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A NEW DISCRETIZATION OF THE EULER EQUATION VIA THE FINITE OPERATOR THEORY

MIGUEL A. RODRÍGUEZ AND PIERGIULIO TEMPESTA

In memoriam of our friend and colleague Decio Levi

ABSTRACT. We propose a novel discretization procedure for the classical Euler equation, based on the theory of Galois differential algebras and the finite operator calculus developed by G.C. Rota and collaborators. This procedure allows us to define algorithmically a new discrete model which inherits from the continuous Euler equation a class of exact solutions.

CONTENTS

1. Introduction	1
2. The finite operator calculus: Background and notation	2
2.1. Delta operators and basic polynomials	3
2.2. A general discretization scheme	4
3. The forward discrete derivative and the associated calculus	4
4. A discretization of the homogeneous Euler equation	6
4.1. The construction of the model	6
4.2. The continuous limit	7
4.3. Exact solutions of the discrete Euler equation	8
5. Future perspectives	8
Appendix A. Some useful combinatorial identities	8
Acknowledgements	10
References	10

1. INTRODUCTION

The study of discrete versions of integrable ODEs has become one of the most remarkable research lines in mathematical physics, especially in the theory of integrable systems, during the last decades. A huge body of literature is available (see e.g. the monographs [1, 7, 10, 20] and the references therein).

More specifically, the problem of discretizing dynamical systems in such a way that their symmetry and integrability properties are preserved has also been widely investigated in the last decades. This has been one of the fields in which the scientific contribution of Decio Levi has been particularly relevant [10]. His beautiful insight in the study of the multiple facets of discrete mathematics and his tireless

scientific activity during several decades have crucially influenced several generations of researchers in mathematical physics - and in particular the authors of this article in a deep way.

Among the many interesting approaches proposed for the discretization of integrable models, we shall focus in this paper on a specific research line where the contribution of Decio Levi has also been crucial. This methodology is based on the *finite operator theory* (also called Umbral Calculus), developed by G. C. Rota and its collaborators [16, 17, 18]. The main idea of this approach is to preserve some crucial properties of a given integrable ODE, in particular its exact solutions, in the discretization process. By preserving some crucial algebraic structures, one can define difference equations which inherit from the continuous counterparts exact solutions and possibly symmetry properties [9]. This philosophy has been introduced in quantum mechanics in [4] and developed in [11, 13, 12, 14] for the general context of difference equations, by requiring that the Heisenberg-Weyl algebra be preserved. This in turn allows one to preserve the Lorentz and Galilei invariance on lattices. An application of this approach to the multiple scale analysis of dynamical systems on a lattice has been proposed in [12].

A related approach is based on the idea of preserving the Leibniz rule on lattices. This idea, under different perspectives has been proposed in [2, 23, 8, 21]. A crucial aspect of this theory is that the standard pointwise product of functions on a lattice is replaced by a suitable nonlocal product, in such a way that discrete derivatives act on functions, or more generally on formal power series, as standard derivations. In this framework, one can define discrete versions of continuous ODEs which share with them a class of exact solutions, namely those expressed in terms of power series.

Another important aspect of this approach is that by choosing the lattice points as the zeroes of the basic polynomials associated with the discrete derivative we are working with, one can avoid the main drawback of the umbral approach to difference equations, namely the appearance of divergent power series for the solutions of the discrete models. Indeed, the series expressing the exact solutions are by construction truncated ones, and therefore convergent in an obvious sense. In [22], this procedure has been generalized to the case of variable coefficients ODEs.

In the following, we shall adopt this discretization procedure in order to produce a new difference equation which can be interpreted as a discrete version of the Euler equation. Indeed, under suitable hypotheses it can be exactly solved and its solutions are obtained in a direct way from those of the standard Euler differential equation.

In this article, for the sake of simplicity we shall keep the mathematical formalism to a minimum. A more formal approach and further technical details can be found in [21, 22]; however, we will try to discuss our results in a self-contained way.

The paper is organized as follows. In section 2, we review some basic notions of finite operator theory. In Section 3, we implement our methodology to the discretization of Euler equation. Some exact solutions of our discrete Euler equation are derived in Section 4. Some open questions are addressed in the final Section 5.

2. THE FINITE OPERATOR CALCULUS: BACKGROUND AND NOTATION

The finite operator theory was proposed in [18] and further developed in [17], [16] as a formal calculus useful to treat combinatorial problems. It can be considered

as the modern version of the umbral calculus introduced by Sylvester, Cayley and Blissard and other authors [3]. In this section, we shall review some basic definition and fundamental results of Rota's approach to difference operators.

2.1. Delta operators and basic polynomials. Let \mathbb{K} be a field of characteristic zero and \mathcal{P} be the space of polynomials in one variable $x \in \mathbb{K}$. Let \mathbb{N} be the set of non-negative integers. The operator $T : \mathcal{P} \rightarrow \mathcal{P}$ such that $Tp(x) = p(x+h)$, where $h > 0$ is a suitable real constant, will be said to be the *shift operator*. For simplicity, we shall restrict to the case $\mathbb{K} = \mathbb{R}$. The notion of delta operator is crucial in the development of our discretization approach.

Definition 1. An operator S is said to be *shift-invariant* if commutes with the shift operator T . A shift-invariant operator Q is called a *delta operator* if $Qx = \text{const} \neq 0$.

A simple consequence comes from the previous definition [18]:

Corollary 2. For every constant $c \in \mathbb{R}$, $Qc = 0$.

Delta operators very commonly used in the applications are the standard derivative D , the forward discrete derivative $\Delta^+ = T - \mathbb{1}$, the backward derivative $\Delta^- = \mathbb{1} - T$ and the symmetric derivative $\Delta^s = \frac{T-T^{-1}}{2}$. Many other interesting examples can be found in the literature (e.g. in [18]).

Definition 3. A polynomial sequence $\{p_n(x)\}_{n \in \mathbb{N}}$ is said to be the *sequence of basic polynomials* for a given delta operator Q if the following conditions are satisfied:

- 1) $p_0(x) = 1$;
- 2) $p_n(0) = 0$ for all $n > 0$;
- 3) $Qp_n(x) = np_{n-1}(x)$.

One can prove that there exists a unique basic sequence of polynomials associated with a given delta operator Q . In the case of the continuous derivative operator, the sequence of basic polynomials is given by $p_n(x) = x^n$. For the forward and backward discrete derivatives Δ^\pm , the associated basic polynomials are, respectively

$$(1) \quad p_0^\mp(x) = 1, \quad p_n^\mp(x) := x(x \pm 1)(x \pm 2) \dots (x \pm (n-1)).$$

The basic polynomials for Δ^s have been explicitly determined in [12]. A general procedure for the construction of basic polynomials for an arbitrary delta operator has been discussed in [14]. For completeness, we shall sketch here the salient aspects of this procedure. Let \mathcal{A} be the algebra of shift-invariant operators, endowed with the usual operations of sum of two operators, product of a scalar with an operator, and product of two operators. Let $x : p(x) \rightarrow xp(x)$ denote the multiplication operator. The *Pincherle derivative* of an operator $U \in \mathcal{A}$ is defined by the relation

$$(2) \quad U' = [U, x]$$

In [18] it has been proved that, if Q is a delta operator, then Q' is invertible. Let us introduce a conjugate operator $\beta \in \mathcal{A}$ such that the Heisenberg-Weyl algebra is satisfied [11]:

$$(3) \quad [Q, x\beta] = \mathbb{1}$$

Thus, the operator β is uniquely determined by the relation $\beta = (U')^{-1}$. In order to illustrate three simple examples, we mention that in the continuous case, $Q = D$,

$\beta = 1$; for $Q = \delta^+$, $\beta = T^{-1}$; for $Q = \delta^-$, $\beta = T$. Generally speaking, the basic polynomials associated with a delta operator Q can be formally computed by means of the relation

$$(4) \quad p_n(x) = (x\beta)^n \cdot 1.$$

In particular, this relation reproduces eqs. (1).

Let \mathcal{F} be the algebra of formal power series in x . Since the polynomials $\{p_n(x)\}_{n \in \mathbb{N}}$ for every choice of Q provide a basis of \mathcal{F} , any $f \in \mathcal{F}$ can be expanded into a formal series of the form $f(x) = \sum_{n=0}^{\infty} a_n p_n(x)$.

2.2. A general discretization scheme. Our discretization procedure is based on the *umbral correspondence* [18, 4, 13, 14]: given a ODE, by replacing the continuous derivative with a delta operator Q , and the basic sequence x^n with the basic polynomials $p_n(x)$ for Q , one can construct a formal discretization of the original ODE. However, in order for this discretization to be effective, it is also crucial to replace the pointwise product of functions with a suitable $*$ product, which is associative and commutative, in such a way that

$$(5) \quad p_n(x) * p_m(x) := p_{n+m}(x).$$

This product for the forward difference operator Δ , has been proposed in [23] and in [8]. Given a choice of Q , one can define the notion of *Rota algebra*, introduced [21]: it is the space $(\mathcal{F}, +, \cdot, *_Q)$, endowed with the composition laws of sum of series, multiplication by a scalar and the $*$ product (5).

If f and g are formal series defined on a lattice of points \mathcal{L} , expanded in terms of a basic sequence, we have that

$$(6) \quad \Delta(f * g) = (\Delta f) * g + f * (\Delta g).$$

In other words, Δ acts as a derivation with respect to the $*$ -product: the Leibniz rule is restored on \mathcal{F} . If the zeroes of the basic polynomials coincide with the set of points of our lattice, then we have an effective discretization, since the series involved truncate. We shall explain this crucial point in detail in the forthcoming discussion.

Remark 4. The difference equations obtained by following this procedure naturally inherit from the continuous counterparts the class of real analytic exact solutions, which are expressed in terms of convergent power series. Indeed, their discrete versions are represented in terms of finite series obtained replacing x^n with the corresponding basic polynomial $p_n(x)$; the coefficients of these expansions coincide with those of the analytic solutions of the original models [23, 21, 22]. In this technical sense, we say that this procedure preserves integrability.

3. THE FORWARD DISCRETE DERIVATIVE AND THE ASSOCIATED CALCULUS

Since in the forthcoming discussion we shall focus on the operator $\Delta^+ := \Delta$, in order to fix the notation we propose here some useful formulas. We introduce a uniform lattice \mathcal{L} in the positive semi-axis \mathbb{R}_+ ,

$$(7) \quad x_n = nh, \quad h > 0, \quad n \in \mathbb{N},$$

and, given a function $u(x)$ of a real variable, we define its discretization as the set of all of its values at the lattice points:

$$(8) \quad u_n := u(x_n) = u(nh).$$

The k -degree basic polynomials for the forward discrete derivative operator on the uniform lattice are

$$(9) \quad (x)_0 := 1, \quad (x)_k := \prod_{j=0}^{k-1} (x - jh), \quad k = 1, 2, \dots$$

These polynomials have zeros at the lattice points x_i , $i = 0, \dots, k-1$. For $n \geq k$, we have:

$$(10) \quad (nh)_k = \prod_{j=0}^{k-1} (nh - jh) = h^k n(n-1) \cdots (n-k+1) = \frac{n!h^k}{(n-k)!}, \quad n \geq k$$

Let Q be a delta operator acting on \mathcal{P} , and $\{p_n(x)\}_{n \in \mathbb{N}}$ be the basic sequence of polynomials of order n uniquely associated with Q . Let \mathcal{F} be the algebra of formal power series in x . Since the polynomials $\{p_n(x)\}_{n \in \mathbb{N}}$ for every choice of Q provide a basis of \mathcal{F} , they allow us to expand a function $u(x)$ in terms of a formal power series [21]:

$$(11) \quad u(x) = \sum_{k=0}^{\infty} \zeta_k p_k(x), \quad \zeta_k \in \mathbb{R}$$

In this way, we extend the action of delta operators on functions. In general, the formal series (11) is divergent. However, in the case of $Q = \Delta$, since the lattice points coincides with the zeroes of $(x)_n$, the series expression for $u(x)$ on the lattice truncates. Indeed, we have:

$$(12) \quad u_n = \sum_{k=0}^{\infty} \zeta_k p_k(nh) = \sum_{k=0}^n \frac{n!h^k}{(n-k)!} \zeta_k.$$

The coefficients ζ_k can be obtained from the values of u_n by solving a system of linear equations [21]. We obtain the formula

$$(13) \quad \zeta_k = \frac{1}{h^k} \sum_{j=0}^k \frac{(-1)^{k-j}}{j!(k-j)!} u_j.$$

In the discretization process that we shall implement below, the continuous derivatives are replaced by the operators:

$$(14) \quad u' \rightarrow \Delta u_n = \frac{1}{h} (u_{n+1} - u_n), \quad u'' \rightarrow \Delta^2 u_n = \frac{1}{h^2} (u_{n+2} - 2u_{n+1} + u_n)$$

etc. The discretization of the derivative at the lattice points (the upper limit in the sums can be taken equal to n , because $p_i(mh) = 0$ when $i > m$) can be written in terms of the basic polynomials¹(we only consider second order derivatives in this article):

$$(15) \quad u'(nh) \rightarrow \sum_{j=0}^{n+1} j \zeta_j p_{j-1}(nh),$$

$$(16) \quad u''(nh) \rightarrow \sum_{j=0}^{n+2} j(j-1) \zeta_j p_{j-2}(nh)$$

¹The bounds in the sums can be adjusted in order that null terms are not present in the sums, but we always take into account that a negative factorial in the denominator yields 0 in the corresponding term

4. A DISCRETIZATION OF THE HOMOGENEOUS EULER EQUATION

In this section, as an application of the theory of discretization outlined before, we shall consider homogeneous linear equations of second order:

$$(17) \quad x^2 u'' + axu' + bu = 0, \quad a, b \in \mathbb{R}$$

We describe the discretization procedure in a step-by-step fashion, in order to illustrate it in a self-consistent way. A more abstract, formal approach based on category theory can also be implemented [22].

4.1. The construction of the model. Given the differential equation (17), we substitute x^k by $p_k(nh)$, the derivatives by their discrete versions (15), (16) and the usual product by the $*$ product. Explicitly, we have

$$(18) \quad p_2(nh) * \sum_{k=0}^{n+2} \frac{k! \zeta_k}{(k-2)!} p_{k-2}(nh) + a p_1(nh) * \sum_{k=0}^{n+1} \frac{k! \zeta_k}{(k-1)!} p_{k-1}(nh) + b \sum_{k=0}^{\infty} \zeta_k p_k(nh) = 0,$$

that is

$$(19) \quad \sum_{k=0}^n \left(\frac{1}{(k-2)!} + \frac{a}{(k-1)!} + \frac{b}{k!} \right) k! \zeta_k p_k(nh) = 0,$$

or,

$$(20) \quad \sum_{k=0}^n (k(k-1) + ak + b) \zeta_k p_k(nh) = \sum_{k=0}^n \Lambda_k \zeta_k p_k(nh) = 0,$$

where

$$(21) \quad \Lambda_k = k(k-1) + ak + b$$

is the indicial polynomial of the Euler differential equation (17).

The difference equation for u_k can be easily written using (10) and (13):

$$(22) \quad \sum_{k=0}^n \sum_{j=0}^k \frac{(-1)^{k-j} n! \Lambda_k}{j!(k-j)!(n-k)!} u_j = \sum_{k=0}^n \sum_{j=0}^k (-1)^{k-j} \Lambda_k \binom{n}{k} \binom{k}{j} u_j = 0.$$

Changing the order of the sums, we obtain the relation

$$(23) \quad \sum_{j=0}^n \left[\sum_{k=j}^n (-1)^{k-j} \Lambda_k \binom{n}{k} \binom{k}{j} \right] u_j = 0$$

which, using some combinatorial identities of Appendix A, becomes:

$$(24) \quad \sum_{j=0}^n \binom{n}{j} \left[\sum_{k=j}^n (-1)^{k-j} \Lambda_k \binom{n-j}{k-j} \right] u_j = \sum_{j=0}^n \binom{n}{j} c_{nj} u_j = 0$$

where,

$$(25) \quad c_{nj} = \sum_{k=j}^n (-1)^{k-j} \binom{n-j}{k-j} \Lambda_k = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \Lambda_{k+j}$$

Using again the results from Appendix A, we get

$$(26) \quad \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \Lambda_{k+j} \\ = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} (k^2 + (a+2j-1)k + j^2 + (a-1)j + b)$$

This formula can be rewritten as

$$(27) \quad \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \Lambda_{k+j} = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} k^2 + (a+2j-1) \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} k \\ + (j^2 + (a-1)j + b) \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \\ = 2\delta_{n-j,2} - \delta_{n-j,1} - (a+2j-1)\delta_{n-j,1} + (j^2 + (a-1)j + b) \delta_{n-j,0} .$$

Finally, for the coefficients of the difference equation (24) we get

$$c_{nj} = 2\delta_{n-2,j} - (a+2j)\delta_{n-1,j} + (j^2 + (a-1)j + b) \delta_{n,j}$$

Consequently, equation (24) is a three-points difference equation and the only coefficients different from zero for a given n are

$$(28) \quad c_{n,n-2} = 2, \quad c_{n,n-1} = -a - 2(n-1), \quad c_{nn} = n^2 + (a-1)n + b .$$

Therefore, our difference equation becomes:

$$(29) \quad \binom{n}{n-2} c_{n,n-2} u_{n-2} + \binom{n}{n-1} c_{n,n-1} u_{n-1} + \binom{n}{n} c_{nn} u_n = 0$$

We arrive at the following result.

Definition 5. *We shall call the difference equation*

$$(30) \quad (n^2 + (a-1)n + b)u_n - n(a+2n-2)u_{n-1} + n(n-1)u_{n-2} = 0$$

the discrete Euler equation.

4.2. The continuous limit. The difference equation (30), which has been obtained by an algorithmic, purely algebraic procedure, can be considered as the discrete version of the continuous model (17), defined on the Rota algebra $(\mathcal{F}, +*_{\Delta})$. However, there is also a direct relation among equations (30) and (17).

Precisely, if $h \rightarrow 0$ and nh remains finite with $nh \equiv x$, the continuous limit of the difference equation can be obtained as follows. Rewrite the equation (30) as:

$$(31) \quad n(n-1)h^2 \frac{u_n - 2u_{n-1} + u_{n-2}}{h^2} + a(nh) \frac{u_n - u_{n-1}}{h} + bu_n = 0$$

Then, in the limit when $n \rightarrow \infty$, our discrete model converts into the original differential equation

$$(32) \quad x^2 u'' + axu' + bu = 0$$

4.3. Exact solutions of the discrete Euler equation. The Euler equation (17) has (in the generic case) two independent solutions of the form x^r where r is a root of the indicial polynomial, $\Lambda_r = 0$. They represent a basis of the space of solutions. Let us assume that $r \in \mathbb{N}$. This restriction on the coefficients a, b in eq. (17) ensures the existence of analytic solutions.

According to our procedure, in order to inherit exact solutions of the discrete equation (30) from the continuous one, we expand the known solutions of eq. (17) in terms of the basic sequence associated with the Δ derivative. By using eqs. (11)-(12), we obtain the discrete function:

$$(33) \quad u_n = p_r(n) = \frac{n!}{(n-r)!}, \quad n \geq r$$

Let us show directly that this expression is an exact solution of the difference equation (30). By substituting it in the equation, we have

$$(34) \quad (n^2 + (a-1)n + b) \frac{n!}{(n-r)!} - n(a+2n-2) \frac{(n-1)!}{(n-1-r)!} + n(n-1) \frac{(n-2)!}{(n-2-r)!} = 0$$

and simplifying:

$$(35) \quad (n^2 + (a-1)n + b) - (a+2n-2)(n-r) + (n-r)(n-r-1) = 0$$

we finally get

$$(36) \quad r(r-1) + ar + b = 0$$

which is an identity since r is a root of the indicial polynomial.

5. FUTURE PERSPECTIVES

This work is part of a research project concerning the discretization of ODEs and PDEs in the algebraic framework of the finite operator calculus. In the case of the Euler equation, we have shown the effectiveness of our discretization procedure, by deriving the new discrete model (30). In particular, a class of exact solutions of this model is inherited by construction from the continuous Euler equation.

For a large class of variable coefficients ODEs, several general results and different examples have been obtained in [22]. A discrete version of the classical Frobenius theorem for ODEs is also in order [15].

The discretization approach proposed deserves further investigation. Among the related open problems, we wish to address the study of the symmetry groups possibly admitted by our discrete models, in the spirit of Lie's approach to ODEs, and the definition of appropriate boundary value problems.

More generally, we plan to extend the discretization procedure based on Rota algebras to the case of evolution equations and partial differential equations, both linear and nonlinear. Several preliminary results show the potential applicability of this technique also to the construction of new integrable quantum models defined on a lattice [5]. This also paves the way to possible applications of the theory in discrete formulations of quantum gravity [6, 19].

APPENDIX A. SOME USEFUL COMBINATORIAL IDENTITIES

The following formula (the Newton binomial) holds for any $a \in \mathbb{C}$:

$$(37) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a^k = (a-1)^n$$

When $a = 1$, it reduces to

$$(38) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} = 0, \quad n > 0$$

If $n = 0$ we can write for any a :

$$(39) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a^k = 1.$$

Then, for $a = 1$ and any $n \in \mathbb{N} \cup \{0\}$ we have the identity

$$(40) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} = (-1)^n \delta_{n0} = \delta_{n0}$$

Taking formally the derivative with respect to a in (37) and then multiplying both sides by a , we obtain

$$(41) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k a^k = n a (a-1)^{n-1}.$$

Thus, if $a = 1$, we get

$$(42) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k = 0, \quad n > 1$$

Again, if $n = 1$, we get 1, then:

$$(43) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} k = (-1)^n \delta_{n1} = -\delta_{n1}$$

Taking the second derivative in (37), we deduce the relation

$$(44) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k(k-1) a^{k-2} = n(n-1)(a-1)^{n-2}$$

which can also be written as:

$$(45) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^2 a^k - \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k a^k = n(n-1) a^2 (a-1)^{n-2}$$

or,

$$(46) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^2 a^k = n a (n a - 1) (a-1)^{n-2}$$

If $a = 1$:

$$(47) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^2 = 0, \quad n > 2$$

However, if $n = 1$

$$(48) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} k^2 = (-1)^n \delta_{n1} = -\delta_{n1}$$

and, if $n = 2$

$$(49) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} k^2 = 2(-1)^n \delta_{n2} = 2\delta_{n2}$$

that is:

$$(50) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} k^2 = 2\delta_{n2} - \delta_{n1} = \begin{cases} 0, & n = 0 \\ -1, & n = 1 \\ 2, & n = 2 \\ 0, & n > 2 \end{cases}$$

In particular, in the case under study:

$$(51) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} \Lambda_k = \sum_{k=0}^n (-1)^k \binom{n}{k} k^2 + (a-1) \sum_{k=0}^n (-1)^k \binom{n}{k} k + b \sum_{k=0}^n (-1)^k \binom{n}{k} \\ = 2\delta_{n2} - \delta_{n1} - (a-1)\delta_{n1} + b\delta_{n0} = 2\delta_{n2} - a\delta_{n1} + b\delta_{n0}$$

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