) to a localized phase $\mathrm{F}(\lambda, a) > 0$ (positive density of contact).

Note also that if $\mathrm{I}_+ = \mathrm{I}_- = 0$, then $\lambda_c(a) = \lambda_c(0) = \lambda_c$ for all $a \geq 0$, which is in particular the case if Assumption 1 holds with $\alpha \in (0, 2)$ and $\varrho \in (1 - \frac{1}{\alpha}, \frac{1}{\alpha})$ (this excludes the totally asymmetric case if $\alpha \in (1, 2)$).

3.2.2. *About the free energy and the phase diagram.* Let us now make more explicit the expressions of the free energy $\mathrm{F}(\lambda, a)$. We also give an expression for the critical line separating the localized phase $\mathcal{L} = \{(\lambda, a), \mathrm{F}(\lambda, a) > 0\}$ from the delocalized phase $\mathcal{D} = \{(\lambda, a), \mathrm{F}(\lambda, a) = 0\}$:

$$a_c(\lambda) = \inf\{a \geq 0 : \mathrm{F}(\lambda, a) = 0\} \quad \text{and} \quad \lambda_c(a) = \sup\{\lambda \geq 0 : \mathrm{F}(\lambda, a) = 0\}.$$

We refer to Figure 4 for an illustration of the phase diagram.

Before we state the results, let us introduce the radii of convergence for Λ_+, Λ_-:

$$t_0^+ := \sup\{t \geq 0, \Lambda_+(t) < +\infty\} \quad \text{and} \quad t_0^- := \sup\{t \geq 0, \Lambda_-(t) < +\infty\}. \tag{3.6}$$

In the case where $t_0^+ = t_0^- = 0$, then we trivially have that $\mathrm{I}_+ = \mathrm{I}_- \equiv 0$ and $\mathrm{F}(\lambda, a) = \mathrm{F}(\lambda)$ for any $a > 0$. Our next results therefore have some interest only if $t_0^+ > 0$ or $t_0^- > 0$. We stress that Λ_+ is increasing and analytic on $[0, t_0^+)$ so it is in fact invertible with analytic inverse Λ_+^{-1} on that interval. We extend this definition on $\mathbb{R}_+$ by letting Λ_+^{-1} be the left-continuous inverse of Λ_+; in particular, $\Lambda_+^{-1}(x) = t_0^+$ for all $x \geq \Lambda_+(t_0^+)$ and $\Lambda_+^{-1} \equiv 0$ if $t_0^+ = 0$. Similar notation holds for Λ_-.

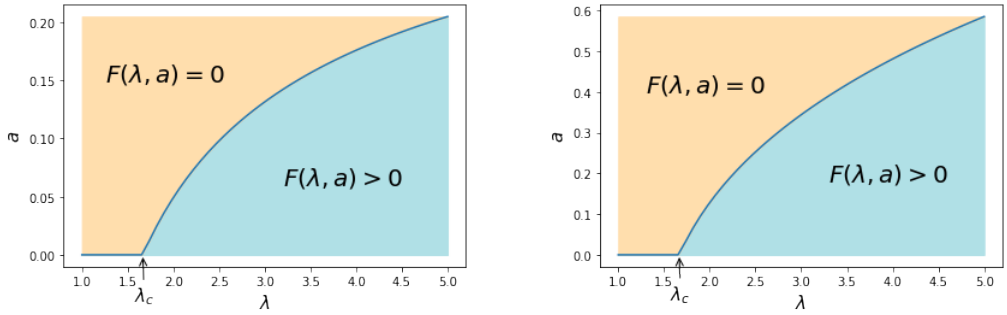


FIGURE 4. Phase diagram for the wetting in a square well for two integrable models (see Appendix A): the symmetric lazy random walk (on the left) and of the symmetric geometric random walk of Section A.2 (on the right). We refer to Sections A.1 and A.2 for explicit formulas derived from Theorem 3.3 (with the parameter $\gamma = 0.4$ in the figures above).

Theorem 3.3. *For any $\lambda > 0$, we have*

$$\mathrm{F}(\lambda, a) = \max\{\psi(\lambda, a), 0\} = \max\left\{\mathrm{F}(\lambda) - a\Lambda_+^{-1}(\mathrm{F}(\lambda)) - a\Lambda_-^{-1}(\mathrm{F}(\lambda)), 0\right\} \tag{3.7}$$

In particular, we have the following formula for $a_c(\lambda)$: if $\lambda > \lambda_c$,

$$a_c(\lambda) = \frac{\mathrm{F}(\lambda)}{\Lambda_+^{-1}(\mathrm{F}(\lambda)) + \Lambda_-^{-1}(\mathrm{F}(\lambda))}\,. \tag{3.8}$$

Let us now give some properties of $\mathrm{F}(\lambda, a)$.

- Similarly as for the free energy of the (standard) wetting model, the function $\lambda \mapsto \mathrm{F}(\lambda, a)$ is non-negative and non-decreasing and $\beta \mapsto \mathrm{F}(e^\beta, a)$ is convex, for any $a \geq 0$.
- The function $a \mapsto \mathrm{F}(\lambda, a)$ is affine by parts.

Now, define $\lambda_+, \lambda_- > 0$ as follows:

$$\begin{cases} \mathrm{F}(\lambda_+) = \Lambda_+(t_0^+) & \text{if } t_0^+ \in (0, +\infty)\,, \\ \mathrm{F}(\lambda_-) = \Lambda_-(t_0^-) & \text{if } t_0^- \in (0, +\infty)\,, \end{cases} \tag{3.9}$$

and $\lambda_+ = +\infty$, resp. $\lambda_- = +\infty$, if $t_0^+ = 0$ or $t_0^+ = +\infty$, resp. if $t_0^- = 0$ or $t_0^- = +\infty$. Naturally we also have $\lambda_\pm = +\infty$ if $\Lambda_\pm(t_0^\pm) = +\infty$. Then, the function $\lambda \mapsto \Lambda_+^{-1}(\mathrm{F}(\lambda))$ is analytic on $(0, \lambda_+)$ and constant (equal to t_0^+) on $[\lambda_+, +\infty)$; and similarly for $\lambda \mapsto \Lambda_-^{-1}(\mathrm{F}(\lambda))$. Let us therefore make two observations:

- If $\lambda > \lambda_+, \lambda_-$, then we have that $\mathrm{F}(\lambda) = \max\{\mathrm{F}(\lambda) - a(t_0^+ + t_0^-), 0\}$ and $a_c(\lambda) = \frac{1}{t_0^+ + t_0^-}\mathrm{F}(\lambda)$.
- The functions $\lambda \mapsto \mathrm{F}(\lambda, a)$ is analytic except at $\lambda_c(a)$ and at λ_+, λ_-; the function $\lambda \mapsto a_c(\lambda)$ is analytic on $(\lambda_c, +\infty)$, except at λ_+, λ_-.

The last observation shows that in addition to the *localization* phase transition at $\lambda_c(a)$, there might be another phase transition (or two others), at $\lambda = \lambda_+, \lambda_-$, provided that $\lambda_+, \lambda_- > \lambda_c(a)$. The phase transition at $\lambda_\pm$ is a *saturation* phase transition, in the sense that the left or right-most point of contact becomes degenerate when $\lambda > \lambda_-$ or $\lambda > \lambda_+$, see Section 3.2.4 (and Figure 5). This second phase transition appears to be absent from other wetting models considered so far in the literature, and essentially comes from the fact that the rate function $\mathrm{I}_\pm$ itself may have a (saturation) phase transition at some $\rho_\pm$, see Section 5 below.

Example 3.4. *Consider a random walk with symmetric increments whose distribution is given by* $\mathbf{P}(X_1 = x) = c_\theta(1+|x|)^{-\theta}e^{-|x|}$ *for* $x \in \mathbb{Z}$*, where* $\theta \in \mathbb{R}$ *is a parameter and* c_θ *is a constant that normalizes* $\mathbf{P}$ *to a probability. Then, we clearly have that* $t_0^+ = t_0^- = 1$*, and* $\Lambda_\pm(1) = +\infty$ *if* $\theta \le 1$*,* $\Lambda_\pm(1) < +\infty$ *if* $\theta > 1$*. Therefore, the critical point* $\lambda_\pm$ *is finite if and only if* $\theta > 1$*.*

3.2.3. *Maximizers of the variational problem* (3.5). As mentioned above, Λ_+ is increasing and analytic on $[0, t_0^+)$, so Λ_+^{-1} and Λ_+' are well defined on that interval: we extend their definition on $\mathbb{R}_+$ by considering respectively the left-continuous inverse and the left-derivative of Λ_+. In the case $t_0^+ = 0$, we set $\Lambda_+' \circ \Lambda_+^{-1} \equiv +\infty$. Similar notation holds for Λ_-. We also denote

$$\rho_+ := \Lambda_+'(t_0^+) = \lim_{t \uparrow t_0^+} \Lambda_+'(t), \qquad \rho_- := \Lambda_-'(t_0^-) = \lim_{t \uparrow t_0^-} \Lambda_-'(t)\,, \tag{3.10}$$

where by convention we set $\rho_\pm = +\infty$ if $t_0^\pm = 0$. Then, for any $\lambda \ge \lambda_c$ and $a \ge 0$, we define

$$w_+^* := \frac{a}{\Lambda_+' \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))} \in \Big[\frac{a}{\rho_+}, +\infty\Big)\,, \qquad w_-^* := \frac{a}{\Lambda_-' \circ \Lambda_-^{-1}(\mathrm{F}(\lambda))} \in \Big[\frac{a}{\rho_-}, +\infty\Big)\,. \tag{3.11}$$

Note that $w_+^* > \frac{a}{\rho_+}$ if $\lambda < \lambda_+$ and $w_+^* = \frac{a}{\rho_+}$ if $\lambda \ge \lambda_+$, and similarly for w_-^*. We then have the following result.

Lemma 3.5. *If* $w_+^* + w_-^* \le 1$*, and in particular if* $\lambda \ge \lambda_c(a)$*, we have*

$$\sup_{0 \le u \le v \le 1} \big\{ g_{\lambda,a}(u,v) \big\} = \mathrm{F}(\lambda) - a\Lambda_-^{-1}(\mathrm{F}(\lambda)) - a\Lambda_+^{-1}(\mathrm{F}(\lambda))\,,$$

and the the supremum is attained on $U^* \times V^* = U^*_{\lambda,a} \times V^*_{\lambda,a}$*, with*

$$U^* = \begin{cases} \{w_-^*\} & \text{if } \lambda < \lambda_-\,, \\ [0, \frac{a}{\rho_-}] & \text{if } \lambda = \lambda_-\,, \\ \{0\} & \text{if } \lambda > \lambda_-\,, \end{cases} \qquad V^* = \begin{cases} \{1 - w_+^*\} & \text{if } \lambda < \lambda_+\,, \\ [1 - \frac{a}{\rho_+}, 1] & \text{if } \lambda = \lambda_+\,, \\ \{1\} & \text{if } \lambda > \lambda_+\,. \end{cases}$$

We show in Lemma 6.5 below that $w_+^* + w_-^* < 1$ whenever $\lambda_c \ge \lambda_c(a)$; this proves in particular the first claim of the lemma.

Remark 3.6. *As far as the wetting model* (2.2) *with non-symmetric elevated boundary conditions* $S_0 = \lfloor aN \rfloor$*,* $S_N = \lfloor bN \rfloor$ *is concerned, one could also adapt Lemma 3.5 (and Theorem 3.8 below): one simply need to replace* a *by* b *in the definition of* w_+^**.*

Remark 3.7. *In the case where* $\Lambda_+(t_0^+), \Lambda_-(t_0^-) = +\infty$*, then* $\lambda_+, \lambda_- = +\infty$ *and the supremum in* (3.5) *is uniquely attained. In fact, the only case when the supremum is not uniquely attained is if* $\lambda = \lambda_-$ *(and* $\rho_- < +\infty$*) or* $\lambda = \lambda_+$ *(and* $\rho_+ < +\infty$*). Considering Example 3.4 as an illustration, we have* $\lambda_\pm < +\infty$ *if and only if* $\theta > 1$ *and we additionally have* $\rho_\pm < +\infty$ *if and only if* $\theta > 2$*.*

3.2.4. *Left and right-most pinned points.* We can extract from Theorem 3.1 and Lemma 3.5 some properties of the path, in analogy with [LT15, Thm. 2.2]. Let us define the left-most and right-most points of contact between the walk and the bottom of the well:

$$L_N = L_N^a := \min\{0 \le n \le N, \lfloor aN \rfloor + S_n = 0\}\,,$$
$$R_N = R_N^a := \max\{0 \le n \le N, \lfloor aN \rfloor + S_n = 0\}\,.$$

By convention, we set $L_N, R_N = +\infty$ if $H_N^a(S) = 0$, *i.e.* if there is no contact with the bottom of the well. Note that we focus on the left and right-most point of contact, but physically, an important quantity to consider is the contact angles between the interface and the bottom of the

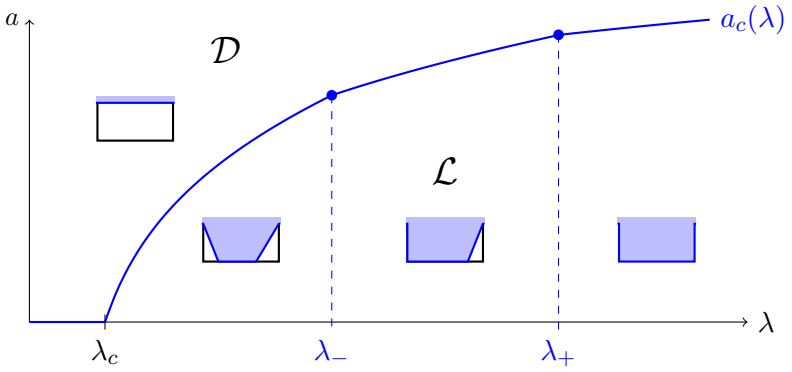


FIGURE 5. Illustration of the phase diagram in the case where $\lambda_c < \lambda_- < \lambda_+ < +\infty$. The critical curve $a_c(\lambda)$ separates a localized phase $\mathcal{L} = \{(\lambda, a), \mathrm{F}(\lambda, a) > 0\}$ and a delocalized phase $\mathcal{D} = \{(\lambda, a), \mathrm{F}(\lambda, a) = 0\}$. We have also represented a typical configuration, depending on the range of parameters: according to Lemma 3.5, the left-most point of contact is either at $w_-^* > \frac{a}{\rho_-}$ for $\lambda < \lambda_-$, in $[0, \frac{a}{\rho_-}]$ for $\lambda = \lambda_-$ or at 0 for $\lambda > \lambda_-$ (and similarly for the right-most point of contact). Also, in view of Theorem 3.3, the formula for $\mathrm{F}(\lambda, a)$ is different in the regions (λ_c, λ_-), (λ_-, λ_+) and $(\lambda_+, +\infty)$.

well (or substrate); we refer to Remark 3.9 below for some comments. We prove the following convergence of (L_N, R_N); we let $\mathrm{dist}(z, A) := \inf_{z' \in A} \|z - z'\|$.

Theorem 3.8. *In the super-critical case, that is for $\lambda > \lambda_c(a)$, we have that for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbf{P}^a_{N,\lambda}\left(\mathrm{dist}\left(\tfrac{1}{N}(L_N, R_N), U^* \times V^*\right) > \varepsilon\right) = 0\,, \tag{3.12}$$

where $U^* \times V^*$ is defined in Lemma 3.5 (and are the maximizers of the variational problem (3.5)). In the subcritical case, that is for $\lambda < \lambda_c(a)$, if we assume that $\lim_{n\to\infty} \frac{1}{n} a_n = +\infty$ (so the bottom of the well is much further than the typical fluctuations of the random walk), then we have

$$\lim_{n\to\infty} \mathbf{P}^a_{N,\lambda}(H^a_N(S) = 0) = 1\,. \tag{3.13}$$

In the case where $\Lambda_+(t_0^+), \Lambda_-(t_0^-) = +\infty$, Lemma 3.5 shows that the supremum in (3.5) is uniquely attained, and from Theorem 3.8 we get that the left and right-most points of contact are located at $(u^*, v^*) = (w_-^*, 1 - w_+^*)$ with $\frac{a}{\rho_-} < u^* < v^* < 1 - \frac{1}{\rho_+}$, for any $a > 0$, $\lambda > \lambda_c(a)$. The most unexpected case occurs when $\Lambda_+(t_0^+) < +\infty$ and/or $\Lambda_-(t_0^-) < +\infty$. Let us focus on the left-most point for simplicity. Recalling the critical value λ_- defined in (3.9), Lemma 3.5-Theorem 3.8 show that the left-most point is located at $u^* = w_-^* > \frac{a}{\rho_+}$ for $\lambda < \lambda_-$ and then drops (or *saturates*) to $u^* = 0$ for $\lambda > \lambda_-$, either continuously if $\rho_- = +\infty$ or discontinuously if $\rho_- < +\infty$; at $\lambda = \lambda_-$, the left-most point is located in an interval $[0, \frac{a}{\rho_-}]$. We refer to Figure 5 for an illustration.

Let us mention that, as far as the critical case $\lambda = \lambda_c(a)$ is concerned, one needs some extra assumption to obtain a result analogous to Theorem 3.8. For instance, if $t_0^+ = t_0^- = 0$ we have $\lambda_c(a) = 0$: then we can easily show that (3.13) holds also at $\lambda = \lambda_c(a)$. On the other hand, if one have $\lambda_c(a) < \min\{\lambda_+, \lambda_-\}$, *i.e.* if one is inside the so-called Cramér's region, we prove in Theorem 3.10 below that (3.12) holds at $\lambda = \lambda_c(a)$, with a unique maximizer (u^*, v^*) verifying $0 < u^* < v^* < 1$.

Remark 3.9. *One could improve Theorem 3.8 to obtain in the super-critical case the convergence of the full trajectory* $(\frac{1}{N}S_{\lfloor tN\rfloor})_{t\in[0,1]}$*, as done for instance in* [BFO09]*. If* $\lambda > \lambda_c(a)$ *and if the maximizers* U^*, V^* *are reduced to one point* w_-^*, w_+^**, one can easily show that the trajectory converges under* $\mathbf{P}^a_{N,\lambda}$ *to three line segments joining the points* $(0,0)$*,* $(w_-^*, -a)$*,* $(w_+^*, -a)$*,* $(1,0)$*; we refer to Figure 5 for an illustration.*

Some interesting quantities considered for instance in [Pat98, PV99] *are the contact angles* θ_-, θ_+ *between the bottom of the well and the first and last segment of the trajectory. Since we have* $\tan\theta_- = a/w_-^*$*,* $\tan\theta_+ = a/w_+^*$*, then in view of the formula* (3.11) *for* w_+^*, w_-^**, we observe* the contact angles θ_-, θ_+ do not depend on a but only on λ *(this remains true for non-symmetric elevated boundary conditions) — note that in the case of a symmetric random walk one additionally has* $\theta_+ = \theta_-$*. This was already noticed in* [Pat98, PV99] *for the pinning of interfaces in the 2D Ising model, the angle verifying the so-called Herring–Young equation.*

3.3. **Cramér's region and Gaussian fluctuations.** We now wish to obtain more precise estimates on the partition function and on the left and right-most pinned point, sharpening in particular Theorem 3.8. For that, we are going to assume that Cramér's condition holds, which ensures in particular that the rate functions $\mathrm{I}_+, \mathrm{I}_-$ are non-trivial.

Assumption 2 (Cramér's condition)**.** *There is some* $t_0 > 0$ *such that* $\mathbf{E}[e^{tX_1}] < +\infty$ *for* $|t| < t_0$*. In other words,* $t_0^+ > 0$ and $t_0^- > 0$.

Under that assumption, we call *Cramér's region* the interval $(-\rho_-, \rho_+)$, where $\rho_-, \rho_+ \in (0, +\infty]$ are defined in (3.10). In particular, one can show that the rate function I is strictly convex inside Cramér's region $(-\rho_-, \rho_+)$; we refer to Section 5 for more comments. In the super-critical and critical case $\lambda \geq \lambda_c(a)$, Assumption 2 allows us to obtain inside Cramér's region a sharp estimate on the partition function and on the distribution of L_N, R_N under $\mathbf{P}^a_{N,\lambda}$.

More precisely, we will assume that $\lambda_c(a) \leq \lambda < \min\{\lambda_+, \lambda_-\}$, which ensures: first that $\lambda \geq \lambda_c(a)$ thanks to Lemma 6.5 below; second that w_+^*, w_-^* defined in (3.11) verify $w_+^* > \frac{a}{\rho_+}$, $w_-^* > \frac{a}{\rho_-}$. In that case, we set

$$u^* = u^*_{\lambda,a} = w_-^* = \frac{a}{\Lambda_-' \circ \Lambda_-^{-1}(\mathrm{F}(\lambda))}, \qquad v^* = v^*_{\lambda,a} = 1 - w_+^* = 1 - \frac{a}{\Lambda_+' \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))}, \tag{3.14}$$

which verify $\frac{a}{\rho_-} < u^* < v^* < 1 - \frac{a}{\rho_+}$; hence $\frac{a}{u^*}$ and $\frac{a}{1-v^*}$ are both inside Cramérs's region.

Theorem 3.10. *Suppose that Assumption 2 holds, that* $a > 0$ *and that* $\lambda_c(a) \leq \lambda < \min\{\lambda_+, \lambda_-\}$*. Let* (u^*, v^*) *be defined as in* (3.14)*.*

(i) There are some constants $c_0 = c_0(a,\lambda) > 0$*,* $c_1 = c_1(a,\lambda) > 0$ *(explicit in the proof) such that*

$$Z^a_{N,\lambda} \sim c_1 e^{\mathrm{F}(\lambda,a)N + c_0\{aN\}} \qquad \text{as } N \to +\infty\,, \tag{3.15}$$

where $\{aN\} = aN - \lfloor aN\rfloor$ *is the fractional part of* aN*.*

(ii) There are some constants $\sigma_1 = \sigma_1(a,\lambda) > 0$*,* $\sigma_2 = \sigma_2(a,\lambda) > 0$ *(explicit in the proof, see* (8.11)*) such that uniformly for* $\delta_\ell := |\frac{1}{N}\ell - u^*|, \delta_r := |\frac{1}{N}r - v^*| \leq \varepsilon_N$ *with* $\varepsilon_N \to 0$*, we have as* $N \to \infty$

$$\mathbf{P}^a_{N,\lambda}\big(L_N = \ell, R_N = r\big) = \frac{1+o(1)}{2\pi N\sigma_1\sigma_2} e^{-\frac{(\ell - u^*N)^2}{2\sigma_1^2 N}(1+\delta_\ell O(1))} e^{-\frac{(r-v^*N)^2}{2\sigma_2^2 N}(1+\delta_r O(1))}. \tag{3.16}$$

As a consequence, as $N \to \infty$*, under* $\mathbf{P}^a_{N,\lambda}$ *we have that* $(\frac{L_N - Nu^*}{\sqrt{N}}, \frac{R_N - Nv^*}{\sqrt{N}})$ *converges weakly to* (Z_1, Z_2)*, where* $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$ *and* $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$ *are independent.*

This result is analogous to Theorem 1.5 in [BFO09], which has been obtained in the context of the pinning or wetting of a d-dimensional *Gaussian* random walk on a subspace M; see also Remark 4.1 in [FO10], which considers the pinning version of the model, under the assumption that the random walk has all finite exponential moments (*i.e.* $t_0^+ = t_0^- = +\infty$). Here we provide a local limit theorem for L_N, R_N (and the number of contacts $H_N(S)$), under much milder conditions.

Notice that under $\mathbf{P}^a_{N,\lambda}$, conditionally on $L_N = \ell, R_N = r$, the number of contacts $H_N = H_N(S)$ with the bottom of the well as the same distribution as under $\mathbf{P}_{r-\ell,\lambda}$. In particular, combining Theorem 3.10 with Proposition 2.6, we prove the following corollary.

Corollary 3.11. *Suppose that Assumption 2 holds, that $a > 0$ and that $\lambda_c(a) \le \lambda < \min\{\lambda_+, \lambda_-\}$. Then, as $N \to \infty$, we have the following joint convergence in distribution under $\mathbf{P}^a_{N,\lambda}$:*

$$\Big(\frac{L_N - u^* N}{\sqrt{N}}, \frac{R_N - v^* N}{\sqrt{N}}, \frac{H_N - m_\lambda^{-1}(v^* - u^*)N}{\sqrt{N}}\Big) \xrightarrow[N\to\infty]{\mathbf{P}^a_{N,\lambda}} (Z_1, Z_2, Z_3) \qquad \text{as } N \to \infty\,, \tag{3.17}$$

where (Z_1, Z_2, Z_3) is a Gaussian vector of density

$$\frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} e^{-\frac{1}{2}Q(z_1,z_2,z_3)} \qquad \text{with} \quad Q(z_1,z_2,z_3) = \frac{z_1^2}{\sigma_1^2} + \frac{z_2^2}{\sigma_2^2} + \frac{(z_3 - m_\lambda^{-1}(z_2 - z_1))^2}{\sigma_3^2}\,,$$

for some constants $\sigma_1, \sigma_2, \sigma_3$ explicit in the proof. In particular, Z_3 is Gaussian, but not independent of (Z_1, Z_2). (We actually obtain a local version of this convergence in distribution, see (8.13).)

As a consequence, the density of contact $\frac{1}{N}H_N(S)$ converges in $\mathbf{P}^a_{N,\lambda}$-probability to $\frac{1}{m_\lambda}(v^* - u^*)$, with $v^* - u^* > 0$, even at the critical point $\lambda = \lambda_c(a)$.

Remark 3.12. *Obtaining detailed results when $\lambda \ge min\{\lambda_+, \lambda_-\}$, i.e. at the border or outside Cramér's region is left open. It would actually require further assumptions on the underlying random walk, in particular to obtain sharp local large deviation estimates, in the spirit of Theorem 8.2 below. We do not pursue the investigation further, but the situation at $\lambda = \lambda_\pm$ when $\rho_\pm < +\infty$ (hence the variational problem has no unique maximizer) is particularly intriguing.*

3.4. **About the phase transitions.** In this section, we also focus on the case where Cramér's condition is satisfied, *i.e.* both $t_0^+, t_0^- > 0$. This is only to simplify the statements, but the results could be adapted to the case where only one of t_0^+, t_0^- is non-zero (up to a factor $1/2$ in some constants). On the other hand, the case $t_0^+ = t_0^- = 0$ is trivial, since $\mathrm{F}(\lambda, a) = \mathrm{F}(\lambda)$ for all $a \ge 0$.

3.4.1. *About the localization (or wetting) phase transition.* As far as the critical behavior near the *localization* transition (at $\lambda_c(a)$) is concerned, we have the following result. It shows in particular that, as soon as $a > 0$, the phase transition in λ is of first order. Note we have already observed that $a \mapsto \mathrm{F}(\lambda, a)$ is affine on $(0, a_c(\lambda))$.

Proposition 3.13. *Suppose that Assumption 2 holds. For any $a > 0$ with $\lambda_c(a) < +\infty$, we have*

$$\mathrm{F}(\lambda_c(a) + u, a) \sim C_a u \qquad \text{as } u \downarrow 0\,.$$

for some explicit *constant $C_a > 0$ given in (6.11). The constant C_a verifies $C_a \sim \sqrt{8c_3}\, a$ as $a \downarrow 0$, with c_3 defined in (2.13).*

3.4.2. *About the saturation phase transition.* Let us also give some information on the saturation transition(s) (at λ_+, λ_-): we show that it is of first order if $\rho_\pm < +\infty$ and of second order if $\rho_\pm = \infty$.

Assume that $\lambda_c(a) < \lambda_- \leq \lambda_+ < +\infty$; the case $\lambda_+ \leq \lambda_-$ is symmetric and the case where $\lambda_+ = +\infty$ is simpler. We introduce the excess free energies as follows: recalling Theorem 3.3 (and the fact that $\Lambda_\pm^{-1}(\mathrm{F}(\lambda_\pm)) = t_0^\pm$), define for $\lambda_c(a) < \lambda$

$$\begin{aligned} \mathrm{F}^*(\lambda, a) &:= \mathrm{F}(\lambda, a) - \big(\mathrm{F}(\lambda) - a\Lambda_+^{-1}(\mathrm{F}(\lambda)) - at_0^-\big)\,, \\ \mathrm{F}^{**}(\lambda, a) &:= \mathrm{F}(\lambda, a) - \big(\mathrm{F}(\lambda) - at_0^+ - at_0^-\big)\,. \end{aligned} \tag{3.18}$$

In particular, we have that $\mathrm{F}^*(\lambda, a) = 0$ for $\lambda \in (\lambda_-, \lambda_+)$ and $\mathrm{F}^{**}(\lambda, a) = 0$ for $\lambda \in (\lambda_+, +\infty)$. In the case where $\lambda_- = \lambda_+$ then $\mathrm{F}^* = \mathrm{F}^{**}$. We then have the following result.

Proposition 3.14. *Suppose that Assumption 2 holds. If $\lambda_- < \lambda_+ < +\infty$, then as $u \downarrow 0$ we have*

$$\mathrm{F}^*(\lambda_- - u, a) = (1 + o(1))\frac{a\mathrm{F}'(\lambda_-)}{\rho_-}\,u\,, \qquad \mathrm{F}^{**}(\lambda_+ - u, a) = (1 + o(1))\frac{a\mathrm{F}'(\lambda_+)}{\rho_+}\,u\,.$$

If $\lambda_- = \lambda_+ < +\infty$, then $\mathrm{F}^*(\lambda_\pm - u, a) = (1 + o(1))a\mathrm{F}'(\lambda_\pm)\big(\frac{1}{\rho_-} + \frac{1}{\rho_+}\big)u$ as $u \downarrow 0$.

In particular, we find that $\mathrm{F}^*(\lambda_\pm - u, a) = o(u)$ if $\rho_\pm = +\infty$.

3.4.3. *About the critical curve.* Let us now provide some information about the critical curve, in particular close to λ_c and λ_-, λ_+ (again, assume $\lambda_- \leq \lambda_+ < +\infty$). To study $a_c(\lambda)$ as λ crosses the values λ_-, λ_+, we define similarly as above (recalling the formulas of Theorem 3.3):

$$a_c^*(\lambda) = a_c(\lambda) - \frac{\mathrm{F}(\lambda)}{\Lambda_+^{-1}(\mathrm{F}(\lambda)) + t_0^-}\,, \qquad a_c^{**}(\lambda) = a_c(\lambda) - \frac{\mathrm{F}(\lambda)}{t_0^+ + t_0^-}\,, \tag{3.19}$$

so that $a_c^*(\lambda) = 0$ for $\lambda \in (\lambda_-, \lambda_+)$ and $a_c^{**}(\lambda) = 0$ for $\lambda \in (\lambda_+, +\infty)$. Also, $a_c^* = a_c^{**}$ in the case where $\lambda_- = \lambda_+$.

Proposition 3.15. *Suppose that Assumption 2 holds.*

(i) We have

$$a_c(\lambda_c + u) \sim \frac{\sigma^2}{2\sqrt{2}}\sqrt{c_3}u \qquad \text{as } u \downarrow 0\,,$$

where σ^2 is the variance of X_1 and c_3 is defined in (2.13).

(ii) If $\lambda_- < \lambda_+ < +\infty$, then as $u \downarrow 0$ we have

$$a_c^*(\lambda_- - u) = (1 + o(1))\frac{\mathrm{F}(\lambda_-)\mathrm{F}'(\lambda_-)}{(\Lambda_+^{-1}(\mathrm{F}(\lambda_-)) + t_0^-)^2\rho_-}\,u\,, \quad a_c^{**}(\lambda_+ - u) = (1 + o(1))\frac{\mathrm{F}(\lambda_+)\mathrm{F}'(\lambda_+)}{(t_0^- + t_0^+)^2\rho_+}\,u\,.$$

If $\lambda_- = \lambda_+ < +\infty$, then $a_c^*(\lambda_\pm - u) = (1 + o(1))\frac{\mathrm{F}(\lambda_\pm)\mathrm{F}'(\lambda_\pm)}{(t_0^+ + t_0^-)^2}\big(\frac{1}{\rho_-} + \frac{1}{\rho_-}\big)u$ as $u \downarrow 0$.

In particular, we find that $\lambda \mapsto a_c(\lambda)$ remains differentiable at $\lambda_\pm$ if $\rho_\pm = +\infty$; if $\rho_\pm < +\infty$, the left and right derivative at $\lambda_\pm$ differ.

4. Some related models and open questions

4.1. **Wetting on a tilted wall or in a convex well.** Let $(X_i)_{i\geq 1}$ be i.i.d. real random variables, with a density $f(\cdot)$ with respect to the Lebesgue measure. For a given function $\varphi : [0,1] \to \mathbb{R}$ we define for $N \geq 1$ the function $\varphi_N : \{0, \dots N\} \to \mathbb{R}$ by setting $\varphi_N(i) = \frac{1}{N}\varphi(\frac{i}{N})$. Then, we introduce the following Gibbs measure, analogously to (2.1):

$$\mathrm{d}\mathbf{P}^{\varphi}_{N,\lambda}(s_1, \dots, s_n) = \frac{1}{Z^{\varphi}_{N,\lambda}} \prod_{i=1}^{N-1} \left(\mathbb{1}_{\{s_i > \varphi_N(i)\}} \mu(\mathrm{d}s_i) + \lambda \delta_{\varphi_N(i)}(\mathrm{d}s_i) \right) \prod_{i=1}^{N} f(s_i - s_{i-1}) \lambda \delta_{\varphi_N(N)}(\mathrm{d}s_N)\,.$$

This corresponds to a wetting model on a wall of shape φ; note that we consider only the continuous model so that there is no restriction on the function φ. Two natural examples are the following

(i) If $\varphi(x) = \mu x$ for some $\mu \in \mathbb{R}$, it corresponds to a wetting model on a wall with slope μ;

(ii) If $\varphi(0) = \varphi(1) = 0$ and φ is convex, this corresponds to a wetting model in a convex well.

We then define the free energy $\mathrm{F}(\lambda, \varphi) = \lim_{N\to\infty} \frac{1}{N} \log Z^{\varphi}_{N,\lambda}$ (one has to show that it exits) and one would like to obtain some explicit expression for it.

(i) For a wall of slope μ. If $\varphi(x) = \mu x$, let us denote $\mathrm{F}(\lambda, \varphi) =: \mathrm{F}_\mu(\lambda)$. In that case, in the Cramér region, that is if $\mu \in (\rho_-, \rho_+)$, then we should have that

$$\mathrm{F}_\mu(\lambda) = \mathrm{F}^{(\mu)}(\lambda) - \mathrm{I}(\mu)\,,$$

where $\mathrm{F}^{(\mu)}(\lambda)$ is the free energy of the wetting model with a wall of slope 0 but with underlying random walk with i.i.d. increments $\widetilde{X}_i = X_i^{(\mu)} - \mu$, where $X_1^{(\mu)}$ has the tilted law $\mathbf{P}^{(\mu)}(\mathrm{d}x) = e^{t_\mu x - \Lambda(t_\mu)}\mathbf{P}(\mathrm{d}x)$ with t_μ chosen so that $\mathbf{E}^{(\mu)}[X_1^{(\mu)}] = \mu$. For instance, in the case of a (standard) Gaussian random walk, since $\mathrm{I}(\mu) = \frac{1}{2}\mu^2$ and $\mathbf{P}^{(\mu)} \sim \mathcal{N}(\mu, 1)$, we should get $\mathrm{F}_\mu(\lambda) = \mathrm{F}(\lambda) - \frac{1}{2}\mu^2$.

(ii) For a convex well. In the case where $\varphi(0) = \varphi(1) = 0$ and φ is convex, then we should get the following. For $0 \leq u \leq v \leq 1$, set

$$g_{\lambda,\varphi}(u, v) := \int_u^v \mathrm{F}_{\varphi'(t)}(\lambda)\mathrm{d}t - u\,\mathrm{I}_-\Big(\frac{|\varphi(u)|}{u}\Big) - (1-v)\,\mathrm{I}_+\Big(\frac{|\varphi(v)|}{1-v}\Big)\,,$$

where $\mathrm{F}_{\varphi'(t)}(\lambda)$ is the free energy of the wetting model on a wall of slope $\varphi'(t)$. Then, setting $\psi(\lambda, \varphi) = \sup_{0\leq u\leq v\leq 1} g_{\lambda,\varphi}(u, v)$, we expect that

$$\mathrm{F}(\lambda, \varphi) = \max\{\psi(\lambda, \varphi), 0\}\,,$$

generalizing the formula found in Theorem 3.1. For instance, the formulas should slightly simplify for a (standard) Gaussian random walk: we should get $g_{\lambda,\varphi}(u, v) = (v - u)\mathrm{F}(\lambda) - E_{u,v}(\varphi)$, with $E_{u,v}(\varphi) := \frac{1}{2}\int_u^v \varphi'(t)^2\mathrm{d}t + \frac{1}{2u}\varphi(u)^2 + \frac{1}{2(1-v)}\varphi(v)^2$.

4.2. **Wetting on a random walk.** Another model that seems natural to consider but has not been studied in the literature (to our knowledge) is the wetting model in the case where the wall is random (but quenched), given by the realization of another random random walk $(Y_n)_{n\geq 0}$.

More precisely, let $Y = (Y_n)_{n\geq 0}$ be a random walk and consider, for a fixed (*i.e.* quenched) realization of Y, the Gibbs measure

$$\mathrm{d}\mathbf{P}^{Y}_{N,\lambda}(s_1, \dots, s_n) = \frac{1}{Z^{Y}_{N,\lambda}} \prod_{i=1}^{N-1} \left(\mathbb{1}_{\{s_i > Y_i\}} \mu(\mathrm{d}s_i) + \lambda \delta_{Y_i}(\mathrm{d}s_i) \right) \prod_{i=1}^{N} f(s_i - s_{i-1}) \lambda \delta_{Y_N}(\mathrm{d}s_N)\,.$$

It corresponds to a wetting model of the random walk $(S_n)_{n\geq 0}$ above the (quenched) wall $(Y_n)_{n\geq 0}$. In the discrete case, this can be rewritten and

$$\frac{\mathrm{d}\mathbf{P}^Y_{N,\lambda}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z^Y_{N,\lambda}} \lambda^{\sum_{n=1}^N \mathbb{1}_{\{S_n=Y_n\}}} \mathbb{1}_{\{S_i\geq Y_i \text{ for all } i\in\{1,\dots,N\}\}}\,. \tag{4.1}$$

By Kingman's super-additive ergodic theorem, one can show that the *quenched* free energy exists and is a.s. constant, *i.e.* almost surely does not depend on the specific realization of Y:

$$\mathrm{F}^{\mathrm{que}}(\lambda) := \lim_{N\to\infty} \frac{1}{N} \log Z^Y_{N,\lambda} = \lim_{N\to\infty} \frac{1}{N} \mathbf{E}_Y\big[\log Z^Y_{N,\lambda}\big] \quad \mathbf{P}_Y\text{-a.s. and in } L^1(\mathbf{P}_Y)\,.$$

Additionally, one may define the *quenched* critical point $\lambda_c^{\mathrm{que}} := \inf\{\lambda, \mathrm{F}^{\mathrm{que}}(\lambda) > 0\}$.

Let us observe that the *annealed* model, with partition function $\mathbf{E}_Y\big[Z^Y_{N,\lambda}\big]$, corresponds to the usual wetting model of Section 2, with underlying random walk $(X_n - Y_n)_{n\geq 0}$. Hence its critical point λ_c^{ann} (and the critical behavior of the annealed free energy) is explicit, see Theorem 3.1. Furthermore, applying Jensen's inequality $\mathbf{E}_Y[\log Z^Y_{N,\lambda}] \leq \log \mathbf{E}_Y[Z^Y_{N,\lambda}]$, one obtains that the annealed free energy dominates the quenched one, and in particular $\lambda_c^{\mathrm{que}} \geq \lambda_c^{\mathrm{ann}}$. The question of then to know whether one has the strict inequality $\lambda_c^{\mathrm{que}} > \lambda_c^{\mathrm{ann}}$ or not.

Quenched vs. annealed critical point. A variant of (4.1), where the constraint that $S_i \geq Y_i$ for all $1 \leq i \leq N$ is removed, is known as the Random Walk Pinning Model and has been studied in several instances, see [BT10, BS10, BS11]. For this model, the question of the strict inequality between the quenched and annealed critical point has some consequences in several interacting stochastic systems, see [BGdH11]. The above references mostly consider the case of the simple symmetric random walk on $\mathbb{Z}^d$: they show that quenched and annealed critical points are equal (to 1) in dimension $d = 1, 2$ and that they differ in dimension $d \geq 3$. The case of symmetric random walks on $\mathbb{Z}$ with Assumption 1 is also (partially) treated in [BGdH11]: they prove that the quenched and annealed critical points are equal if $\alpha \geq 1$ and different if $\alpha < \frac{1}{2}$.

The question of the strict inequality between the quenched and annealed critical point for the wetting on a random walk (4.1) is however slightly different and remains open. Abiding by Harris' predictions [Har74] for disorder relevance in physical systems[2], since the annealed model has a critical exponent $\nu = \min(1, \alpha)$, which is smaller than 2 in the case $\alpha < 2$, disorder should then be relevant, meaning that $\lambda_c^{\mathrm{que}} > \lambda_c^{\mathrm{ann}}$. The case $\alpha = 2$ (of random walks in the Gaussian domain of attraction) is *marginal* for the question of disorder relevance and there is no clear prediction; one might expect that in analogy with the pinning models [BL18, GLT10], one has $\lambda_c^{\mathrm{que}} > \lambda_c^{\mathrm{ann}}$ also in that case, but this is a questionable conjecture here.

4.3. **Wetting in higher dimension.** Another natural model to consider is the wetting of a square well in dimension $d \geq 2$. For some compact domain $B \subset \mathbb{R}^d$ (of connected interior), we could consider the wetting of an interface in a well of shape B. Let $B_N := (NB) \cap \mathbb{Z}^d$ and consider the following model of wetting in a well of depth aN:

$$\mathrm{d}\mathbf{P}^a_{B_N,\lambda} := \frac{1}{Z^a_{B_N,\lambda}} e^{-H^0_{B_N}(\phi)} \prod_{x\in B_N} \big(\mu(\mathrm{d}\phi_x)\mathbb{1}_{\{\phi_x > -\lfloor aN\rfloor\}} + \lambda\delta_{-\lfloor aN\rfloor}(\mathrm{d}\phi_x)\big)\,,$$

[2]The prediction is made for a disorder which is i.i.d., but it has been confirmed in the random walk pinning model, see [BT10, BS10, BS11] and also [AB18].

where $H^0_{B_N}(\phi)$ is a Hamiltonian associated with a random interface $\phi : B_N \to \mathbb{R}^d$ (or $\mathbb{Z}^d$), with zero boundary condition (that is such that $\phi_x \equiv 0$ on B_N^c).

A natural choice for the underlying interface model is a gradient model, *i.e.* a Hamiltonian

$$H^0_{B_N}(\phi) := \sum_{\substack{x,y\in B_N\\ x\sim y}} V(\phi_x - \phi_y) + \sum_{\substack{x\in B_N, y\notin B_N\\ x\sim y}} V(\phi_x)\,,$$

for some potential $V(\cdot)$; we refer to [Fun05] for an overview. The choice $V(x) = \frac{1}{2}x^2$ corresponds to the massless Gaussian Free Field (GFF); the choice $V(x) = |x|$ corresponds to the Solid-On-Solid (SOS) model. The wetting of the GFF and SOS has been considered in [CV00], where it is shown that $\lambda_c > 1$ for the GFF in dimension $d = 2$ or for the SOS model in any dimension $d \geq 2$; on the other hand, one has that $\lambda_c = 1$ for the GFF in dimension $d \geq 3$, see [BDZ00, GL18].

A first goal would be to obtain a formula for the free energy of the model, defined as

$$\mathrm{F}_B(\lambda, a) := \lim_{N\to\infty} \frac{1}{N^d} \log Z^a_{B_N,\lambda}\,.$$

In the spirit of Theorem 3.1, we expect that one has $\mathrm{F}_B(\lambda, a) = \max\{\psi_B(\lambda, a), 0\}$, with

$$\psi_B(\lambda, a) := \sup_{D\subset B} \big\{\mathrm{F}(\lambda)|D| - I_B(a, D)\big\}\,. \tag{4.2}$$

Here, $\mathrm{F}(\lambda) := \lim_{N\to\infty} \frac{1}{|D_N|} \log Z^{a=0}_{D_N,\lambda}$ is the free energy per unit volume of the (usual) wetting model and $I_B(a, D)$ is the rate function for the large deviations of the interface:

$$I_B(a, D) := \lim_{N\to\infty} -\frac{1}{N^d} \log \mathbf{P}_{B_N}\big(\phi_x \leq -aN \ \forall x \in D_N\big)\,,$$

where $\mathbf{P}_{B_N}$ is the law of the underlying interface, that is $\mathrm{d}\mathbf{P}_{B_N}(\phi) := \frac{1}{Z_{B_N}} e^{-H^0_{B_N}(\phi)} \prod_{x\in B_N} \mu(\mathrm{d}\phi_x)$. The main difficulty in proving (4.2) should come from the large number of choices for the possible microscopic "wetted" regions; in analogy with [BI97], a coarse graining of these regions might be necessary.

We stress that for strictly convex potentials $V(\cdot)$ with bounded second derivative (in particular for the GFF), [DGI00] proves a strong large deviation principle for the rescaled surface $\frac{1}{N}\phi$, with speed N^d and rate function $\Sigma(u) = \int_B \sigma(\nabla u(\theta))\mathrm{d}\theta$, where $\sigma(\cdot)$ is the surface tension of the model. In particular, for the GFF one has $\sigma(v) = \frac{1}{2}\|v\|^2$, so one obtains that the above rate function is $I_B(a, D) = da^2\mathrm{Cap}_B(D)$, where $\mathrm{Cap}_B(D)$ is the Newtonian capacity of D with respect to B, namely

$$\mathrm{Cap}_B(D) := \sup\Big\{\frac{1}{2d}\int_D \|\nabla f(x)\|^2 \mathrm{d}x\,, f \in \mathcal{C}^\infty_c, f(x) \geq 1 \text{ for } x \in D, f(x) = 0 \text{ for } x \in B^c\Big\}\,.$$

In general, if $V(x) = c_0\|x\|^p$ for some $p \geq 1$ (in particular for the SOS model if $p = 1$), one should also obtain that $I_B(a, D) = 2dc_0a^p\mathrm{Cap}^{(p)}_B(D)$, where $\mathrm{Cap}^{(p)}_B(D)$ is the p-capacity of D with respect to B, namely

$$\mathrm{Cap}^{(p)}_B(D) := \sup\Big\{\frac{1}{2d}\int_D \|\nabla f(x)\|^p \mathrm{d}x\,, f \in \mathcal{C}^\infty_c, f(x) \geq 1 \text{ for } x \in D, f(x) = 0 \text{ for } x \in B^c\Big\}\,.$$

Remark 4.1. *One could also consider other type of interfaces, for instance with Laplacian interactions (also known as the* membrane *model), where the Hamiltonian is $H^0_{B_N}(\phi) = \sum_{x\in D_N} V(\Delta\phi_x)$, with Δ the discrete Laplacian. This is already an interesting question in dimension $d = 1$, where the wetting model has been considered in* [CD08] *(it is shown that $\lambda_c > 1$). The δ-pinning of the*

membrane model has also been considered in [BCK16, Sch22] *in dimension* $d \geq 4$*, but many open questions remain, especially in dimension* $d = 2, 3$ *where it is not known whether the critical point is strictly positive or not. We refer to the above references for more details on this model.*

5. Large deviations: notation and preliminary observations

Recall that the rate functions $\mathrm{I}_+, \mathrm{I}_-$ defined in (3.4) are (upward and downward) large deviation rate functions for $(S_n)_{n\geq 0}$, see [DZ09, Thm. 2.2.3], namely

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(S_n \geq xn) = -\mathrm{I}_+(x)\,, \qquad \lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(S_n \leq -xn) = -\mathrm{I}_-(x)\,. \tag{5.1}$$

Note that the remark after [DZ09, Thm. 2.2.5] (or simply using Chernov's exponential inequality) tells that we have the following useful upper bounds: for any $n \geq 0$ and $x \geq 0$

$$\mathbf{P}(S_n \geq xn) \leq e^{-n\mathrm{I}_+(x)}\,, \quad \mathbf{P}(S_n \leq -xn) \leq e^{-n\mathrm{I}_-(x)}\,. \tag{5.2}$$

5.1. **Some remarks on the rate functions.** Let us recall that for $t \geq 0$ we have defined $\Lambda_+(t) = \log \mathbf{E}[e^{tX_1}]$ and $\Lambda_-(t) = \log \mathbf{E}[e^{-tX_1}]$ and their respective radius of convergence:

$$t_0^+ := \sup\{t \geq 0, \Lambda_+(t) < +\infty\} \in [0, +\infty]\,, \quad t_0^- := \sup\{t \geq 0, \Lambda_-(t) < +\infty\} \in [0, +\infty]\,.$$

Some standard properties of the log-moment generating functions $\Lambda_\pm$ and of the rate functions $\mathrm{I}_\pm$ are given in [DZ09, Lem. 2.2.5]. Let us summarize some of the properties here for Λ_+ and I_+ (similar statements hold for Λ_- and I_-):

- Λ_+ is non-decreasing and convex and I_+ is a convex non-decreasing rate function;
- If $t_0^+ = 0$ then $\mathrm{I}_+(x) = 0$ for all $x \geq 0$;
- If $t_0^+ > 0$, then Λ_+ is differentiable and strictly convex on $[0, t_0^+)$ and I_+ is strictly increasing on $\{x \geq 0, \mathrm{I}_+(x) < +\infty\}$. Additionally, $\Lambda'_+(s) = y$ implies that $\mathrm{I}_+(y) = sy - \Lambda_+(s)$.

Lets us recall the definition of ρ_+, ρ_-, the limiting slope of Λ when approaching the radius of convergence, as in (3.10):

$$\rho_+ := \lim_{t\uparrow t_0^+} \Lambda'_+(t)\,, \qquad \rho_- := \lim_{t\uparrow t_0^-} \Lambda'_-(t)\,.$$

Let us also introduce the extremal points of the support of the random walk increments:

$$\bar{x}_+ := \sup\{x \in \mathbb{N}, \mathbf{P}(X_1 = x) > 0\}\,, \qquad \bar{x}_- := \sup\{x \in \mathbb{N}, \mathbf{P}(X_1 = -x) > 0\}\,. \tag{5.3}$$

There are mostly four cases that we need to consider (we focus on the "+" case but similar considerations hold for the "−" case): we summarize them in Table 1 below.

Let us introduce, for $t \in [0, t_0^+)$, the tilted measure $\mathbf{P}_t$ as

$$\frac{\mathrm{d}\mathbf{P}_t}{\mathrm{d}\mathbf{P}}(x) = \frac{e^{tx}}{\mathbf{E}[e^{tX_1}]} = e^{tx - \Lambda_+(x)}\,. \tag{5.4}$$

Notice that Λ_+ is analytic on $[0, t_0^+)$ and that $\Lambda'_+(t) = \mathbf{E}_t[X_1]$ is strictly increasing from $[0, t_0^+)$ to $[0, \rho_+)$ (Assumption 1 implies that $\mathbf{E}[X_1] = 0$ whenever X_1 has a finite expectation) and that $\Lambda''(t) = \mathbf{V}\mathrm{ar}_t(X_1) > 0$ (otherwise X_1 would be a.s. constant).

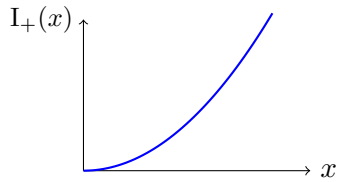
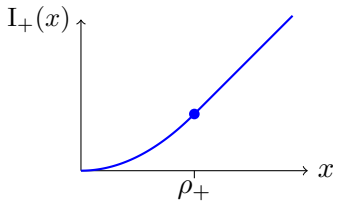
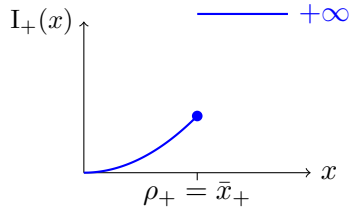
<table>
<tr><td>$t_0^+ = +\infty$
$\bar{x}_+ = +\infty$</td><td>$t_0^+ < +\infty$
$\bar{x}_+ = +\infty$</td><td colspan="2">$t_0^+ < +\infty$
$\bar{x}_+ = +\infty$</td><td>$t_0^+ = +\infty$
$\bar{x}_+ = \rho_+ < +\infty$</td></tr>
<tr><td colspan="2">$\rho_+ = +\infty$</td><td colspan="3">$\rho_+ < +\infty$</td></tr>
<tr><td colspan="2">I_+ is strictly convex on $\mathbb{R}_+$
and $\lim_{x\to\infty} \mathrm{I}_+(x) = +\infty$</td><td colspan="2">$\mathrm{I}_+$ is strictly convex and finite on $[0, \rho_+]$ and
is affine on $(\rho_+, +\infty)$</td><td>is infinite on $(\rho_+, +\infty)$</td></tr>
<tr><td colspan="2"> 
</td><td colspan="2"> 
</td><td> 
</td></tr>
</table>

TABLE 1. Summary of the different possibilities if $t_0^+ > 0$. In all cases, I_+ is strictly convex on $[0, \rho_+)$. Similar statements hold true for I_-.

For all $x \in [0, \rho_+)$, the supremum in the definition (3.4) is attained at $t_x := (\Lambda'_+)^{-1}(x)$, such that $\Lambda'_+(t_x) = x$. We therefore get, after some classical calculations

$$\begin{aligned} &\mathrm{I}_+(x) = xt_x - \Lambda_+(t_x) = x(\Lambda'_+)^{-1}(x) - \Lambda_+\left((\Lambda'_+)^{-1}(x)\right) \\ &\mathrm{I}'_+(x) = t_x = (\Lambda'_+)^{-1}(x)\,, \qquad \mathrm{I}''_+(x) = \frac{1}{\Lambda''((\Lambda'_+)^{-1}(x))} = \frac{1}{\mathbf{V}\mathrm{ar}_{t_x}(X_1)} < +\infty\,. \end{aligned} \tag{5.5}$$

Note also that if $\rho_+ < \infty$ and $t_0^+ < \infty$, then I_+ is defined (and affine) on $[\rho_+, +\infty)$, see Table 1: we have $\mathrm{I}_+(x) = xt_0^+ - \Lambda_+(t_0^+)$. In that case, I_+ is differentiable on $\mathbb{R}_+$, with $\mathrm{I}'_+(x) = t_0^+ = (\Lambda'_+)^{-1}(\rho_+)$ for $x \geq \rho_+$,

To summarize, we have that I_+ is differentiable on $[0, \bar{x}_+)$ and left-differentiable at $\bar{x}_+$ with

$$\mathrm{I}'_+(x) = \begin{cases} (\Lambda'_+)^{-1}(x) & \text{if } x \in (0, \rho_+)\,, \\ t_0^+ & \text{if } x \geq \rho_+\,, \end{cases} \tag{5.6}$$

which is continuous in the case where $\rho_+ < +\infty$. As another useful formula that derives from the above, we have

$$\mathrm{I}_+(x) - x\mathrm{I}'_+(x) = -\begin{cases} \Lambda_+ \circ (\Lambda'_+)^{-1}(x) & \text{if } x \in (0, \rho_+)\,, \\ \Lambda_+(t_0^+) & \text{if } x \geq \rho_+\,. \end{cases} \tag{5.7}$$

5.2. **A local large deviations result.** We complement here Cramér's large deviations (5.1) with a local version; we were not able to find a reference for it, so we prove it in Appendix C.

Lemma 5.1. *Suppose that Assumption 1 holds. For any any sequence of non-negative integers $(x_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \frac{1}{n} x_n = x < \bar{x}_+$, we have the following local large deviation behavior:*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(S_n = x_n) = -\mathrm{I}_+(x) \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(S_n = -x_n) = -\mathrm{I}_-(x)\,.$$

Note that if $x = \bar{x}_+ < +\infty$, then we have $\mathbf{P}(S_n = \bar{x}_+ n) = \mathbf{P}(X_1 = \bar{x}_+)^n$, which also gives $\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(S_n = x_n) = -\mathrm{I}_+(x)$ since one can easily check that $\mathrm{I}_+(\bar{x}_+) = -\log \mathbf{P}(X_1 = \bar{x}_+)$.

However, the result cannot hold for any sequence x_n such that $\lim_{n\to\infty} \frac{1}{n} x_n = \bar{x}_+$; for instance $\mathbf{P}(S_n = x_n) = 0$ for any $x_n > \bar{x}_+ n$.

6. Optimizers of $\psi(\lambda, a)$, formula for the free energy and phase transitions

In this section, we prove Lemma 3.5 that gives the location of the maximizers of $\psi(\lambda, a)$. We then derive from it the formulas for the free energy $\mathrm{F}(\lambda)$ and the critical line $a_c(\lambda)$ given in Theorem 3.3, and we deduce the critical behavior of $\mathrm{F}(\lambda, a)$ and $a_c(\lambda)$ of Proposition 3.13.

Before that, let us make one simple observation on the limitation of the depth of the well.

Lemma 6.1. *Recall the definition (5.3) of $\bar{x}_+, \bar{x}_-$; notice also that $\bar{x}_\pm = \sup\{x\,, \mathrm{I}_\pm(x) < +\infty\}$. Let us define*

$$\bar{a} := \Big(\frac{1}{\bar{x}_-} + \frac{1}{\bar{x}_+}\Big)^{-1} = \frac{\bar{x}_+ \bar{x}_-}{\bar{x}_+ + \bar{x}_-} \in (0, +\infty]\,, \tag{6.1}$$

with the convention $\frac{1}{\infty} = 0$ and $\frac{1}{0} = +\infty$ in cases where $\bar{x}_+$ and/or $\bar{x}_-$ are infinite. Then, if $a > \bar{a}$, the walk cannot reach the bottom of the well, i.e. $\mathbf{P}(H_N^a(S) \geq 1) = 0$.

Proof. If the walk has a contact with the bottom of the well, *i.e.* $H_N^a(S) \geq 1$, then by definition of $\bar{x}_+$, $\bar{x}_-$, the left-most contact point $\mathbf{P}$-a.s. verifies $L_N \geq \frac{\lfloor aN \rfloor}{\bar{x}_-}$ and the right-most one $N - R_N \geq \frac{\lfloor aN \rfloor}{\bar{x}_+}$. Since we must have $L_N + N - R_N \leq N$, this implies that $a(\frac{1}{\bar{x}_-} + \frac{1}{\bar{x}_+}) \leq 1$, that is $a \leq \bar{a}$. □

Remark 6.2. *In the case $a = \bar{a}$, since $H_N(S) \leq R_N - L_N + 1$, the same reasoning gives that $H_N(S) \leq N - \lfloor \bar{a} N \rfloor (\frac{1}{\bar{x}_-} + \frac{1}{\bar{x}_+}) + 1 \leq \frac{1}{\bar{x}_-} + \frac{1}{\bar{x}_+} + 1$, where we have used that $\lfloor \bar{a} N \rfloor \geq \bar{a} N - 1$ with $\bar{a}(\frac{1}{\bar{x}_-} + \frac{1}{\bar{x}_+}) = 1$. This shows that in the case $a = \bar{a}$, the number of contacts is bounded by $\bar{a}^{-1} + 1$.*

Remark 6.3. *The reasoning of Lemma 6.1 also translates into the fact that we have $\psi(\lambda, a) = -\infty$ for any $a > \bar{a}$. Indeed, having $u\mathrm{I}_-(\frac{a}{u}) + (1-v)\mathrm{I}_+(\frac{a}{1-v}) < +\infty$ with $u \leq v$ means that $\frac{a}{u} \leq \bar{x}_-$ and $\frac{a}{1-u} \leq \frac{a}{1-v} \leq \bar{x}_+$, since $\mathrm{I}_\pm(x) = +\infty$ if $x > \bar{x}_\pm$. We therefore end up with $\frac{1}{a} = \frac{u}{a} + \frac{1-u}{a} \geq \frac{1}{\bar{x}_-} + \frac{1}{\bar{x}_+}$, which implies that $a \leq \bar{a}$.*

In the case $a = \bar{a} < +\infty$, the only non-zero term in the supremum (3.5) is at $u_0 = \frac{\bar{x}_-}{\bar{a}}$ and $v_0 = 1 - \frac{\bar{x}_+}{\bar{a}} = u_0$, so that

$$\psi(\lambda, \bar{a}) = g_{\lambda,\bar{a}}(u_0, v_0) = -\frac{\bar{a}}{\bar{x}_-}\mathrm{I}_-(\bar{x}_-) - \frac{\bar{a}}{\bar{x}_+}\mathrm{I}_+(\bar{x}_+) \in (-\infty, 0)\,. \tag{6.2}$$

6.1. Optimizers of $\psi(\lambda, a)$: proof of Lemma 3.5.

Let us fix $\lambda, a \leq \bar{a}$ and recall the definition of $g_{\lambda,a}$ in (3.5): for $(u, v) \in \mathcal{D} = \{(u, v)\,, 0 \leq u \leq v \leq 1\}$,

$$g_{\lambda,a}(u, v) := (v - u)\mathrm{F}(\lambda) - u\,\mathrm{I}_-\Big(\frac{a}{u}\Big) - (1 - v)\,\mathrm{I}_+\Big(\frac{a}{1 - v}\Big)\,, \tag{6.3}$$

so that $\psi(\lambda, a) := \sup_{(u,v)\in\mathcal{D}} g_{\lambda,a}(u, v)$. Notice that since $\mathrm{I}_\pm(x) = +\infty$ if $x > \bar{x}_\pm$, then if there is a maximum for $g_{\lambda,a}$ it must be reached for $(u, v) \in \mathcal{D}_a = \{\frac{a}{\bar{x}_-} \leq u \leq v \leq 1 - \frac{a}{\bar{x}_+}\}$.

Let us first deal with the degenerate cases. Recall the definition (3.6) of t_0^+, t_0^-.

Case $t_0^+ = t_0^- = 0$. In that case, we have $\mathrm{I}_+ = \mathrm{I}_- \equiv 0$, so $g_{\lambda,a}(u, v) = (v - u)\mathrm{F}(\lambda)$, which is clearly maximized for $(u^*, v^*) = (0, 1)$. Note that the value of (u^*, v^*) matches the formula for $U^* \times V^*$ since $w_-^* = w_+^* = 0$ in that case.

Case $a = \bar a < +\infty$. Since $\frac{\bar a}{\bar x_-} = 1 - \frac{\bar a}{\bar x_+}$, then as noticed in Remark 6.3, the supremum in (3.5) is attained at the unique value (u_0, v_0) and $\psi(\lambda, \bar a) = g_{\lambda,\bar a}(u_0, v_0) < 0$, see (6.2). This in turn shows that $\lambda < \lambda_c(a)$, and in particular $\mathrm{F}(\lambda, \bar a) = 0$. Note that the value (u_0, v_0) does not match here the formula for $U^* \times V^*$ since $w^*_- + w^*_+ > 1$ in that case (note that we have $t_0^+ = +\infty$ if $\bar x_+ < +\infty$, see Table 1, so in view of the formula (3.11) we have $w^*_+ > \frac{\bar a}{\rho_+} = \frac{\bar a}{\bar x_+}$).

Case $a < \bar a$. In that case we have $\psi(\lambda, a) > -\infty$. Note that we can write

$$g_{\lambda,a}(u, v) = \mathrm{F}(\lambda) - g^-_{\lambda,a}(u) - g^+_{\lambda,a}(1 - v)\,, \tag{6.4}$$

with

$$g^-_{\lambda,a}(w_1) = w_1 \mathrm{F}(\lambda) + w_1 \mathrm{I}_-\Big(\frac{a}{w_1}\Big)\,, \qquad g^+_{\lambda,a}(w_2) = w_2 \mathrm{F}(\lambda) + w_2 \mathrm{I}_+\Big(\frac{a}{w_2}\Big)\,.$$

As a first (and central) step, we therefore identify the minimizers of $g^+_{\lambda,a}(\cdot)$, resp. $g^-_{\lambda,a}(\cdot)$.

Lemma 6.4. *We have* $\inf_{w\ge 0} g^+_{\lambda,a}(w) = a\Lambda_+^{-1}(\mathrm{F}(\lambda))$, *and the infimum is attained on* W^*_+, *with*

$$W^*_+ := \begin{cases} \{w^*_+\} & \text{if } \mathrm{F}(\lambda) < \Lambda_+(t_0^+)\,, \\ [0, \frac{a}{\rho_+}] & \text{if } \mathrm{F}(\lambda) = \Lambda_+(t_0^+)\,, \\ \{0\} & \text{if } \mathrm{F}(\lambda) > \Lambda_+(t_0^+)\,, \end{cases} \qquad \text{with } w^*_+ := \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))} \in \Big[\frac{a}{\rho_+}, +\infty\Big)\,.$$

Proof. Recalling (5.5)-(5.6) and (5.7), we have that for any $w > \frac{a}{\bar x_+}$

$$\frac{\partial}{\partial w} g^+_{\lambda,a}(w) = \mathrm{F}(\lambda) + \mathrm{I}_+\Big(\frac{a}{w}\Big) - \frac{a}{w}\mathrm{I}'_+\Big(\frac{a}{w}\Big) = \mathrm{F}(\lambda) - \begin{cases} \Lambda_+ \circ (\Lambda'_+)^{-1}\big(\frac{a}{w}\big) & \text{if } \frac{a}{w} < \rho_+\,, \\ \Lambda_+(t_0^+) & \text{if } \frac{a}{w} \ge \rho_+\,. \end{cases} \tag{6.5}$$

Let us now consider different cases; we refer to Table 1 for an overview, see also (5.7).

• If $\rho_+ = +\infty$, then

$$\frac{\partial}{\partial w} g^+_{\lambda,a}(w) = \mathrm{F}(\lambda) - \Lambda_+ \circ (\Lambda'_+)^{-1}\Big(\frac{a}{w}\Big) \qquad \text{for all } w > 0\,.$$

∗ If $t_0^+ = +\infty$, then Λ_+ and Λ'_+ are increasing bijections from $(0, +\infty)$ to $(0, +\infty)$. From (6.5) we get that $\frac{\partial}{\partial w} g^+_{\lambda,a}(w) > 0$ for $\frac{a}{w} < \Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))$ and $\frac{\partial}{\partial w} g^+_{\lambda,a} < 0$ for $\frac{a}{w} > \Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))$. Letting $w^*_+ := \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))}$, this shows that $g^+_{\lambda,a}$ is decreasing on $[0, w^*_+]$ and increasing on $[w^*_+, +\infty)$. We therefore get that $g^+_{\lambda,a}$ has a unique minimum at $w^*_+ \in (0, +\infty)$.

Using the formulas (5.5) for $\mathrm{I}_+(\frac{a}{w^*_+})$ (and the fact that $a/w^*_+ < \rho_+ = +\infty$), the minimal value of $g^+_{\lambda,a}$ is then

$$\inf_{w\ge 0} g^+_{\lambda,a}(w) = w^*_+ \mathrm{F}(\lambda) + w^*_+\Big(\frac{a}{w^*_+}(\Lambda'_+)^{-1}\Big(\frac{a}{w^*_+}\Big) - \Lambda_+ \circ (\Lambda'_+)^{-1}\Big(\frac{a}{w^*_+}\Big)\Big) = a\Lambda_+^{-1}(\mathrm{F}(\lambda))\,. \tag{6.6}$$

where we have also used the formula for w^*_+ together with the fact that Λ_+, Λ'_+ are proper bijections.

∗ If $t_0^+ < +\infty$, then Λ_+ is an increasing bijections from $(0, t_0^+)$ to $(0, x\Lambda_+(t_0^+))$ and Λ'_+ is an increasing bijection from $(0, t_0^+)$ to $(0, +\infty)$ (recall $\rho_+ = +\infty$). In view of (6.5), we distinguish two subcases.

If $\mathrm{F}(\lambda) < \Lambda_+(t_0^+)$, then $\Lambda_+ \circ (\Lambda'_+)^{-1}(x) = \mathrm{F}(\lambda)$ if and only if $x = \Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda)) < \Lambda'_+(t_0^+) = +\infty$: we then get that $w^*_+ := \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))} > 0$ is the unique minimizer of $g^+_{\lambda,a}$.

If $\mathrm{F}(\lambda) \geq \Lambda_+(t_0^+)$, then $\mathrm{F}(\lambda) > \Lambda_+ \circ (\Lambda'_+)^{-1}(x)$ for any $x < +\infty$ (since $(\Lambda'_+)^{-1}(x) < t_0^+$): this shows that $g^+_{\lambda,a}$ is increasing on $(0,+\infty)$ so it attains its unique minimum at $w_+^* = 0$. Note that it also corresponds to the formula $w_+^* := \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))}$ since $\Lambda_+^{-1}(\mathrm{F}(\lambda)) = t_0^+$ and $\Lambda'_+(t_0^+) = \rho_+ = +\infty$.

The minimal value of $g^+_{\lambda,a}$ is then

$$\inf_{w\geq 0} g^+_{\lambda,a}(w) = \left\{ \begin{array}{ll} a\Lambda_+^{-1}(\mathrm{F}(\lambda)) & \text{if } \mathrm{F}(\lambda) < \Lambda_+(t_0^+) \\ \lim\limits_{w\downarrow 0} w\mathrm{I}_+\big(\frac{a}{w}\big) = at_0^+ & \text{if } \mathrm{F}(\lambda) \geq \Lambda_+(t_0^+) \end{array} \right\} = a\Lambda_+^{-1}(\mathrm{F}(\lambda))\,. \tag{6.7}$$

Here, we have used in the second line that $\liminf_{x\to\infty} \frac{1}{x}\mathrm{I}_+(x) = t_0^+$, which follows from the fact that $\frac{1}{\Lambda'_+(y)}\mathrm{I}_+(\Lambda'_+(y)) = y - \frac{\Lambda_+(y)}{\Lambda'_+(y)}$, which goes to t_0^+ as $y \uparrow t_0^+$ (indeed, $\frac{\Lambda_+(y)}{\Lambda'_+(y)}$ goes to 0 since we are only concerned with the case $\Lambda_+(t_0) \leq \mathrm{F}(\lambda) < +\infty$ and $\Lambda'_+(t_0^+) = \rho_+ = +\infty$).

• If $\rho_+ < +\infty$, then we again have two subcases.

∗ If $t_0^+ = +\infty$. Then $\rho_+ = \bar{x}_+ < +\infty$, and $g^+_{\lambda,a}(w) = +\infty$ for $w < a/\bar{x}_+$. On the other hand, Λ'_+ an increasing bijection from $[0,\bar{x}_+]$ to $[0,\rho_+]$ and Λ_+ an increasing bijection from $(0,+\infty)$ to $(0,+\infty)$. From (6.5), we get as above that $g^+_{\lambda,a}$ is decreasing on $[0,w_+^*]$ and increasing on $[w_+^*,+\infty)$, with $w_+^* := \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))} \in [\frac{a}{\bar{x}_+}, +\infty)$.

As in (6.6), we have that the minimal value of $g^+_{\lambda,a}$ is

$$\inf_{w\geq 0} g^+_{\lambda,a}(w) = a\Lambda_+^{-1}(\mathrm{F}(\lambda))\,. \tag{6.8}$$

∗ If $t_0^+ < +\infty$. Then $\rho_+ < \bar{x}_+ = +\infty$. Then in view of (6.5), we distinguish as above in several subcases.

First, if $\mathrm{F}(\lambda) < \Lambda_+(t_0^+)$, then $\Lambda_+ \circ (\Lambda'_+)^{-1}(x) = \mathrm{F}(\lambda)$ if and only if $x = \Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda)) < \Lambda'_+(t_0^+) = +\infty$: we then get that $w_+^* := \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))} > \frac{a}{\rho_+}$ is the unique minimizer of $g^+_{\lambda,a}$.

Second, if $\mathrm{F}(\lambda) = \Lambda_+(t_0^+)$, then $g^+_{\lambda,a}$ is constant on $[0,w_+^*]$ and then increasing, where $w_+^* = \frac{a}{\rho_+} = \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))} > 0$. In that case the minimum of $g^+_{\lambda,a}$ is attained on the whole interval $[0, \frac{a}{\rho_+}]$.

Third, if $\mathrm{F}(\lambda) > \Lambda_+(t_0^+)$, then $g^+_{\lambda,a}$ is increasing on $(0,+\infty)$ so it attains its unique minimum at $w_+^* = 0$. Note that *it does not* corresponds to the formula $w_+^* := \frac{a}{\Lambda'_+ \circ \Lambda_+^{-1}(\mathrm{F}(\lambda))}$ since $\Lambda_+^{-1}(\mathrm{F}(\lambda)) = t_0^+$ and $\Lambda'_+(t_0^+) = \rho_+ < +\infty$.

The minimal value of $g^+_{\lambda,a}$ is then

$$\inf_{w\geq 0} g^+_{\lambda,a}(w) = \left\{ \begin{array}{ll} a\Lambda_+^{-1}(\mathrm{F}(\lambda)) & \text{if } \mathrm{F}(\lambda) < \Lambda_+(t_0^+) \\ \lim\limits_{w\downarrow 0} w\mathrm{I}_+\big(\frac{a}{w}\big) = at_0^+ & \text{if } \mathrm{F}(\lambda) \geq \Lambda_+(t_0^+) \end{array} \right\} = a\Lambda_+^{-1}(\mathrm{F}(\lambda))\,. \tag{6.9}$$

Here, we have used that $\lim_{x\to\infty} \frac{1}{x}\mathrm{I}_+(x) = \mathrm{I}'_+(\rho_+) = t_0^+$, since $\mathrm{I}_+(x)$ is affine on $[\rho_+,+\infty)$. □

We can now use Lemma 6.4 into (6.4), to conclude the proof of Lemma 3.5. Recall the definition (3.11) of w_+^*, w_-^*. Now, in the case where $w_+^* + w_-^* \leq 1$, then thanks to Lemma 6.4 (since $w \mapsto g^\pm_{\lambda,a}(w)$ is increasing for $w < w_\pm$), we obtain that

$$\sup_{0\leq u\leq v\leq 1} g_{\lambda,a}(u,v) = \mathrm{F}(\lambda) - a\Lambda_-^{-1}(\mathrm{F}(\lambda)) - a\Lambda_+^{-1}(\mathrm{F}(\lambda))\,,$$

with the supremum attained on $U^* \times V^*$, with

$$U^* = \begin{cases} \{w_-^*\} & \text{if } \mathrm{F}(\lambda) < \Lambda_-(t_0^-)\,, \\ [0, \frac{a}{\rho_-}] & \text{if } \mathrm{F}(\lambda) = \Lambda_-(t_0^-)\,, \\ \{0\} & \text{if } \mathrm{F}(\lambda) > \Lambda_-(t_0^-)\,, \end{cases} \qquad V^* = \begin{cases} \{1 - w_+^*\} & \text{if } \mathrm{F}(\lambda) < \Lambda_+(t_0^+)\,, \\ [1 - \frac{a}{\rho_+}, 1] & \text{if } \mathrm{F}(\lambda) = \Lambda_+(t_0^+)\,, \\ \{1\} & \text{if } \mathrm{F}(\lambda) > \Lambda_+(t_0^+)\,. \end{cases}$$

The last part of Lemma 3.5 tells that the condition $w_+^* + w_-^* \le 1$ is ensured by having $\lambda \ge \lambda_c(a)$. This is a consequence of the following lemma.

Lemma 6.5. *If $\lambda \ge \lambda_c(a)$, then we have $w_+^* + w_-^* < 1$.*

Proof. First of all, this is trivial in the case where $t_0^+ = t_0^- = 0$ since in that case we have $w_+^* = w_-^* = 0$ as noticed above. Also, if $a = \bar{a} < +\infty$, we have that $\psi(\lambda, a) < 0$ for any $\lambda \ge 0$ (see (6.2)), so $\lambda_c(a) = +\infty$ and this case has to be excluded.

Let us now focus on the case where $a < \bar{a}$ and $t_0^+ + t_0^- > 0$, so in particular one of I_+ or I_- is non-degenerate. Let us assume that $w_+^* + w_-^* \ge 1$. Then we show that

$$\inf_{(u,v)\in\mathcal{D}} \left\{ g_{\lambda,a}^-(u) + g_{\lambda,a}^+(1-v) \right\} = \inf_{u\in[0,1]} \left\{ g_{\lambda,a}^-(u) + g_{\lambda,a}^+(1-u) \right\}, \tag{6.10}$$

namely the infimum in (3.5) is attained on the diagonal $\{(u,u), u \in [0,1]\}$. In the end, this shows that

$$\psi(\lambda, a) = -\inf_{u\in[0,1]} \left\{ u\,\mathrm{I}_-\Big(\frac{a}{u}\Big) + (1-u)\,\mathrm{I}_+\Big(\frac{a}{1-u}\Big) \right\} < 0\,,$$

because at least one of $\mathrm{I}_+, \mathrm{I}_-$ is non-degenerate. Since $\lambda_c(a) = \sup_{\lambda\ge 0}\{\psi(\lambda, a) < 0\}$ and $\lambda \mapsto \psi(\lambda, a)$ is continuous and non-decreasing, this proves that $\lambda < \lambda_c(a)$, which is a contradiction.

It therefore remains to show that if $w_+^* + w_-^* \ge 1$, then (6.10) holds. Let $(u, v) \in \mathcal{D}$ with $u < v$. Then necessarily, either $u < w_-^*$ or $1 - v > w_+^*$; let us assume $u \ne w_-^*$, the other case is treated analogously. If $u < w_-^*$, then recalling that $g_{\lambda,a}^-$ is decreasing on $[0, w_+^*]$, we have

$$g_{\lambda,a}^-(u) + g_{\lambda,a}^+(1-v) > g_{\lambda,a}^-(u') + g_{\lambda,a}^+(1-v)\,,$$

with $u' = \min\{w_+^*, 1-v\}$. This shows (6.10) and concludes the proof. □

As a corollary from the proof of Lemma 3.5, let us extract the following lemma for future use. It is a direct consequence from the rewriting (6.4) and the computation of the derivative of $g_{\lambda,a}^{\pm}$, see (6.5).

Lemma 6.6. *Assume that $\lambda \ge \lambda_c(a)$ and that $\mathrm{F}(\lambda) < \min\{\Lambda_+(t_0^+), \Lambda_-(t_0^-)\}$, so that w_+^*, w_-^* verifies $w_+^* > \frac{a}{\rho_+}$, $w_-^* > \frac{a}{\rho_-}$ and $w_+^* + w_-^* < 1$. Let $(u^*, v^*) = (w_-^*, 1 - w+^*)$ be the unique maximizer in (3.5), see Lemma 3.5. Then we have that*

$$\mathrm{F}(\lambda) + \mathrm{I}_-\Big(\frac{a}{u^*}\Big) - \frac{a}{u^*}\mathrm{I}_-'\Big(\frac{a}{u^*}\Big) = 0 \quad \text{and} \quad \mathrm{F}(\lambda) + \mathrm{I}_+\Big(\frac{a}{1-v^*}\Big) - \frac{a}{1-v^*}\mathrm{I}_+'\Big(\frac{a}{1-v^*}\Big) = 0\,.$$

6.2. **Formula for the free energy and phase transitions.** In this section, we derive a formula for $\psi(\lambda, a)$ and the critical point (Theorem 3.3), and we deduce the properties of the phase transitions and the critical curve (Propositions 3.13, 3.14 and 3.15).

6.2.1. *Formula for the free energy: proof of Theorem 3.3.* The formula (3.7) is a direct corollary of the proof of Lemma 3.5. For $\lambda < \lambda_c(a)$, we have that $\mathrm{F}(\lambda, a) = 0$. On the other hand, for $\lambda \geq \lambda_c(a)$, we have from Lemma 6.5 that $w_+^* + w_-^* < 1$ so we can apply Lemma 3.5 to get that

$$\psi(\lambda, a) = \mathrm{F}(\lambda) - a\Lambda_-^{-1}(\mathrm{F}(\lambda)) - a\Lambda_+^{-1}(\mathrm{F}(\lambda))\,.$$

This concludes the proof, since $\mathrm{F}(\lambda, a) = \psi(\lambda, a)$ for $\lambda \geq \lambda_c(a)$. The formula (3.8) for $a_c(\lambda)$ directly follows from (3.7), since the critical line is characterized by $\psi(\lambda, a) = 0$. □

6.2.2. *Localization phase transition: proof of Proposition 3.13.* Let $a > 0$ be such that $\lambda_c(a) < +\infty$, *i.e.* $a < \bar{a}$; we have $\mathrm{F}(\lambda_c(a), a) = 0$. Looking at the expression (3.7), we can compute the derivative of $\mathrm{F}(\lambda, a)$ at a given $\lambda > \lambda_c(a)$. Recalling that we defined λ_+ such that $\mathrm{F}(\lambda_+) = \Lambda_+(t_0^+)$, we have

$$\frac{\partial}{\partial\lambda}\Lambda_+^{-1}(\mathrm{F}(\lambda)) = \begin{cases} \frac{\mathrm{F}'(\lambda)}{\Lambda_+'\circ\Lambda_-^{-1}(\mathrm{F}(\lambda))} & \text{if } \lambda < \lambda_+\,, \\ 0 & \text{if } \lambda > \lambda_+\,. \end{cases}$$

Therefore, by taking the limit $\lambda \downarrow \lambda_c(a) > 0$, $\mathrm{F}(\cdot, a)$ has a right derivative at $\lambda_c(a)$, given by:

$$C_a := \lim_{\lambda\downarrow\lambda_c(a)} \frac{\partial \mathrm{F}(\lambda, a)}{\partial\lambda} = \mathrm{F}'(\lambda_c(a))\Big(1 - a\Big(\frac{\mathbb{1}_{\{\lambda_c(a)<\lambda_-\}}}{\Lambda_-' \circ \Lambda_-^{-1}(\mathrm{F}(\lambda_c(a)))} + \frac{\mathbb{1}_{\{\lambda_c(a)<\lambda_+\}}}{\Lambda_+' \circ \Lambda_+^{-1}(\mathrm{F}(\lambda_c(a)))}\Big)\Big)\,. \tag{6.11}$$

We therefore get, as $u \downarrow 0$,

$$\mathrm{F}(\lambda_c(a) + u, a) = \mathrm{F}(\lambda_c(a), a) + C_a\, u + o(u) \sim C_a u\,.$$

This concludes the first part of the proof. We deal with the behavior of C_a as $a \downarrow 0$ below (we deduce it from the behavior of $a_c(\lambda)$ as $\lambda \downarrow \lambda_c$). □

6.2.3. *Saturation phase transition: proof of Proposition 3.14.* Assume that $\lambda_- < \lambda_+ < +\infty$ and recall the definition (3.18) of the excess free energies $\mathrm{F}^*, \mathrm{F}^{**}$. Thanks to the formulas of Theorem 3.3, we have that

$$\mathrm{F}^*(\lambda, a) = a\big(t_0^- - \Lambda_-^{-1}(\mathrm{F}(\lambda))\big)\,, \qquad \mathrm{F}^{**}(\lambda, a) = a\big(t_0^+ - \Lambda_+^{-1}(\mathrm{F}(\lambda))\big)\,.$$

The statement simply follows once one observes that, by a Taylor expansion, since $t_0^- = \Lambda_-^{-1}(\mathrm{F}(\lambda_-))$

$$t_0^- - \Lambda_-^{-1}(\mathrm{F}(\lambda_- - u)) = (1 + o(1))\frac{\mathrm{F}'(\lambda_-)}{\Lambda_-'(\Lambda_-^{-1}(\mathrm{F}(\lambda_-)))}u = (1 + o(1))\frac{\mathrm{F}'(\lambda_-)}{\rho_-}u \quad \text{as } u \downarrow 0\,. \tag{6.12}$$

This gives the behavior of $\mathrm{F}^*(\lambda_- - u, a)$ as $u \downarrow 0$.

When $\lambda_- = \lambda_+$, we have $\mathrm{F}^*(\lambda, a) = \mathrm{F}^{**}(\lambda, a) = a(t_0^-\Lambda_-^{-1}(\mathrm{F}(\lambda))) + a(t_0^+ - \Lambda_+^{-1}(\mathrm{F}(\lambda)))$ so the result follows in an identical manner. □

6.2.4. *Critical curve: proof of Proposition 3.15.* There are two parts in the statement.

Behavior as $\lambda \downarrow \lambda_c$. We use the formula (3.8), together with the fact that under Assumption 2 we have $\Lambda_\pm(x) \sim \frac{1}{2}\sigma^2 x^2$ as $x \downarrow 0$, where $\sigma^2 := \mathbf{E}[X_1^2]$ is the variance of X_1; hence $\Lambda_\pm^{-1}(x) \sim \sqrt{2x/\sigma^2}$. Using that $\mathrm{F}(\lambda) \downarrow 0$ as $\lambda \downarrow \lambda_c$, we get that

$$a_c(\lambda) \sim \frac{\sigma}{2\sqrt{2}}\mathrm{F}(\lambda)^{1/2}\,. \tag{6.13}$$

The conclusion follows by applying (2.13), using also the definition of c_3.

Behavior as $\lambda \uparrow \lambda_-, \lambda_+$. Assume $\lambda_- < \lambda_+ < \infty$. Similarly to the proof of Proposition 3.14, recalling the definition (3.19) of $a_c^*(\lambda)$ we have from the formulas of Theorem 3.3 that

$$a_c^*(\lambda) = \frac{\mathrm{F}(\lambda)}{(\Lambda_+^{-1}(\mathrm{F}(\lambda)) + \Lambda_-^{-1}(\mathrm{F}(\lambda)))(\Lambda_+^{-1}(\mathrm{F}(\lambda)) + t_0^-)} \big(t_0^- - \Lambda_-^{-1}(\mathrm{F}(\lambda))\big)\,.$$

The conclusion follows exactly as above, using (6.12); similarly for $a_c^{**}(\lambda)$. The case where $\lambda_- = \lambda_+ < +\infty$ is treated identically. □

Behavior of the constant C_a *in* (6.11) *as* $a \downarrow 0$. Let us now conclude the proof of Proposition 3.13, regarding the behavior of the constant C_a as $a \downarrow 0$.

First of all, notice that $\lambda_c(a) \downarrow \lambda_c$ as $a \downarrow 0$, so $\mathrm{F}(\lambda_c(a)) \downarrow 0$ as $a \downarrow 0$. In particular, we get that $\lambda_c(a) < \lambda_+, \lambda_-$ for a sufficiently small so we will use the formula (6.11) without the indicator functions.

From the asymptotic $a_c(\lambda) \sim \frac{\sigma^2}{2\sqrt{2}}\sqrt{c_3}(\lambda - \lambda_c)$ as $\lambda \downarrow \lambda_c$, we get that $\lambda_c(a) - \lambda_c \sim \frac{2\sqrt{2}}{\sigma^2\sqrt{c_3}}a$ as $a \downarrow 0$. Therefore, using (6.13), we get that

$$\mathrm{F}(\lambda_c(a)) \sim \frac{8a^2}{\sigma^2} \qquad \text{as } a \downarrow 0\,.$$

Now, under Cramér's condition, we have that $\Lambda_\pm' \circ \Lambda_\pm^{-1}(x) \sim \Lambda_\pm'(\sqrt{2x/\sigma^2}) \sim \sqrt{2\sigma^2 x}$ as $x \downarrow 0$, so that from the behavior of $\mathrm{F}(\lambda_c(a))$ that we just obtained, we get

$$\lim_{a\downarrow 0} \frac{a}{\Lambda_\pm' \circ \Lambda_\pm^{-1}(\mathrm{F}(\lambda_c(a)))} = \frac{1}{4}\,.$$

In view of (6.11), this gives that $C_a \sim \frac{1}{2}\mathrm{F}'(\lambda_c(a))$ as $a \downarrow 0$.

Now, using that $\mathrm{F}'(\lambda_c(a)) \sim 2c_3\sigma^2(\lambda_c(a) - \lambda_c)$ thanks to (2.13) (and convexity) and the fact that $\lambda_c(a) - \lambda_c \sim \frac{2\sqrt{2}}{\sigma^2\sqrt{c_3}}a$ as seen above, we end up with $C_a \sim 2\sqrt{2c_3}\,a$ as $a \downarrow 0$, which is what is claimed in Proposition 3.13.

7. Proof of Theorems 3.1 and 3.8

Recall the definition (3.5) of $\psi(\lambda, a)$. Note that when $a = 0$, we have $\psi(\lambda, 0) = \mathrm{F}(\lambda)$ since $\mathrm{I}_\pm(0) = 0$ (the supremum in (3.5) is attained for $u = 0, v = 1$), we therefore focus on the case $a > 0$.

7.1. Some preliminary notation.

For an event A, we denote

$$Z_{N,\lambda}^a(A) = \mathbf{E}\left[\lambda^{H_N(S)} \mathbb{1}_{\Omega_N^+(S)} \mathbb{1}_{\{S_N=0\}} \mathbb{1}_A\right]\,, \tag{7.1}$$

so that $\mathbf{P}_{N,\lambda}^a(A) = \frac{Z_{N,\lambda}^a(A)}{Z_{N,\lambda}^a}$. We also define, using the same notation as in [LT15],

$$\bar{Z}_{N,\lambda}^a := Z_{N,\lambda}^a\big(H_n^a(S) = 0\big)\,, \qquad \check{Z}_{N,\lambda}^a := Z_{N,\lambda}^a\big(H_n^a(S) \geq 1\big)\,, \tag{7.2}$$

so that $Z_{N,\lambda}^a = \bar{Z}_{N,\lambda}^a + \check{Z}_{N,\lambda}^a$. (The notation is chosen so that the superscripts $\bar{}$ and $\check{}$ mimic the shape of random walk trajectories in both cases.) Let us also define

$$\check{Z}_{\ell,r} := Z_{N,\lambda}^a\big(L_N = \ell, R_N = r\big)\,, \tag{7.3}$$

so in particular $\check{Z}_{N,\lambda}^a = \sum_{0\leq\ell\leq r\leq N} \check{Z}_{\ell,r}$.

Let us stress that from Lemma 6.1, we have that $\check{Z}^a_{N,\lambda}$ whenever $a > \bar{a}$. For this reason, we will focus on the case $a \le \bar{a}$.

7.2. **Proof of Theorem 3.1.** We have that $Z^a_{N,\lambda} = \bar{Z}^a_{N,\lambda} + \check{Z}^a_{N,\lambda}$, so in particular $Z^a_{N,\lambda}$

$$\max\{\bar{Z}^a_{N,\lambda}, \check{Z}^a_{N,\lambda}\} \le Z^a_{N,\lambda} \le 2\max\{\bar{Z}^a_{N,\lambda}, \check{Z}^a_{N,\lambda}\}\,,$$

and therefore

$$\mathrm{F}(\lambda, a) = \max\Big\{ \lim_{N\to\infty} \frac{1}{N} \log \bar{Z}^a_{N,\lambda}, \lim_{N\to\infty} \frac{1}{N} \log \check{Z}^a_{N,\lambda} \Big\}.$$

We now prove the following two lemmas.

Lemma 7.1. *For any $\lambda > 0$ and $a > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \bar{Z}^a_{N,\lambda} = 0\,.$$

For later purposes, we also give a more precise statement under some further assumption: if one has $\lim_{N\to\infty} \frac{1}{N} a_N = 0$, then

$$\bar{Z}^a_{N,\lambda} = \mathbf{P}\Big(\min_{0\le n\le N} S_n > -\lfloor aN \rfloor, S_N = 0 \Big) \sim \mathbf{P}(S_N = 0) \sim \frac{f_\alpha(0)}{a_N} \qquad \text{as } N \to \infty\,, \tag{7.4}$$

where f_α is the density of the limiting α-stable law.

Lemma 7.2. *For any $\lambda > 0$ and $a < \bar{a}$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \check{Z}^a_{N,\lambda} = \psi(\lambda, a)\,.$$

If $a = \bar{a} < +\infty$, we have $\limsup_{N\to\infty} \frac{1}{N} \log \check{Z}^a_{N,\lambda} \le \psi(\lambda, \bar{a}) < 0$.

These two lemmas readily conclude the proof. Note in particular that when $\lambda \le \lambda_c$ or $a = \bar{a}$, we have $\psi(\lambda, a) \le 0$, so $\mathrm{F}(\lambda, a) = 0$. □

Proof of Lemma 7.1. First of all, let us prove a general bound:

$$1 \ge \bar{Z}^a_{N,\lambda} \ge \mathbf{P}(S_1 > 0, \dots, S_{N-1} > 0, S_N = 0) \sim \frac{c_0}{N a_N} \qquad \text{as } N \to \infty.$$

The lower bound is obvious and the last asymptotic come from Lemma 2.3. This readily shows that $\lim_{N\to\infty} \frac{1}{N} \log \bar{Z}^a_{N,\lambda} = 0$.

As far as the more precise asymptotic (7.4) is concerned, note that the last part is simply the local central limit theorem, see *e.g.* [GK54, Ch. 9 §50]. Let us therefore focus on the first asymptotic, for which we need to show that

$$\mathbf{P}(S_N = 0, \min_{0\le n\le N} S_n \le -\lfloor aN \rfloor) = o(1/a_N) \qquad \text{as } N \to \infty. \tag{7.5}$$

Decomposing according to the instant $T_a := \min\{n, S_n \le -\lfloor aN \rfloor\}$, we have that the above probability is equal to

$$\mathbf{P}\big(S_N = 0, T_a \le N\big) = \mathbf{P}\big(S_N = 0, T_a \le N/2\big) + \mathbf{P}\big(S_N = 0, N/2 < T_a \le N\big)\,.$$

We treat only the first term; the second one is treated identically. We have

$$\mathbf{P}\big(S_N = 0, T_a \le N/2\big) = \sum_{k=1}^{N/2} \sum_{x \ge \lfloor aN \rfloor} \mathbf{P}(T_a = k, S_k = -x)\mathbf{P}(S_{N-k} = x)$$
$$\le \frac{C}{a_N} \sum_{k=1}^{N/2} \sum_{x \ge \lfloor aN \rfloor} \mathbf{P}(T_a = k, S_k = -x) = \frac{C}{a_N} \mathbf{P}\Big(\min_{1 \le k \le N/2} S_k \le -\lfloor aN \rfloor \Big)\,,$$

where we have used the local central limit theorem to get that $\mathbf{P}(S_{n-k} = x) \le C/a_{N-k} \le C'/a_N$ uniformly for $k \le N/2$. Now, since $(a_N^{-1} S_{\lfloor tN \rfloor})_{t \in [0,1]}$ converges in distribution to an α-stable Lévy process (for the Skorokhod topology), we get that

$$\lim_{N \to \infty} \mathbf{P}\Big(\min_{0 \le k \le N/2} S_k \le -\lfloor aN \rfloor \Big) = 0\,,$$

provided that $\lim_{N\to\infty} \frac{\lfloor aN \rfloor}{a_N} = +\infty$. This concludes the proof of (7.5), since we assumed that $\lim_{N\to\infty} \frac{1}{N} a_N = 0$. □

Proof of Lemma 7.2. As far as $\check{Z}^a_{N,\lambda}$ is concerned, we have

$$\check{Z}^a_{N,\lambda} = \sum_{0 < \ell \le r < N} \check{Z}_{\ell,r}\,, \tag{7.6}$$

and in particular $\max_{0<\ell\le r<N}\{\check{Z}_{\ell,r}\} \le \check{Z}^a_{N,\lambda} \le N^2 \max_{0<\ell\le r<N}\{\check{Z}_{\ell,r}\}$, so we focus on $\check{Z}_{\ell,r}$.

For $n \in \mathbb{N}$ and $x \in \mathbb{N}$, we define

$$Q_+(n,x) = \mathbf{P}\big(S_1 > 0, \dots, S_{n-1} > 0, S_n = x\big)\,,$$
$$Q_-(n,x) = \mathbf{P}\big(S_1 < 0, \dots, S_n < 0, S_{n-1} = -x\big)\,, \tag{7.7}$$

so we can write

$$\check{Z}_{\ell,r} = Q_-(\ell, \lfloor aN \rfloor)\, Z_{r-\ell,\lambda}\, Q_+(N-r, \lfloor aN \rfloor)\,, \tag{7.8}$$

where $Z_{r-\ell,\lambda} = Z^0_{r-\ell,\lambda}$ is the partition function of the original wetting model (from Section 2), of length $r - \ell$.

Upper bound. As far as the upper bound is concerned, we can use (5.2) to obtain

$$Q_+(n,x) \le \mathbf{P}(S_n \ge x) \le e^{-n\mathrm{I}_+(\frac{x}{n})}\,, \qquad Q_-(n,x) \le \mathbf{P}(S_n \ge x) \le e^{-n\mathrm{I}_-(\frac{x}{n})}\,.$$

Together with (2.14), we therefore get that, for any $0 < \ell \le r < N$,

$$\check{Z}_{\ell,r} \le \exp\left(-\ell \mathrm{I}_-\Big(\frac{\lfloor aN \rfloor}{\ell}\Big) + (r-\ell)\mathrm{F}(\lambda) - (N-r)\mathrm{I}_+\Big(\frac{\lfloor aN \rfloor}{N-r}\Big)\right)$$
$$\le \exp\left(-N u_\ell \mathrm{I}_-\Big(\frac{a}{u_\ell}\Big) + N(v_r - u_\ell)\mathrm{F}(\lambda) - N(1-v_r)\mathrm{I}_+\Big(\frac{a}{1-v_r}\Big)\right) \le e^{N\psi(\lambda,a)}\,, \tag{7.9}$$

where we have set $u_\ell = \frac{\ell}{N}$ and $v_r = \frac{r}{N}$ and used that $\mathrm{I}_\pm$ is non-decreasing, with $\lfloor aN \rfloor \ge aN$. The last inequality comes from the fact that u_ℓ, v_r is a subset of $0 \le u \le v \le 1$, together with the definition (3.5) of $\psi(\lambda, a)$. We have therefore proven that

$$\check{Z}^a_{N,\lambda} \le N^2 e^{N\psi(\lambda,a)}\,,$$

so $\limsup_{N\to\infty} \frac{1}{N} \log \check{Z}^a_{N,\lambda} \le \psi(\lambda, a)$.

Lower bound. For the lower bound, we only consider the case $a < \bar{a}$. Let us fix (u, v) such that $\frac{a}{\bar{x}_-} < u < v < 1 - \frac{a}{\bar{x}_+}$. We define $\ell_N := \lfloor uN \rfloor$ and $r_N := \lfloor vN \rfloor$. Since $\check{Z}^a_{N,\lambda} \geq \check{Z}_{\ell_N, r_N}$, we simply need to show that

$$\liminf_{N\to\infty} \frac{1}{N} \log \check{Z}_{\ell_N, r_N} \geq g_{\lambda,a}(u,v) := (v-u)\mathrm{F}(\lambda) - u\,\mathrm{I}_-\Big(\frac{a}{u}\Big) - (1-v)\,\mathrm{I}_+\Big(\frac{a}{1-v}\Big)\,,$$

to ensure that

$$\liminf_{N\to\infty} \frac{1}{N} \log \check{Z}^a_{N,\lambda} \geq \sup_{\frac{a}{\bar{x}_-} < u < v < 1 - \frac{a}{\bar{x}_+}} g_{\lambda,a}(u,v) = \psi(\lambda, a)\,,$$

which concludes the lower bound.

Using the same decomposition as in (7.8) and the definition (2.8) of the free energy of the standard wetting model, we therefore simply have to prove that

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \log Q_-(\ell_N, \lfloor aN \rfloor) &\geq -u\mathrm{I}_-\Big(\frac{a}{u}\Big)\,, \\ \liminf_{N\to\infty} \frac{1}{N} \log Q_+(N - r_N, \lfloor aN \rfloor) &\geq -(1-v)\mathrm{I}_+\Big(\frac{a}{1-v}\Big)\,. \end{aligned} \tag{7.10}$$

We therefore rely on the following lemma, whose proof is similar to that of Lemma 5.1 and postponed to Appendix C.

Lemma 7.3. *Suppose that Assumption 1 holds. For any $0 \leq x < \bar{x}_+$, for any sequence of integers $(x_n)_{n\geq 0}$ such that $\lim_{n\to\infty} \frac{1}{n} x_n = x$, we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log Q_+(n, x_n) \geq -\mathrm{I}_+(x)\,.$$

A similar statement holds for Q_-.

This lemma readily proves (7.10), since $\frac{a}{u} < \bar{x}_-$ and $\frac{a}{1-v} < \bar{x}_+$. This concludes the proof of Lemma 7.2. □

7.3. **Proof of Theorem 3.8.** First of all, let us note that in the supercritical case $\lambda > \lambda_c(a)$, we have $\psi(\lambda, a) > 0$. Hence, thanks to Lemmas 7.1-7.2 we have $\lim_{N\to\infty} \bar{Z}^a_{N,\lambda} / \check{Z}^a_{N,\lambda} = 0$. We therefore obtain that

$$\lim_{N\to\infty} \mathbf{P}^a_{N,\lambda}\big(H^a_N(S) > 0\big) = \lim_{N\to\infty} \frac{\check{Z}^a_{N,\lambda}}{\bar{Z}^a_{N,\lambda} + \check{Z}^a_{N,\lambda}} = 1\,.$$

Let us now show something slightly more general than needed: for any $\lambda > 0$ and $a < \bar{a}$, for any fixed $\varepsilon > 0$ we have

$$\lim_{N\to\infty} \mathbf{P}^a_{N,\lambda}\Big(\mathrm{dist}\Big(\tfrac{1}{N}(L_N, R_N), \operatorname{argmax}\psi\Big) > \varepsilon \;\Big|\; H^a_N(S) > 0\Big) = 0\,, \tag{7.11}$$

where we used the shorthand notation $\operatorname{argmax}\psi = \{(u,v) \in \mathcal{D}, g_{\lambda,a}(u,v) = \psi(\lambda,a)\}$ for the set of maximizers of $\psi(\lambda, a)$ in (3.5); note that the supremum is attained, see Lemma 3.5 if $w^*_+ + w^*_- \leq 1$ or (6.10) if $w^*_+ + w^*_- \geq 1$ (recall that $g^-_{\lambda,a}, g^+_{\lambda,a}$ are continuous). Together with Lemma 3.5 (and Lemma 6.5), this concludes the proof of (3.12) in the supercritical case $\lambda > \lambda_c(a)$.

The proof of (7.11) is easy, since the probability is the ratio

$$\frac{1}{\check{Z}^a_{N,\lambda}} Z^a_{N,\lambda}\Big(\mathrm{dist}\Big(\tfrac{1}{N}(L_N, R_N), \operatorname{argmin}\psi\Big) > \varepsilon\Big) = \frac{1}{\check{Z}^a_{N,\lambda}} \sum_{\mathrm{dist}((\frac{\ell}{N}, \frac{r}{N}), \operatorname{argmin}\psi) > \varepsilon} \check{Z}_{\ell,r}\,.$$

Using the same upper bound as in (7.9), we get that the sum is bounded by $N^2 e^{N\psi_\varepsilon(\lambda,a)}$, with

$$\psi_\varepsilon(\lambda,a) := \sup_{\substack{(u,v)\in\mathcal{D}\\ \mathrm{dist}((u,v),\mathrm{argmin}\,\psi)>\varepsilon}} g_{\lambda,a}(u,v)\,.$$

Together with to Lemma 7.2, we get that

$$\limsup_{N\to\infty}\frac{1}{N}\log \mathbf{P}^a_{N,\lambda}\Big(\mathrm{dist}\Big(\tfrac{1}{N}(L_N,R_N),\mathrm{argmin}\,\psi\Big)>\varepsilon \;\Big|\; H^a_N(S)>0\Big)\le \psi_\varepsilon(\lambda,a)-\psi(\lambda,a)<0\,,$$

where we have used the continuity of $(u,v)\mapsto g_{\lambda,a}(u,v)$ to obtain the strict inequality at the end. This concludes the proof of (7.11) and of (3.12) in Theorem 3.8.

As far as the subcritical case $\lambda<\lambda_c(a)$ is concerned, let us distinguish two possibilities. First, if either one of $\mathrm{I}_+,\mathrm{I}_-$ is not identically equal to 0. Then $\lambda\mapsto\psi(\lambda,a)$ is strictly negative on $[0,\lambda_c(0)]$ where $\mathrm{F}(\lambda)=0$ and then strictly increasing: we therefore get that $\psi(\lambda,a)<0$ for any $\lambda<\lambda_c(a)$ since $\psi(\lambda_c(a),a)=0$ by continuity. From Lemmas 7.1-7.2, we get that $\lim_{N\to\infty}\check{Z}^a_{N,\lambda}/\bar{Z}^a_{N,\lambda}=0$, which readily implies that

$$\lim_{N\to\infty}\mathbf{P}^a_{N,\lambda}\big(H^a_N(S)=0\big)=\lim_{N\to\infty}\frac{\bar{Z}^a_{N,\lambda}}{\bar{Z}^a_{N,\lambda}+\check{Z}^a_{N,\lambda}}=1\,.$$

Second, if we have that $\mathrm{I}_+,\mathrm{I}_-$ are both identically equal to 0. Then we have $\psi(\lambda,a)=\mathrm{F}(\lambda)$ for any $\lambda>0$, hence $\lambda_c(a)=\lambda_c$ for any $a>0$. We show below that for any $\lambda<\lambda_c=\lambda_c(a)$, there is a constant $C>0$ such that

$$\check{Z}^a_{N,\lambda}\le C\mathbf{P}\big(S_N=0,\min_{1\le k\le N}S_k\le-\lfloor aN\rfloor\big)\,, \tag{7.12}$$

so we get that $\check{Z}^a_{N,\lambda}=o(1/a_N)$ thanks to (7.5). Recalling Lemma 7.1-(7.4), we get that $\lim_{N\to\infty}\check{Z}^a_{N,\lambda}/\bar{Z}^a_{N,\lambda}=0$, which allows us to conclude as above.

To prove (7.12), we use that for any $\lambda<\lambda_c$, there is a constant C_λ such that $Z_{n,\lambda}\sim C_\lambda K(n)$ as $n\to\infty$, see [Gia07, Thm. 2.2], where $K(n)=\kappa^{-1}f^+_n(0)$ is defined in (2.4). Therefore, using also Lemma 2.3 and the local central limit theorem to get that $f^+_n(0)\le C/a_n\le\mathbf{P}(S_n=0)$, we get that $Z_{n,\lambda}\le C'\mathbf{P}(S_n=0)$ for all $n\ge0$, for some constant C'. All together, we have that $\check{Z}^a_{N,\lambda}$ is bounded by a constant times

$$\sum_{1\le\ell\le r\le N}Q_-(\ell,\lfloor aN\rfloor)\mathbf{P}(S_{r-\ell}=0)Q_+(n-r,\lfloor aN\rfloor)\le\mathbf{P}\big(S_N=0,\min_{1\le k\le N}S_k\le-\lfloor aN\rfloor\big)\,,$$

which is exactly what is claimed in (7.12). □

8. Sharp asymptotic behavior of the partition function, fluctuations of L_N, R_N

In this section, we prove Theorem 3.10, *i.e.* we provide, in Cramér's region, an exact asymptotic behavior of $\check{Z}_{N,\lambda}$ and a local central limit theorem for L_N,R_N.

8.1. Local large deviation with a positivity constraint.

Let us start with the proof of a technical estimate which is key in our proof. It is a precise estimate of $Q_\pm(n,x_n)$ inside Cramér's region; recall that $Q_\pm(n,x_n)$ is defined in (7.7).

Proposition 8.1. *Let $x \in (0, \rho_+)$ and let $(\delta_n)_{n\geq 0}$ be a sequence such that $\lim_{n\to\infty} \delta_n = 0$. Then, there exists a constant $p_x \in (0,1)$ such that, uniformly for non-negative sequences $(x_n)_{n\geq 0}$ with $|\frac{1}{n}x_n - x| \leq \delta_n$, we have*

$$Q_+(n, x_n) \sim \frac{p_x \sqrt{\mathrm{I}_+''(x)}}{\sqrt{2\pi n}}\, e^{-n\mathrm{I}_+\left(\frac{x_n}{n}\right)}, \qquad \text{as } n \to \infty .$$

A similar statement holds for $Q_-(n, x_n)$.

This result is a version of a local large deviation estimate with the additional positivity constraint. Our proof is quite standard, using a change of measure argument, but we also have to deal with the positivity constraint, which makes things slightly more technical.

Proof. For $t \in [0, t_0^+)$, we define the tilted law of X_1:

$$\frac{\mathrm{d}\mathbf{P}_t}{\mathrm{d}\mathbf{P}}(x) = \frac{e^{tx}}{\mathbf{E}[e^{tX_1}]} = e^{tx-\Lambda_+(x)} . \tag{8.1}$$

With a slight abuse of notation, we will also write $\mathbf{P}_t$ for the law of i.i.d. copies. Let us denote, for $t < t_0^+$,

$$m(t) := \mathbf{E}_t[X_1] = \Lambda_+'(t) , \qquad \sigma^2(t) := \mathbf{Var}_t(X_1) = \Lambda''(t) . \tag{8.2}$$

Notice that $m = \Lambda'$ is strictly increasing and continuous from $[0, t_0^+)$ to $[0, \rho_+)$, so we may define $t_x := m^{-1}(x)$ for any $x \in [0, \rho_+)$, in such a way that $\mathbf{E}_{t_x}(X_1) = x$. Note that we have $\Lambda''(t_x) = \mathrm{I}_+(x)$, see Section 5; we also stress that for any $x \in [0, \rho_+)$, the supremum in $\sup_{t\geq 0}\{tx - \Lambda_+(t)\}$ is attained at $t = t_x$, so $\mathrm{I}_+(x) = xt_x - \Lambda_+(t_x)$.

For simplicity, let us denote $t_n = t_{x_n/n}$ and $\sigma_n := \sigma^2(t_n)$ so in particular $x_n t_n - n\Lambda_+(t_n) = \mathrm{I}_+(\frac{x_n}{n})$. Let also

$$A_k^+ := \{S_1 > 0, \dots, S_k > 0\} \qquad \text{for } k \geq 1 . \tag{8.3}$$

Then, using the definition (8.1) of $\mathbf{P}_t$, we may rewrite

$$\begin{aligned} Q_+(n, x_n) := \mathbf{P}\big(A_n^+, S_n = x_n\big) &= \mathbf{E}_{t_n}\Big[e^{n\Lambda_+(t_n)-t_nS_n}\mathbb{1}_{\{A_n^+, S_n = x_n\}}\Big] \\ &= e^{-n\mathrm{I}_+(\frac{x_n}{n})}\mathbf{P}_{t_n}\big(A_n^+, S_n = x_n\big) . \end{aligned}$$

We now let $(k_n)_{n\geq 1}$ be a sequence of integers such that $\lim_{n\to\infty} k_n = +\infty$ and $\lim_{n\to\infty} \frac{1}{n}k_n = 0$. We also take k_n such that $\lim_{n\to\infty} k_n\delta_n^2 = 0$, which ensures in particular that $\lim_{n\to\infty} k_n(t_n - t_x)^2 = 0$ (as will be used below). Then, we need to control

$$\mathbf{P}_{t_n}\big(A_n^+, S_n = x_n\big) = \sum_{y\geq 0} \mathbf{P}_{t_n}(A_{k_n}^+, S_{k_n} = y)\mathbf{P}_{t_n}\big(S_{n-k_n} = x_n - y, S_1 > -y, \dots, S_{n-k_n} > -y\big) . \tag{8.4}$$

We split our proof in an upper and a lower bound.

Upper bound on (8.4). As an upper bound, we remove the condition $S_i > -y$ for $1 \leq i \leq n - k_n$ in the last probability, to obtain the following:

$$\mathbf{P}_{t_n}\big(A_n^+, S_n = x_n\big) \leq \sum_{y\geq 0} \mathbf{P}_{t_n}(A_{k_n}^+, S_{k_n} = y)\mathbf{P}_{t_n}\big(S_{n-k_n} = x_n - y\big) .$$

At this point, we need a local central limit theorem for a family of probability distributions: we refer to Theorem 8.7.1A in [Bor13], that we now state for completeness.

Theorem 8.2 (Thm. 8.7.3A, [Bor13]). *Let $(\mathbf{P}_t)_{t\in[0,t_0]}$ be a family of distributions indexed by a parameter $t\in[0,t_0]$; denote also $\mathbf{P}_t$ the law of i.i.d. random variables $(X_i)_{i\geq 1}$ with law $\mathbf{P}_t$. We assume that for any $t\in[0,t_0]$, under $\mathbf{P}_t$ the X_i's are $\mathbb{Z}$-valued and $S_n=\sum_{i=1}^n X_i$ is aperiodic. Let $m(t):=\mathbf{E}_t(X_1)$ and $\sigma^2(t):=\mathbf{V}\mathrm{ar}_t(X_1)$. Assume that*

$$0<\inf_{t\in[0,t_0]}\sigma^2(t)\leq\sup_{t\in[0,t_0]}\sigma^2(t)<+\infty\,,\qquad\text{and}\qquad\sup_{t\in[0,t_0]}\sup_{\theta\in[\varepsilon,\frac{\pi}{2}]}|\mathbf{E}_t[e^{i\theta X}]|<1\ \text{for any }\varepsilon>0\,.$$

(The latter corresponds to having some "uniform aperiodicity".) Then we have a local limit theorem which is uniform *in t:*

$$\sup_{t\in[0,t_0]}\sup_{x\in\mathbb{Z}}\left|\sqrt{n}\mathbf{P}_t(S_n-nm(t)=x)-g_{\sigma(t)}\Big(\frac{x}{\sqrt{n}}\Big)\right|\xrightarrow{n\to\infty}0\,,$$

where $g_\sigma(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$ is the density of the Gaussian $\mathcal{N}(0,\sigma^2)$.

We may apply this theorem for parameters $t\in[\varepsilon,t_0^+-\varepsilon]$ for some $\varepsilon>0$ fixed, since then we will have that $\sigma^2(t)\in[\sigma^2(\varepsilon),\sigma^2(t_0^+-\varepsilon)]$ with $\sigma^2(\varepsilon)>0$ and $\sigma^2(t_0^+-\varepsilon)<+\infty$; the second condition on the uniform aperiodicity follows similarly. Since $m(t_n)=\frac{1}{n}x_n$ and $\lim_{n\to\infty}t_n=t_x\in(0,t_0^+)$, we can apply this result to obtain that, uniformly on $y\in\mathbb{Z}$,

$$\mathbf{P}_{t_n}\big(S_{n-k_n}=x_n-y\big)\leq\frac{1}{\sqrt{2\pi(n-k_n)\sigma_n^2}}e^{-\frac{(k_nm(t_n)-y)^2}{(n-k_n)^2\sigma_n^2}}+o\Big(\frac{1}{\sqrt{n}}\Big)\leq\frac{1+o(1)}{\sqrt{2\pi n}\sigma(x)}\,,$$

where the $o(1)$ is uniform in $|\frac{1}{n}x_n-x|\leq\delta_n$.

We therefore obtain that

$$\mathbf{P}_{t_n}\big(A_n^+,S_n=x_n\big)\leq\frac{1+o(1)}{\sqrt{2\pi n}\sigma(x)}\,\mathbf{P}_{t_n}\big(A_{k_n}^+\big)\,.$$

We now use the following lemma, which concludes the upper bound; we postpone its proof to the end of the section.

Lemma 8.3. *Let $(t_n)_{n\geq 0}$ be a sequence such that $\lim_{n\to\infty}t_n=t_x\in(0,t_0^+)$. Then, for any sequence $(k_n)_{n\geq 1}$ such that $\lim_{n\to\infty}k_n=+\infty$ and $\lim_{n\to\infty}k_n(t_n-t_x)^2=0$, we have*

$$\lim_{n\to\infty}\mathbf{P}_{t_n}\left(S_1>0,\dots,S_{k_n}>0\right)=\mathbf{P}_{t_x}(S_i>0\ \text{ for all } i\geq 1)=:p_x\in(0,1)\,. \tag{8.5}$$

The rate of convergence depends only on $k_n(t_n-t_x)^2$.

Lower bound on (8.4). For a lower bound, we restrict the sum to $y\in[\frac{1}{2}m(t_n)k_n,\frac{3}{2}m(t_n)k_n]$. Let us introduce, for $y\geq 0$,

$$B_n(y):=\mathbf{P}_{t_n}\big(S_{n-k_n}=x_n-y,S_1>-y,\dots,S_{n-k_n}>-y\big)\,, \tag{8.6}$$

so we obtain the lower bound

$$\mathbf{P}_{t_n}\big(A_n^+,S_n=x_n\big)\geq\mathbf{P}\Big(A_{k_n}^+,S_{k_n}\in[\tfrac{1}{2}m(t_n)k_n,\tfrac{3}{2}m(t_n)k_n]\Big)\inf_{y\in[\frac{1}{2}m(t_n)k_n,\frac{3}{2}m(t_n)k_n]}B_n(y)\,.$$

For the first probability, we write

$$\mathbf{P}\Big(A_{k_n}^+,S_n\in[\tfrac{1}{2}m(t_n)k_n,\tfrac{3}{2}m(t_n)k_n]\Big)\geq\mathbf{P}\big(A_{k_n}^+\big)-\mathbf{P}\Big(S_{k_n}\notin[\tfrac{1}{2}m(t_n)k_n,\tfrac{3}{2}m(t_n)k_n]\Big)\,.$$

The first term converges to p_x thanks to Lemma 8.3, and it remains to see that the second one goes to 0. But since $\mathbf{E}_{t_n}[X_i] = \frac{1}{n}x_n = m(t_n)$, we get by Chebyshev's inequality that

$$\mathbf{P}\Big(S_{k_n} \notin [\tfrac{1}{2}m(t_n)k_n, \tfrac{3}{2}m(t_n)k_n]\Big) \leq \frac{k_n\sigma_n^2}{\frac{1}{4}m(t_n)k_n^2} \xrightarrow{n\to\infty} 0$$

since $\lim_{n\to\infty} k_n = +\infty$, using also that $\lim_{n\to\infty}\sigma_n^2 = \sigma^2(x)$ and $\lim_{n\to\infty} m(t_n) = x > 0$.

It therefore remains to show that

$$\inf_{y\in[\frac{1}{2}m(t_n)k_n, \frac{3}{2}m(t_n)k_n]} B_n(y) \geq \frac{1+o(1)}{\sqrt{2\pi n}\sigma(x)}\,. \tag{8.7}$$

Recalling the definition (8.6), we write $B_n(y) \geq B_n^{(1)}(y) - B_n^{(2)}(y)$, with

$$B_n^{(1)}(y) := \mathbf{P}_{t_n}\big(S_{n-k_n} = x_n - y\big)\,, \quad B_n^{(2)}(y) := \mathbf{P}_{t_n}\big(S_{n-k_n} = x_n - y, \min_{1\leq i\leq n-k_n} S_i \leq -\tfrac{1}{2}m(t_n)k_n\big)\,.$$

The first term is again controlled thanks to Theorem 8.2: we have that

$$B_n^{(1)}(y) \geq \frac{1}{\sqrt{2\pi(n-k_n)}\sigma_n^2} e^{-\frac{(k_n m(t_n)-y)^2}{(n-k_n)^2\sigma_n^2}} - o\Big(\frac{1}{\sqrt{n}}\Big) \geq \frac{1+o(1)}{\sqrt{2\pi n}\sigma^2(x)}\,,$$

where all inequalities are uniform in $y \in [\frac{1}{2}m(t_n)k_n, \frac{3}{2}m(t_n)k_n]$. It only remains to show that $B_n^{(2)}(y) = o(1/\sqrt{n})$ uniformly in $y \in [\frac{1}{2}m(t_n)k_n, \frac{3}{2}m(t_n)k_n]$. Let $T_{k_n} := \min\{i \geq 1, S_i \leq -\frac{1}{2}m(t_n)k_n\}$, so we can write that

$$B_n^{(2)}(y) = \sum_{j=1}^{n-k_n} \sum_{z\leq -\frac{1}{2}m(t_n)k_n} \mathbf{P}_{t_n}\big(T_{k_n} = j, S_j = z\big)\mathbf{P}_{t_n}\big(S_{n-k_n-j} = x_n - y - z\big)\,.$$

We show just below that there is a constant $C > 0$ such that

$$\sup_{1\leq j\leq n-k_n} \sup_{w\leq m(t_n)k_n} \mathbf{P}_{t_n}\big(S_{n-k_n-j} = x_n - w\big) \leq \frac{C}{\sqrt{n}}\,. \tag{8.8}$$

From the above display, we therefore deduce that

$$\sup_{y\in[\frac{1}{2}m(t_n)k_n, \frac{3}{2}m(t_n)k_n]} B_n^{(2)}(y) \leq \frac{C}{\sqrt{n}}\mathbf{P}_{t_n}\big(T_{k_n} \leq n-k_n\big) = \frac{C}{\sqrt{n}}\mathbf{P}_{t_n}\big(\min_{1\leq i\leq n-k_n} S_i \leq -\tfrac{1}{2}m(t_n)k_n\big)\,,$$

with the last probability going to 0 as $n\to\infty$, since $\mathbf{E}_{t_n}[X_i] = m(t_n) \geq \frac{1}{2}x > 0$ (at least for n sufficiently large).

We are left with showing (8.8). First of all, if $j \leq n/2 - k_n$ (so $n - k_n - j \geq n/2$), the local limit theorem from Theorem 8.2 gives that

$$\mathbf{P}_{t_n}\big(S_{n-k_n-j} = w'\big) \leq \frac{C}{\sqrt{n}}\,,$$

uniformly in $w' \in \mathbb{Z}$. If on the other hand $j \leq n/2 - k_n$, we bound

$$\sup_{k\leq n/2} \sup_{w\leq m(t_n)k_n} \mathbf{P}_{t_n}\big(S_k = x_n - w\big) \leq \sup_{k\geq n/2} \mathbf{P}_{t_n}\big(S_k \geq x_n - m(t_n)k_n\big) \leq \sup_{k\leq n/2} \frac{k\sigma_n^2}{(x_n - m(t_n)(k_n+k))^2}\,,$$

where we have used Chebyshev's inequality for the last bound. Since we have $m(t_n)k \leq x_n/2$, $\lim_{n\to\infty} m(t_n)k_n = 0$, $\lim_{n\to\infty}\frac{1}{n}x_n = x > 0$ and also $\lim_{n\to\infty}\sigma_n^2 = \sigma^2(x) < +\infty$, we get that this

is bounded by $\frac{4\sigma^2(x)}{nx}$ for n large enough. This concludes the proof of (8.8), hence of the upper bound. □

Proof of Lemma 8.3. Recall that we denoted $A^+_{k_n} := \{S_1 > 0, \dots, S_{k_n} > 0\}$. Recalling also the definition (8.1) of $\mathbf{P}_{t_n}$, we have

$$\mathbf{P}_{t_n}\big(A^+_{k_n}\big) = \mathbf{E}_{t_x}\Big[\mathbb{1}_{A^+_{k_n}} e^{(t_n - t_x)S_{k_n}}\Big] e^{(\Lambda(t_x)-\Lambda(t_n))k_n}.$$

Letting $\varepsilon > 0$ be fixed, we can apply Hölder inequality, to get

$$\begin{aligned}\mathbf{P}_{t_n}\big(A^+_{k_n}\big) &\le \mathbf{P}_{t_x}\big(A^+_{k_n}\big)^{1-\varepsilon}\mathbf{E}_{t_x}\Big[e^{\varepsilon^{-1}(t_n-t_x)S_{k_n}}\Big]^{\varepsilon} e^{(\Lambda(t_x)-\Lambda(t_n))k_n}\\ &\le \mathbf{P}_{t_x}\big(A^+_{k_n}\big)^{1-\varepsilon}\exp\Big(k_n\Big(\varepsilon\big(\Lambda\big(t_x+\varepsilon^{-1}(t_n-t_x)\big)-\Lambda(t_x)\big)-\big(\Lambda(t_n)-\Lambda(t_x)\big)\Big)\Big).\end{aligned}$$

Now, by a Taylor expansion in $t_n - t_x$, we have that

$$\varepsilon\big(\Lambda\big(t_x+\varepsilon^{-1}(t_n-t_x)\big)-\Lambda(t_x)\big)+\Lambda(t_x)-\Lambda(t_n) = (1+o(1))\frac{1}{2}(\varepsilon^{-1}-1)\Lambda''(t_x)(t_n-t_x)^2.$$

Hence, since $\lim_{n\to\infty} k_n(t_n-t_x)^2 = 0$, we get that

$$\limsup_{n\to\infty}\mathbf{P}_{t_n}\big(A^+_{k_n}\big) \le \limsup_{n\to\infty}\mathbf{P}_{t_x}\big(A^+_{k_n}\big)^{1-\varepsilon} = (p_x)^{1-\varepsilon},$$

with $p_x := \mathbf{P}_{t_x}(S_i > 0 \text{ for all } i \ge 1) \in (0,1)$ (note that $p_x > 0$ since $\mathbf{E}_{t_x}[X_1] = x > 0$). We stress that the rate of convergence depends on $k_n(t_n - t_x)^2$.

On the other hand, we also have a lower bound on $\mathbf{P}_{t_n}(A^+_{k_n})$ using Hölder's inequality the other way around:

$$\begin{aligned}\mathbf{P}_{t_x}\big(A^+_{k_n}\big) &= \mathbf{E}_{t_n}\Big[\mathbb{1}_{A^+_{k_n}} e^{(t_x-t_n)S_{k_n}}\Big] e^{(\Lambda(t_n)-\Lambda(t_x))k_n}\\ &\le \mathbf{P}_{t_n}\big(A^+_{k_n}\big)^{1-\varepsilon}\mathbf{E}_{t_n}\Big[e^{\varepsilon^{-1}(t_x-t_n)S_{k_n}}\Big]^{\varepsilon} e^{(\Lambda(t_n)-\Lambda(t_x))k_n}.\end{aligned}$$

Similarly as above, we get that

$$p_x = \lim_{n\to\infty}\mathbf{P}_{t_x}\big(A^+_{k_n}\big) \le \liminf_{n\to\infty}\mathbf{P}_{t_n}\big(A^+_{k_n}\big)^{1-\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, this concludes the proof. □

8.2. **Asymptotic behavior of $\check{Z}_{\ell,r}$.** From Proposition 8.1, we are able to obtain the following result on the partition function $\check{Z}_{\ell,r} := Z^a_{N,\lambda}(L_N = \ell, R_N = r)$.

Proposition 8.4. *Let $\lambda > \lambda_c$, let $\frac{a}{\rho_+} < u < v < 1 - \frac{a}{\rho_+}$ and let $(\delta_N)_{N\ge 0}$ be a vanishing sequence. Then uniformly for (ℓ, r) with and $|\frac{1}{N}\ell - u| \le \delta_N$ and $|\frac{1}{N}r - v| \le \delta_N$, we have*

$$\check{Z}_{\ell,r} \sim \frac{p_{x_u}p_{x_{1-v}}\sqrt{\mathrm{I}''_-(x_u)\mathrm{I}''_+(x_{1-v})}}{2\pi N m_\lambda\sqrt{u(1-v)}}\exp\Big(N\Big(\big(\tfrac{r}{N}-\tfrac{\ell}{N}\big)\mathrm{F}(\lambda) - \tfrac{\ell}{N}\mathrm{I}_-\big(\tfrac{\lfloor aN\rfloor}{\ell}\big) - \big(1-\tfrac{r}{N}\big)\mathrm{I}_-\big(\tfrac{\lfloor aN\rfloor}{N-r}\big)\Big)\Big),$$

with $x_s = a/s$ and m_λ from (2.17).

Proof. We simply need to use (7.8), that is:

$$\check{Z}_{\ell,r} = Q_-(\ell, \lfloor aN\rfloor)Z_{r-\ell,\lambda}Q_+(N-r, \lfloor aN\rfloor).$$

Then, using Proposition 8.1 we get that

$$Q_-(\ell, \lfloor aN \rfloor) \sim \frac{p_{x_u}\sqrt{\mathrm{I}_-''(x_u)}}{\sqrt{2\pi N u}} \exp\Big(- \ell \mathrm{I}_-\big(\tfrac{\lfloor aN \rfloor}{\ell}\big)\Big)\,,$$
$$Q_+(N-r, \lfloor aN \rfloor) \sim \frac{p_{x_{1-v}}\sqrt{\mathrm{I}_+''(x_{1-v})}}{\sqrt{2\pi N (1-v)}} \exp\Big(- (N-r) \mathrm{I}_+\big(\tfrac{\lfloor aN \rfloor}{N-r}\big)\Big)\,.$$

Combined with the asymptotic (2.17) for $Z_{r-\ell,\lambda}$, this concludes the proof □

8.3. **Proof Theorem 3.10, part (i).** We prove part (i) of Theorem 3.10, namely the sharp asymptotic (3.15) for $Z^a_{N,\lambda}$. Let $\lambda \geq \lambda_c(a) > \lambda_c$. Recall that we assume that Cramér's condition holds, so in particular X_i have a finite variance: we can choose the normalizing sequence $a_N = \sigma\sqrt{N}$, and we get that $\bar{Z}^a_{N,\lambda} \sim \frac{1}{\sigma\sqrt{2\pi N}}$ from Lemma 7.1.

We can therefore focus on $\check{Z}^a_{N,\lambda}$. Moreover, from (7.11), we get that there is a sequence $(\varepsilon_n)_{n\geq 0}$ with $\lim_{n\to\infty} \varepsilon_n = 0$, such that we have

$$\lim_{n\to\infty} \frac{1}{\check{Z}^a_{N,\lambda}} Z^a_{N,\lambda}\Big(\Big| \frac{1}{N}(L_N, R_N) - (u^*, v^*)\Big| < \varepsilon_n\Big) = 1\,,$$

where we recall that (u^*, v^*) is (in Cramér's region) the unique maximizer of $\psi(\lambda, a)$ in (3.5), see (3.14). In other words, we have that

$$\check{Z}^a_{N,\lambda} \sim \sum_{|\frac{1}{N}\ell - u^*| \leq \varepsilon_n} \sum_{|\frac{1}{N} r - v^*| \leq \varepsilon_n} \check{Z}_{\ell,r}\,.$$

Therefore, from Proposition 8.4 and since $\frac{a}{\rho_+} < u^* < v^* < 1 - \frac{a}{\rho_+}$ (see (3.14)), we get that there is some explicit constant $C_{\lambda,a}$, such that

$$\check{Z}^a_{N,\lambda} \sim \frac{C_{\lambda,a}}{2\pi N} \sum_{|\frac{1}{N}\ell - u^*| \leq \varepsilon_n} \sum_{|\frac{1}{N} r - v^*| \leq \varepsilon_n} \exp\Big(N g_{\lambda, \frac{\lfloor aN \rfloor}{N}}\Big(\frac{\ell}{N}, \frac{r}{N}\Big)\Big)\,, \tag{8.9}$$

where we used the definition (6.3) of $g_{\lambda,a}$ to simplify notation. Let us stress that the constant $C_{\lambda,a}$ is explicit, $C_{\lambda,a} := \frac{1}{m_\lambda} p_{x_{u^*}} p_{x_{\bar{v}^*}} \frac{a^2 u^* \bar{v}^*}{\sigma_1 \sigma_2}$, with σ_1, σ_2 defined below in (8.11) (and $\bar{v}^* := 1 - v^*$).

Then, we will use Taylor expansions inside $g_{\lambda, \frac{\lfloor aN \rfloor}{N}}$. Let us denote, for ℓ, r in the range of the sum in (8.9),

$$\varepsilon_a := 1 - \frac{\lfloor aN \rfloor}{aN} = \frac{\{aN\}}{aN} = O(N^{-1})\,, \quad \varepsilon_\ell := \frac{\ell}{u^* N} - 1 = O(\varepsilon_N)\,, \quad \varepsilon_r := \frac{N-r}{\bar{v}^* N} - 1 = O(\varepsilon_N)\,,$$

with $\bar{v}^* = 1 - v^*$. Then, thanks to a Taylor expansion of I_- around $\frac{a}{u^*}$, we get

$$\begin{aligned}\frac{\ell}{N}\mathrm{I}_-\Big(\frac{\lfloor aN \rfloor}{\ell}\Big) &= (1+\varepsilon_\ell) u^* \mathrm{I}_-\Big(\frac{a}{u^*}\frac{1-\varepsilon_a}{1+\varepsilon_\ell}\Big) \\ &= (1+\varepsilon_\ell) u^* \mathrm{I}_-\Big(\frac{a}{u^*}\Big) - a(\varepsilon_\ell + \varepsilon_a)\mathrm{I}_-'\Big(\frac{a}{u^*}\Big) + \frac{a^2}{2u^*}\frac{(\varepsilon_\ell+\varepsilon_a)^2}{1+\varepsilon_\ell}\mathrm{I}_-''\Big(\frac{a}{u^*}\Big) + (\varepsilon_\ell+\varepsilon_a)^3 O(1)\,,\end{aligned}$$

with the $O(1)$ uniform in $|\frac{1}{N}\ell - u^*| \leq \varepsilon_N$. Hence we can write $\frac{\ell}{N}\mathrm{I}_-\big(\frac{\lfloor aN \rfloor}{\ell}\big)$ as

$$u^*\mathrm{I}_-\Big(\frac{a}{u^*}\Big) + \varepsilon_\ell u^*\Big(\mathrm{I}_-\Big(\frac{a}{u^*}\Big) - \frac{a}{u^*}\mathrm{I}_-'\Big(\frac{a}{u^*}\Big)\Big) + \frac{a^2}{2u^*}\varepsilon_\ell^2 \mathrm{I}_-''\Big(\frac{a}{u^*}\Big) - a\varepsilon_a \mathrm{I}_-\Big(\frac{a}{u^*}\Big) + O(\varepsilon_\ell^3) + o\Big(\frac{1}{N}\Big)\,.$$

Using Lemma 6.6 and denoting $\sigma_1^2 := \frac{(u^*)^3}{a^2 \mathrm{I}_-''(a/u^*)}$, this can be rewritten as

$$\frac{\ell}{N}\mathrm{I}_-\Big(\frac{\lfloor aN\rfloor}{\ell}\Big) = u^*\mathrm{I}_-\Big(\frac{a}{u^*}\Big) - \varepsilon_\ell u^* \mathrm{F}(\lambda) + \frac{(u^*\varepsilon_\ell)^2}{2\sigma_1^2} - a\varepsilon_a \mathrm{I}_-\Big(\frac{a}{u^*}\Big) + O(\varepsilon_\ell^3) + o\Big(\frac{1}{N}\Big)\,.$$

Similarly, denoting $\sigma_2^2 := \frac{(\bar v^*)^3}{a^2 \mathrm{I}_+''(a/\bar v^*)}$ (and also $\bar v^* = 1 - v^*$ to simplify notation), we have

$$\frac{N-r}{N}\mathrm{I}_+\Big(\frac{\lfloor aN\rfloor}{N-r}\Big) = \bar v^*\mathrm{I}_+\Big(\frac{a}{\bar v^*}\Big) - \varepsilon_r \bar v^* \mathrm{F}(\lambda) + \frac{(\bar v^*\varepsilon_r)^2}{2\sigma_2^2} - a\varepsilon_a \mathrm{I}_+\Big(\frac{a}{\bar v^*}\Big) + O(\varepsilon_r^3) + o\Big(\frac{1}{N}\Big)\,.$$

All together, recalling the definition (3.5) of $g_{\lambda,a}$, we obtain

$$g_{\lambda,\frac{\lfloor aN\rfloor}{N}}\Big(\frac{\ell}{N},\frac{r}{N}\Big) = \mathrm{F}(\lambda)\Big(\frac{r}{N} - \frac{\ell}{N} + \varepsilon_\ell u^* + \varepsilon_r \bar v^*\Big) - u^*\mathrm{I}_-\Big(\frac{a}{u^*}\Big) - \bar v^*\mathrm{I}_+\Big(\frac{a}{\bar v^*}\Big) \\ - \frac{(u^*\varepsilon_\ell)^2}{2\sigma_1^2} - \frac{(\bar v^*\varepsilon_r)^2}{2\sigma_2^2} + c_0 a\varepsilon_a + O(\varepsilon_\ell^3+\varepsilon_r^3) + o\Big(\frac{1}{N}\Big)\,,$$

with $c_0 := \mathrm{I}_-\big(\frac{a}{u^*}\big) + \mathrm{I}_+\big(\frac{a}{\bar v^*}\big)$. Recalling the definition of $\varepsilon_\ell, \varepsilon_r$, we have that $\frac{\ell}{N} - \varepsilon_\ell u^* = u^*$ and $\frac{r}{N} + \varepsilon_r \bar v^*$; also, $a\varepsilon_a = \frac{1}{N}\{aN\}$. Hence, since we have $\psi(\lambda,a) = (v^*-u^*)\mathrm{F}(\lambda) - u^*\mathrm{I}_-\big(\frac{a}{u^*}\big) - \bar v^*\mathrm{I}_+\big(\frac{a}{\bar v^*}\big)$, we end up with

$$g_{\lambda,\frac{\lfloor aN\rfloor}{N}}\Big(\frac{\ell}{N},\frac{r}{N}\Big) = \psi(\lambda,a) - \frac{(u^*\varepsilon_\ell)^2}{2\sigma_1^2} - \frac{(\bar v^*\varepsilon_r)^2}{2\sigma_2^2} + c_0\frac{1}{N}\{aN\} + (\varepsilon_\ell^3+\varepsilon_r^3)O(1) + o\Big(\frac{1}{N}\Big)\,, \tag{8.10}$$

where the constants are

$$\sigma_1^2 := \frac{(u^*)^3}{a^2\mathrm{I}_-''(a/u^*)}, \qquad \sigma_2^2 := \frac{(\bar v^*)^3}{a^2\mathrm{I}_+''(a/\bar v^*)}, \qquad c_0 := \mathrm{I}_-\Big(\frac{a}{u^*}\Big) + \mathrm{I}_+\Big(\frac{a}{\bar v^*}\Big)\,. \tag{8.11}$$

Note that, in (8.10), the $O(1)$ and $o(\frac{1}{N})$ are uniform (depending only on ε_N). Going back to (8.9), we obtain that

$$\check Z^a_{N,\lambda} = (1+o(1))\frac{C_{\lambda,a}}{2\pi N} e^{N\psi(\lambda,a)+c_0\{aN\}} \sum_{|\frac{1}{N}\ell - u^*|\le \varepsilon_N} e^{-(1+o(1))\frac{(u^*\varepsilon_\ell)^2 N}{2\sigma_1^2}} \sum_{|\frac{1}{N}r - v^*|\le \varepsilon_N} e^{-(1+o(1))\frac{(\bar v^*\varepsilon_r)^2 N}{2\sigma_2^2}}\,.$$

Recalling that $u^*\varepsilon_\ell = \frac{\ell}{N} - 1$, we get that

$$\sum_{|\frac{1}{N}\ell - u^*|\le\varepsilon_N} \frac{1}{\sqrt N} e^{-(1+o(1))\frac{(u^*\varepsilon_\ell)^2}{2\sigma_1^2}N} = \sum_{|j|\le N\varepsilon_N} \frac{1}{\sqrt N} e^{-(1+o(1))\frac{j^2}{2\sigma_1^2 N}} \xrightarrow{N\to\infty} \int_{\mathbb{R}} e^{-\frac{s^2}{2\sigma_1^2}}\mathrm{d}s = \sqrt{2\pi}\sigma_1\,,$$

by a Riemann sum approximation. A similar convergence holds for the other sum. We therefore end up with

$$\check Z^a_{N,\lambda} \sim \sigma_1\sigma_2 C_{\lambda,a} e^{N\psi(\lambda,a)+c_0\{aN\}}\,.$$

Since we have seen above that $\bar Z^a_{N,\lambda} = O(1/\sqrt N)$, we get that whenever $\psi(\lambda,a)\ge 0$, *i.e.* when $\lambda \ge \lambda_c(a)$, which in particular entails that $\mathrm{F}(\lambda,a) = \psi(\lambda,a)$, we get

$$Z^a_{N,\lambda} = \bar Z^a_{N,lambda} + \check Z^a_{N,\lambda} = (1+o(1))\check Z^a_{N,\lambda} \sim \sigma_1\sigma_1 2 C_{\lambda,a} e^{N\psi(\lambda,a)+c_0\{aN\}}\,. \tag{8.12}$$

This concludes the proof of (3.15). Note also that in view of the definition of $C_{\lambda,a}$ above, we have $c_1 := \frac{1}{m_\lambda} a^2 u^*\bar v^* p_{a/u^*} p_{a/\bar v^*}$ in (3.15). □

8.4. **Proof Theorem 3.10, part (ii).** Let us now turn to the proof of the local central limit theorem, that is (3.16). Notice that we have

$$\mathbf{P}^a_{N,\lambda}\big(L_N = \ell, R_N = r\big) = \frac{1}{Z_{N,\lambda}} \check{Z}_{\ell,r}\,.$$

From the first part of Theorem 3.10 (see in particular (8.12)) and thanks to Proposition 8.4, we obtain, similarly to (8.9),

$$\mathbf{P}^a_{N,\lambda}\big(L_N = \ell, R_N = r\big) = (1+o(1))\frac{1}{2\pi N\sigma_1\sigma_2} \exp\Big(N g_{\lambda,\frac{\lfloor aN\rfloor}{N}}\Big(\frac{\ell}{N},\frac{r}{N}\Big) - N\psi(\lambda,a) - c_0\{aN\}\Big)\,,$$

with the $o(1)$ uniform over $|\frac{1}{N}\ell - u^*| \le \varepsilon_N$, $|\frac{1}{N}r - u^*| \le \varepsilon_N$. Then, applying the Taylor expansion of (8.10) and recalling that $\varepsilon_\ell = \frac{\ell}{u^*N} - 1$ and $\varepsilon_r = \frac{N-r}{N\bar{v}^*} - 1$, we get

$$N g_{\lambda,\frac{\lfloor aN\rfloor}{N}}\Big(\frac{\ell}{N},\frac{r}{N}\Big) - N\psi(\lambda,a) - c_0\{aN\} = -\frac{(\ell - u^*N)^2}{2\sigma_1^2 N}(1+\varepsilon_\ell O(1)) - \frac{(r-v^*N)^2}{2\sigma_1^2 N}(1+\varepsilon_r O(1))\,,$$

with the $O(1)$ uniform. This concludes the proof of (3.16).

To obtain the convergence in distribution, we simply notice that for any $z_1 < z_2$ and $z_1' < z_2'$, we can write

$$\begin{aligned}\mathbf{P}^a_{N,\lambda}\big(z_1\sqrt{N} \le L_N - u^*N \le z_2\sqrt{N}, z_1'\sqrt{N} \le R_N - v^* \le z_2'\sqrt{N}\big)\\ = \sum_{z_1\sqrt{N}\le \ell - u^*N \le z_2\sqrt{N}} \ \sum_{z_1'\sqrt{N}\le r - v^*N \le z_2'\sqrt{N}} \mathbf{P}^a_{N,\lambda}\big(L_N = \ell, R_N = r\big)\,.\end{aligned}$$

Then, using the local central limit theorem from (3.16) and a Riemann sum approximation, we get that the above probability converges to

$$\int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{s_1^2}{2\sigma_1^2}}\mathrm{d}s_1 \int_{z_1'}^{z_2'} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{s_2^2}{2\sigma_1^2}}\mathrm{d}s_2\,,$$

which concludes the proof. □

8.5. **Proof of Corollary 3.11.** Let us prove a more general result, *i.e.* a joint local central limit theorem for L_N, R_N, H_N. First, let us observe that $\mathbf{P}^a_{N,\lambda}(H_N = k \,\big|\, L_N = \ell, R_N = r) = \mathbf{P}_{r-\ell,\lambda}(H_{r-\ell} = k)$, where $\mathbf{P}_{n,\lambda}$ is the (standard) wetting measure (2.2). Then, and using the local limit theorem of Proposition 2.6, we get that, as $r - \ell \to \infty$

$$\mathbf{P}^a_{N,\lambda}\big(H_N = k \,\big|\, L_N = \ell, R_N = r\big) = \frac{m_\lambda}{\sqrt{2\pi}\sigma_\lambda} e^{-\frac{(r-\ell-m_\lambda k)^2}{2\sigma_\lambda^2(r-\ell)}} + o\Big(\frac{1}{\sqrt{r-\ell}}\Big)\,.$$

Let us now set

$$\Delta_\ell := |\ell - u^*N|, \quad \Delta_r := |r - v^*N|, \quad \Delta_k := |k - m_\lambda^{-1}(u^* - v^*)N|\,.$$

Then, using the local central limit theorem of Theorem 3.10-(ii), see (3.16), we obtain that for any $A > 0$, uniformly for $\Delta_\ell, \Delta_r, \Delta_k \le A\sqrt{N}$ (in particular $|(r-\ell) - (v^* - u^*)N| \le 2A\sqrt{N}$), we have

$$\mathbf{P}^a_{N,\lambda}\big(L_N = \ell, R_N = r, H_N = k\big) = \frac{(1+o(1))m_\lambda}{(2\pi N)^{3/2}\sqrt{v^* - u^*}\sigma_1\sigma_2\sigma_\lambda} e^{-\frac{\Delta_\ell^2}{2\sigma_1^2 N}} e^{-\frac{\Delta_r^2}{2\sigma_1^2 N}} e^{-\frac{(\Delta_r - \Delta_n - m_\lambda\Delta_k)^2}{2\sigma_\lambda^2(v^*-u^*)N}}\,.$$

In other words, letting $\sigma_3 := \frac{\sqrt{v^*-u^*}}{m_\lambda}\sigma_\lambda$, we have

$$\mathbf{P}^a_{N,\lambda}\big(L_N=\ell, R_N=r, H_N=k\big) = \frac{(1+o(1))}{(2\pi N)^{3/2}\sigma_1\sigma_2\sigma_3} e^{-\frac12 Q\left(\frac{\Delta_\ell}{\sqrt N},\frac{\Delta_r}{\sqrt N},\frac{\Delta_k}{\sqrt N}\right)}, \tag{8.13}$$

where $Q(z_1,z_2,z_3) = \frac{z_1^2}{\sigma_1^2} + \frac{z_2^2}{\sigma_2^2} + \frac{(z_3 - m_\lambda^{-1}(z_2-z_1))^2}{\sigma_3^2}$ as defined below (8.13); note that the $o(1)$ is uniform in $\Delta_\ell, \Delta_r, \Delta_k \le A\sqrt{N}$. The convergence in distribution stated in (3.17) then follows directly from (8.13) by a Riemann sum approximation. □

Appendix A. A few examples of integrable wetting models

To complement the study of the standard wetting model, we collect a few examples for which the free energy $\mathrm{F}(\lambda)$ (or the critical point) of the admits an explicit formula. We start with discrete examples, that have been more studied in the literature, before turning to some continuous cases.

We then compute explicitly the free energy $\mathrm{F}(\lambda,a)$ and the critical curve $a_c(\lambda)$ of the wetting model with elevated boundary condition. In all examples, the underlying random walk is symmetric, so we denote $\Lambda(t) := \Lambda_+(t) = \Lambda_-(t)$. Recall that, by Theorem 3.3 we have that the free energy and the critical curve are given by

$$\mathrm{F}(\lambda,a) = \Big(\mathrm{F}(\lambda) - 2a\Lambda^{-1}(\mathrm{F}(\lambda))\Big)_+, \qquad a_c(\lambda) = \frac{\mathrm{F}(\lambda)}{2\Lambda^{-1}(\mathrm{F}(\lambda))}.$$

A.1. **Symmetric lazy random walk.** Let $\gamma \in (0,\frac12)$ and consider $(X_i)_{i\ge1}$ i.i.d. random variables, with symmetric distribution given by $\mathbf{P}(X_i=1) = \mathbf{P}(X_i=-1) = \gamma$ and $\mathbf{P}(X_i=0) = 1-2\gamma > 0$; the case $\gamma \uparrow \frac12$ corresponds to the simple symmetric random walk.

This model has been studied in details, see e.g. [Fis84, IY01]. For the critical point, using Lemma 2.2 we have $\lambda_c = \kappa^{-1} = \mathbf{P}(\bar H_1 = 0)^{-1}$. Since the steps are only ±1 or 0, the (weak) ladder height is always zero except if $X_1 = -1$: we therefore get $\lambda_c = \frac{1}{\mathbf{P}(\bar H_1=0)} = \frac{1}{1-\gamma}$. It turns out that the free energy can also be computed explicitly, see e.g. [IY01, Eq. (1.7)]:

$$\mathrm{F}(\lambda) = \ln x_\lambda \quad \text{ for } \lambda \ge \lambda_c = \frac{1}{1-\gamma},$$

where x_λ is the positive solution of $(\lambda-1)x^2 - \lambda(\lambda-1)(1-2\gamma)x - \lambda^2\gamma^2 = 0$; one can easily check that $x_\lambda \ge 1$ if $\lambda \ge \lambda_c$. As far as the critical behavior is concerned, we leave as an exercise to check that $\mathrm{F}(\lambda_c+u) \sim \gamma(3-4\gamma)u^2$ as $u \downarrow 0$.

Note that in the limit $\gamma \uparrow \frac12$, we find the free energy of the wetting model for the simple random walk found in [LT15, Eq. (1.6)] (see also [Hol07, Ch. 7]): $\mathrm{F}(\lambda) = \ln\big(\frac{\lambda}{2\sqrt{\lambda-1}}\big)\mathbb{1}_{\{\lambda>2\}}$. Another remarkable value is in the case $\gamma = \frac14$, we find: $\mathrm{F}(\lambda) = \ln\big(\frac{\lambda}{4\sqrt{\lambda-1}}(\sqrt{\lambda-1}+\sqrt{\lambda})\big)\mathbb{1}_{\{\lambda>\frac43\}}$.

With elevated boundary conditions. We use that $\Lambda(t) = \ln\big(1+2\gamma(\cosh(t)-1)\big)$ for any $t \in \mathbb{R}_+$ so $\Lambda^{-1}(x) = \cosh^{-1}\big(1+\frac{1}{2\gamma}(e^x-1)\big)$. Together with the formula $\mathrm{F}(\lambda) = \ln x_\lambda$, we get that $\Lambda^{-1}(\mathrm{F}(\lambda)) = \cosh^{-1}\big(1+\frac{1}{2\gamma}(x_\lambda-1)\big)$, so we end up with

$$\mathrm{F}(\lambda,a) = \Big(\ln x_\lambda - 2a\cosh^{-1}\big(1+\tfrac{1}{2\gamma}(x_\lambda-1)\big)\Big)_+, \qquad a_c(\lambda) = \frac{\ln x_\lambda}{2\cosh^{-1}\big(1+\frac{1}{2\gamma}(x_\lambda-1)\big)},$$

where x_λ is the positive solution of $(\lambda-1)x^2 - \lambda(\lambda-1)(1-2\gamma)x - \lambda^2\gamma^2 = 0$. Note that in the case $\gamma \uparrow \frac12$, we obtain a free energy $\mathrm{F}(\lambda,a) = \big(\ln(\frac{\lambda}{2\sqrt{\lambda-1}}) - 2a\cosh^{-1}(\frac{\lambda}{2\sqrt{\lambda-1}})\big)_+$.

A.2. **Symmetric geometric random walk, one-dimensional (discrete) SOS.** Let $\gamma \in (0,1)$ and consider $(X_i)_{i\geq 1}$ i.i.d. random variables, with symmetric distribution given by $\mathbf{P}(X_i = k) = c_\gamma \gamma^{|k|}$, with $c_\gamma = \frac{1-\gamma}{1+\gamma}$. This geometric random walk arises naturally in the context of the (discrete) Solid-On-Solid (SOS) model, which is a gradient interface model with potential $V(x) = -\log f(x) = |x|$, used is an effective model for interfaces in the Ising model, see [Vel06, IV18] for reviews. It also appears in the context of Interacting Partially Directed Self-avoiding Walk, see [CNPT18] for a review.

As far as the critical point is concerned, we also use Lemma 2.2 to get that $\lambda_c = \mathbf{P}(\bar{H}_1 = 0)^{-1}$. Here, thanks to the memoryless property of the geometric distribution, one easily gets that the ladder height $\bar{H}_1$ has distribution $\mathbf{P}(\bar{H}_1 = k) = (1-\gamma)\gamma^k$ for $k \geq 0$. We therefore get that $\lambda_c = \frac{1}{\mathbf{P}(\bar{H}_1=0)} = \frac{1}{1-\gamma}$. The computation of the Laplace transform (2.7) of $K(\cdot)$ (hence of the free energy) has been made in [LP22, Prop. A.1]: we have

$$\mathrm{F}(\lambda) = \ln\Big(\frac{\lambda(\lambda-1)(1-\gamma)^2}{\lambda(1-\gamma^2)-1}\Big) \qquad \text{for } \lambda \geq \lambda_c = \frac{1}{1-\gamma}\,.$$

As far as the critical behavior is concerned, we find that $\mathrm{F}(\lambda_c + u) \sim \frac{(1-\gamma)^2}{\gamma} u^2$ as $u \downarrow 0$.

With elevated boundary conditions. We use that $\Lambda(t) = -\ln\big(1 - \frac{2\gamma}{(1-\gamma)^2}(\cosh(t)-1)\big)$ for any $|t| < \ln\frac{1}{\gamma}$, so $\Lambda^{-1}(x) = \cosh^{-1}\big(1 + \frac{(1-\gamma)^2}{2\gamma}(1-e^{-x})\big)$. Then, using the formula above for $\mathrm{F}(\lambda)$, we get that $\Lambda^{-1}(\mathrm{F}(\lambda)) = \cosh^{-1}\big(\frac{(\lambda-1)^2+\gamma^2\lambda^2}{2\gamma\lambda(\lambda-1)}\big)$, so we end up with

$$\mathrm{F}(\lambda, a) = \left(\ln\Big(\frac{\lambda(\lambda-1)(1-\gamma)^2}{\lambda(1-\gamma^2)-1}\Big) - 2a\cosh^{-1}\Big(\frac{(\lambda-1)^2+\gamma^2\lambda^2}{2\gamma\lambda(\lambda-1)}\Big)\right)_+,$$

and the value of $a_c(\lambda)$ can be read from the above.

A.3. **Symmetric Laplace random walk, one-dimensional (continuous) SOS.** Let $\gamma > 0$ and consider $(X_i)_{i\geq 1}$ i.i.d. random variables with symmetric Laplace distribution of parameter γ, that is with density $f(x) = \frac{1}{2}\gamma e^{-\gamma|x|}$; this is a symmetrized $\mathrm{Exp}(\gamma)$ distribution. The measure (2.1) then corresponds to the (δ-pinning) wetting of the continuous Solid-On-Solid (SOS) model in dimension $d = 1$, see [CV00]. As for the geometric random walk, the memoryless property of the exponential gives that $\bar{H}_1$ has an $\mathrm{Exp}(\gamma)$ distribution: we get that $\bar{H}_1$ has density $f_{\bar{H}_1}(x) = \gamma e^{-\gamma x}\mathbb{1}_{\{x\geq 0\}}$. Using again Lemma 2.2, we obtain $\lambda_c = \frac{1}{f_{\bar{H}_1}(0)} = \frac{1}{\gamma}$. As far as the free energy is concerned, we can also compute the Laplace transform $\mathcal{K}(\vartheta)$ explicitly (this is done in Appendix B.3 and matches the formula [dCDH11, Eq. (3.6)]): we obtain

$$\mathrm{F}(\lambda) = \ln\Big(\frac{\gamma^2\lambda^2}{2\gamma\lambda-1}\Big) \qquad \text{for } \lambda \geq \lambda_c = \frac{1}{\gamma}\,.$$

As far as the critical behavior is concerned, we find that $\mathrm{F}(\lambda_c + u) \sim \gamma^2 u^2$ as $u \downarrow 0$.

With elevated boundary conditions. We use that $\Lambda(t) = -\ln(1 - t^2/\gamma^2)$ for any $|t| < \gamma$, so we obtain $\Lambda^{-1}(x) = \gamma\sqrt{1-e^{-x}}$. Using the formula above for $\mathrm{F}(\lambda)$, we get that $\Lambda^{-1}(\mathrm{F}(\lambda)) = \frac{\gamma\lambda-1}{\lambda}$, so we end up with

$$\mathrm{F}(\lambda, a) = \left(\ln\Big(\frac{\gamma^2\lambda^2}{2\gamma\lambda-1}\Big) - \frac{2a(\gamma\lambda-1)}{\lambda}\right)_+, \qquad a_c(\lambda) = \frac{\lambda}{2(\gamma\lambda-1)}\ln\Big(\frac{\gamma^2\lambda^2}{2\gamma\lambda-1}\Big)\,.$$

A.4. **Strictly α-stable random walk, one-dimensional Gaussian free field.** Consider $(X_i)_{i\geq 1}$ i.i.d. random variables with strictly α-stable distribution, $\alpha \in (0,2]$, *i.e.* such that $n^{-1/\alpha}S_n$ has the same law as X_1. In other words, we have $X_i \sim \mathcal{Z}$ and $a_n = n^{1/\alpha}$ in Assumption 1; examples include standard Cauchy and Normal distributions. In the Gaussian case, the measure (2.1) corresponds to the (δ-pinning) wetting of the massless Gaussian free field in dimension $d = 1$, see [CV00].

This class of wetting models is not completely integrable, in the sense that there is no closed formula for the free energy, but still, the critical point is explicit. Indeed, relying on the relation (B.1) and the strict stability, we obtain that $f_n^+(0) = f_\alpha(0) n^{-(1+\frac{1}{\alpha})}$ for all $n \geq 1$, where f_α is the density of $X_1 \sim \mathcal{Z}$; note that an explicit expression for $f_\alpha(0)$ can be found in [Nol20, Cor. 3.1]. Thanks to Theorem 2.4, we therefore get

$$\lambda_c = \frac{1}{\kappa} := \frac{1}{\sum_{n=1}^\infty \kappa_n} = \frac{1}{f_\alpha(0)\zeta(1+\frac{1}{\alpha})}\,.$$

For instance: we get $\lambda_c = \sqrt{2\pi}/\zeta(3/2)$ if $X_i \sim \mathcal{N}(0,1)$; we get $\lambda_c = \pi/\zeta(2) = 6/\pi$ if $X_i \sim \mathcal{C}(0,1)$.

In view of the formula for $f_n^+(0)$, we obtain that $K(n) = n^{-s}\zeta(s)^{-1}$, so the renewal τ has a zeta(s) inter-arrival distribution, with $s := 1 + \frac{1}{\alpha}$. Hence, the Laplace transform (2.7) is $\mathcal{K}(\vartheta) = \zeta(s)^{-1}\mathrm{Li}_s(e^{-\vartheta})$ where $\mathrm{Li}_s(z) := \sum_{n\geq 1} z^n n^{-s}$ is the so-called polylogarithmic function. The free energy is then given by the relation (2.8), *i.e.* $\mathrm{Li}_s(\mathrm{F}(\lambda)) = \zeta(s)(\kappa\lambda)^{-1} = (f_\alpha(0)\lambda)^{-1}$, which cannot be explicitly inverted. On the other hand, after some calculation, one gets that $\mathrm{F}(\lambda_c + u) \sim c_\alpha u^\alpha$ as $u \downarrow 0$, with $c_\alpha := (f_\alpha(0)/\alpha\Gamma(1/\alpha))^\alpha$; see also Proposition 2.5.

Remark A.1. *In the continuous case, the relation* (B.1) *gives that* $f_n^+(0) = \frac{1}{n} f_n(0)$, *where* f_n *is the density of* S_n. *Therefore, if the density of* S_n *at* 0 *is explicit, one obtains an explicit formula for* $f_n^+(0)$ *and the model could also be (at least partially) integrable.*

With elevated boundary conditions. Let us focus on the (standard) Gaussian random walk. We have that $\Lambda(t) = \frac{1}{2}t^2$, so $\Lambda^{-1}(x) = \sqrt{2x}$. We therefore get that

$$\mathrm{F}(\lambda, a) = \Big(\mathrm{F}(\lambda) - 2a\sqrt{2\mathrm{F}(\lambda)}\Big)_+\,, \qquad a_c(\lambda) = \frac{1}{2\sqrt{2}}\sqrt{\mathrm{F}(\lambda)}\,,$$

but with no explicit expression for the free energy $\mathrm{F}(\lambda)$.

A.5. **Generalized Laplace distribution.** In view of Remark A.1, we include in the list of examples the case of (symmetric) generalized Laplace distributions, also known as *variance-gamma* or *Bessel function* distributions; we refer to [KKP01, Sec. 4.1] for an overview. The density of a symmetric generalized Laplace distribution of parameter (ν, σ) (we write $X \sim$ g-Laplace(ν,σ) for short) has an explicit distribution, see [KKP01, §4.1.4.2], given by $f_{\nu,\sigma}(x) = \frac{\sigma\sqrt{2}}{\Gamma(\nu)\sqrt{\pi}}\Big(\frac{|x|}{\sigma\sqrt{2}}\Big)^{\nu-1/2}\mathrm{K}_{\nu-1/2}\big(\sqrt{2}|x|/\sigma\big)$, where $\mathrm{K}_{\nu-1/2}$ is the modified Bessel function of the third kind. Additionally, the behavior of the density at $x = 0$ is known: it goes to infinity if $\nu \leq \frac{1}{2}$ and $f_{\nu,\sigma}(0) = \frac{1}{\sigma\sqrt{2\pi}}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}$ if $\nu > \frac{1}{2}$.

A random variable $X \sim$ g-Laplace(ν,σ) can be represented as a mixture of centered normal distribution with random variance $\sigma^2 W$, with $W \sim$ Gamma$(\nu, 1)$; alternatively, it can be represented as the difference of two independent Gamma(ν,γ) random variables with $\gamma = \sqrt{2}/\sigma$. Additionally, such distributions are stable under convolution: the sum of two independent g-Laplace random variables with respective parameters (ν,σ), (ν',σ) has itself a g-Laplace distribution, with parameters $(\nu+\nu',\sigma)$.

Therefore, if $(X_i)_{i\geq 1}$ are i.i.d. with g-Laplace(ν,σ) distribution (with $\sigma=\sqrt{2}/\gamma$ if one wants to match the parametrization of Section A.3), then $S_n \sim$ g-Laplace$(n\nu,\sigma)$. Therefore, taking $\nu > \frac{1}{2}$ so that $f_n(0) < +\infty$ for all n and using the relation (B.1) together with the value for $f_{n\nu,1}(0)$ above, we get that

$$f_n^+(0) = \frac{1}{n} f_n(0) = \frac{1}{\sigma\sqrt{2\pi}} \frac{\Gamma(n\nu - \frac{1}{2})}{\Gamma(n\nu+1)}\,.$$

Hence, the critical point is "explicit", since $\lambda_c = \kappa^{-1}$ with $\kappa = \sum_{n=1}^\infty f_n^+(0)$. It turns out that the Laplace transform (or generating function) of $f_n^+(0)$ has a simple form if $\nu = 1$ (this is the example of Section A.3), but also if $\nu = 2$ (it also has a closed form if $\nu = 3$): for all $x \in [-1,1]$,

$$\sum_{n=1}^\infty x^n f_n^+(0) = \begin{cases} \frac{\sqrt{2}}{\sigma}(1-\sqrt{1-x}) & \text{if } \nu = 1\,, \\ \frac{\sqrt{2}}{\sigma}(1-\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1-x}}) & \text{if } \nu = 2\,. \end{cases}$$

Inverting the formula of the Laplace transform gives the expression of the free energy (and of the critical point), thanks to (2.8). When $\nu = 2$, setting $\gamma = \sqrt{2}/\sigma$, we get

$$\mathrm{F}(\lambda) = -\ln\Big(\frac{4(\gamma\lambda-1)^2(2\gamma\lambda-1)}{(\gamma\lambda)^4}\Big) \qquad \text{for } \lambda \geq \lambda_c = \frac{1}{\gamma}\frac{\sqrt{2}}{\sqrt{2}-1}\,.$$

To conclude, let us mention that when ν is an integer, the generating function of $f_n^+(0)$ is equal to the generalized hypergeometric function ${}_{\nu+1}F_\nu(\frac{-1}{2\nu}, \frac{1}{2\nu}, \dots, \frac{2\nu-3}{2\nu}; \frac{1}{\nu}, \dots, \frac{\nu-1}{\nu}; x)$, up to centering (by 1) and normalizing (by $\frac{1}{\sigma\sqrt{2\pi}}$).

With elevated boundary conditions. We use that $\Lambda(t) = -\nu\ln(1-\frac{1}{2}\sigma^2t^2)$, so that $\Lambda^{-1}(x) = \frac{\sqrt{2}}{\sigma}\sqrt{1-e^{-x/\nu}}$; for instance using the representation as a difference of independent $\Gamma(\nu,\gamma)$ random variables with $\gamma = \sqrt{2}/\sigma$. Similarly to the Gaussian case, we obtain that the critical line is $a_c(\lambda) = \frac{\sigma}{2\sqrt{2}}\mathrm{F}(\lambda)(1-e^{-\mathrm{F}(\lambda)/\nu})^{-1/2}$, with in general no explicit expression for $\mathrm{F}(\lambda)$.

Appendix B. A few results related to the (standard) wetting model

B.1. **Proof of Lemmas 2.2 and 2.3.** We start with the proof of Lemma 2.2, which uses a simple rewritting of $f_n^+(x)$ in terms of (weak) ladder epochs and heights.

Proof of Lemma 2.2. Let us introduce the first (weak) ladder epoch and height

$$\bar{T}_1 := \min\{n \geq 1, S_n \leq 0\} \quad \text{and} \quad \bar{H}_1 = -S_{\bar{T}_1}\,,$$

and the following density with respect to μ:

$$f_n^+(x) = \frac{1}{\mu(\mathrm{d}x)}\mathbf{P}(\bar{T}_1 = n, \bar{H}_1 \in \mathrm{d}x) \quad \text{for } x \geq 0\,.$$

Hence, summing over n we get that $\kappa = \kappa(0)$ where $\kappa(x) = \sum_{n=1}^\infty f_n^+(x) = f_{\bar{H}_1}(x)$, with $f_{\bar{H}_1}$ the density of $\bar{H}_1$ with respect to μ. In particular, $\kappa = \mathbf{P}(\bar{H}_1 = 0)$ in the discrete case and $\kappa = f_{\bar{H}_1}(0)$ in the continuous case. This concludes the proof of Lemma 2.2. □

We now turn to the proof of Lemma 2.3: the result is given in [CC13] (Proposition 4.1-(4.5) for the discrete case and Theorem 5.1-(5.2) for the continuous case); we give here a simpler proof for the sake of completeness since we do not aim for the level of generality in [CC13].

Proof of Lemma 2.3, continuous case. In the continuous case, by using [CGZ06, App. A.2], we obtain that

$$f_n^+(0) := \frac{1}{n} f_n(0) \tag{B.1}$$

where f_n is the density of S_n w.r.t. the Lebesgue measure; see also [AD99, Eq. (3)] for a more general statement[3].

Then the local limit theorem for densities [IL71, Thm. 4.3.1] gives that if f_n is bounded for some $n \geq 1$ (which is given by Assumption 1), then $f_n(0) \sim \frac{1}{a_n} f_\alpha(0)$. where f_α is the density of the limiting α-stable distribution. This shows Lemma 2.3 in the continuous case, recalling that $\mathbf{P}(\bar{H}_1 > 0) = 1$. □

Proof of Lemma 2.3, discrete case. In the discrete case, we start from the relation [AD99, Eq. (3)], which reads

$$f_n^+(0) = \mathbf{P}(\bar{T}_1 = n, \bar{H}_1 = 0) = \frac{1}{n}\mathbf{P}(S_n = 0, \bar{H}_1 > 0)\,. \tag{B.2}$$

(The strict inequality $>$ in place of $\geq$ is due to the fact that we are considering *weak* rather than *strong* ladder heights.) The reasoning in [AD99, Prop. 6] cannot be reproduced identically to obtain that $f_n^+(0) \sim \frac{1}{n}\mathbf{P}(S_n = 0)\mathbf{P}(\bar{H}_1 > 0)$ but we can easily adapt the proof; we simply need to update the use of Iglehart's lemma.

Our goal is to show that $\lim_{n\to\infty} \mathbf{P}(\bar{H}_1 > 0 \mid S_n = 0) = \mathbf{P}(\bar{H}_1 > 0)$ or equivalently

$$\lim_{n\to\infty} \mathbf{P}(\bar{H}_1 = 0 \mid S_n = 0) = \mathbf{P}(\bar{H}_1 = 0)\,. \tag{B.3}$$

Since $\mathbf{P}(S_n = 0) \sim \frac{1}{a_n} f_\alpha(0)$ as $n \to \infty$ by the local limit theorem, see e.g. [IL71, Thm. 4.2.1], combined with (B.2), this would end the proof of Lemma 2.3.

To obtain (B.3), let us start by writing

$$\mathbf{P}(\bar{H}_1 = 0 \mid S_n = 0) = \sum_{k=1}^{n} \mathbf{P}(\bar{T}_1 = k, S_k = 0 \mid S_n = 0) = \sum_{k=1}^{n} \mathbf{P}(\bar{T}_1 = k, S_k = 0)\frac{\mathbf{P}(S_{n-k} = 0)}{\mathbf{P}(S_n = 0)}\,.$$

To use similar notation as in [AD99], denote $d_k := \mathbf{P}(\bar{T}_1 = k, S_k = 0) = \mathbf{P}(\bar{T}_1 = k, \bar{H}_1 = 0)$ and $c_j := \mathbf{P}(S_j = 0)$ so we need to show that $\lim_{n\to\infty} \sum_{k=1}^{n} d_k \frac{c_{n-k}}{c_n} = \sum_{k=1}^{\infty} d_k$ since it is clear that $\sum_{k=1}^{\infty} = \mathbf{P}(\bar{H}_1 = 0)$.

Note that $c_n \sim f_\alpha(0)/a_n$ by the local limit theorem, with a_n regularly varying with exponent $1/\alpha$. Therefore, we have $c_{n-k}/c_n \to 1$ for any fixed k and $c_{n-k}/c_n \leq C$ uniformly for $k \geq n/2$: applying the dominated convergence theorem we get that

$$\lim_{n\to\infty} \sum_{k=1}^{n/2} d_k \frac{c_{n-k}}{c_n} = \sum_{k=1}^{\infty} d_k = \mathbf{P}(\bar{H}_1 = 0)\,.$$

For the remaining term, using that $d_k = f_k^+(0) = \frac{1}{k}\mathbf{P}(S_k = 0, \bar{H}_1 > 0)$ by (B.2), we have $d_k \leq \frac{1}{k} c_k$ for all $k \geq 1$, so

$$\sum_{k=n/2}^{n} d_k \frac{c_{n-k}}{c_n} \leq \sum_{k=n/2}^{n} \frac{1}{k}\frac{c_k}{c_n} c_{n-k} \leq \frac{C'}{n} \sum_{j=0}^{n/2} c_j\,,$$

[3]Note that [AD99, Eq. (3)] considers *strong* rather than *weak* ladder epochs and heights: in our context, it reads $\mathbf{P}(\bar{T}_1 = n, \bar{H}_1 \in \mathrm{d}x) = \frac{1}{n}\mathbf{P}(S_n \in \mathrm{d}x, \bar{H}_1 > x)$ for any $n \geq 1$ and $x \geq 0$.

where we have used that $\frac{1}{k}\frac{c_k}{c_n} \le C'\frac{1}{n}$ uniformly for $k \ge n/2$, again thanks to the regular variation of c_n. Now, we can simply use that $c_j \to 0$ as $j \to \infty$ to get that the Cesàro mean $\frac{1}{n}\sum_{j=0}^{n/2} c_j$ goes to 0. This ends the proof of (B.3) and hence of Lemma 2.3. □

B.2. **About the critical behavior of the free energy of the wetting model.** Our goal here is to prove the asymptotic for the free energy that have been collected in Proposition 2.5. First of all, let us observe that Assumption 1 is equivalent to having the following properties on the distribution of $(X_i)_{i\ge1}$; we refer to [Fel71, IX.8].

∗ When $\alpha \in (0,2)$, there is some slowly varying function $\varphi(\cdot)$ and constants $p, q > 0$ (take $p+q=1$ for normalization purposes) such that, as $x \to \infty$,

$$\mathbf{P}(X_i > x) \sim p\varphi(x)x^{-\alpha} \qquad \text{and} \qquad \mathbf{P}(X_i < -x) \sim q\varphi(x)x^{-\alpha}\,. \tag{B.4}$$

This is equivalent to the fact that there exist sequences $(a_n)_{n\ge0}$, $(b_n)_{n\ge0}$ such that $\frac{1}{a_n}(S_n - b_n)$ converges in distribution to some α-stable random variable. In order to have $b_n \equiv 0$, one must have $\mathbf{E}[X_i] = 0$ in the case $\alpha \in (1,2)$ and $p = q = \frac{1}{2}$ in the case $\alpha \in (0,1]$; in the case $\alpha = 1$, one also needs to have that $L(x)^{-1}\mathbf{E}[X_1 \mathbb{1}_{\{|X_1|\le x\}}]$ converges as $x \to \infty$.

∗ For $\alpha = 2$, then setting

$$\sigma^2(x) := \mathbf{E}\big[(X_i)^2 \mathbb{1}_{\{|X_i|\le x\}}\big]\,, \tag{B.5}$$

we have that Assumption 1 holds if and only if $\mathbf{E}[X_i] = 0$ and σ^2 is slowly varying at $+\infty$.

We may define $(a_n)_{n\ge0}$ up to asymptotic equivalence by the following relations

$$\mathbf{P}(|X_1| > a_n) \sim \varphi(a_n)a_n^{-\alpha} \sim \frac{1}{n} \quad \text{if } \alpha \in (0,2)\,, \qquad \sigma^2(a_n)a_n^{-2} \sim \frac{1}{n} \quad \text{if } \alpha = 2\,. \tag{B.6}$$

Then, we have that $\frac{1}{a_n S_n}$ converges in distribution to an α-stable random variable $\mathcal{Z}$.

Asymptotic of $\mathcal{K}(\vartheta)$. We now use Lemma 2.3 (or (2.6)) to obtain the asymptotic behavior of $\mathcal{K}(\vartheta)$ as $\vartheta \downarrow 0$. The proof is standard and follows the lines of [Gia07, Thm. 2.1], but we provide it for the sake of completeness.

Lemma B.1. *Recall that $\mathcal{K}(\vartheta) = \sum_{n=1}^{\infty} e^{-\vartheta n}K(n)$, with $K(n) = \frac{1}{\kappa}f_n^+(0) = L(n)n^{-(1+\frac{1}{\alpha})}$. Then we have, as $\vartheta \downarrow 0$*

$$1 - \mathcal{K}(\vartheta) \sim \begin{cases} \vartheta\, m_{1/\vartheta} & \text{if } \alpha \in (0,1]\,, \\ \alpha\Gamma(\frac{\alpha-1}{\alpha})\vartheta K(1/\vartheta) & \text{if } \alpha \in (1,2]\,, \end{cases}$$

where $m_x = \sum_{n=1}^{x} nK(n)$ converges if $\alpha < 1$ and is slowly varying if $\alpha = 1$.

Proof. Since $\sum_{n\ge1} K(n) = 1$, we simply write

$$1 - \mathcal{K}(\vartheta) = \sum_{n=1}^{\infty}(1 - e^{-\vartheta n})K(n) = \sum_{n=1}^{\infty}(1 - e^{-\vartheta n})L(n)n^{-(1+\frac{1}{\alpha})}\,.$$

If $\sum_{n=1}^{\infty} nK(n) < +\infty$, which contains the case $\alpha < 1$ in view of the fact that $K(n) = L(n)n^{-(1+\frac{1}{\alpha})}$, we get by dominated convergence that

$$\lim_{\vartheta\downarrow0}\sum_{n=1}^{\infty}\frac{1-e^{-\vartheta n}}{\vartheta}K(n) = \sum_{n=1}^{\infty} nK(n) = \kappa^{-1}\sum_{n=1}^{\infty} nf_n^+(0) = \lim_{\vartheta\downarrow0} m_{1/\vartheta}\,.$$

If $\sum_{n=1}^{\infty} n f_n^+(0) = +\infty$ and $\alpha = 1$, then $m_x := \sum_{n=1}^{x} L(n) n^{-1}$ is slowly varying at $+\infty$ and verifies $m_x / L(x) \to \infty$, see [BGT87, Prop. 1.5.9a]. Then, we get that

$$1 - \mathcal{K}(\vartheta) \geq \sum_{n=1}^{1/\vartheta} (1 - e^{-\vartheta n}) K(n) \geq \vartheta \sum_{n=1}^{1/\vartheta} L(n) n^{-1} - c\vartheta^2 \sum_{n=1}^{1/\vartheta} L(n) = (1 + o(1)) \vartheta \, m_{1/\vartheta} \,,$$

where we have used that $\sum_{n=1}^{1/\vartheta} L(n) \sim \vartheta^{-1} L(1/\vartheta)$, with $L(1/\vartheta) = o(m_{1/\vartheta})$. On the other hand,

$$1 - \mathcal{K}(\vartheta) \leq \sum_{n=1}^{1/\vartheta} (1 - e^{-\vartheta n}) K(n) + \sum_{n > 1/\vartheta} K(n) \leq \vartheta \sum_{n=1}^{1/\vartheta} L(n) n^{-1} + \sum_{n > 1/\vartheta} L(n) n^{-2} = (1 + o(1)) \vartheta \, m_{1/\vartheta} \,,$$

where we have used that $\sum_{n > 1/\vartheta} L(n) n^{-2} \sim L(1/\vartheta) \vartheta$ with $L(1/\vartheta) = o(m_{1/\vartheta})$.

It remains to treat the case $\alpha \in (1, 2]$. This time, we use a Riemann sum approximation (see also [BGT87, Thm. 1.7.1 and Cor. 8.1.7]) to get that

$$1 - \mathcal{K}(\vartheta) = L(1/\vartheta) \vartheta^{\frac{1}{\alpha}} \sum_{n=1}^{\infty} \vartheta \frac{1 - e^{-\vartheta n}}{(\vartheta n)^{1 + \frac{1}{\alpha}}} \frac{L(n)}{L(1/\vartheta)} \sim L(1/\vartheta) \vartheta^{\frac{1}{\alpha}} \int_0^{\infty} \frac{1 - e^{-u}}{u^{1 + \frac{1}{\alpha}}} \mathrm{d}u \,,$$

with $\int_0^{\infty} \frac{1 - e^{-u}}{u^{1 + \frac{1}{\alpha}}} \mathrm{d}u = \alpha \Gamma(1 - \frac{1}{\alpha})$ by some integration by parts. □

Asymptotic of the free energy: proof of Proposition 2.5. Now we are ready to prove Proposition 2.5. From (2.8), we get that for $\lambda > \lambda_c := 1/\kappa$,

$$1 - \mathcal{K}(\mathrm{F}(\lambda)) = 1 - (\kappa \lambda)^{-1} = \frac{\lambda - \lambda_c}{\lambda} \,.$$

Since we know that $\mathrm{F}(\lambda) \downarrow 0$ as $\lambda \downarrow \lambda_c$, we may use Lemma B.1 to obtain the following. To simplify notation, we write $\mathrm{F}_u := \mathrm{F}(\lambda_c + u)$, which goes to 0 as $u \downarrow 0$.

∗ If $\sum_{n=1}^{\infty} n K(n) < +\infty$, or equivalently (by Lemma 2.3) if $\sum_{n=1}^{\infty} \frac{1}{a_n} < +\infty$ that is if $(S_n)_{n \geq 0}$ is transient, then, as $u \downarrow 0$,

$$\mathrm{F}_u \sum_{n=1}^{\infty} n K(n) \sim \frac{u}{\lambda_c} = \kappa u \,.$$

Since $K(n) = \kappa^{-1} f_n^+(0)$ and $\lambda_c = \kappa^{-1}$, we get (2.10).

∗ If $\alpha \in (1, 2]$, then as $u \downarrow 0$,

$$\alpha \Gamma(\tfrac{\alpha - 1}{\alpha}) \, \mathrm{F}_u \, K(1/\mathrm{F}_u) \sim \kappa^{-1} c_0 \alpha \Gamma(\tfrac{\alpha - 1}{\alpha}) \frac{1}{a_{1/\mathrm{F}_u}} \sim \kappa u \,,$$

where we also have used Lemma 2.3 to get that $n K(n) \sim \kappa^{-1} c_0 / a_n$ as $n \to \infty$. Using the relation (B.6) that defines $(a_n)_{n \geq 0}$, we get that:

$$\begin{aligned} \mathrm{F}_u &\sim \mathbf{P}(|X_1| > \tfrac{c_2}{u}) \quad && \text{if } \alpha \in (1, 2) \,, \\ \mathrm{F}_u &\sim \sigma^2 (\tfrac{c_2}{u}) \big(\tfrac{c_2}{u}\big)^{-2} && \text{if } \alpha = 2 \,, \end{aligned}$$

with $c_2 := c_0 \alpha \Gamma(\frac{\alpha - 1}{\alpha}) / \kappa^2$. This gives (2.12)-(2.13), using that $c_0 = \frac{1}{\sqrt{2\pi}} \mathbf{P}(\bar{H}_1 > 0)$ in the case $\alpha = 2$, so that $c_2 = \sqrt{2} \mathbf{P}(\bar{H}_1 > 0) / \kappa^2$.

∗ If $\alpha = 1$ and $\sum_{n=1}^{\infty} n K(n) = \sum_{n=1}^{\infty} L(n) n^{-1} = +\infty$, we obtain that $\mathrm{F}_u m_{1/\mathrm{F}_u} \sim \kappa u$, and we have to invert this relation. Let us introduce v_t such that $L(v_t) v_t^{-1} \sim t^{-1}$ and $w_t := t m_{v_t}$: then

by [Ber19, Lem. 4.3] we get that $m_{w_t} \sim m_{v_t}$ as $t \to \infty$, so that we have $m_{w_t} v_t^{-1} \sim t^{-1}$ as $t \to \infty$. Using this relation, we get that

$$1/\mathrm{F}_u \sim w_{1/\kappa u} \qquad \text{so} \qquad \mathrm{F}_u \sim \frac{\kappa u}{m_{v_{1/u}}},$$

using also that m is slowly varying and that v is regularly varying. Now, we can use Lemma 2.3 to get that

$$m_{v_{1/u}} \sim \kappa^{-1} c_0 \sum_{n=1}^{v_{1/u}} \frac{1}{a_n} \sim \kappa^{-1} c_0 \int_1^{v_{1/u}} \frac{1}{a_t} \mathrm{d}t \sim \kappa^{-1} c_0 \int_1^{a_{v_{1/u}}} \frac{1}{s\varphi(s)} \mathrm{d}s,$$

where we have used a change of variable $s = a_t$ (so $t \sim s/\varphi(s)$, see (B.6)), in the spirit of [Ber19, p. 36]. Now, notice that by Lemma 2.3 we have $a_n \sim \kappa c_0 / nK(n) \sim \kappa c_0 n / L(n)$, so by definition of v_t we get that $a_{v_{1/u}} \sim \kappa c_0 / u$. Since $u \mapsto \int_1^{1/u} \frac{1}{s\varphi(s)} \mathrm{d}s$ is slowly varying, we get that

$$m_{v_{1/u}} \sim \kappa^{-1} c_0 \int_1^{1/u} \frac{1}{s\varphi(s)} \mathrm{d}s \sim \kappa^{-1} c_0 \int_1^{1/u} \frac{\mathrm{d}s}{s^2 \mathbf{P}(|X_1| > s)}.$$

This concludes the proof of (2.11). □

B.3. **About the SOS (or Laplace) wetting model.** In this section, we derive the free energy for the wetting model of Section A.3. We only deal with the exponential random walk (or SOS model) since the proof does not appear clearly in the literature, but it could also be adapted to the lazy and geometric random walks of Sections A.1-A.2, only with more tedious calculations.

First of all, let us notice that, by the memoryless property of the exponential random variable, we have that $\bar{T}_1$ and $\bar{H}_1$ are independent; and $\bar{H}_1$ follows an exponential distribution with $f_{\bar{H}_1}(0) = \gamma$. Therefore, we have that

$$f_n^+(0) = \gamma \mathbf{P}(\bar{T}_1 = n), \qquad K(n) = \mathbf{P}(\bar{T}_1 = n).$$

Let us compute the Laplace transform $\mathcal{K}(\vartheta) = \sum_{n=1}^{\infty} e^{-\vartheta n} K(n)$. For $\vartheta > 0$ fixed, we introduce the martingale $(M_n)_{n \geq 0}$ defined by

$$M_n := e^{-\theta S_n - \vartheta n},$$

where $\theta = \gamma \sqrt{1 - e^{-\vartheta}} < \gamma$ is chosen so that[4] $\mathbf{E}[e^{-\theta X_1}] = \frac{1}{1 - \theta^2/\gamma^2} = e^{-\vartheta}$.

Applying the stopping time theorem, together with dominated convergence (since $S_{n \wedge \bar{T}_1} \geq S_{\bar{T}_1}$, with $-S_{\bar{T}_1} = \bar{H}_1$ an exponential distribution), we obtain that

$$1 = \lim_{n \to \infty} \mathbf{E}[M_{n \wedge \bar{T}_1}] = \mathbf{E}\big[e^{-\theta S_{\bar{T}_1} - \vartheta \bar{T}_1}\big] = \mathbf{E}\big[e^{\theta \bar{H}_1}\big] \mathbf{E}\big[e^{-\vartheta \bar{T}_1}\big],$$

where we have used again that $\bar{H}_1 = -S_{\bar{T}_1}$ is independent of $\bar{T}_1$ in the last identity. We therefore end up with

$$\mathcal{K}(\vartheta) = \mathbf{E}\big[e^{\theta \bar{H}_1}\big]^{-1} = 1 - \frac{\theta}{\gamma} = 1 - \sqrt{1 - e^{-\vartheta}}.$$

The relation $\mathcal{K}(\mathrm{F}(\lambda)) = (\kappa\lambda)^{-1}$ from Theorem 2.4 (with $\kappa = \gamma$) therefore gives

$$\mathrm{F}(\lambda) = \ln\Big(\frac{(\gamma\lambda)^2}{2\gamma\lambda - 1}\Big).$$

[4] To compute $\mathbf{E}[e^{-\theta X_1}]$, we can use the representation $X_1 = Y - Y'$ with Y, Y' independent $\mathrm{Exp}(\gamma)$ random variables, so $\mathbf{E}[e^{uY}] = \frac{1}{1 - u/\gamma}$ for any $|u| < \gamma$.

Appendix C. Local large deviations: proof of Lemmas 5.1 and 7.3

Proof of Lemma 5.1. Let us focus on the upper local large deviation; the lower counterpart is proven in an identical manner. We need to show that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}(S_n = x_n) \geq -\mathrm{I}_+(x)\,,$$

the other inequality being given by (5.2) and the fact that I_+ is continuous at x.

First of all, if $t_0^+ = 0$, *i.e.* if $\Lambda_+(t) = +\infty$ for all $t > 0$, then $\mathrm{I}_+(x) = 0$ for all $x \geq 0$. In that case, we get that $\limsup_{k\to\infty} \frac{1}{k} \log \mathbf{P}(X_1 = k) = 0$ (one can easily check that otherwise $t_0^+ > 0$), so we can choose some k_0 such that $\mathbf{P}(X_1 = k_0) \geq e^{-\varepsilon k_0}$ and $k_0 \geq 3x$. Then, setting $n_0 = \lfloor x_n/k_0 \rfloor$, we have that

$$\mathbf{P}\big(S_n = x_n\big) \geq \mathbf{P}\big(X_i = k_0 \text{ for all } 1 \leq i \leq n_0\big)\mathbf{P}\big(S_{n-n_0} = x_n - k_0 n_0\big)\,. \tag{C.1}$$

Notice that $n_0 \leq n/2$ (provided that n is large enough) and that $x_n - k_0 n_0 \in \{0, \dots, k_0\}$. Using the local central limit theorem, see e.g. [GK54, Ch. 9, §50], we therefore get that

$$\mathbf{P}\big(S_n = x_n\big) \geq \big(e^{-\varepsilon k_0}\big)^{n_0} \frac{c}{a_n} \geq \frac{c}{a_n} e^{-\varepsilon x_n}\,,$$

so $\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}(S_n = x_n) \geq -\varepsilon x$, recalling that $\lim_{n\to\infty} \frac{1}{n} x_n = x$. Since ε is arbitrary, this concludes the proof in the case $t_0^+ = 0$.

Let us now turn to the case where $t_0^+ > 0$. Then, fixing ε small enough so that $\frac{x}{1-\varepsilon} < \bar{x}_+$, we have that

$$\begin{aligned}\mathbf{P}\big(S_{(1-\varepsilon)n} \in [(1-\varepsilon^2)x_n, x_n]\big) &= \mathbf{P}\big(S_{(1-\varepsilon)n} \geq (1-\varepsilon^2)x_n\big) - \mathbf{P}\big(S_{(1-\varepsilon)n} > x_n\big) \\ &= e^{-(1+o(1))\mathrm{I}_+((1+\varepsilon)x)(1-\varepsilon)n} - e^{-(1+o(1))\mathrm{I}_+(\frac{x}{1-\varepsilon})(1-\varepsilon)n}\,,\end{aligned}$$

where we have used that $\lim_{n\to\infty} \frac{1}{n} x_n = x$ and that I_+ is continuous at $(1+\varepsilon)x$ and $x/(1-\varepsilon)$. Then the second term is negligible compared to the first one since $\mathrm{I}_+(\frac{x}{1-\varepsilon}) > \mathrm{I}_+((1+\varepsilon)x)$, because I_+ is increasing on $[0, \bar{x}_+)$. We therefore get that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(S_{(1-\varepsilon)n} \in [(1-\varepsilon^2)x_n, x_n]\big) = -(1-\varepsilon)\mathrm{I}_+((1+\varepsilon)x)\,.$$

We now write

$$\begin{aligned}\mathbf{P}\big(S_n = x_n\big) &\geq \sum_{y=(1-\varepsilon^2)x_n}^{x_n} \mathbf{P}\big(S_{(1-\varepsilon)n} = y\big)\mathbf{P}(S_{\varepsilon n} = x_n - y) \\ &\geq \mathbf{P}\big(S_{(1-\varepsilon)n} \in [(1-\varepsilon^2)x_n, x_n)\big) \times \inf_{z\in[0,\varepsilon^2 x_n]} \mathbf{P}(S_{\varepsilon n} = z)\,.\end{aligned}$$

It remains to get a lower bound on the last term. Let $k_0' := \inf\{k \geq 1, \mathbf{P}(X_1 = k) > 0\}$. Then, similarly as in (C.1), setting $n_0 = n_0(z) = \lfloor z/k_0' \rfloor$, we get that

$$\mathbf{P}(S_{\varepsilon n} = z) \geq \mathbf{P}\big(X_i = k_0' \text{ for all } 1 \leq i \leq n_0\big)\mathbf{P}(S_{\varepsilon n - n_0} = z - k_0' n_0)\,.$$

Again, $n_0 \leq z \leq \varepsilon^2 x_n$, so $\varepsilon n - n_0 \geq \varepsilon n \left(1 - \varepsilon \frac{x_n}{n}\right)$. For n big enough, we have $\frac{x_n}{n} \leq 2x$, then if $\varepsilon \leq \frac{1}{4x}$ we have $\varepsilon n - n_0 \geq \varepsilon \frac{n}{2}$. Since $z - k_0' n_0 \in \{0, \dots, k_0'\}$ we can use the local central limit theorem to bound the last term by a constant times $a_{\varepsilon n}^{-1}$, uniformly in $z \in [0, \varepsilon^2 x_n]$. We therefore end up with

$$\inf_{z\in[0,\varepsilon^2 x_n]} \mathbf{P}(S_{\varepsilon n} = z) \geq \frac{c}{a_{\varepsilon n}} \mathbf{P}(X_1 = k_0')^{\varepsilon^2 n}\,.$$

All together, we get that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(S_n = x_n\big) \geq -(1-\varepsilon)\mathrm{I}_+\big((1+\varepsilon)x\big) + \varepsilon^2 \log \mathbf{P}(X_1 = k_0')\,.$$

Since ε is arbitrary and I_+ is continuous on $[0, \bar{x}_+]$, this concludes the proof. □

Proof of Lemma 7.3. The proof follows the same lines as that of Lemma 5.1.

First of all, let us treat the case $t_0^+ = 0$. Then, as in (C.1), with the same definition for k_0 and with $n_0 = \lfloor \frac{x_n}{k_0} \rfloor$, we have that

$$Q_+(n, x_n) \geq \mathbf{P}\big(X_i = k_0\big)^{n_0} \mathbf{P}\big(S_{n-n_0} = x_n - k_0 n_0, S_i > 0 \text{ for all } 1 \leq i \leq n - n_0 - 1\big)\,.$$

Now, let $k_1 > 0$ be such that $\mathbf{P}(S_1 > 0, \dots, S_{k_1-1} > 0, S_{k_1} = y) > 0$ for all $y \in \{0, \dots, k_0\}$, which exists by aperiodicity and let $c = \inf_{0\leq y\leq k_0} \mathbf{P}(S_1 > 0, \dots, S_{k_1-1} > 0, S_{k_1} = y) > 0$, which depends only on k_0. Recalling that $x_n - k_0 n_0 \in \{0, \dots, k_0\}$, we therefore get that

$$\mathbf{P}\big(S_{n-n_0} = x_n - k_0 n_0, S_i > 0 \text{ for all } 1 \leq i \leq n - n_0 - 1\big) \geq c f^+_{n-n_0-k_1}(0)\,,$$

with $f_n^+(0) = \mathbf{P}(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0)$ defined in (2.3). Now, thanks to Lemma 2.3, and since $n - n_0 - k_1 \geq n/4$ for n large enough, we get that

$$Q_+(n, x_n) \geq \frac{c'}{n a_n} e^{-\varepsilon k_0 n_0} \geq \frac{c'}{n a_n} e^{-\varepsilon x_n}\,.$$

We therefore end up with $\liminf_{n\to\infty} \frac{1}{n} \log Q_+(n, \lfloor xn \rfloor) \geq -\varepsilon x$, which gives the result since ε is arbitrary, recalling that $\mathrm{I}_+(x) = 0$.

Let us now turn to the case where $t_0^+ > 0$. We focus first on the case $\lim_{n\to\infty} \frac{1}{n} x_n = x \in (0, \bar{x}_+)$. Denoting again $k_0' = \min\{k \geq 1, \mathbf{P}(X_1 = k) > 0\}$, we have that

$$Q_+(n, x_n) \geq \mathbf{P}(X_1 = k_0')^{\varepsilon n} \mathbf{P}\big(S_{(1-\varepsilon)n} = x_n - \varepsilon n k_0', S_i > -\varepsilon n k_0' \text{ for all } i \leq (1-\varepsilon)n\big)\,.$$

Then, we write

$$\begin{aligned} &\mathbf{P}\big(S_{(1-\varepsilon)n} = x_n - \varepsilon n k_0', S_i > -\varepsilon n k_0' \text{ for all } i \leq (1-\varepsilon n)\big) \\ &\qquad = \mathbf{P}\big(S_{(1-\varepsilon)n} = x_n - \varepsilon n k_0'\big) - \mathbf{P}\big(S_{(1-\varepsilon)n} = x_n - \varepsilon n k_0', \min_{1\leq i\leq (1-\varepsilon)n} S_i \leq -\varepsilon n k_0'\big)\,. \end{aligned} \tag{C.2}$$

For the first term, we can use Lemma 5.1 to get that it is $\exp(-(1+o(1))(1-\varepsilon)n\mathrm{I}_+(\frac{x-\varepsilon k_0'}{1-\varepsilon})n)$. For the other term, decomposing over the first time where $S_i \leq -\varepsilon n k_0'$, we easily get by the strong Markov property that

$$\mathbf{P}\big(S_{(1-\varepsilon)n} = x_n - \varepsilon n k_0', \min_{1\leq i\leq (1-\varepsilon)n} S_i \leq -\varepsilon n k_0'\big) \leq \sup_{1\leq j\leq (1-\varepsilon)n, y\geq 0} \mathbf{P}\big(S_j = x_n + y\big)\,.$$

Then, using (5.2) and the fact that I_+ is non-decreasing, we get that this is bounded by

$$\sup_{1\leq j\leq (1-\varepsilon)n} e^{-j\mathrm{I}_+(\frac{x_n}{j})} \leq e^{-(1-\varepsilon)n\mathrm{I}_+(\frac{x_n}{(1-\varepsilon)n})} \leq e^{-(1-\varepsilon)n\mathrm{I}_+(\frac{x-\varepsilon^2}{1-\varepsilon})}\,,$$

where the first identity holds by convexity of I_+, and the second one provided that n is large enough so that $\frac{1}{n} x_n \geq x - \varepsilon^2$.

Since $(1-\varepsilon)\mathrm{I}_+(\frac{x-\varepsilon^2}{1-\varepsilon}) > (1-\varepsilon)\mathrm{I}_+(\frac{x-\varepsilon k_0'}{1-\varepsilon})$ because I_+ is increasing, we get that the second term in (C.2) is negligible, so that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(S_{(1-\varepsilon)n} = x_n - \varepsilon n k_0', \min_{1\leq i\leq (1-\varepsilon)n} S_i \leq -\varepsilon n k_0'\big) = (1-\varepsilon)\mathrm{I}_+\Big(\frac{x - \varepsilon k_0'}{1-\varepsilon}\Big)\,.$$

All together, this gives that

$$\liminf_{n\to\infty} \frac{1}{n} \log Q_+(n, x_n) \geq \varepsilon \log \mathbf{P}(X_1 = k_0') - (1-\varepsilon)\mathrm{I}_+\Big(\frac{x - \varepsilon k_0'}{1-\varepsilon}\Big)\,,$$

which concludes the proof since ε is arbitrary and I_+ is continuous at $x < \bar{x}_+$.

For the case $\lim_{n\to\infty} \frac{1}{n} x_n = 0$, we fix $\varepsilon > 0$ and we use the following lower bound (omitting the integer parts for simplicity):

$$Q_+(n, x_n) \geq \mathbf{P}(S_{n/2} = \varepsilon n, S_n = x_n, S_1 > 0, \dots, S_{n-1} > 0) \geq Q_+(n/2, \varepsilon n) Q_-(n/2, \varepsilon n - x_n)\,,$$

where we have used Markov's property and the duality property

$$\begin{aligned}\mathbf{P}(S_{n/2} > x_n, \dots, S_{n-1} > x_n, S_n = x_n \mid S_{n/2} = \varepsilon n)& \\ = \mathbf{P}(S_1 < 0, \dots, S_{n/2} < 0, S_{n/2} = \varepsilon n - x_n) = Q_-(n/2, \varepsilon n - x_n).&\end{aligned}$$

Therefore, from the case $x > 0$ above we get that $\liminf_{n\to\infty} \frac{1}{n} \log Q_+(n, x_n) \geq -\mathrm{I}_+(\varepsilon/2) - \mathrm{I}_-(\varepsilon/2)$. This concludes the proof, by taking $\varepsilon \downarrow 0$, since I_+, I_- are continuous at 0. □

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