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CÉDRIC DE LACROIX, GREGORY CHICHERY, AND LUIGI SANTOCANALE

Abstract. For a functor Q from a category C to the category Pos of ordered sets and order-preserving functions, we study liftings of various kind of structures from the base category C to the total (or Grothendieck) category $\int Q$. That lifting a monoidal structure corresponds to giving some lax natural transformation making Q almost monoidal, might be part of folklore in category theory. We rely on and generalize the tools supporting this correspondence so to provide exact conditions for lifting symmetric monoidal closed and star-autonomous structures.

A corollary of these characterizations is that, if Q factors as a monoidal functor through SLatt, the category SLatt of complete lattices and sup-preserving functions, then $\int Q$ is always symmetric monoidal closed. In this case, we also provide a method, based on the double negation nucleus from quantale theory, to turn $\int Q$ into a star-autonomous category. The theory developed, originally motivated from the categories P-Set of Schalk and de Paiva, yields a wide generalization of Hyland and Schalk construction of star-autonomous categories by means of orthogonality structures.

Keywords. Grothendieck construction, total category, op-fibration, $*$-autonomous category, dualizing object, Girard quantale, double negation nucleus.

1. Introduction

Categorical models of proofs of multiplicative classical linear logic are $*$-autonomous categories [4]. Many of these categories are built from a given $*$-autonomous category, usually a degenerate one such as the category of sets and relations, by attaching to each object a structure and by requiring the maps to be compatible with the structures. The new category is usually better behaved, at least, for the semantics of proofs. Most often, the collection of all the structures that we can attach to an object is a poset. This is the case, for example, of Schalk and de Paiva categories P_F-Set [29] and of Hyland and Schalk orthogonality categories [14, 11].

From a categorical perspective, attaching to each object a structure amounts to considering the total category (or Grothendieck category) $\int Q$ of a (monoidal) functor $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$, where C is some symmetric monoidal category and Pos is the category of posets and order-preserving functions. This construction yields the canonical (op-)fibration $\pi : \int Q \longrightarrow \mathsf{C}$ which strictly preserves the monoidal structure. The theory of monoidal fibrations was firstly developed in [30] and later, from a different perspective more relevant here, in [21]. Roughly speaking, we investigate in this paper variants of the following questions: given a monoidal

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functor $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$ (so that we are ensured that the total category $\int Q$ is monoidal), when is $\int Q$ a closed category, and when is it $*$-autonomous?

More precisely, we give exact answers (and characterizations) to the following questions: given a monoidal functor $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$, where C is monoidal with some additional structure, when has $\int Q$ this structure in a such way that the canonical op-fibration π strictly preserves it? We call this the lifting problem for a structure. The structures considered here are being closed, being $*$-autonomous, and having terminal coalgebras (and initial algebras) of functors. Yet the tools developed in this paper can, in principle, be used to lift other kind of structures, if not all the structures.

When the monoidal functor Q takes values in the category SLatt of complete lattices and sup-preserving functions, our characterizations yield remarkable consequences. In this case, $\int Q$ turns out to always be closed. Moreover, assuming that C is $*$-autonomous with dualizing object 0, it is possible to turn $\int Q$ into a $*$-autonomous category by choosing an element $\omega \in Q(0)$, which might be thought of as a sort of global falsity, and by considering a double negation quotient of $\int Q$. While this construction yields a generalization of focused orthogonality categories [14, 11], the way we discovered it was through the analogy with quantales, that is, provability models of intuitioninistic linear logic, see [23]. In order to turn a quantale into a model of classical linear logic, a so called Girard quantale, see [22], it suffices to choose a candidate falsity (a candidate dualizing element) and consider the fixed points of the double negation nucleus it gives rise.

This research was partly motivated by recent research on the algebraic and categorical semantics of linear logic with fixed points [8, 6, 15, 11] extending to linear logic previous work by one of authors on the categorical semantics of fixed-point logics and circular proof systems [27, 26, 25, 12]. An important model of proofs of linear logic with fixed points, that has been considered in those works, is the category Nuts of non uniform totality spaces, which indeed arises as $\int Q$ for a functor $Q : \mathsf{Rel} \longrightarrow \mathsf{Pos}$. Besides considering models of proofs of linear logic, many other reasons have triggered us to this research. Persuaded that ordered structures are pervasive and essential both in logic and computation, we started investigating Frobenius quantales as Frobenius monoids in the $*$-autonomous category SLatt and, later, Frobenius monoids in arbitrary $*$-autonomous categories. We could prove in [7] the equivalence between an object X being nuclear and the monoid $X \multimap X$ being Frobenius under the hypothesis that the tensor unit embeds into X as a retract. To argue that this hypothesis is necessary, we built a counterexample by resorting to Schalk and de Paiva categories P-Set [29] for a well-chosen Girard quantale P. These categories, also of the form $\int Q$ for a functor $Q : \mathsf{Rel} \longrightarrow \mathsf{Pos}$, turned out to be extremely interesting for other reasons. For example—also considering the possible generalizations of these categories that we hint to in this paper—they can accommodate in a uniform way many categories of fuzzy sets and/or relations that have been considered in the literature, see e.g. [31, 13, 20]. These categories rely on the unit interval $[0, 1]$ and on some of its quantale structures (the Łukasiewicz quantale, its Heyting algebra structure, the Lawvere quantale, see e.g. [2]). The categories of the form P-Set deeply exemplify an interplay between quantales, that is, provability models of linear logic, and symmetric monoidal closed category, the models of proofs of the same logic. For intuitionistic logic, this interplay is well known, the connections between Heyting algebras, Cartesian closed categories, and

topoi being the object of several monographs. Whether this interplay is still relevant for linear logic is, in our opinion, unclear. The results presented in this paper can be understood as providing evidence and suggesting a positive answer to this question.

This paper is organised as follows. We give in Section 2 some background on posets, complete lattices, and monoidal categories; by doing so, we also settle the notation. We also define in this section the total category $\int Q$ and recall some of its properties. In Section 3 we characterise how to lift a functor of several variables, some contravariants and some covariants, from C to $\int Q$. Section 4, relying on the tools developed in the previous section, develops a methodology to devise exact conditions for lifting structures; the methodology is illustrated with the symmetric monoidal structure. The following Section 5 gives incremental conditions for lifting the closed structure, and Section 6 gives the conditions for lifting a dualizing object, thus the $*$-autonomous structure. In Section 7 we consider the case of a monoidal functor Q into SLatt, for which the results in the previous sections ensure that $\int Q$ is symmetric monoidal closed; we exhibit then a double negation construction, similar to the one in quantale theory, by which Q is transformed into a functor $Q^{\jmath^\omega}$ such that $\int Q^{\jmath^\omega}$ is $*$-autonomous. We also give a representation theorem for those monoidal functors Q into SLatt for which $\int Q$ is $*$-autonomous via a sort of phase semantics. In Section 8 we study the categories of algebras and coalgebras (as well as their initial or terminal objects) of an endofunctor $\overline{F}$ of $\int Q$ that is the lifting of an endofunctor F of C. Section 9 exemplifies the scope of the theory, while the last Section 10 gives concluding remarks and hints for future research. Due to the page limit, we present the proofs in the appendix.

2. Background, and the Grothendieck construction

Pos shall denote the category of partially ordered sets (posets) and order-preserving (or monotone) maps. The category SLatt has complete lattices as objects and sup-preserving maps as arrows. It is a subcategory of Pos. We refer the reader to standard monographs, for example [5] and [9], for an introduction to posets and complete lattices. A monotone map $f : X \longrightarrow Y$ between posets has $g : Y \longrightarrow X$ as a *right adjoint* adjoint if and only if, for each $x \in X$ and $y \in Y$, $f(x) \leq y$ is equivalent to $x \leq g(y)$. Such a map f can have at most one right adjoint for which we shall use the notation f^*. If f has a right adjoint, then f is sup-preserving and f^* inf-preserving. If X and Y are complete lattices, then f has a right adjoint if and only if it is sup-preserving. An explicit formula for f^* is given by $f^*(y) = \bigvee\{\, x \in X \mid f(x) \leq y \,\}$. If $f : X \longrightarrow Y$ is a map in SLatt, that is, if f is sup-preserving and X, Y are complete lattices, then we can see its right adjoint as a sup-preserving map $f^* : Y^{\mathrm{op}} \longrightarrow X^{\mathrm{op}}$. A *closure operator* on a complete lattice L is an order-preserving map $\jmath : L \longrightarrow L$ such that $x \leq \jmath(x)$ and $\jmath(\jmath(x)) = \jmath(x)$, for all $x \in L$. A *quantale* $(Q, e, *)$ is an ordered monoid whose underlying poset is complete and whose multiplication is sup-preserving in each variable. Said otherwise, it is monoid in the monoidal category SLatt, or a complete posetal symmetric monoidal closed category, see [9]. We assume that the reader is familiar with elementary category theory as exposed, for example, in the monograph [19]. We will mainly focus on symmetric monoidal categories $(\mathsf{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$, where I is the unit of the tensor $\otimes$, α is the associator, λ and ρ are respectively the left and right unitors, and σ is the symmetry. Moreover, if C is symmetric monoidal closed, then

we will denote by $- \multimap - : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \longrightarrow \mathsf{C}$ its internal hom. Given an object 0 we define a contravariant functor $(-)^* := - \multimap 0$ and C is a $*$-autonomous category if the canonical arrow $j_X : X \longrightarrow X^{**}$ is invertible for every object X.

In this paper we shall consider functors of the form $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$.

Definition 1. The Grothendieck (or total) category of Q, noted $\int Q$, is defined as follows:

- an object is a pair (X, α) with X an object of C and $\alpha \in Q(X)$,
- an arrow $(X, \alpha) \longrightarrow (Y, \beta)$ is an arrow $f : X \longrightarrow Y$ such that $Q(f)(\alpha) \leq \beta$.

The following lemma has an easy proof, that we skip:

Lemma 2. *A morphism $f : (X, \alpha) \longrightarrow (Y, \beta)$ is invertible if and only if $f : X \longrightarrow Y$ is invertible and $Q(f)(\alpha) = \beta$.*

The first projection $(X, \alpha) \mapsto X$ yields a functor

$$\pi : \int Q \longrightarrow \mathsf{C}$$

which is the usual example of an *op-fibration*. We avoid defining here this notion which, while pervasive in many cases, tuns out to be peripheral in our development.

The main aim of this paper is to describe the conditions for which some structures on the category C (monoidal, symmetric monoidal closed, $*$-autonomous, ...) can be lifted to $\int Q$ in a way that the projection functor strictly preserves the structure.

3. Lifting functors from C to $\int Q$

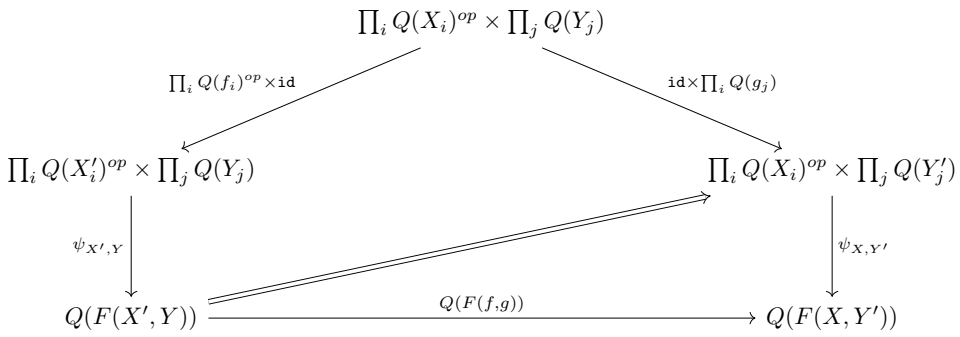
Let $F : (\mathsf{C}^{op})^n \times \mathsf{C}^m \longrightarrow \mathsf{C}$ be a functor. A *lifting* of F to $\int Q$ is a functor $\overline{F} : (\int Q^{op})^n \times \int Q^m \longrightarrow \int Q$ such that the following diagram strictly commutes:

$$\begin{array}{ccc} (\int Q^{op})^n \times \int Q^m & \xrightarrow{\overline{F}} & \int Q \\ \downarrow{\scriptstyle \pi^n \times \pi^m} & & \downarrow{\scriptstyle \pi} \\ (\mathsf{C}^{op})^n \times \mathsf{C}^m & \xrightarrow{F} & \mathsf{C} \end{array}$$

A *lax extranatural transformation* $\psi : \prod Q^{n+m} \longrightarrow Q \circ F$ is a collection of order-preserving maps

$$\psi_{X,Y} : \prod_i Q(X_i)^{op} \times \prod_j Q(Y_j) \longrightarrow Q(F(X, Y))$$

indexed by objects (X, Y) of $\mathsf{C}^n \times \mathsf{C}^m$, such that, for each pair of maps $f : X \longrightarrow X'$ in C^n and $g : Y \longrightarrow Y'$ in C^m, the following diagram semi-commutes:



(1)

By saying that the diagram above semi-commutes we mean that the lower leg of the diagram is below the upper leg in the pointwise ordering.

Remark that the notation $\psi : \prod Q^{n+m} \longrightarrow Q \circ F$ is just a matter of convenience, since for $\prod Q^{n+m} : \mathsf{C}^{n+m} \longrightarrow \mathsf{Pos}$ while $Q \circ F : (\mathsf{C}^{op})^n \times \mathsf{C}^m \longrightarrow \mathsf{Pos}$. Indeed, $\prod$ does not make a functor from $(C^{op})^n \times \mathsf{C}^n \longrightarrow \mathsf{C}$.

Theorem 3. *There is a bijection beweeen liftings a functor F from C to $\int Q$ and lax extranatural transformations $\psi : \prod Q^{n+m} \longrightarrow Q \circ F$.*

Proof. Let us first introduce some notation. For $X = (X_1, \ldots, X_k)$ an object of C^k, we write $Q(X)$ for the product $\prod_{i=1,\ldots,k} Q(X_i)$, and then, for $x \in Q(X) = \prod_i Q(X_i)$, we write (X, x) to denote the object $((X_1, x_1), \ldots, (X_k, x_k))$ of $(\int Q)^k$. Suppose that we have a lifting $\overline{F}$ of F. We can build $\overline{F}((X, x), (Y, y)) = (F(X, Y), z)$ with $z \in Q(F(X, Y))$. Calling then $z = \psi_{X,Y}(x, y)$, this defines a map $\psi_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(F(X, Y))$. We claim that $\psi_{X,Y}$ is monotone in $y \in Q(Y)$ and antitone in $x \in Q(X)$. For example, if $x, x' \in Q(X)$ and $x' \leq x$, then $\mathtt{id}_X$ is a map in $(\int Q)^n$ from (X, x') to (X, x); then, by functoriality and since $\overline{F}$ is contravariant in X, $\overline{F}(\mathtt{id}_X, (Y, y))$ is a map in $\int Q$ from $\overline{F}((X, x), (Y, y))$ to $\overline{F}((X, x'), (Y, y))$, which amounts to the inequality

$$\psi_{X,Y}(x, y) = Q(F(\mathtt{id}_X, \mathtt{id}_Y))(\psi_{X,Y}(x, y)) \leq \psi_{X,Y}(x', y)\,.$$

Next, if $f = (f_1, \ldots, f_n) : (X, x) \longrightarrow (X', x')$ and $g = (g_1, \ldots, g_m) : (Y, y) \longrightarrow (Y', y')$ are maps in $(\int Q)^n$ and $(\int Q)^m$, then $\overline{F}(f, g) : \overline{F}((X', x'), (Y, y)) \longrightarrow \overline{F}((X, x), (Y', y'))$ is a map of $\int Q$. This statement, an entailment, is equivalent to the statement $Q(f)(x) \leq x'$ and $Q(g)(y) \leq y'$ implies $Q(F(f, g))(\psi_{X',Y}(x', y)) \leq \psi_{X,Y'}(x, y')$ for all x, x', y, y'. Letting $x' = Q(f)(x)$ and $y' = Q(g)(y)$ in the last inequality, we deduce the inclusion

$$Q(F(f, g))(\psi_{X',Y}(Q(f)(x), y)) \leq \psi_{X,Y'}(x, Q(g)(y))\,,$$

which amounts to the semi-commutativity of (1). On the other hand, this inclusion is sufficient as well: if $Q(f)(x) \leq x'$ and $Q(g)(y) \leq y'$, then, by the order properties of ψ, we have

$$\begin{aligned} Q(F(f, g))(\psi_{X',Y}(x', y)) &\leq Q(F(f, g))(\psi_{X,Y}(Q(f)(x), y)) \\ &\leq \psi_{X,Y'}(x, Q(g)(y)) \leq \psi_{X,Y'}(x, y')\,. \end{aligned}$$

Consequently, the contruction of ψ from $\overline{F}$ can be reversed, thus building a lifting of F to $\int Q$ from a collection ψ with the stated properties. □

We shall say that Q *factors through* SLatt if

(a) each $Q(X)$ is complete lattice,
(b) for $f : X \longrightarrow Y$, $Q(f) : Q(X) \longrightarrow Q(Y)$ is sup-preserving.

The more advanced reader will have recognised that Q factors through SLatt means indeed that we can write $Q = U \circ Q_0$ for a functor $Q_0 : \mathsf{C} \longrightarrow \mathsf{SLatt}$, where $U : \mathsf{SLatt} \longrightarrow \mathsf{Pos}$ is the inclusion functor. If Q factors through SLatt, then (lax) extranaturality can be replaced by (lax) naturality, as stated in the next Lemma.

Lemma 4. *Suppose Q factors through SLatt. Then any lax extranatural transformations $\psi : \prod Q^{n+m} \longrightarrow Q \circ F$ makes the diagram below semi-commutative.*

$$\begin{array}{ccc} \prod_i Q(X_i')^{op} \times \prod_j Q(Y_j) & \xrightarrow{\prod_i Q(f_i)^* \times \prod_i Q(g_j)} & \prod_i Q(X_i)^{op} \times \prod_j Q(Y_j') \\ \downarrow{\scriptstyle \psi_{X',Y}} & \swarrow & \downarrow{\scriptstyle \psi_{X,Y'}} \\ Q(F(X',Y)) & \xrightarrow{Q(F(f,g))} & Q(F(X,Y')) \end{array}$$

Proof. Just like before, the maps $\psi_{X,Y}$ are antitone on the first variable and monotone on the second. If $f : (X,x) \longrightarrow (X',x')$ is in $\int Q$, then $Q(f)(x) \leq x'$ which as $Q(f)$ is a sup-preserving map is equivalent to $x \leq Q(f)^*(x')$. If moreover $g : (Y,y) \longrightarrow (Y',y')$ – ie $Q(g)(y) \leq y'$ – then $Q(F(f,g))(\psi_{X',Y}(x',y)) \leq \psi_{X,Y'}(x,y')$ for all x, x', y, y'. Again, by instancing this last statement on $x = Q(f)^*(x')$ and $y' = Q(g)(y)$ and by the order properties of $\psi_{X,Y}$, we have that

$$Q(F(f,g))(\psi_{X',Y}(x',y)) \leq \psi_{X,Y'}(Q(f)^*(x'), Q(g)(y))\,.$$ □

Note that the $\psi_{X,Y}$ need not be sup-preserving, even when Q factors through SLatt.

4. Lifting the monoidal structure

Theorem 3 immediately tells how to lift the tensor $\otimes$, as a simple bifunctor, from C to $\int Q$.

Proposition 5. *There is a bijection between the following kind of data:*

- *a lifting of a bifunctor $\otimes$ from C to $\int Q$,*
- *a collection of order-preserving maps*

$$\{\, \mu_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(X \otimes Y) \mid X, Y \in \mathsf{Obj}(\mathsf{C}) \,\}\,,$$

such that, for $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$, the following diagram semi-commutes:

$$\begin{array}{ccc} Q(X) \times Q(Y) & \xrightarrow{Q(f) \times Q(g)} & Q(X') \times Q(Y') \\ \downarrow{\scriptstyle \mu_{X,Y}} & \swarrow & \downarrow{\scriptstyle \mu_{X',Y'}} \\ Q(X \otimes Y) & \xrightarrow{Q(f \otimes g)} & Q(X' \otimes Y') \end{array} \tag{2}$$

Let us remark that the maps $\mu_{X,Y}$ are natural (i.e. diagram (2) fully commutes) if and only if the lifted bifunctor preserves op-cartesian arrows, that is the projection functor $\int Q \longrightarrow \mathsf{C}$ is a monoidal op-fibration as defined in [21]. In a similar way,

lifting the unit (of a tensor structure) as a constant functor yields $u \in Q(I)$ or, equivalently, $u : 1 \longrightarrow Q(I)$.

Next, lifting the maps $(\lambda, \rho, \alpha, \sigma)$ of a symmetric monoidal structure on C simply amounts to requiring the following inclusions:

$$Q(\lambda)(\mu_{I,Y}(u,y)) \leq y\,, \qquad Q(\rho)(\mu_{X,I}(x,u)) \leq x\,,$$
$$Q(\alpha)(\mu_{X\otimes Y,Z}(\mu_{X,Y}(x,y),z)) \leq \mu_{X,Y\otimes Z}(x,\mu_{Y,Z}(y,z))\,,$$
$$Q(\sigma)(\mu_{X,Y}(x,y)) \leq \mu_{Y,X}(y,x)\,.$$

Considering that we actually need these lifted maps to be invertible, we require (by Lemma 2) the above inclusions to be equalities. Thus we have $u : 1 \longrightarrow Q(I)$ and $\mu_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(X \otimes Y)$, where the latter is lax natural, making commutative the usual diagrams for a monoidal functor, see Figure 1.

$$\begin{array}{ccc} 1 \times Q(X) & \xrightarrow{u\times id} & Q(I) \times Q(X) \\ \downarrow{\scriptstyle \lambda_Q} & & \downarrow{\scriptstyle \mu} \\ Q(X) & \xleftarrow[Q(\lambda)]{} & Q(I \otimes X) \end{array} \qquad \begin{array}{ccc} Q(X) \times 1 & \xrightarrow{id\times u} & Q(X) \times Q(I) \\ \downarrow{\scriptstyle \rho_Q} & & \downarrow{\scriptstyle \mu} \\ Q(X) & \xleftarrow[Q(\rho)]{} & Q(X \otimes I) \end{array}$$

$$\begin{array}{ccc} (Q(X) \times Q(Y)) \times Q(Z) & \overset{\alpha_Q}{\rightarrow} & Q(X) \times (Q(Y) \times Q(Z)) \\ \downarrow{\scriptstyle \mu\times id} & & \downarrow{\scriptstyle id\times\mu} \\ Q(X \otimes Y) \times Q(Z) & & Q(X) \times Q(Y \otimes Z) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ Q((X \otimes Y) \otimes Z) & \xrightarrow{Q(\alpha)} & Q(X \otimes (Y \otimes Z)) \end{array} \qquad \begin{array}{ccc} Q(X) \times Q(Y) & \xrightarrow{\sigma_Q} & Q(Y) \times Q(X) \\ \downarrow{\scriptstyle \mu_{X,Y}} & & \downarrow{\scriptstyle \mu_{Y,X}} \\ Q(X \otimes Y) & \xrightarrow{Q(\sigma)} & Q(Y \otimes X) \end{array}$$

FIGURE 1. Coherence diagrams for a monoidal functor.

5. Lifting the closed structure

We suppose next that we have lifted the symmetric monoidal structure from C to $\int Q$ via a lax natural μ as in the previous section, and that C is closed as well. Let us recall that C is closed if, for each object X of C, the functor $X \otimes -$ (tensoring with X) has a right adjoint, noted here $X \multimap -$. As from elementary theory, the right adjoint is made into a bifunctor $\multimap\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \longrightarrow \mathsf{C}$. Moreover, for each pair of objects X and Y of C, we have maps (the units and counits of the adjunction)

$$\eta_{X,Y} : Y \longrightarrow X \multimap (X \otimes Y)\,, \qquad \mathtt{ev}_{X,Y} : X \otimes (X \multimap Y) \longrightarrow Y\,,$$

natural in Y, satisfying the usual pasting diagrams for adjunctions:

$$(X \multimap \mathtt{ev}_{X,Y}) \circ \eta_{X,X\multimap Y} = \mathtt{id}_{X\multimap Y}\,, \quad \mathtt{ev}_{X,X\otimes Y} \circ (X \otimes \eta_{X,Y}) = \mathtt{id}_{X\otimes Y}\,. \tag{3}$$

The following statement, allowing to lift the bifunctor $\multimap$, is a direct consequence of Theorem 3.

Proposition 6. *There is a bijection between the following kind of data:*

- *a lifting of a bifunctor* $\multimap\colon \mathsf{C}^{op} \times \mathsf{C} \longrightarrow \mathsf{C}$ *to a functor* $(\int Q)^{op} \times \int Q \longrightarrow \int Q$,

- *a collection of order-preserving maps*

$$\{\, \iota_{X,Y} : Q(X)^{op} \times Q(Y) \longrightarrow Q(X \multimap Y) \mid X, Y \in \mathsf{Obj}(\mathsf{C}) \,\},$$

such that, for $f : X \longrightarrow X'$ *and* $g : Y \longrightarrow Y'$*, the following diagram semi-commutes:*

$$\begin{array}{ccc} Q(X)^{op} \times Q(Y) & \xrightarrow{\mathtt{id} \times Q(g)} & Q(X)^{op} \times Q(Y') \\ {\scriptstyle Q(f)^{op} \times \mathtt{id}}\downarrow & \nearrow & \\ Q(X')^{op} \times Q(Y) & & \downarrow{\scriptstyle \iota_{X,Y'}} \\ {\scriptstyle \iota_{X',Y}}\downarrow & & \\ Q(X' \multimap Y) & \xrightarrow{Q(f \multimap g)} & Q(X \multimap Y') \end{array} \tag{4}$$

Once the bifunctor $\multimap$ has been lifted, in order to lift the adjunction $X \otimes - \dashv X \multimap -$, we require the unit and counit to be maps of $\int Q$. This is achieved by requiring the following inequalities to hold, for each pair of objects X, Y of C, each $x \in Q(X)$ and $y \in Q(Y)$:

$$Q(\eta_{X,Y})(y) \leq \iota_{X,X\otimes Y}(x, \mu_{X,Y}(x,y))\,, \tag{5}$$

$$Q(\mathtt{ev}_{X,Y})(\mu_{X,X\multimap Y}(x, \iota_{X,Y}(x,y)) \leq y\,. \tag{6}$$

Due to its central role in this paper, let us introduce an explicit notation for the map

$$\langle -, - \rangle_{X,Y} := Q(\mathtt{ev}_{X,Y}) \circ \mu_{X,X\multimap Y} : Q(X) \times Q(X \multimap Y) \longrightarrow Q(Y)\,.$$

Notice that the inclusion (6) is then written as $\langle x, \iota_{X,Y}(x,y)\rangle_{X,Y} \leq y$.

Theorem 7. *The following conditions are equivalent:*

(i) C *and* $\int Q$ *are symmetric monoidal closed and* $\pi : \int Q \longrightarrow \mathsf{C}$ *strictly preserves this structure.*

(ii) For each pair of objects X, Y *of* C*, and each* $x \in Q(X)$ *and* $y \in Q(Y)$*, the equality*

$$\mu_{X,Y}(x,y) = \langle\, x,\, Q(\eta_{X,Y})(y) \,\rangle_{X,X\otimes Y} \tag{7}$$

holds and, for each $\alpha \in Q(X)$*, the map*

$$\langle \alpha, - \rangle_{X,Y} : Q(X \multimap Y) \longrightarrow Q(Y) \tag{8}$$

has a right adjoint $\iota_{X,Y}(\alpha, -)$*.*

Proof. Firstly, let us recall that a map $\iota^{\alpha}_{Y} : Q(Y) \longrightarrow Q(X \multimap Y)$ is the right adjoint of $\langle \alpha, \cdot \rangle_{X,Y}$ if and only if the usual unit-counit relations

$$\beta \leq \iota^{\alpha}_{Y}(\, \langle \alpha, \beta \rangle_{X,Y}\,)\,, \qquad\qquad \langle\, \alpha,\, \iota^{\alpha}_{Y}(\gamma)\,\rangle_{X,Y} \leq \gamma\,, \tag{9}$$

hold, for each $\beta \in Q(X \multimap Y)$ and $\gamma \in Q(Y)$.
Assume (i). Let $\iota_{X,Y} : Q(X)^{op} \times Q(Y) \longrightarrow Q(X \multimap Y)$ satisfying (5) and (6) be given. Let us show first that (7) holds. Recall that $Q(X \otimes \eta_{X,Y})(\mu_{X,Y}(x,y)) \leq \mu_{X,X\multimap(X\otimes Y)}(x, Q(\eta_{X,Y})(y))$. Then

$$\begin{aligned} \mu_{X,Y}(x,y) &= Q(\mathtt{ev}_{X,X\otimes Y})(\, Q(X \otimes \eta_{X,Y})(\mu_{X,Y}(x,y))\,) \\ &\leq Q(\mathtt{ev}_{X,X\otimes Y})(\, \mu_{X,X\multimap(X\otimes Y)}(x, Q(\eta_{X,Y})(y))\,) = \langle x, Q(\eta_{X,Y})(y)\rangle_{X,X\otimes Y}\,. \end{aligned}$$

For the opposite inclusion, observe that, using (5) and (6),

$$\langle\, x,\, Q(\eta_{X,Y})(y)\,\rangle_{X,X\otimes Y} \leq \langle\, x,\, \iota_{X,X\otimes Y}(x, \mu_{X,Y}(x,y))\,\rangle_{X,X\otimes Y} \leq \mu_{X,Y}(x,y)\,.$$

Next, let $\alpha \in Q(X)$ and define $\iota^{\alpha}_{Y}(\beta)$ as $\iota_{X,Y}(\alpha,\beta)$. Then (6) immediately yields the counit of (9). For the unit, we argue as follows:

$$\begin{aligned}
\beta &= (Q(X \multimap \mathsf{ev}_{X,Y}) \circ Q(\eta_{X,X\multimap Y}))(\beta) \\
&\leq Q(X \multimap \mathsf{ev}_{X,Y})(\iota_{X,X\otimes(X\multimap Y)}(\alpha, \mu_{X,X\multimap Y}(\alpha,\beta)))\,, && \text{using (5)},\\
&\leq \iota_{X,Y}(\alpha, Q(\mathsf{ev}_{X,Y})(\mu_{X,X\multimap Y}(\alpha,\beta)))\,, && \text{using (4)},\\
&= \iota^{\alpha}_{Y}(\langle \alpha,\beta\rangle_{X,Y})\,.
\end{aligned}$$

Assume now (ii), so ι^{α}_{Y} is given for each $\alpha \in Q(X)$. We define $\iota_{X,Y}(\alpha,\beta)$ as $\iota^{\alpha}_{Y}(\beta)$ and verify that (6) and (5) hold. Again, the co-unit in (9) is exactly equation (6). Thus, we only need to derive (5), which we do as follows:

$$\begin{aligned}
Q(\eta_{X,Y})(y) &\leq \iota^{\alpha}_{X\otimes Y}(\,\langle \alpha, Q(\eta_{X,Y})(y)\rangle_{X,X\otimes Y}\,)\,,\\
&\qquad \text{letting } \beta = Q(\eta_{X,Y})(y) \in Q(X \multimap (X\otimes Y)) \text{ in the unit of (9)},\\
&= \iota^{\alpha}_{X\otimes Y}(\,\mu_{X,Y}(\alpha,y)\,) = \iota_{X,X\otimes Y}(\alpha, \mu_{X,Y}(\alpha,y))\,, \qquad \text{using (7)}. \quad \square
\end{aligned}$$

We notice that if μ is natural (and not merely lax natural) in its right variable, then equation (7) automatically holds, as evident from the proof of Theorem 7.

We say that Q *monoidally factors through* SLatt if Q factors through SLatt and, moreover, $\mu_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(X\otimes Y)$ is natural (and not merely lax natural) and bilinear, that is, sup-preserving in each variable, separately. That is, writing $Q = U \circ Q_0$, the extension of $\mu_{X,Y}$ to the tensor product $Q_0(X) \otimes Q_0(Y)$ in SLatt makes $Q_0 : \mathsf{C} \longrightarrow \mathsf{SLatt}$ into a monoidal functor. Theorem 7 immediately implies the following statement.

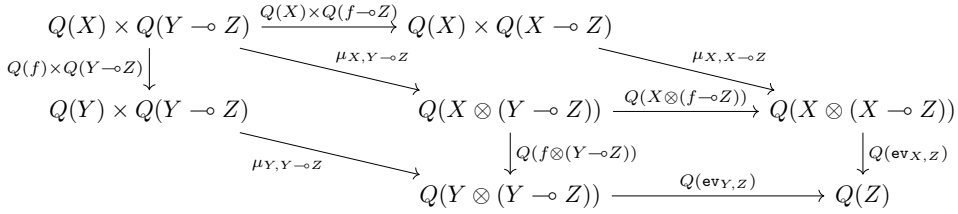
Theorem 8. *If Q monoidally factors through SLatt, then $\int Q$ is symmetric monoidal closed and the projection functor $\pi : \int Q \longrightarrow \mathsf{C}$ strictly preserves this structure.*

It can be expected that if μ is natural, then ι has also some kind of naturality. This is the content of the next lemma and corollary.

Lemma 9. *Suppose that μ is natural. Let $f : X \longrightarrow Y$ and suppose that $Q(f \multimap Z)$ has a right adjoint $Q(f \multimap Z)^*$. Then, for $\alpha \in Q(X)$ and $\gamma \in Q(Z)$, the following holds:*

$$\iota_{Y,Z}(Q(f)(\alpha),\gamma) = Q(f \multimap X)^*(\iota_{X,Z}(\alpha,\gamma))\,. \tag{10}$$

Proof. Observe first that the following diagram commutes:



With this in mind, let us compute as follows (with $\beta \in Q(Y \multimap Z)$):

$$\begin{aligned} \beta \leq \iota_{Y,Z}(Q(f)(\alpha), \gamma) \text{ iff } & Q(\mathtt{ev}_{Y,Z})(\mu(Q(f)(\alpha), \beta)) \leq \gamma \\ \text{iff } & Q(\mathtt{ev}_{X,Z})(\mu(\alpha, Q(f \multimap Z)(\beta))) \leq \gamma \\ \text{iff } & Q(f \multimap Z)(\beta) \leq \iota_{X,Z}(\alpha, \gamma) \\ \text{iff } & \beta \leq Q(f \multimap Z)^*(\iota_{X,Z}(\alpha, \gamma))\,. \end{aligned}$$ □

Notice that Lemma 9 applies when $f : X \longrightarrow Y$ is invertible, in which case $Q(f \multimap X)^* = Q(f^{-1} \multimap X)$.

In the following statement, we fix $0 \in \mathsf{C}$ and $\omega \in Q(0)$, let $X^* := X \multimap 0$ and $\omega_X : Q(X) \longrightarrow Q(X^*)^{\mathrm{op}}$ be defined by $\omega_X(\alpha) := \iota_{X,0}(\alpha, \omega)$. We use $(-)^{\mathrm{op}} : \mathsf{SLatt}^{op} \longrightarrow \mathsf{SLatt}$ to denote the functor sending a complete lattice to its opposite and a sup-preserving map to its right adjoint. Equation (10) implies commutativity of the diagram

$$\begin{array}{ccc} Q(X) & \xrightarrow{\quad Q(f) \quad} & Q(Y) \\ {\scriptstyle \omega_X = \iota_{X,0}(-,\omega)}\downarrow & & \downarrow{\scriptstyle \omega_Y = \iota_{Y,0}(-,\omega)} \\ Q(X^*)^{\mathrm{op}} & \xrightarrow{\quad Q(f^*)^{\mathrm{op}} \quad} & Q(Y^*)^{\mathrm{op}}\,. \end{array}$$

Corollary 10. *Suppose that Q monoidally factors through SLatt. Then, considering Q as a functor $\mathsf{C} \longrightarrow \mathsf{SLatt}$, ω_X is a natural transformation from $Q(X)$ to $Q(X^*)^{\mathrm{op}}$.*

6. Lifting dualizing objects

We incrementally assume that $\int Q$ and C are symmetric monoidal closed and that the projection strictly preserves all the structure. We use the notation introduced before. Namely, for a fixed object 0 of a symmetric monoidal closed category, we let $X^* := X \multimap 0$. If 0 and X are objects of C and $\omega \in Q(0)$, $\omega_X : Q(X)^{op} \longrightarrow Q(X^*)$ is defined by $\omega_X(\alpha) := \iota_{X,0}(\alpha, \omega)$.

Let us recall that an object 0 of a symmetric monoidal closed category $\mathcal{V}$ is *dualizing* if, for each object X of $\mathcal{V}$, the canonical arrow

$$j_X : X \longrightarrow X^{**}\,,$$

arising as the transpose the map $\mathtt{ev}_{X,0} \circ \sigma_{X^*,X} : X^* \otimes X \longrightarrow 0$, is an isomorphism. A symmetric monoidal closed category $\mathcal{V}$ is $*$-*autonomous* if it comes with a dualizing object 0.

We devise in this section conditions for $(0, \omega)$ being a dualizing object of $\int Q$. Recall from Section 5 that $(X, \alpha)^* := (X, \alpha) \multimap (0, \omega) = (X^*, \omega_X(\alpha))$, and notice that since $\int Q$ is closed the canonical arrow $j_{(X,\alpha)}$ is part of this structure and projects to j_X. As this might not be evident, we explicitly record this fact below.

Lemma 11. *For each X object X of C and each $\alpha \in Q(X)$, the canonical arrow $j_X : X \longrightarrow X^{**}$ lifts to an arrow $j_{(X,\alpha)} : (X, \alpha) \longrightarrow (X, \alpha)^{**} = (X^{**}, \omega_{X^*}(\omega_X(\alpha)))$ in $\int Q$. That is, we have the inequality $Q(j_X)(\alpha) \leq \omega_{X^*}(\omega_X(\alpha))$.*

Proof. Writing $j_X = (X^* \multimap (\mathtt{ev}_{X,0} \circ \sigma)) \circ \eta_{X^*,X}$, that is general formula for transpose of an adjunction, we have for $\alpha \in Q(X)$

$$\begin{aligned}
Q(j_X)(\alpha) &= Q((X^* \multimap (\mathtt{ev}_{X,0} \circ \sigma)) \circ \eta_{X^*,X})(\alpha) \\
&\leq Q(X^* \multimap (\mathtt{ev}_{X,0} \circ \sigma))(\iota_{X^*,X^*\otimes X}(x, \mu_{X^*,X}(x,\alpha))) \\
&\qquad \text{for all } x\text{, by functoriality and the inclusion (5)} \\
&\leq \iota_{X^*,0}(x, Q(\mathtt{ev}_{X,0} \circ \sigma)(\mu_{X^*,X}(x,\alpha))) \qquad \text{by diagram (4)} \\
&= \iota_{X^*,0}(x, Q(\mathtt{ev}_{X,0})(\mu_{X,X^*}(\alpha,x))) \qquad \text{by functoriality and figure (1)} \\
&\leq \iota_{X^*,0}(\iota_{X,0}(\alpha,\omega),\omega) \qquad \text{with } x = \iota_{X,0}(\alpha,\omega) \text{ and equation (6)} \\
&= \omega_{X^*}(\omega_X(\alpha))\,.
\end{aligned}$$

□

We study next under which conditions the maps $j_{(X,\alpha)}$ are isomorphisms. Using Lemma 2 and assuming that $Q(X) \neq \emptyset$, for each object X of C, we infer the following:

Proposition 12. *An object $(0,\omega)$ of $\int Q$ is dualizing if and only if 0 is a dualizing object of C and , for each object X of C, the following diagram commutes.*

$$\begin{array}{ccc} Q(X) & \xrightarrow{\omega_X} & Q(X^*)^{op} \\ & \searrow^{Q(j_X)} & \downarrow{\scriptstyle \omega_{X^*}} \\ & & Q(X^{**}) \end{array} \tag{11}$$

Proof. Asking for the object $(0,\omega)$ in $\int Q$ to be dualizing means that for each object X of C and $\alpha \in Q(X)$, the arrow $j_{(X,\alpha)} : (X,\alpha) \longrightarrow (X,\alpha)^{**}$ is invertible. By Lemma 2 and since we can always find $\alpha \in Q(X)$, this is equivalent to asking j_X to be invertible, for each object X of C, and that $Q(j_X)(\alpha) = \omega_{X^*}(\omega_X(\alpha))$, for each object X of C and $\alpha \in Q(X)$. □

Lemma 13. *If $(0,\omega)$ is dualizing, then, for each object X of C, $\omega_X : Q(X)^{op} \longrightarrow Q(X^*)$ is an isomorphism.*

Proof. In diagram (11), since $Q(j_X)$ is invertible, ω_X is split mono, and ω_{X^*} is split epi, for each object X. Thus, ω_{X^*} is both split epi and split mono and therefore it is invertible. It follows that ω_X is inverted by $Q(j_X)^{-1} \circ \omega_{X^*}$. □

It might be difficult to directly make use of Proposition 12. The following Theorem 16 yields a more direct way to verify whether some $(0,\omega)$ is dualizing by simply looking at the maps ω_X. Recall from Theorem 7 that any map of the form $\langle \alpha, - \rangle_{X,Y}$ has a right adjoint, namely $\iota_{X,Y}(\alpha,-)$. In the case $Y = 0$, let us simply write $\langle \alpha, - \rangle_X$ for $\langle \alpha, - \rangle_{X,0}$, so to have the equivalence:

$$\langle \alpha, \beta \rangle_X \leq \omega \;\text{ iff }\; \beta \leq \omega_X(\alpha)\,, \quad \text{for all } \beta \in Q(X^*)\,.$$

Suppose now that for each $\beta \in Q(X^*)$ we can find ${}^{\perp}\beta \in Q(X)$ such that

$$\alpha \leq {}^{\perp}\beta \;\text{ iff }\; \langle \alpha, \beta \rangle_X \leq \omega\,, \quad \text{for all } \alpha \in Q(X)\,.$$

This happens, for example, when $\langle -, \beta \rangle_X$ has a right adjoint and, in particular, when Q monoidally factors through SLatt. Then, the equivalences

$$\alpha \leq {}^{\perp}\beta \;\text{ iff }\; \langle \alpha, \beta \rangle_X \leq \omega \;\text{ iff }\; \beta \leq \omega_X(\alpha)\,, \quad \text{for all } \alpha \in Q(X) \text{ and } \beta \in Q(X^*)\,, \tag{12}$$

hold and exhibit ω_X and ${}^{\perp}(-)$ as a Galois connection. In particular ${}^{\perp}\beta$ is uniquely determined.

Lemma 14. *If $\omega_X : Q(X)^{op} \longrightarrow Q(X^*)$ is invertible, then ${}^{\perp}\beta = \omega_X^{-1}(\beta)$.*

Proof. We have $\langle \alpha, \beta \rangle_X \leq \omega$ if and only if $\beta \leq \omega_X(\alpha)$, if and only if $\alpha \leq \omega_X^{-1}(\beta)$. □

Lemma 15. *If μ is natural, then the following relation holds*

$$\langle \alpha, \beta \rangle_X = \langle \beta, Q(j_X)(\alpha) \rangle_{X^*} , \tag{13}$$

for each $\alpha \in Q(X)$ and $\beta \in Q(X^*)$. Consequently:

(1) If $Q(j_X)$ has a right adjoint $Q(j_X)^*$, then ${}^{\perp}\beta = Q(j_X)^*(\omega_{X^*}(\beta))$.
(2) If j_X is invertible, then ${}^{\perp}\beta = Q(j_X^{-1})(\omega_{X^*}(\beta))$.

Proof. We obtain (13) by diagram chasing:

$$\begin{array}{ccccc}
Q(X) \times Q(X^*) & \xrightarrow{\sigma_{X,X^*}} & Q(X^*) \times Q(X) & \xrightarrow{\mathtt{id} \times Q(j_X)} & Q(X^*) \times Q(X^{**}) \\
\downarrow{\scriptstyle \mu_{X,X^*}} & & \downarrow{\scriptstyle \mu_{X^*,X}} & & \downarrow{\scriptstyle \mu_{X^*,X^{**}}} \\
Q(X \otimes X^*) & \xrightarrow{Q(\sigma_{X,X^*})} & Q(X^* \otimes X) & \xrightarrow{Q(\mathtt{id} \otimes j_X)} & Q(X^* \otimes X^{**}) \\
& {\scriptstyle Q(\mathtt{ev}_{X,0})} \searrow & & \swarrow {\scriptstyle Q(\mathtt{ev}_{X^*,0})} & \\
& & Q(0) & &
\end{array}$$

Using (13), we argue as follows:

$$\begin{aligned}
\langle \alpha, \beta \rangle_X \leq \omega \ \text{ iff } \ & \langle \beta, Q(j_X)(\alpha) \rangle_X \leq \omega \\
\text{iff } \ & Q(j_X)(\alpha) \leq \omega_{X^*}(\beta) \\
\text{iff } \ & \alpha \leq Q(j_X)^*(\omega_{X^*}(\beta)) = Q(j_X^{-1})(\,\omega_{X^*}(\beta)\,) ,
\end{aligned}$$

where the equality $Q(j_X)^*(\omega_{X^*}(\beta)) = Q(j_X^{-1})(\omega_{X^*}(\beta))$ holds when j_X is invertible. □

Theorem 16. *If μ is natural, then, for each $\omega \in Q(0)$, $(0, \omega)$ is dualizing if and only if 0 is dualizing in C and, for each object X of C, ω_X is invertible.*

Proof. We only need to prove the converse of Lemma 13, namely that diagram (11) commutes.
By Lemmas 14 and 15, we have $Q(j_X^{-1})(\omega_{X^*}(\beta)) = {}^{\perp}\beta = \omega_X^{-1}(\beta)$, for each $\beta \in Q(X^*)$, thus $Q(j_X^{-1}) \circ \omega_{X^*} = \omega_X^{-1}$, from which commutativity of (11) follows. □

7. The double negation nucleus

It is well known, see for instance [22, Theorem 1], that given an element ω of a unital commutative quantale $(Q, e, *)$, the map $(- \multimap \omega) \multimap \omega$ is a closure operator, actually a nucleus. Taking the fixed points of this nucleus yields a quantale $(Q_j, j(e), *_j)$ in which every element is equal to its double negation. Otherwise said, $(Q_j, j(e), *_j)$ is a $*$-autonomous complete poset, also called a Girard quantale. In this section we show how to reproduce this process with the total category in place of the quantale.

We suppose from now on that C is $*$-autonomous with 0 as dualizing element, and that $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$ is a monoidal functor, so that, in particular, μ is

natural. Thus, as stated in Theorem 8, $\int Q$ is symmetric monoidal closed. We fix $\omega \in Q(0)$.

Proposition 17. *Define, for each $X \in \mathsf{C}$ and $\alpha \in Q(X)$, $\jmath^{\omega}_X(\alpha) := {}^{\perp}(\omega_X(\alpha))$. Then $\jmath^{\omega}_X$ is a closure operator on $Q(X)$. For X an object of C and $f : X \longrightarrow Y$ an arrow of C, let*

$$Q^{\jmath^{\omega}}(X) := \{\, \alpha \in Q(X) \mid \jmath^{\omega}_X(\alpha) = \alpha \,\}\,, \qquad Q^{\jmath^{\omega}}(f) := \jmath^{\omega}_Y \circ Q(f)\,.$$

Then $Q^{\jmath^{\omega}} : \mathsf{C} \longrightarrow \mathsf{SLatt}$ is a functor and $\jmath^{\omega}_X : Q(X) \longrightarrow Q^{\jmath^{\omega}}(X)$ is a surjective sup-preserving map, natural in X. If $f : X \longrightarrow Y$ is invertible, then $Q(f)$ restricts to a map from $Q^{\jmath^{\omega}}(X) \longrightarrow Q^{\jmath^{\omega}}(Y)$.

Proof. In view of the relations in (12), ${}^{\perp}(\cdot)$ and ω_X form a Galois connection and, by general theory, see e.g. [5, Chapter 7], their composition $\jmath^{\omega}_X$ is a closure operator on $Q(X)$. Then, $Q^{\jmath^{\omega}}(X)$ is a complete lattice where the supremum of a family $\{\, \alpha_i \mid i \in I \,\}$ is $\jmath^{\omega}_X(\bigvee_i \alpha_i)$ (we shall denote this supremum by $\bigsqcup_{i\in I} \alpha_i$). Moreover $\jmath^{\omega}_X : Q(X) \longrightarrow Q^{\jmath^{\omega}}(X)$ is sup-preserving and surjective.
It is obvious that $Q^{\jmath^{\omega}}$ preserves the identities. The fact that it preserves the composition and the naturality of $\jmath^{\omega}$ are directly induced by the equality

$$\jmath^{\omega}_Y(Q(f)(\jmath^{\omega}_X(\alpha))) = \jmath^{\omega}_Y(Q(f)(\alpha))\,, \tag{14}$$

valid for each $\alpha \in Q(X)$. This equality is proved by the following argument. By Corollary 10, for $f : X \longrightarrow Y$ in C, the following diagram commutes:

$$\begin{array}{ccc} Q(X) & \xrightarrow{\omega_X} & Q(X^*)^{\mathrm{op}} \\ \downarrow{\scriptstyle Q(f)} & & \downarrow{\scriptstyle Q(f^*)^{\mathrm{op}}} \\ Q(Y) & \xrightarrow{\omega_Y} & Q(Y^*)^{\mathrm{op}}\,. \end{array} \tag{15}$$

Recall now that if a diagram in Pos

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D\,. \end{array}$$

commutes and is such that f and k have right adjoints $\rho(f)$ and $\rho(k)$, respectively, then

$$h(\rho(f)(b)) \leq \rho(k)(g(b))\,, \tag{16}$$

holds for each $b \in B$. The Galois connection expressed in (12) can also be understood by saying that ${}^{\perp}(-) : Q(X^*)^{\mathrm{op}} \longrightarrow Q(X)$ is right adjoint to $\omega_X : Q(X) \longrightarrow Q(X^*)^{\mathrm{op}}$.
Thus, applying (16) to diagram (16), we obtain

$$Q(f)({}^{\perp}\beta) \leq {}^{\perp}(Q(f^*)^{\mathrm{op}}(\beta))$$

for each $\beta \in Q(X^*)$. And therefore, by the naturality of ω,

$$\begin{aligned} Q(f)(\jmath^{\omega}_X(\alpha)) &= Q(f)({}^{\perp}(\omega_X(\alpha))) \\ &\leq {}^{\perp}((Q(f^*)^{\mathrm{op}})(\omega_X(\alpha))) = {}^{\perp}(\omega_Y(Q(f)(\alpha))) = \jmath^{\omega}_Y(Q(f)(\alpha))\,. \end{aligned} \tag{17}$$

As $\jmath^{\omega}_Y$ is a closure operator, we also obtain that

$$\jmath^{\omega}_Y(Q(f)(\alpha)) \leq \jmath^{\omega}_Y(Q(f)(\jmath^{\omega}_X(\alpha))) \leq \jmath^{\omega}_Y(\jmath^{\omega}_Y(Q(f)(\alpha)) = \jmath^{\omega}_Y(Q(f)(\alpha))\,.$$

That is, we have derived (14). This equality also allows us to prove that $Q^{\jmath^\omega}(f)$ is sup-preserving:

$$Q^{\jmath^\omega}(f)(\bigsqcup_i \alpha_i) = \jmath^\omega_Y(Q(f)(\jmath^\omega_X(\bigvee_i \alpha_i)))] = \jmath^\omega_Y(Q(f)(\bigvee_i \alpha_i)) = \jmath^\omega_Y(\bigvee_i Q(f)(\alpha_i))$$
$$\leq \jmath^\omega_Y(\bigvee_i \jmath^\omega_Y(Q(f)(\alpha_i))) = \bigsqcup_i Q^{\jmath^\omega}(f)(\alpha_i)\,,$$

where the opposite inclusion follows from monotonicity of $Q^{\jmath^\omega}(f)$. Finally, suppose $f : X \longrightarrow Y$ invertible. We aim at showing that $\jmath^\omega_Y(Q(f)(\alpha)) \leq Q(f)(\jmath^\omega_X(\alpha))$ (which then is an equality by equation (17)). Indeed, we have

$$Q(f^{-1})(\jmath^\omega_Y(Q(f)(\alpha))) \leq \jmath^\omega_X(Q(f^{-1})(Q(f)(\alpha))) = \jmath^\omega_X(\alpha)\,,$$

and we obtain the desired inequality by applying $Q(f)$. □

Before arguing that $Q^{\jmath^\omega} : \mathsf{C} \longrightarrow \mathsf{SLatt}$ is monoidal, we first give a variant of Theorem 16, based on closure operators $\jmath^\omega_X$.

Proposition 18. *$(0, \omega)$ is a dualizing element of $\int Q$ if and only if 0 is a dualizing element of C and, for each object X of C and each $\alpha \in Q(X)$, $\jmath^\omega_X(\alpha) = \alpha$.*

Proof. Recall that we assume that μ is natural. Thus, we rely on Theorem 16 and argue that ω_X is invertible if and only if $Q^{\jmath^\omega}(X) = Q(X)$, for each object X of C. If ω_X is invertible, then it is inverted by its adjoint ${}^\perp{-}$, thus $\jmath^\omega_X(\alpha) = \alpha$ for each $\alpha \in Q(X)$.
Conversely, suppose that, for each object X of C, $Q^{\jmath^\omega}(X) = Q(X)$. This amounts to saying that $\mathtt{id}_{Q(X)} = \jmath^\omega_X = Q(j_X^{-1}) \circ \omega_{X^*} \circ \omega_X$. Therefore, ω_X is split mono, and ω_{X^*} is split epi, for each for each object X of C. Therefore ω_{X^*} is both split mono and split epi, and therefore it is an an isomorphism. It follows that ω_X is an isomorphism as well. □

Lemma 19. *The following relation holds:*

$$\langle\, \mu_{X,Y}(\alpha, \beta),\, \gamma \rangle_{X\otimes Y, Z} = \langle\, \alpha,\, \langle \beta, Q(\psi)(\gamma)\,\rangle_{Y, X\multimap Z}\,\rangle_{X,Z}\,, \tag{18}$$

for each $\alpha \in Q(X)$, $\beta \in Q(Y)$, and $\gamma \in Q((X \otimes Y) \multimap Z)$.

Proof. Recall that we assume in this Section that μ is natural. Equation (18) is argued via the diagram chasing in Figure 2. □

Recall that the double negation closure operator $j(-) := (- \multimap \omega) \multimap \omega$ on a unital commutative quantale $(Q, e, *)$ is a *nucleus*, meaning that it satisfies the following inclusion:

$$j(j(\alpha) * \beta) \leq j(\alpha * \beta)\,.$$

The above inclusion is the key property allowing to construct a quantale structure on the set Q_j of fixed point of j. We show that a similar property allows to transform $Q^{\jmath^\omega}$ into a monoidal functor.

Proposition 20. *The collection of maps $\jmath^\omega_X$ forms a nucleus w.r.t. μ. That is, the following inequation holds:*

$$\mu_{X,Y}(\,\jmath^\omega_X(\alpha),\, \beta\,) \leq \jmath^\omega_{X\otimes Y}(\,\mu_{X,Y}(\,\alpha,\, \beta\,)\,)\,, \tag{19}$$

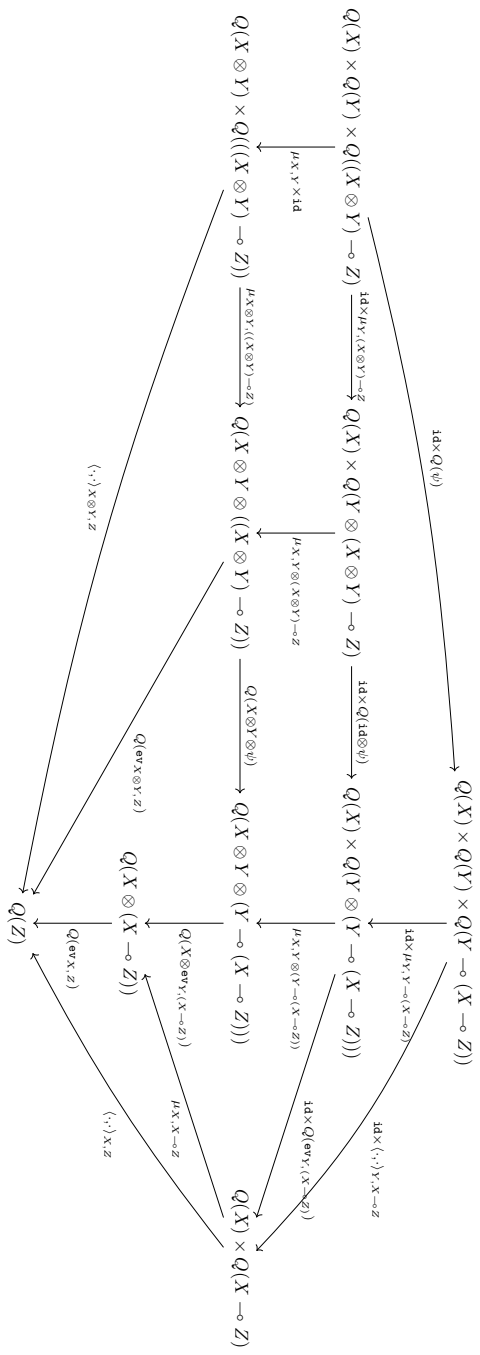


FIGURE 2. Proof of equation (18)

for each $\alpha \in Q(X)$ *and* $\beta \in Q(Y)$*. Consequently, by defining* $u^{\jmath^\omega} := \jmath^\omega_I(u)$ *and*

$$\mu^{\jmath^\omega}_{X,Y} := Q^{\jmath^\omega}(X) \times Q^{\jmath^\omega}(Y) \xrightarrow{\mu_{X,Y}} Q(X \otimes Y) \xrightarrow{\jmath^\omega_{X\otimes Y}} Q^{\jmath^\omega}(X \otimes Y)\,,$$

these maps turn $Q^{\jmath^\omega} : \mathsf{C} \longrightarrow \mathsf{SLatt}$ *into a monoidal functor.*

Proof. Assume (19). Then, by symmetry, we also have

$$\mu_{X,Y}(\alpha, \jmath^\omega_Y(\beta)) \leq \jmath^\omega_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta))\,.$$

Moreover, by monotonicity, $\jmath^\omega_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta)) \leq \jmath^\omega_{X\otimes Y}(\mu_{X,Y}(\jmath^\omega_X(\alpha), \jmath^\omega_Y(\beta)))$. Altogether we deduce:

$$\jmath^\omega_{X\otimes Y}(\mu_{X,Y}(\jmath^\omega_X(\alpha), \jmath^\omega_Y(\beta))) = \jmath^\omega_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta))\,. \tag{20}$$

From (20) it immediately follows that $\mu^{\jmath^\omega}$ is bilinear (as we assume that μ is bilinear) and that it makes commutative the diagrams required for monoidality, see Figure (1). Also, (20) yields naturality of $\mu^{\jmath^\omega}$:

$$\begin{aligned}
Q^{\jmath^\omega}(f \otimes g)(\mu^{\jmath^\omega}_{X,Y}(\alpha,\beta)) &= \jmath^\omega_{X'\otimes Y'}(Q(f\otimes g)(\jmath^\omega_{X,Y}(\mu_{X,Y}(\alpha,\beta))))\\
&= \jmath^\omega_{X'\otimes Y'}(Q(f\otimes g)(\mu_{X,Y}(\alpha,\beta)))\,, && \text{by equation (14),}\\
&= \jmath^\omega_{X'\otimes Y'}(\mu_{X',Y'}(Q(f)(\alpha), Q(g)(\beta)))\,, && \text{since } \mu \text{ is natural,}\\
&= \jmath^\omega_{X'\otimes Y'}(\mu_{X',Y'}(\jmath^\omega_{X'}(Q(f)(\alpha)), \jmath^\omega_{Y'}(Q(g)(\beta))))\,, && \text{by (20),}\\
&= \mu^{\jmath^\omega}_{X',Y'}(Q^{\jmath^\omega}(f)(\alpha), Q^{\jmath^\omega}(g)(\beta))\,.
\end{aligned}$$

Let us tackle the proof of relation (19), which is equivalent to

$$\langle\, \mu_{X,Y}(\jmath^\omega_X(\alpha),\beta),\, \omega_X(\mu_{X,Y}(\alpha,\beta))\,\rangle_{X\otimes Y} \leq \omega\,.$$

We derive the latter relation by means of the folllowing computations:

$$\begin{aligned}
\langle\, \mu_{X,Y}(\jmath^\omega_X(\alpha),\beta), &\omega_X(\mu_{X,Y}(\alpha,\beta))\,\rangle_{X\otimes Y,0}\\
&= \langle \jmath^\omega_X(\alpha), \langle \beta,\, Q(\psi)(\omega_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta)))\,\rangle_{Y,X^*}\rangle_{X,0}\,,\\
&\qquad\qquad \text{by equation (18),}\\
&= \langle \jmath^\omega_X(\alpha), \langle \beta, \iota_{Y,X^*}(\beta, \omega_X(\alpha))\rangle_{Y,X^*}\rangle_{X,0}\,,\\
&\qquad\qquad \text{by equation (21), to be argued for,}\\
&\leq \langle \jmath^\omega_X(\alpha), \omega_X(\alpha)\rangle_{X,0}\,, && \text{by equation (6),}\\
&= \langle {}^\perp(\omega_X(\alpha)), \omega_X(\alpha)\rangle_{X,0} \leq \omega\,.
\end{aligned}$$

To complete the proof, we need to asses the equality

$$Q(\psi)(\omega_{X\otimes Y}(\mu_{X,Y}(\alpha,\beta))) = \iota_{Y,X^*}(\beta, \omega_X(\alpha))\,. \tag{21}$$

The canonical isomorphism $\psi : (X \otimes Y) \multimap Z \longrightarrow Y \multimap (X \multimap Z)$ lifts from C to $\int Q$, which means that the following diagram commutes:

$$\begin{array}{ccc}
Q(X)^{op} \times Q(Y)^{op} \times Q(Z) & \xrightarrow{\sigma_{Q(X)^{op},Q(Y)^{op}}\times Q(Z)} & Q(Y)^{op} \times Q(X)^{op} \times Q(Z)\\
\downarrow{\scriptstyle \mu^{op}_{X,Y}\times Q(Z)} & & \downarrow{\scriptstyle Q(Y)^{op}\times \iota_{X,Z}}\\
Q(X\otimes Y)^{op} \times Q(Z) & & Q(Y)^{op} \times Q(X \multimap Z)\\
\downarrow{\scriptstyle \iota_{X\otimes Y,Z}} & & \downarrow{\scriptstyle \iota_{Y,X\multimap Z}}\\
Q((X\otimes Y)\multimap Z) & \xrightarrow{\qquad Q(\psi) \qquad} & Q(Y \multimap (X \multimap Z))
\end{array}$$

Equation (21) is just an instance of this commutativity, with $Z = 0$. □

Next, the important feature of the unital commutative quantale Q_j is that the implication in Q_j is computed exactly as in Q and, moreover, that $\omega \in Q_j$. We collect analogous remarks for the functor Q^{j^ω}.

Lemma 21.

(1) The following inclusion holds:

$$j^\omega_{X\multimap Y}(\iota_{X,Y}(\alpha,\beta)) \leq \iota_{X,Y}(j^\omega_X(\alpha), j^\omega_Y(\beta))\,. \tag{22}$$

Consequently, if $\beta \in Q^{j^\omega}(Y)$, then $\iota_{X,Y}(\alpha,\beta) \in Q^{j^\omega}(X \multimap Y)$, for each $\alpha \in Q(X)$.

(2) Every element of the form $\omega_X(\alpha)$, $\alpha \in Q(X)$, belongs to $Q^{j^\omega}(X^*)$.

(3) The element $\omega \in Q(0)$ is a fixed point of j^ω_0.

Proof. (1) We have

$$\begin{aligned}
\langle j^\omega_X(\alpha), j^\omega_{X\multimap Y}(\iota_{X,Y}(\alpha,\beta))\rangle_{X,Y} &= Q(\mathtt{ev}_{X,Y})(\mu_{X,X\multimap Y}(j^\omega_X(\alpha), j^\omega_{X\multimap Y}(\iota_{X,Y}(\alpha,\beta))))\\
&\leq Q(\mathtt{ev}_{X,Y})(j^\omega_{X\otimes(X\multimap Y)}(\mu_{X,X\multimap Y}(\alpha,\iota_{X,Y}(\alpha,\beta))))\\
&\qquad\qquad \text{using equation (19)}\\
&\leq j^\omega_Y(Q(\mathtt{ev}_{X,Y})(\mu_{X,X\multimap Y}(\alpha,\iota_{X,Y}(\alpha,\beta))))\\
&\qquad\qquad \text{by equation (17)}\\
&\leq j^\omega_Y(\beta)\,, \qquad\qquad \text{by equation (6).}
\end{aligned}$$

From the above inequality, equation (22) follows using adjointness. Next, assume $\beta \in Q^{j^\omega}(Y)$, let $\alpha \in Q(X)$, and observe that

$$\begin{aligned}
\iota_{X,Y}(\alpha,\beta) &\leq j^\omega_{X\multimap Y}(\iota_{X,Y}(\alpha,\beta))\\
&\leq \iota_{X,Y}(j^\omega_X(\alpha), j^\omega_Y(\beta))\,, && \text{by (22),}\\
&= \iota_{X,Y}(j^\omega_X(\alpha),\beta)\,, && \text{since } \beta \in Q^{j^\omega}(Y),\\
&\leq \iota_{X,Y}(\alpha,\beta)\,, && \text{since } \iota \text{ reverses the order in its first variable,}
\end{aligned}$$

and therefore $\iota_{X,Y}(\alpha,\beta) = j^\omega_{X\multimap Y}(\iota_{X,Y}(\alpha,\beta))$.

(2) By Corollary 10, the following diagram commutes:

$$\begin{array}{ccc}
Q(X) & \xrightarrow{\omega_X} & Q(X^*)^{op}\\
\Big\downarrow{\scriptstyle Q(j_X)} & & \Big\downarrow{\scriptstyle Q((j_X)^*)^{op} = Q((j_X)^*)^{-1} = Q((j_X)^{*-1}) = Q(j_{X^*})}\\
Q(X^{**}) & \xrightarrow{\omega_{X^{**}}} & Q(X^{***})^{op}
\end{array}$$

where the equality $(j_X)^{*-1} = j_{X^*}$ follows from the relation $(j_X)^* \circ j_{X^*} = \mathtt{id}_{X^*}$, valid in every symmetric monoidal closed category, and since $(j_X)^*$ is invertible whence mono. Using this observation, we can compute as follows:

$$j^\omega_{X^*} \circ \omega_X = Q(j^{-1}_{X^*}) \circ \omega_{X^{**}} \circ \omega_{X^*} \circ \omega_X = \omega_X \circ Q(j^{-1}_X) \circ \omega_{X^*} \circ \omega_X = \omega_X \circ j^\omega_X = \omega_X\,.$$

(3) Since μ is monoidal, $Q(I)$ is a monoid in SLatt (that is, a quantale) and, for each object X of C, $Q(X)$ is a $Q(I)$-module. For $q \in Q(I)$ and $\alpha \in Q(X)$, the action is given by $q \cdot \alpha = Q(\lambda_X)(\mu_{I,X}(q,\alpha))$.

Let $\lambda_0^\sharp : 0 \longrightarrow I \multimap 0$ the transpose of $\lambda_0 : I \otimes 0 \longrightarrow 0$ and recall $\lambda_0^\sharp$ is an isomorphsm. Commutativity of the following diagram

$$\begin{array}{ccc} Q(I)\times Q(0) & \xrightarrow{Q(I)\times Q(\lambda_0^\sharp)} & Q(I)\times Q(I\multimap 0) \\ \Big\downarrow{\scriptstyle \mu_{I,0}} & & \Big\downarrow{\scriptstyle \mu_{I,I\multimap 0}} \\ & & Q(I\otimes (I\multimap 0)) \\ & & \Big\downarrow{\scriptstyle Q(\mathtt{ev}_{I,0})} \\ Q(I\otimes 0) & \xrightarrow{\qquad Q(\lambda_0)\qquad} & Q(0) \end{array}$$

yields the relation

$$q\cdot\alpha = \langle q, Q(\lambda_0^\sharp)(\alpha)\rangle_{I,0}\,.$$

Let next e be the unit of $Q(I)$ and compute as follows:

$$\begin{aligned} Q(\lambda_0^\sharp)(\omega) &= Q(\lambda_0^\sharp)(\bigvee\{\,\beta\in Q(0)\mid e\cdot\beta\le\omega\,\}) \\ &= \bigvee\{\,Q(\lambda_0^\sharp)(\beta)\mid \beta\in Q(0),\ \langle e, Q(\lambda_0^\sharp)(\beta)\rangle_{I,0}\le\omega\,\} \\ &= \bigvee\{\,\gamma\in Q(I\multimap 0)\mid \langle e,\gamma\rangle_{I,0}\le\omega\,\} = \omega_I(e)\,. \end{aligned}$$

It follows that $\omega = Q((\lambda_0^\sharp)^{-1})(\omega_I(e))$. By item 2, $\omega_I(e)\in Q^{\jmath^\omega}(I\multimap 0)$ and since morpshims of the form $Q(f)$ with $f : X\longrightarrow Y$ invertible restrict to maps from $Q^{\jmath^\omega}(X)\longrightarrow Q^{\jmath^\omega}(Y)$, see Proposition 17, $\omega = Q((\lambda_0^\sharp)^{-1})(\omega_I(e))\in Q^{\jmath^\omega}(0)$. □

The following theorem is a direct consequence of these observations and of Proposition 18.

Theorem 22. *Let* C *be a* $*$*-autonomous category and* $Q : \mathsf{C}\longrightarrow \mathsf{SLatt}$ *be a monoidal functor. Let* 0 *be a dualizing object of* C *and pick* $\omega\in Q(0)$*. Then* $(0,\omega)$ *is a dualizing object of* $\int Q^{\jmath^\omega}$*.*

Proof. As we are dealing with two monoidal functors, Q and $Q^{\jmath^\omega}$, we shall distinguish the structures making $\int Q$ and $\int Q^{\jmath^\omega}$ by adding $\jmath^\omega$ in superscript. For example, we have the following charactetrisation if $\langle -,-\rangle^{\jmath^\omega}_{X,Y}$:

$$\begin{aligned} \langle\alpha,\beta\rangle^{\jmath^\omega}_{X,Y} &= Q^{\jmath^\omega}(\mathtt{ev}_{X,Y})(\,\mu^{\jmath^\omega}(\alpha,\beta)\,) \\ &= \jmath^\omega_Y(\,Q(\mathtt{ev}_{X,Y})(\jmath^\omega_{X\otimes(X\multimap Y)}(\mu_{X,X\multimap Y}(\alpha,\beta)))\,) \\ &= \jmath^\omega_Y(Q(\mathtt{ev}_{X,Y})(\mu_{X,X\multimap Y}(\alpha,\beta))) \qquad \text{by equation (14)} \\ &= \jmath^\omega_Y(\langle\alpha,\beta\rangle_{X,Y})\,. \end{aligned}$$

Using this, we see that the right adjoint to $\langle\alpha,\cdot\rangle^{\jmath^\omega}$ is $\iota_{X,Y}(\alpha,\cdot)$. Indeed, for $\alpha\in Q^{\jmath^\omega}(X), \beta\in Q^{\jmath^\omega}(X\multimap Y)$, and $\gamma\in Q^{\jmath^\omega}(Y)$, we have $\iota_{X,Y}(\alpha,\gamma)\in Q^{\jmath^\omega}(Y)$ and

$$\langle\alpha,\beta\rangle^{\jmath^\omega}_{X,Y} = \jmath^\omega_Y(\langle\alpha,\beta\rangle_{X,Y})\le\gamma \ \text{ iff }\ \langle\alpha,\beta\rangle_{X,Y}\le\gamma \ \text{ iff }\ \beta\le\iota_{X,Y}(\alpha,\gamma)\,.$$

Since $\omega\in Q^{\jmath^\omega}(0)$, then $\omega^{\jmath^\omega}_X(\alpha) = \iota_{X,0}(\alpha,\omega) = \omega_X(\alpha)$. We also have ${}^{\perp^{\jmath^\omega}}\beta = {}^{\perp}\beta$. Indeed, $\beta\in Q^{\jmath^\omega}(X^*)$, since ${}^{\perp}\beta = Q(j_X^{-1})(\omega_{X^*}(\beta))\in Q^{\jmath^\omega}(X)$ by Lemma 15, $\omega_{X^*}(\beta)\in Q^{\jmath^\omega}(X^{**})$ by item 2 of Lemma 21, $Q(j_X^{-1})$ restricts to a map $Q^{\jmath^\omega}(X^{**})\longrightarrow Q^{\jmath^\omega}(X)$ by Proposition 17.
Thus ${}^{\perp^{\jmath^\omega}}(\omega^{\jmath^\omega}_X(\alpha)) = {}^{\perp}(\omega_X(\alpha)) = \jmath^\omega_X(\alpha) = \alpha$, for every object X of C and every $\alpha\in Q^{\jmath^\omega}(X)$, and therefore $\int Q^{\jmath^\omega}$ is $*$-autonomous, using Proposition 18. □

We end this section by giving a representation theorem analogous to the well-known representation theorem for Girard quantales that yields the completeness of phase semantics of linear logic. The theorem is also meant to illustrate how our construction relate to the focused orthogonality categories introduced in [14] and further studied in [11], see also Example 30.

Theorem 23. *Let $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$ be a monoidal functor and suppose that $(0,\omega)$ is a dualizing element of $\int Q$. Consider the functor $P \circ UQ$, where UQ is the composition $\mathsf{C} \xrightarrow{Q} \mathsf{SLatt} \xrightarrow{U} \mathsf{Set}$ of Q and the forgetful functor to Set, and where $P : \mathsf{Set} \longrightarrow \mathsf{SLatt}$ is the covariant powerset functor (free complete sup-lattice functor). Define*

$$\perp\!\!\!\perp := \{\, x \in Q(0) \mid x \leq \omega \,\}.$$

Then, the maps sending $x \in Q(X)$ to $\downarrow_X(x) := \{\, x' \in Q(X) \mid x' \leq x \,\} \subseteq Q(X)$ form a natural isomorphism between the functors $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$ and $(P \circ UQ)^{\jmath^{\perp\!\!\!\perp}} : \mathsf{C} \longrightarrow \mathsf{SLatt}$.

Proof. Let us argue that a set $A \subseteq Q(X)$ belongs to $(P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(X)$ if and only if it is of the form $\downarrow_X(\alpha)$ for some $\alpha \in Q(X)$. To this goal, observe that, for $B \in (P \circ UQ)(X^*)$,

$$^{\perp}B = \{\, \alpha \in Q(X) \mid \langle \alpha, \beta \rangle \leq \omega, \text{ for all } \beta \in B \,\}.$$

Moreover, by general theory of Galois connections, all the elements of $(P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(X)$ are of this form. Now, it is immediate that if $\alpha \in {}^{\perp}B$ and $\alpha' \leq \alpha$, then $\alpha' \in {}^{\perp}B$. Moreover, since $\langle -, \beta \rangle_X$ is sup-preserving, $\bigvee({}^{\perp}B) \in {}^{\perp}B$. Thus, if $A \in (P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(X)$, then $A = \downarrow_X(\bigvee A)$.
Next, for $\alpha \in Q(X)$, $\downarrow_X(\alpha) \in (P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(X)$, since

$$^{\perp}\{\, \omega_X(\alpha) \,\} = \{\, \alpha' \in Q(X) \mid \langle \alpha', \omega_X(\alpha) \rangle_X \leq \omega \,\} = \downarrow_X(\jmath^{\omega}_X(\alpha)) = \downarrow_X(\alpha),$$

since we are assume that $Q(X) = Q^{\jmath^{\omega}}(X)$. Thus, $(P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(X)$ is the set of all the principal downsets of $Q(X)$ and $\downarrow_X(-) : Q(X) \longrightarrow (P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(X)$ is an order isomorphism, thus an isomorphism in SLatt.
For any closure operator j on a complete lattice L, $j(x)$ is the least element of $\{\, y \in L \mid j(y) = y \,\}$. It follows then that

$$\jmath^{\perp\!\!\!\perp}_X(A) = \downarrow_X(\bigvee A), \quad \text{for each } A \subseteq Q(X).$$

For $f : X \longrightarrow Y$ in C, we have

$$\begin{aligned}(P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(f)(\downarrow_X(\alpha)) &= \downarrow_Y(\bigvee\{\, Q(f)(\alpha') \mid \alpha' \leq \alpha \,\}) \\ &= \downarrow_Y(Q(f)(\bigvee\{\, \alpha' \mid \alpha' \leq \alpha \,\}) = \downarrow_Y(Q(f)(\alpha)),\end{aligned}$$

showing that the maps sending $\alpha \in Q(X)$ to $\downarrow_X(\alpha) \in (P \circ UQ)^{\jmath^{\perp\!\!\!\perp}}(X)$ are natural in X. □

8. Lifting fixed points of functors

This research was also motivated by ongoing research on the algebraic and categorical semantics of linear logic with fixed points [8, 6, 15]. In these works, several questions about initial and terminal (co)algebras of endofunctors of the category Nuts were raised. We shall argue in Example 29 that this category also arises as

$\int Q$ for some functor Q into SLatt. This was realised in [11] where these questions were also settled. Nonetheless, we believe pertinent to pinpoint here how this kind of results relate to the general theory developed in Section 3.

Recall that, given a category C and an endofunctor $F : \mathsf{C} \longrightarrow \mathsf{C}$, an *$F$-algebra* (resp. *$F$-coalgebra*) is an object X together with a morphism (called structure) $\gamma : F(X) \longrightarrow X$ (resp. $\gamma : X \longrightarrow F(X)$). A morphism of algebras (resp. coalgebras) $(X, \gamma) \longrightarrow (Y, \delta)$ is simply a morphism $f : X \longrightarrow Y$ such that the following diagram commutes (respectively):

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\delta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Algebras (resp. coalgebras) and their morphisms form a category $\mathrm{Alg}_C(F)$ (resp. $\mathrm{CoAlg}_C(F)$). Notice that we have $\mathrm{CoAlg}_C(F) = \mathrm{Alg}_{C^{\mathrm{op}}}(F^{\mathrm{op}})^{\mathrm{op}}$, where $F^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{C}^{\mathrm{op}}$ is the same functor as F, just considered as an endofunctor of C^{op}.

An *initial algebra* (resp. *terminal coalgebra*) is an initial (resp. terminal) object of $\mathrm{Alg}_C(F)$ ($\mathrm{CoAlg}_C(F)$). It is well-known that the structure of an initial algebra (terminal coalgebra) is invertible [18]. For this reason, initial and terminal (co)algebras of functors have been used to give semantics (of proofs) to logics with least and greatest fixed point-operators.

Let $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$ and $F : \mathsf{C} \longrightarrow \mathsf{C}$ be functors and suppose that there is a lifting $\overline{F} : \int Q \longrightarrow \int Q$. Denote by $\psi : Q \longrightarrow QF$ the lax natural transformation corresponding to $\overline{F}$, as given in Theorem 3.

Proposition 24. *For an F-coalgebra (X, γ), define*

$$Q^\nu(X, \gamma) := \{\, \alpha \in Q(X) \mid Q(\gamma)(\alpha) \leq \psi_X(\alpha) \,\}\,.$$

Then, for $f : (X, \gamma) \longrightarrow (Y, \delta)$ a coalgebra morphism, $Q(f)$ restricts from $Q^\nu(X, \gamma)$ to $Q^\nu(Y, \delta)$. Thus Q^ν is a functor $\mathrm{CoAlg}_{\mathsf{C}}(F) \longrightarrow \mathsf{Pos}$ *and we have an isomorphism*

$$\mathrm{CoAlg}_{\int Q}(\overline{F}) \simeq \int Q^\nu\,. \tag{23}$$

Proof. Let $f : (X, \gamma) \longrightarrow (Y, \delta)$ be a coalgebra morphism and assume $Q(\gamma)(\alpha) \leq \psi_X(\alpha)$. Verification that $Q(\delta)(Q(f)(\alpha)) \leq \psi_Y(Q(f)(\alpha)$ is achieved as follows:

$$\begin{aligned} Q(\delta)(Q(f)(\alpha)) &= Q(Ff)(Q(\gamma)(\alpha))\,, && \text{since } f \text{ is a coalgebra morphism,} \\ &\leq Q(Ff)(\psi_X(\alpha))\,, && \text{by assumption on } \alpha, \\ &\leq (\psi_Y(Q(Ff)(\alpha))\,, && \text{by lax-naturality of } \psi. \end{aligned}$$

For $(X, \alpha) \in \int Q$, recall that $\overline{F}(X, \alpha) = (F(X), \psi_X(\alpha))$. Therefore, an $\overline{F}$-coalgebra $((X, \alpha), \gamma)$ is given by an F-coalgebra (X, γ) such that

$$Q(\gamma)(\alpha) \leq \psi_X(\alpha)\,.$$

A morphism of $\overline{F}$-coalgebras $f : ((X, \alpha), \gamma) \longrightarrow ((Y, \beta), \delta)$ is just a morphism of F-coalgebra $f : X \longrightarrow Y$ in C such that $Q(f)(\alpha) \leq \beta$.
Then, the mapping sending an object $((X, \alpha), \gamma)$ of $\mathrm{CoAlg}_{\int Q}(\overline{F})$ to the object

$((X,\gamma),\alpha)$ of $\int Q^\nu$ (and a map f to itself) is the desired isomorphism $\mathrm{CoAlg}_{\int Q}(\overline{F}) \simeq \int Q^\nu$. □

We can give a similar result for algebras if we assume that $\psi : Q \longrightarrow Q \circ F$ is natural or that Q factors through SLatt. For the latter case, we need to develop some tools, illustrating the intrinsic dualities of the category SLatt. Suppose therefore that Q factors through SLatt, so $Q(X)$ is a complete lattice and $Q(f)$ has a right adjoint $Q(f)^*$. Denote by $Q^* : C^{\mathrm{op}} \longrightarrow \mathsf{Pos}$ the functor such that $Q^*(X) := Q(X)^{\mathrm{op}}$ and $Q^*(f) := Q(f)^*$. Obviously, also Q^* factors through SLatt.

Lemma 25. *If Q factors through SLatt, then the following statements hold:*

(1) The functor Q^ν factors through SLatt.

(2) We have $(\int Q)^{\mathrm{op}} = \int Q^*$ and the functor $\overline{F}^{\mathrm{op}}$ is a lifting of F^{op} to $\int Q^*$.

Proof. (1) First, observe that for $(b_i)_i \subset Q^\nu(B,\beta)$—that is $b_i \in Q(B)$ is such that $Q(\beta)(b_i) \leq \psi_B(b_i)$, for each i—we have

$$Q(\beta)(\bigvee_i b_i) \leq \bigvee_i Q(\beta)(b_i) \leq \bigvee_i \psi_B(b_i) \leq \psi_B(\bigvee b_i)$$

because $Q(\beta)$ is sup-preserving and ψ_B is order-preserving. This shows that $Q^\nu(B,\beta)$ is a subset of $Q(B)$ closed under suprema. Necessarily, for a coalgebra morphism $f : (B,\delta) \longrightarrow (A,\gamma)$, the restriction $Q(f)$ to $Q(B,\delta)$ preserves these suprema. So $Q^\nu(B,\beta)$ is a complete lattice and Q^ν is functor because it is the restriction of Q for morphism.

(2) Objects of $\int Q^*$ are of the form (X,α) where $X \in \mathsf{C}$ and $\alpha \in Q(X)^{\mathrm{op}}$, where we can simply write $\alpha \in Q(X)$. Morphisms $(X,\alpha) \longrightarrow (Y,\beta)$ are maps $f : Y \longrightarrow X$ in C such that $Q(f)^*(\alpha) \leq \beta$ in $Q(Y)^{\mathrm{op}}$, that is, $\beta \leq Q(f)^*(\alpha)$ in $Q(Y)$. Under the adjunction, this is equivalent to $Q(f)(\beta) \leq \alpha$ in $Q(X)$. Let us spell out what is $(\int Q)^{\mathrm{op}}$. Objects of $(\int Q)^{\mathrm{op}}$ are pairs (X,α) where $X \in C$ and $\alpha \in Q(X)$. A morphism $(X,\alpha) \longrightarrow (Y,\beta)$ in $(\int Q)^{\mathrm{op}}$ is a morphism $f : Y \longrightarrow X$ in C such that $Q(f)(\beta) \leq \alpha$. Moreover the following diagram commutes, thus showing the very last statement:

$$\begin{array}{ccc} \int Q^* & \xrightarrow{\overline{F}^{\mathrm{op}}} & \int Q^* \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ C^{\mathrm{op}} & \xrightarrow[F^{\mathrm{op}}]{} & C^{\mathrm{op}} \end{array}$$

□

Proposition 26. *Assume that $\psi : Q \longrightarrow Q \circ F$ is natural or that Q factors through SLatt. For an F-algebra (X,γ), define*

$$Q^\mu(X,\gamma) := \{\, \alpha \in Q(X) \mid Q(\gamma)(\psi_X(\alpha)) \leq \alpha \,\}.$$

If ψ is natural, then, for $f : (X,\gamma) \longrightarrow (Y,\delta)$ an algebra morphism, $Q(f)$ restricts to a map from $Q^\mu(X,\gamma)$ to $Q^\mu(Y,\delta)$. If Q factors through SLatt, then Q^μ is extended to a functor so to have $Q^\mu = Q^{*\nu *}$. In both cases, Q^μ is made into a functor $\mathrm{Alg}_{\mathsf{C}}(F) \longrightarrow \mathsf{Pos}$ and we have an isomorphism

$$\mathrm{Alg}_{\int Q}(\overline{F}) \simeq \int Q^\mu \,. \tag{24}$$

Proof. Suppose that ψ is natural, let $f : (X,\gamma) \longrightarrow (Y,\delta)$ be an algebra morphism and assume $Q(\gamma)(\psi_X(\alpha)) \leq \alpha$. Verification that $Q(f)(\alpha) \in Q^\mu(Y,\delta)$, that is, $Q(\delta)(\psi_Y(Q(f)(\alpha))) \leq Q(f)(\alpha)$, is achieved as follows:

$$\begin{aligned} Q(\delta)(\psi_Y(Q(f)(\alpha))) &= Q(\delta)(Q(Ff)(\psi_X(\alpha)))\,, && \text{by naturality of } \psi,\\ &= Q(f)(Q(\gamma)(\psi_X(\alpha)))\,, && \text{since } f \text{ is an algebra morphism},\\ &\leq Q(f)(\alpha)\,, && \text{by assumption on } \alpha. \end{aligned}$$

An $\overline{F}$-algebra $((X,\alpha),\gamma)$ is given by an F-algebra (X,γ) such that

$$Q(\gamma)(\psi_X(\alpha)) \leq \alpha\,.$$

A morphism of $\overline{F}$-algebras $f : ((X,\alpha),\gamma) \longrightarrow ((Y,\beta),\delta)$ is just a morphism of F-algebras $f : (X,\gamma) \longrightarrow (Y,\delta)$ in C such that $Q(f)(\alpha) \leq \beta$. Then, the mapping sending an object $((X,\alpha),\gamma)$ of $\mathrm{Alg}_{\int Q}(\overline{F})$ to the object $((X,\gamma),\alpha)$ of $\int Q^\mu$ (and a map f to itself) is the desired isomorphism $\mathrm{Alg}_{\int Q}(\overline{F}) \simeq \int Q^\mu$.

Next, suppose that Q factors through SLatt. As we no more assume that ψ is natural, for an algebra morphism $f : (X,\gamma) \longrightarrow (Y,\delta)$ and $\alpha \in Q^\mu(X,\gamma)$, $Q(f)(\alpha)$ might not belong to $Q^\mu(Y,\delta)$. Yet, since $Q^\mu(Y,\delta)$ is a subset of $Q(Y)$ closed under infima, we can define

$$Q^\mu(f)(\alpha) := \bigwedge \{\, \beta \in Q^\mu(Y,\delta) \mid Q(f)(\alpha) \leq \beta \,\}\,. \tag{25}$$

The definition in (25) doesn't make explicit that Q^μ is a functor. We claim that this is the case, since we actually have the equality

$$Q^\mu = Q^{*\nu *}\,. \tag{26}$$

Assuming this equality, and recalling we can describe algebras in term of coalgebras, we obtain

$$\begin{aligned} \mathrm{Alg}_{\int Q}(\overline{F}) &= \left[\mathrm{CoAlg}_{(\int Q)^{\mathrm{op}}}(\overline{F}^{\mathrm{op}})\right]^{\mathrm{op}} = \left[\mathrm{CoAlg}_{\int Q^*}(\overline{F^{\mathrm{op}}})\right]^{\mathrm{op}}\\ &\simeq \left[\int Q^{*\nu}\right]^{\mathrm{op}} = \int Q^{*\nu *} = \int Q^\mu\,, \end{aligned}$$

where we have used, in the order, Lemma 25, Proposition 24, again Lemma 25, and then (26).

Next, let us show that $Q^\mu = Q^{*\nu *}$. We have $Q^{*\nu} : \mathrm{CoAlg}_{\mathsf{C}^{\mathrm{op}}}(F^{\mathrm{op}}) \longrightarrow \mathsf{Pos}$, so $Q^{*\nu *} : [\mathrm{CoAlg}_{\mathsf{C}^{\mathrm{op}}}(F^{\mathrm{op}})]^{\mathrm{op}} = \mathrm{Alg}_{\mathsf{C}}(F) \longrightarrow \mathsf{Pos}$. Let $(X,\gamma) \in \mathrm{Alg}_{\mathsf{C}}(F)$, we have $Q^{*\nu *}(X,\gamma) = Q^{*\nu}(X,\gamma)^{\mathrm{op}}$ and

$$Q^{*\nu}(X,\gamma) = \{\, \alpha \in Q(X)^{\mathrm{op}} \mid Q^*(\gamma)(\alpha) \leq \psi_X(\alpha) \text{ in } QF(X)^{\mathrm{op}} \,\}\,.$$

Because the order of this poset is the one from $Q(X)^{\mathrm{op}}$, the order of the opposite poset is the one from $Q(X)$. Also the inequality condition above is equivalent, by transposition, to $Q(\gamma)(\psi_X(\alpha)) \leq \alpha$. Then

$$Q^{*\nu *}(X,\gamma) = Q^{*\nu}(X,\gamma)^{\mathrm{op}} = \{\, \alpha \in Q(X) \mid Q(\gamma)(\psi_X(\alpha)) \leq \alpha \text{ in } Q(X) \,\} = Q^\mu(X,\gamma)\,.$$

That is, the equality (26) holds on objects.

For morphisms, consider that $f : (X,\gamma) \longrightarrow (Y,\delta)$ in $\mathrm{Alg}_{\mathsf{C}}(F)$. Recall that, as a

morphism of $\mathrm{CoAlg}_{\mathsf{C}^{\mathrm{op}}}(F^{\mathrm{op}})$, f is typed the opposite way, $f : (Y,\delta) \longrightarrow (X,\gamma)$. Let $\alpha \in Q^{\mu}(X,\gamma)$, $\beta \in Q^{\mu}(Y,\delta)$, and observe the following chain of equivalences:

$$\begin{aligned}
Q^{\mu}(f)(\alpha) \leq \beta \quad \text{in } Q^{\mu}(Y,\delta) \quad &\text{iff } \; Q(f)(\alpha) \leq \beta \quad \text{in } Q(Y)\,, \\
&\qquad\qquad \text{by definition of } Q^{\mu}(f)(\alpha), \\
&\text{iff } \; \alpha \leq Q(f)^*(\beta) \quad \text{in } Q(X) \\
&\text{iff } \; Q^*(f)(\beta) = Q(f)^*(\beta) \leq \alpha \quad \text{in } Q^*(X) = Q(X)^{\mathrm{op}} \\
&\text{iff } \; Q^{*\nu}(f)(\beta) \leq \alpha \quad \text{in } Q^{*\nu}(X,\gamma) \\
&\text{since } \alpha,\beta \in Q^{*\nu}(X) = Q^{\mu}(X,\gamma)^{\mathrm{op}} \text{ and } Q^*(f) \text{ restricts from } Q^* \text{ to } Q^{*\nu} \\
&\text{iff } \; \beta \leq Q^{*\nu}(f)^*(\beta) \quad \text{in } Q^{*\nu}(Y,\delta) \\
&\text{iff } \; Q^{*\nu *}(f)(\alpha) = Q^{*\nu}(f)^*(\alpha) \leq \beta
\end{aligned}$$

where the last inequality is taken in $Q^{*\nu}(Y,\delta)^{\mathrm{op}} = Q^{*\nu *}(Y,\delta) = Q^{\mu}(Y,\delta)$. From these equivalences we obtain the equality $Q^{\mu}(f)(\alpha) = Q^{*\nu *}(f)(\alpha)$, for each $\alpha \in Q^{\mu}(X,\gamma)$, as aimed. □

The isomorphisms in (23) and (24) allow to reduce the existence (or lifting) of a terminal coalgebra (initial algebra) to the existence of a terminal (initial) object in a generic op-fibration $\int Q \longrightarrow \mathsf{C}$. It is straightforward that if 1 a terminal object of C and $\top_1$ is the supremum of $Q(1)$, then $(1,\top_1)$ is a terminal object of $\int Q$. Similarly, if 0 is an initial object of C, $\bot_0$ is the least element of $Q(0)$, and for each object X, the map $Q(!_X) : Q(0) \longrightarrow Q(X)$ sends $\bot_0$ to a least element of $Q(X)$, then $(0,\bot_0)$ is an initial object of $\int Q$. Thus we have the following proposition, that for convenience we state directly for functors Q factoring through SLatt, leaving the possible generalizations to the reader.

Proposition 27. *Let $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$ and suppose that an endofunctor F of C has a lifting $\overline{F}$ to $\int Q$.*

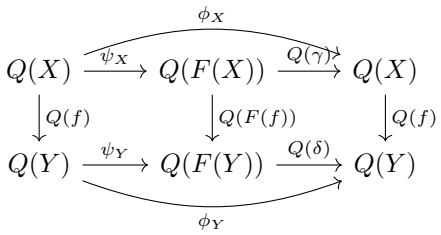
1. *If F has a terminal F-coalgebra $(\nu.F,\chi)$, then the object $(\nu.F,\chi,\nu.\phi)$ of $\int Q^{\nu} \simeq \mathrm{CoAlg}_{\int Q}(\overline{F})$ is terminal, where $\nu.\phi$ is the greatest fixed point of the map ϕ defined as the composition $Q(\nu.F) \xrightarrow{\psi_{\nu.F}} Q(F(\nu.F)) \xrightarrow{Q(\chi^{-1})} Q(\nu.F)$.*
2. *Suppose that $\psi : Q \longrightarrow Q \circ F$ is natural and that F has an initial algebra $(\mu.F,\chi)$. For each F-algebra (X,γ), let $\mu.\phi_X$ denote the least fixed point of the map ϕ_X defined as the composition $Q(X) \xrightarrow{\psi_X} Q(F(X)) \xrightarrow{Q(\gamma)} Q(X)$. Then the object $(\mu.F,\chi,\mu.\phi_{\mu.F})$ of $\int Q^{\mu} \simeq \mathrm{Alg}_{\int Q}(\overline{F})$ is initial.*

Notice that, in the second item of the above statement, the maps ϕ_X need not be sup-preserving, since we do not require the maps ψ_X to be sup-preserving. Therefore, the least fixed point of $\phi_{\mu.F}$ might not coincide with the least element of $Q(\mu.F)$.

Proof. For the first statement, consider that the structure $\chi : \nu.F \longrightarrow F(\nu.F)$ is invertible, so elements in $Q^{\nu}(\nu.F,\chi)$ are exaclty post-fixed points of the map ϕ. The statement is then an obvious application of the sufficient condition given for the extisence of a terminal object in $\int Q$

The second statement is also an application of the sufficient condition given for the existence of an initial object in $\int Q$. For this, we need to argue that least fixed points of the maps $\phi_X : Q(X) \longrightarrow Q(X)$ are preserved by maps of the form

$Q(f)$ with f and algebra morphism. This is a standard consequence in fixed poiont theory of commutativity of the following diagram:



and of the fact that $Q(f)$ has a right adjoint. □

9. Examples

Example 28. Categories of poset-valued sets. A main reason for developing this research was to understand the structure of categories of the form Q-Set introduced in [29] and, possibly, to devise generalization of this kind of constructions.

For $Q = (Q, e, *)$ a unital commutative quantale, the category Q-Set is defined as follows:

- An object is a pair (X, α) with X a set and $\alpha : X \longrightarrow Q$ a function.
- An arrow $R : (X, \alpha) \longrightarrow (Y, \beta)$ is a relation $R : X \longrightarrow Y$ such that, for all $x \in X$ and $y \in Y$, xRy implies $\alpha(x) \leq \beta(y)$.

Let $Q(X) := Q^X$ be the set of functions from X to Q, pointwise ordered and, for a relation $R : X \longrightarrow Y$, let $Q(R) : Q^X \longrightarrow Q^Y$ be defined by

$$Q(R)(\alpha)(y) := \bigvee_{xRy} \alpha(x)\,, \quad \text{for each } \alpha \in Q^X \text{ and } y \in Y.$$

It is easily verified that $Q(X)$ is a complete lattice, that $Q(R)$ is sup-preserving, and that Q is a functor $\mathsf{Rel} \longrightarrow \mathsf{Pos}$ factoring through SLatt. Clearly, Q-Set arises as the total category $\int Q$. While, for each set X, $Q(X)$ is a quantale (with the product structure), for a relation $R : X \longrightarrow Y$, only the inclusions $Q(R)(1) \leq 1$ and $Q(R)(\alpha * \beta) \leq Q(R)(\alpha) * Q(R)(\beta)$ hold—that is, $Q(R)$ is comonoidal. Nonetheless, and recalling that $Q(\{*\})$ can be identified with Q, if we define

$$u := e \in Q(\{*\})\,, \qquad \mu_{X,Y}(\alpha, \beta)(x, y) := \alpha(x) * \beta(y)\,,$$

then $\mu_{X,Y}$ is natural (and not merely lax-natural), u and μ make Q into a monoidal functor, and $\mu_{X,Y}$ is bilinear. Thus Q *monoidally* factors though SLatt. The monoidal structure of Q-Set exhibited in [29] is the one induced by $\mu_{X,Y}$. Recall that Rel is, w.r.t. the monoidal structure given by $\times$ and $\{*\}$, compact closed category, that is, we have $X \multimap Y = X \times Y$. It is verified that

$$\iota_{X,Y}(\alpha, \beta)(x, y) = \alpha(x) \multimap \beta(y)\,, \quad \text{thus, for } \omega \in Q \quad \omega_X(\alpha)(x) = \alpha(x) \multimap \omega\,. \tag{27}$$

In [29] this construction was also generalized as follows. Let $F : \mathsf{Rel} \longrightarrow \mathsf{Rel}$ be a 2-functor (equivalently, the relation lifiting of weak-pullback preserving endofunctor of Set, see [17]) that is comonoidal and whose natural arrows $\nu_{X,Y} : F(X \times Y) \longrightarrow F(X) \times F(Y)$ are functional. An object of Q_F-Set is a pair (X, α) with $\alpha \in Q(F(X))(= Q^{F(X)})$. An arrow in Q-Set from (X, α) to (Y, β) is a relation $R : X \longrightarrow Y$ such that, for each $x \in F(X)$ and $y \in F(Y)$, if

$xF(R)y$, then $\alpha(x) \leq \beta(y)$. Let $\mathsf{c} : \mathsf{Rel} \longrightarrow \mathsf{Rel}^{op}$ be the converse functor, identity on objects and picking the converse of a given relation. Given F as above, $\nu^{\mathsf{c}}_{X,Y} : F(X) \times F(Y) \longrightarrow F(X \times Y)$ makes F into a monoidal functor. Recall next that if $F : \mathsf{Rel} \longrightarrow \mathsf{Rel}$ and $Q : \mathsf{Rel} \longrightarrow \mathsf{Pos}$ are monoidal, then their composition $Q \circ F$ is monoidal as well, with natural transformation

$$\mu^{Q\circ F}_{X,Y} = Q(\mu^{F}_{X,Y}) \circ \mu^{Q}_{F(X),F(Y)} : Q(F(X)) \times Q(F(Y)) \longrightarrow Q(F(X \times Y)) \,.$$

Clealry $Q \circ F$ monoidally factors through SLatt and the equality $Q_F\text{-}\mathsf{Set} = \int Q \circ F$ holds. The monoidal structure described in [29] is the one arising from the monoidal $\mu^{Q\circ F}$.

A wide generalization of the category Q-Set arises if we replace Rel with $\mathsf{Rel}(Q)$, the category of relations or matrices over Q. An object of the category of $\mathsf{Rel}(Q)$ is a set, while an arrow $X \longrightarrow Y$ is a map $\psi : X \times Y \longrightarrow Q$. Identity is the Kronecker δ, while composition of $\psi : X \longrightarrow Y$ and $\chi : Y \longrightarrow Z$ is defined as follows:

$$(\psi \cdot \chi)(x, z) := \bigvee_{y \in Y} \psi(x, y) * \chi(y, z) \,.$$

The category $\mathsf{Rel}(Q)$ is compact closed [16, 24], with $X \otimes Y = X \multimap Y = X \times Y$, $I = 0 = \{*\}$, and $X^* = X$ on objects. $\mathsf{Rel}(Q)$ is also a quantaloid, that is, a category enriched over SLatt. From this it follows that $\mathsf{Rel}(Q)(\{*\}, -)$ is a monoidal functor from $\mathsf{Rel}(Q)$ to SLatt. Thus, $\int \mathsf{Rel}(Q)(\{*\}, (-))$ is monoidal closed. Since formulas as in (27) still hold in this wider setting, then dualizing objects of $\int \mathsf{Rel}(Q)(\{*\}, -)$ correspond to dualizing elements in $\mathsf{Rel}(Q)(\{*\}, \{*\}) \simeq Q$.

Recall that a fuzzy set is a map $\alpha : X \longrightarrow [0, 1]$. The unit interval $[0, 1]$ has several quantale structures, for example, it is a complete Heyting algebra. A *norm* on $[0, 1]$ is indeed a continuous multiplication making $[0, 1]$ into a quantale. Fuzzy sets have been organised in several categories of fuzzy relations, see e.g. [31, 13, 20]. For $Q = [0, 1]$ (with a norm), the category $\int \mathsf{Rel}(Q)(\{*\}, -)$ accommodates most of these constructions.

Example 29. Non-uniform totality spaces. Let us recall some definitions from [15, §1.2.2]. A *totality space* is a pair (X, α) with α is an upward closed (w.r.t. in containement) collection of susbets X. For a relation $R \subseteq X \times Y$ and $A \subseteq X$, let us define $R^\sharp(A) := \{\, y \in Y \mid \exists x \in A \;\text{ s.t. }\; xRy \,\}$. A *morphism of totality spaces* from (X, α) to (Y, β) is a relation $R \subseteq X \times Y$ such that, for each $A \in \alpha$, $R^\sharp(A) \in \beta$, see [15, Lemma 33].

Let $\mathcal{UP}(X)$ be the set of upsets of $\mathcal{P}(X)$ (so, in particular, $\mathcal{UP}(X)$ is a complete lattice). We extend $\mathcal{UP}$ to a functor from Rel to SLatt as follows: for $R \subseteq X \times Y$ and $\alpha \in \mathcal{UP}(X)$, we let

$$\mathcal{UP}(R)(\alpha) := \uparrow \exists_{R^\sharp}(\alpha) \,, \tag{28}$$

that is, $\mathcal{UP}(R)$ is the upward closure of the direct image of $R^\sharp$. Notice that

$$\uparrow \exists_{R^\sharp}(\alpha) \subseteq \beta \;\text{ iff }\; \alpha \subseteq (R^\sharp)^{-1}(\beta) \,, \tag{29}$$

so $\mathcal{UP}(R)$, having a right adjoint, is sup-preserving. Let us consider the category $\int \mathcal{UP}$. It is easily seen that the equivalent conditions in (29) amount to the statement that $R^\sharp(A) \in \beta$, for each $A \in \alpha$. That is, Nuts arises as the total category of

the functor $\mathcal{UP}$. The monoidal structure of $\mathsf{Nuts} = \int \mathcal{UP}$ is obtained by means of

$$\{\{*\}\} \in \mathcal{UP}(\{*\}) \quad \text{and} \quad \mu_{X,Y}(\alpha,\beta) = \uparrow \{\, a \times b \mid a \in \alpha,\, b \in \beta \,\},$$

where the latter is natural and bilinear.

Example 30: Focused orthogonality categories. The category Nuts is an example of $*$-autonomous category $\mathcal{O}_{\perp\!\!\!\perp}(\mathsf{C})$ arising from a focused orthogonality structure [14]. It was recently recognised [11] that these categories arise as total categories of some functor. Let us reconsider these categories with the help of the theory we have developed.

For a monoidal category C, the hom functor $\mathsf{C}(I,-) : \mathsf{C} \longrightarrow \mathsf{Set}$ is monoidal, since we can associate to $x : I \longrightarrow X$ and $y : I \longrightarrow Y$, the arrow $x \bullet y := I \xrightarrow{\simeq} I \otimes I \xrightarrow{x \otimes y} X \otimes Y$. The powerset functor $P : \mathsf{Set} \longrightarrow \mathsf{SLatt}$ is also monoidal. By composing these functors, we obtain the functor $P(\mathsf{C}(I,-)) : \mathsf{C} \longrightarrow \mathsf{SLatt}$ whose monoidal structure is given by

$$\mu_{X,Y}(\alpha,\beta) = \{\, x \bullet y \mid x \in \alpha,\, y \in \beta \,\}.$$

Next, if C is $*$-autonomous, then there is a natural isomorphism $\sharp : \mathsf{C}(X,0) \longrightarrow \mathsf{C}(I,X^*)$ with the property that, for $x : I \longrightarrow X$ and $y : X \longrightarrow 0$, $y \circ x = \mathsf{ev}_{X,0} \circ (x \bullet y^\sharp)$. Let now $\perp\!\!\!\perp \subseteq \mathsf{C}(I,0)$. Using the notation of [11], for $\alpha \subseteq \mathsf{C}(I,X)$, we have

$$\begin{aligned}
P(\sharp)(\alpha^\perp) &= \{\, y^\sharp \mid y \in \mathsf{C}(0,X),\, y \circ x \in \perp\!\!\!\perp,\ \text{for all } x \in \alpha \,\} \\
&= \{\, y \in \mathsf{C}(I,X^*) \mid \mathsf{ev}_{X,0} \circ (x \bullet y) \in \perp\!\!\!\perp,\ \text{for all } x \in \alpha \,\} \\
&= \bigcup \{\, y \in \mathsf{C}(I,X^*) \mid P(\mathsf{C}(I,\mathsf{ev}_{X,0}))(\mu_{X,X^*}(\alpha,\{\, y \,\})) \subseteq \perp\!\!\!\perp \,\} \\
&= \bigcup \{\, \beta \subseteq \mathsf{C}(I,X^*) \mid P(\mathsf{C}(I,\mathsf{ev}_{X,0}))(\mu_{X,X^*}(\alpha,\beta)) \subseteq \perp\!\!\!\perp \,\} = \iota_{X,0}(\alpha, \perp\!\!\!\perp).
\end{aligned}$$

That is, the Galois connection described in [14, 11] is, up to the isomorphism $P(\sharp) : P(C(X,0)) \simeq P(\mathsf{C}(I,X^*))$, the one we describe in Section 7. The category $\mathcal{O}_{\perp\!\!\!\perp}(\mathsf{C})$ is, therefore, a special case of the given construction $\int Q^{\jmath^\omega}$, where Q has been instantiated with $P(C(I,-))$ and $\omega = \perp\!\!\!\perp \in P(C(I,0))$.

10. Conclusions

We have given exact conditions that allow to lift a functor from C to $\int Q$. In particular, lifting the monoidal structure of C is almost equivalent to Q being a monoidal functor, as naturality of the arrows $\mu_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(X \otimes Y)$ is replaced by lax naturality. Similarly, lifting the internal hom to $\int Q$ was argued to be equivalent to giving a lax extranatural transformation with components $\iota_{X,Y} : Q(X)^{\mathrm{op}} \times Q(Y) \longrightarrow Q(X \multimap Y)$ making the unit and counit of the adjunction in C maps in $\int Q$. These conditions were found to be equivalent to the fact that the maps $\langle \alpha, - \rangle_{X,Y}$ have a right adjoint and that μ may be defined in terms of these maps. We argued that these conditions are automatically satisfied when Q is actually a monoidal functor into SLatt—thus, in this case, $\int Q$ is always symmetric monoidal closed. We investigated then dualizing objects in $\int Q$. We proved that, when Q is monoidal, a necessary and sufficient condition for $(0,\omega)$ to be dualizing is that C is $*$-autonomous and that the maps $\iota_{X,0}(-,\omega)$ are invertible. One of our most fascinating result was to construct a functor $Q^{\jmath^\omega}$ in an analogous way of the double negation construction in quantale theory. This gives another total category $\int Q^{\jmath^\omega}$ where $(0,\omega)$ is a dualizing object. Moreover, we gave a representation theorem for

monoidal functors Q into SLatt such that $\int Q$ is $*$-autonomous, thus giving a sort of phase semantics of proofs, and a completeness theorem for this semantics. We ended by studying lifting of intial algebras and terminal coalgebras, that is, of fixed points of endofunctors. We gave a useful description of categories of algebras and coalgebras in terms of other total categories. In particular, we argued that we can always lift initial algebras and terminal coalgebras when Q is a functor having SLatt as codomain.

We focused in this paper on lifting structures—the symmetric monoidal closed one, being star-autonomous, initial and terminal coalgebras of functors—that arise in the semantics of proofs for fragments of linear logic augmented with least and greatest fixed points. The categorical machinery developed in Section 3 and exemplified all along this paper can be applied for lifting different kind of categorical structures. For example, one might wish to lift products from C. It is then verified that a product is lifted from C to $\int Q$ via μ if and only if $\mu_{X,Y} : Q(X)\times Q(Y) \longrightarrow Q(X\times Y)$ is right adjoint to the canonical map $Q(X\times Y) \longrightarrow Q(X)\times Q(Y)$. This observation is implicit in the literature, see for example [28, 1, 10]. Notice that, in principle, the methodology applies for all kind of categorical structures and, due to its generality, can be instantiated to several concrete categories. Categories of generalized metric spaces, which can be seen as fibrations over Set, are an example. The problem of lifting set theoretic functors to these categories is also investigated in [3].

The research developed here also raises many natural mathematical questions that we aim to answer in a close future. For example, it can be shown that if C is $*$-autonomous with the tensor unit I being dualizing, and if (I,ω) is a dualizing object of $\int Q$, then ω is a dualizing element of the quantale $Q(I)$. Whether the converse hold, we do not know yet. We expect the representation Theorem 23 to yield hints on this problem. Last but not least, we have emphasized throughout this paper the analogy of a monoidal functor $Q : \mathsf{C} \longrightarrow \mathsf{SLatt}$ with a quantale. Such a monoidal functor is naturally enriched over SLatt, and the convolution product makes monoidal the category $[\mathsf{C},\mathsf{SLatt}]$ of functors enriched over SLatt. It is not difficult to observe that if C is $*$-autonomous, then so is $[\mathsf{C},\mathsf{SLatt}]$. Monoidal functors into SLatt are actually monoids in $[\mathsf{C},\mathsf{SLatt}]$. We studied in [7] Frobenius monoids in arbitrary $*$-autonomous categories. A natural conjecture is that the structure described in Sections 6 and 7, making $\int Q$ into a $*$-autonomous category, is the one of a Frobenius monoid in $[\mathsf{C},\mathsf{SLatt}]$.

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References

[1] S. Abramsky and G. Carù. Non-locality, contextuality and valuation algebras: a general theory of disagreement. *Philos. Trans. R. Soc. Lond., A, Math. Phys. Eng. Sci.*, 377(2157):22, 2019. Id/No 20190036.

[2] G. Bacci, R. Mardare, P. Panangaden, and G. Plotkin. Propositional logics for the Lawvere quantale. Available at https://arxiv.org/abs/2302.01224, 2023.

[3] A. Balan, A. Kurz, and J. Velebil. Extending set functors to generalised metric spaces. *Log. Methods Comput. Sci.*, 15(1):57, 2019. Id/No 5.

[4] M. Barr. **-autonomous categories. With an appendix by Po-Hsiang Chu*, volume 752 of *Lect. Notes Math.* Springer, Cham, 1979.

[5] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, Cambridge, 1990.

[6] A. De, F. Jafarrahmani, and A. Saurin. Phase semantics for linear logic with least and greatest fixed points. In A. Dawar and V. Guruswami, editors, *42nd IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2022, December 18-20, 2022, IIT Madras, Chennai, India*, volume 250 of *LIPIcs*, pages 35:1–35:23. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

[7] C. de Lacroix and L. Santocanale. Frobenius Structures in Star-Autonomous Categories. In B. Klin and E. Pimentel, editors, *31st EACSL Annual Conference on Computer Science Logic (CSL 2023)*, volume 252 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 18:1–18:20, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[8] T. Ehrhard and F. Jafarrahmani. Categorical models of linear logic with fixed points of formulas. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, pages 1–13. IEEE, 2021.

[9] P. Eklund, J. García, U. Höhle, and J. Kortelainen. *Semigroups in Complete Lattices: Quantales, Modules and Related Topics*. Developments in Mathematics. Springer International Publishing, 2018.

[10] N. Evangelou-Oost, C. Bannister, and I. J. Hayes. Contextuality in distributed systems. In R. Glück, L. Santocanale, and M. Winter, editors, *Relational and Algebraic Methods in Computer Science - 20th International Conference, RAMiCS 2023, Augsburg, Germany, April 3-6, 2023, Proceedings*, volume 13896 of *Lecture Notes in Computer Science*, pages 52–68. Springer, 2023.

[11] M. Fiore, Z. Galal, and F. Jafarrahmani. Fixpoint constructions in focused orthogonality models of linear logic. To appear in the proceeedings of the conference MFPS 2023, Electronic Notes in Theoretical Informatics and Computer Science, 2023.

[12] J. Fortier and L. Santocanale. Cuts for circular proofs: semantics and cut-elimination. In S. R. D. Rocca, editor, *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy*, volume 23 of *LIPIcs*, pages 248–262. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013.

[13] J. Harding, C. Walker, and E. Walker. Categories with fuzzy sets and relations. *Fuzzy Sets Syst.*, 256:149–165, 2014.

[14] M. Hyland and A. Schalk. Glueing and orthogonality for models of linear logic. *Theor. Comput. Sci.*, 294(1-2):183–231, 2003.

[15] F. Jafarrahmani. *Fixpoints of Types in Linear Logic from a Curry-Howard-Lambek Perspective*. PhD thesis, Université Paris Cité, 2023.

[16] G. M. Kelly and M. L. Laplaza. Coherence for compact closed categories. *J. Pure Appl. Algebra*, 19:193–213, 1980.

[17] A. Kurz and J. Velebil. Relation lifting, a survey. *J. Log. Algebr. Methods Program.*, 85(4):475–499, 2016.

[18] J. Lambek. A fixpoint theorem for complete categories. *Math. Z.*, 103:151–161, 1968.

[19] S. Mac Lane. *Categories for the working mathematician.*, volume 5 of *Grad. Texts Math.* New York, NY: Springer, 2nd ed edition, 1998.

[20] J. Močkoř. Categories of q-sets and fuzzy logic models. *Adv. Fuzzy Sets Syst.*, 20(1):47–78, 2015.

[21] J. Moeller and C. Vasilakopoulou. Monoidal Grothendieck construction. *Theory Appl. Categ.*, 35:1159–1207, 2020.

[22] K. I. Rosenthal. A note on Girard quantales. *Cah. Topologie Géom. Différ. Catégoriques*, 31(1):3–11, 1990.

[23] K. I. Rosenthal. *Quantales and their applications*, volume 234 of *Pitman Res. Notes Math. Ser.* Harlow: Longman Scientific & Technical; New York: John Wiley & Sons, Inc., 1990.

[24] K. I. Rosenthal. $*$-autonomous categories of bimodules. *J. Pure Appl. Algebra*, 97(2):189–202, 1994.

[25] L. Santocanale. μ-bicomplete categories and parity games. *RAIRO Theor. Informatics Appl.*, 36(2):195–227, 2002.

[26] L. Santocanale. A calculus of circular proofs and its categorical semantics. In M. Nielsen and U. Engberg, editors, *Foundations of Software Science and Computation Structures, 5th International Conference, FOSSACS 2002. Held as Part of the Joint European Conferences*

on Theory and Practice of Software, ETAPS 2002 Grenoble, France, April 8-12, 2002, Proceedings, volume 2303 of *Lecture Notes in Computer Science*, pages 357–371. Springer, 2002.

[27] L. Santocanale. Free μ-lattices. *J. Pure Appl. Algebra*, 168(2-3):227–264, 2002.

[28] L. Santocanale. The equational theory of the natural join and inner union is decidable. In C. Baier and U. D. Lago, editors, *Foundations of Software Science and Computation Structures - 21st International Conference, FOSSACS 2018, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2018, Thessaloniki, Greece, April 14-20, 2018, Proceedings*, volume 10803 of *Lecture Notes in Computer Science*, pages 494–510. Springer, 2018.

[29] A. Schalk and V. de Paiva. Poset-valued sets or how to build models for linear logics. *Theor. Comput. Sci.*, 315(1):83–107, 2004.

[30] M. Shulman. Framed bicategories and monoidal fibrations. *Theory Appl. Categ.*, 20:650–738, 2008.

[31] M. Winter. *Goguen categories. A categorical approach to L-fuzzy relations*, volume 25 of *Trends Log. Stud. Log. Libr.* Berlin: Springer, 2007.