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# Stochastic differential equations and stochastic parallel translations in the Wasserstein space

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## Abstract

We will develop some elements in stochastic analysis in the Wasserstein space  $\mathbb{P}_2(M)$  over a compact Riemannian manifold  $M$ , such as intrinsic Itô formulae, stochastic regular curves and parallel translations along them. We will establish the existence of parallel translations along regular curves, or stochastic regular curves in case of  $\mathbb{P}_2(\mathbb{T})$ . Surprisingly enough, in this last case, the equation defining stochastic parallel translations is a SDE on a Hilbert space, instead of a SPDE.

**MSC 2010:** 58B20, 60J45

**Keywords:** constant vector fields, stochastic regular curves, Itô's formula, stochastic parallel translations, internal energy functional.

## 1 Introduction

During recent decades, the study of the optimal transportation problem has obtained great successes. In particular, in a seminar paper [6], Y. Brenier proved that the optimal transport map for the long standing Monge-Kantorovich problem with the quadratic distance cost function on Euclidean spaces is given by the gradient of a convex function, which satisfies a Monge-Ampère equation. In [28], R. McCann extended Brenier's result to compact Riemannian manifolds. In [7], L. A. Caffarelli et al. obtained the  $C^{2,\alpha}$  regularity estimates for the solutions of Brenier Monge-Ampère equation. Here we will not enter the details of this topic, but only refer to two monographs by C. Villani [34, 35]. For infinite dimensional case, see [15, 14].

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On the other hand, R. Jordan, D. Kinderlehrer and F. Otto [18] gave a steepest descent formulation of the Fokker-Planck equation on Euclidean space. In [29], F. Otto introduced an infinite dimensional Riemannian structure on the Wasserstein space of probability measures with finite second moments, and interpreted the Fokker-Planck equation and the porous medium equation as the gradient flows associated to the Boltzmann or Rényi type entropy functionals on the Wasserstein space  $\mathbb{P}_{2,\infty}(M)$ , which is the space of probability measures having positive smooth density. To obtain the long time behavior of above equations, he developed heuristically the Hessian calculus on the Wasserstein space  $\mathbb{P}_{2,\infty}(M)$ . A striking result is that the Hessian of the Boltzmann entropy functional is given by the Bakry-Emery Ricci curvature [4]. See Otto [29], Otto-Villani [30], Renesse-Sturm [32] and Lott [24]. In [33, 26, 32], J. Lott, C. Villani and K.T. Sturm developed a synthetic geometric analysis on metric measured spaces. In [2], L. Ambrosio, N. Gigli and G. Savaré developed the theory of generalized gradient flows on the Wasserstein space over metric measured spaces. In [21, 22], S. Li and the third named author of this paper introduced the Langevin deformation between the Wasserstein geodesic flow and the gradient flows on  $\mathbb{P}_{2,\infty}(M)$  and proved some  $W$ -entropy-information formulas.

The definition of Levi-Civita covariant derivatives on  $\mathbb{P}_{2,\infty}(M)$  was introduced by J. Lott in [24]. He also derived the linear partial differential equation for parallel translations and identified the equation of the associated Riemannian geodesics with those introduced by Benamou and Brenier [5] for the  $L^2$ -(Monge-Kantorovich-Rubinstein-)Wasserstein distance from a hydrodynamic point of view. However, the issue of existence and uniqueness of parallel transports has not been studied in [24]. In [1], L. Ambrosio and N. Gigli established the existence of parallel translations along regular curves induced by flows of diffeomorphisms, but in a weak sense; the question whether the strong solutions for parallel translations exist remains open. Concerning parallel translations along geodesics in  $\mathbb{P}_{2,\infty}(M)$ , the situation seems much more complicated, see [25]. In [10], the first two authors revisited these works by proving that vector fields along a regular curve are restrictions on this curve of some vector fields defined on the whole space; this allowed us to introduce parallel translations as in differential geometry. The main purpose of this work is to introduce parallel translations along stochastic regular curves, we will establish their existence and uniqueness in the case of  $\mathbb{P}_{2,\infty}(\mathbb{T})$ . Finally we remark that M.K Von Renesse and K.T. Sturm rigorously constructed in [31] a quasi-invariant measure on  $\mathbb{P}_2(\mathbb{T})$ .

## 2 Framework

In this work, we consider a connected compact Riemannian manifold  $M$ , of the Riemannian distance  $d_M$ , together with the normalized Riemannian measure  $dx$ :  $\int_M dx = 1$ . As usual, we denote by  $\mathbb{P}_2(M)$  the space of probability measures on  $M$ , endowed with the Wasserstein distance  $W_2$  defined by

$$W_2^2(\mu_1, \mu_2) = \inf \left\{ \int_{M \times M} d_M^2(x, y) \pi(dx, dy), \quad \pi \in \mathcal{C}(\mu_1, \mu_2) \right\},$$

where  $\mathcal{C}(\mu_1, \mu_2)$  is the set of probability measures  $\pi$  on  $M \times M$ , having  $\mu_1, \mu_2$  as two marginal laws. It is well known that  $\mathbb{P}_2(M)$  endowed with  $W_2$  is a compact space.

For tangent spaces  $\bar{\mathbf{T}}_\mu$  of  $\mathbb{P}_2(M)$  at  $\mu$ , we adopt the definition given in [2], that is,

$$\bar{\mathbf{T}}_\mu = \overline{\{\nabla\psi, \psi \in C^\infty(M)\}}^{L^2(\mu)}, \quad (2.1)$$

the closure of gradients of smooth functions in the space  $L^2(\mu)$  of vector fields on  $M$ . A curve  $\{c(t); t \in [0, 1]\}$  in  $\mathbb{P}_2(M)$  is said to be absolutely continuous if there exists  $k \in L^2([0, 1])$  such that

$$W_2(c(t_1), c(t_2)) \leq \int_{t_1}^{t_2} k(s) ds, \quad t_1 < t_2.$$

For such a curve, Ambrosio, Gigli and Savaré [2] proved that there exists a Borel vector field  $Z_t$  on  $M$  in  $L^2([0, 1] \times M)$ , that is,

$$\int_0^1 \left[ \int_M |Z_t(x)|_{T_x M}^2 c_t(dx) \right] dt < +\infty,$$

which satisfies the continuity equation

$$\frac{dc_t}{dt} + \nabla \cdot (Z_t c_t) = 0. \quad (2.2)$$

The uniqueness of solutions to (2.2) holds if  $Z_t \in \bar{\mathbf{T}}_{c_t}$  for almost all  $t \in [0, 1]$ . We say that  $Z_t$  is the intrinsic derivative of  $\{c_t\}$  and denote it by

$$\frac{d^I c_t}{dt}. \quad (2.3)$$

For reader's convenience, we introduce now two classes of absolutely continuous curves in  $\mathbb{P}_2(M)$ . The first example is those  $\{c_t\}$  generated by flows of maps. More precisely,  $c_t = (U_t)_\# c_0$  with

$$\frac{dU_t}{dt} = \nabla \psi(U_t), \quad U_0(x) = x, \quad (2.4)$$

for regular function  $\psi$  on  $M$ . In this case,  $\frac{d^I c_t}{dt} = \nabla \psi$ .

In what follows, we use the notation  $V_\psi$  when  $\nabla \psi$  is seen as a vector field on  $\mathbb{P}_2(M)$ , just as J. Lott did in [24] in order to clarify different roles played by  $\nabla \psi$ . For a functional  $F$  on  $\mathbb{P}_2(M)$ , we say that  $F$  is derivable along  $V_\psi$  if

$$(\bar{D}_{V_\psi} F)(\mu) = \left\{ \frac{d}{dt} F((U_t)_\# \mu) \right\}_{t=0} \text{ exists.} \quad (2.5)$$

We say that the gradient  $\bar{\nabla} F(\mu) \in \bar{\mathbf{T}}_\mu$  exists if for each  $\psi \in C^\infty(M)$ ,  $\bar{D}_{V_\psi} F$  exists and  $(\bar{D}_{V_\psi} F)(\mu) = \langle \bar{\nabla} F, V_\psi \rangle_{\bar{\mathbf{T}}_\mu}$ .

The second class of examples of such  $\{c_t\}$  is the class of geodesics in  $\mathbb{P}_2(M)$ . More precisely, for two probability measures  $c_0, c_1$  on  $M$  having density, according to R. McCann [28], there is a function  $\phi$  in the Sobolev space  $\mathbb{D}_1^2(M)$  such that the optimal transport map  $\mathcal{T}$  pushing  $c_0$  to  $c_1$ , admits the expression

$$\mathcal{T}(x) = \exp_x(\nabla \phi(x)). \quad (2.6)$$

Then  $\{c_t\}$  is defined by  $c_t = (\mathcal{T}_t)_\# c_0$  where  $\mathcal{T}_t(x) = \exp_x(t \nabla \phi(x))$ , and  $\frac{d^I c_t}{dt} = \nabla \phi(\mathcal{T}_t^{-1}(x))$ .

In order to intrinsically formulate stochastic differential equations (SDE) on  $\mathbb{P}_2(M)$ , we need to introduce suitable functionals. Here are usual functionals considered in literature (see for example [2]).

1) Potential energy functional. For any  $\varphi \in C^2(M)$ , we set  $F_\varphi(\mu) = \int_M \varphi \mu(dx)$ .

2) Internal energy functional. Let  $\chi : [0, +\infty[ \rightarrow ]-\infty, +\infty]$  be a proper, continuous convex function satisfying  $\chi(0) = 0$ ,  $\liminf_{s \rightarrow 0} \frac{\chi(s)}{s^\alpha} > -\infty$  for some  $\alpha > \frac{d}{d+2}$ , where  $d$  is the dimension of  $M$ . The internal energy  $\mathcal{F}$  is defined as follows

$$\mathcal{F}(\mu) = \int_M \chi(\rho(x)) dx, \quad \text{if } d\mu = \rho dx,$$

and  $\mathcal{F}(\mu) = +\infty$  otherwise. Two important examples are  $\chi(s) = s \log(s)$  and  $\chi(s) = \frac{s^m}{m-1}$  for  $m > 1$ .

3) Interaction energy functional. Let  $W : M^2 \rightarrow ]-\infty, +\infty]$  be a l.s.c function, we define

$$\mathcal{W}(\mu) = \int_{M \times M} W(x, y) \mu(dx) \mu(dy).$$

The results below are not new, but for the sake of self-contained of our paper, we give a complete proof in what follows. We collect first and second order derivatives of these functionals in the following

**Proposition 2.1.** *Let  $V_\psi$  be a constant vector field on  $\mathbb{P}_2(M)$ , we have*

(i)  $(\bar{D}_{V_\psi} F_\varphi)(\mu) = \langle V_\psi, \bar{\nabla} F_\varphi \rangle_{\bar{\mathbb{T}}_\mu}$  and

$$(\bar{D}_{V_\psi} \bar{D}_{V_\psi} F_\varphi)(\mu) = \int_M \mathcal{L}_{\bar{\nabla} V_\psi}^2 \varphi \mu(dx). \quad (2.7)$$

(ii) For  $\chi \in C^2(\mathbb{R}^*)$  such that  $|\chi(s)| + s|\chi'(s)| + s^2|\chi''(s)|$  is bounded over  $[0, 1]$ , we have

$(\bar{D}_{V_\psi} \mathcal{F})(\mu) = - \int_M (\chi'(\rho)\rho - \chi(\rho)) \Delta\psi dx$  and

$$(\bar{D}_{V_\psi} \bar{D}_{V_\psi} \mathcal{F})(\mu) = \int_M p'(\rho) (\Delta\psi)^2 \rho^2 dx - \int_M p(\rho) \langle \nabla\psi, \nabla\Delta\psi \rangle \rho dx, \quad (2.8)$$

where  $p(s) = \chi'(s) - \frac{\chi(s)}{s}$ . For  $\chi(s) = s \log s$ ,  $(\bar{D}_{V_\psi} \bar{D}_{V_\psi} \mathcal{F})(\mu) = - \int_M \langle \nabla\psi, \nabla\Delta\psi \rangle \rho dx$ .

(iii) Set

$$\Phi(x, \mu) = \int_M (W(x, y) + W(y, x)) \mu(dy).$$

Then  $(\bar{D}_{V_\psi} \mathcal{W})(\mu) = \int_M \langle \nabla\Phi(x, ; \mu), \nabla\psi(x) \rangle \mu(dx)$ , and

$$(\bar{D}_{V_\psi} \bar{D}_{V_\psi} \mathcal{W})(\mu) = \int_M ((\bar{D}_{V_\psi} \mathcal{L}_{\nabla\psi} \Phi)(x, \mu) + (\mathcal{L}_{\bar{\nabla} V_\psi}^2 \Phi)(x, \mu)) \mu(dx).$$

Where  $\mathcal{L}_{\nabla\psi}$  denotes the Lie derivative with respect to  $\nabla\psi$  on  $M$ .

*Proof.* Item (i) is well-known. For proving (ii), we consider the flow  $(U_t)$  associated to  $\nabla\psi$  (see (2.4)). Let  $\mu_t = (U_t)_\#(\rho dx) = \rho_t dx$ . It is known (see [8]) that

$$\rho_t = \rho(U_{-t}) e^{-\int_0^t \Delta\psi(U_{-s}) ds} \quad \text{or} \quad \rho_t(U_t) = \rho e^{-\int_0^t \Delta\psi(U_{t-s}) ds}.$$

It follows that

$$\left\{ \frac{d}{dt} \rho_t(U_t) \right\}_{t=0} = -(\Delta\psi) \rho. \quad (2.9)$$

We have  $\mathcal{F}(\mu_t) = \int_M \chi(\rho_t) dx = \int_M \frac{\chi(\rho_t)}{\rho_t} \rho_t dx = \int_M \hat{\chi}(\rho_t(U_t)) \rho dx$ , where  $\hat{\chi}(s) = \chi(s)/s$ . Therefore by Relation (2.9), we get

$$\left\{ \frac{d}{dt} \mathcal{F}(\mu_t) \right\}_{t=0} = - \int_M \hat{\chi}'(\rho) (\Delta \psi \rho) \rho dx = - \int_M (\chi'(\rho) \rho - \chi(\rho)) \Delta \psi dx.$$

For the sake of simplicity, we denote for a moment by  $\tilde{\mathcal{F}}$  the right hand of above relation. Then

$$\tilde{\mathcal{F}}(\mu_t) = - \int_M (\chi'(\rho_t) \rho_t - \chi(\rho_t)) \Delta \psi dx = - \int_M p(\rho_t) \Delta \psi \rho_t dx$$

which is equal to

$$- \int_M p(\rho_t(U_t)) \Delta \psi(U_t) \rho dx.$$

Again using Relation (2.9), we get

$$\left\{ \frac{d}{dt} \tilde{\mathcal{F}}(\mu_t) \right\}_{t=0} = \int_M p'(\rho) (\Delta \psi \rho) \Delta \psi \rho dx - \int_M p(\rho) \langle \nabla \Delta \psi, \nabla \psi \rangle \rho dx,$$

which is nothing but the second formula in (ii). Item (iii) comes from direct calculation.  $\square$

### 3 Stochastic differential equations on $\mathbb{P}_2(M)$

Let's first say a few words on SDE on a Riemannian manifold  $M$ . Given a family of vector fields  $\{A_0(t, \cdot), A_1(t, \cdot), \dots, A_N(t, \cdot)\}$  on  $M$ , and  $B_t^0 = t$  and a standard Brownian motion  $t \rightarrow (B_t^1, \dots, B_t^N)$  on  $\mathbb{R}^N$ , how to understand the following SDE on  $M$

$$dX_{t,s} = \sum_{i=0}^N A_i(t, X_{t,s}) \circ dB_t^i, \quad X_{s,s}(x) = x ? \quad (3.1)$$

The equality (3.1) formally holds in the tangent space  $T_{X_{t,s}}M$ . Rigorously, a stochastic process  $\{X_{t,s}, t \geq s\}$  is a solution to SDE (3.1) if for any test function  $f \in C^2(M)$ , it holds

$$\begin{aligned} f(X_{t,s}) = & f(x) + \sum_{i=1}^N \int_s^t (\mathcal{L}_{A_i(u)} f)(X_{u,s}) dB_u^i \\ & + \int_s^t \left( (\mathcal{L}_{A_0(u)} f + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{A_i(u)}^2 f)(X_{u,s}) \right) du, \end{aligned} \quad (3.2)$$

where  $\mathcal{L}_{A_i(u)}$  denotes the Lie derivative on  $M$  with respect to  $x \rightarrow A_i(u, x)$ , see [11, 17, 27].

In what follows, we develop this concept and use three types of functionals in the preceding section to introducing SDE on  $\mathbb{P}_2(M)$ .

Let  $\{\phi_0, \phi_1, \dots, \phi_N\}$  be a family of functions on  $[0, 1] \times M$ , continuous in  $t \in [0, 1]$  and smooth in  $x \in M$ . In this work,  $\nabla$  always denotes the gradient operator on  $M$ . First we consider the following Stratanovich SDE on  $M$ :

$$dX_{t,s} = \sum_{i=0}^N \nabla \phi_i(t, X_{t,s}) \circ dB_t^i, \quad t \geq s, \quad X_{s,s}(x) = x. \quad (3.3)$$

Let  $d\mu = \rho dx$  be a probability measure on  $M$ , we set  $\mu_t(\omega) = (X_{t,0}(\omega))_{\#}\mu$ . Let  $\varphi \in C^2(M)$ . After first using Itô's formula to  $\varphi(X_{t,0})$ , then integrating the two hand sides respect to  $d\mu$ , we get

$$\circ d_t F_\varphi(\mu_t) = \sum_{i=0}^N \left( \int_M \langle \nabla \varphi, \nabla \phi_i(t, \cdot) \rangle \mu_t(dx) \right) \circ dB_t^i = \sum_{i=0}^N \langle V_\varphi, V_{\phi_i(t, \cdot)} \rangle_{\bar{\mathbf{T}}_{\mu_t}} \circ dB_t^i.$$

**Definition 3.1.** We say that the intrinsic Itô stochastic differential of  $\mu_t$ , denoted by  $\circ d_t^I \mu_t$ , admits the following expression

$$\circ d_t^I \mu_t = \sum_{i=0}^N V_{\phi_i(t, \cdot)} \circ dB_t^i. \quad (3.4)$$

Recall that  $\bar{\nabla} F_\varphi = V_\varphi$ ; using notation (3.4),  $\circ d_t F_\varphi(\mu_t)$  can be written in the form

$$\circ d_t F_\varphi(\mu_t) = \langle \bar{\nabla} F_\varphi, \circ d_t^I \mu_t \rangle_{\bar{\mathbf{T}}_{\mu_t}},$$

symbolically read in the inner product of  $\bar{\mathbf{T}}_{\mu_t}$ , in the same way as we did on Riemannian manifolds.

By Itô's formula (3.2),

$$d_t \varphi(X_{t,0}) = \sum_{i=0}^N (\mathcal{L}_{\nabla \phi_i(t)} \varphi)(X_{t,0}) dB_t^i + \frac{1}{2} \sum_{i=1}^N (\mathcal{L}_{\nabla \phi_i(t)}^2 \varphi)(X_{t,0}) dt.$$

According to Proposition 2.1, above relation yields

$$d_t F_\varphi(\mu_t) = \sum_{i=0}^N \langle \bar{\nabla} F_\varphi, V_{\phi_i(t)} \rangle_{\bar{\mathbf{T}}_{\mu_t}} dB_t^i + \frac{1}{2} \sum_{i=1}^n (\bar{D}_{V_{\phi_i(t)}}^2 F_\varphi)(\mu_t) dt.$$

**Proposition 3.2.** For any polynomial  $F$  on  $\mathbb{P}_2(M)$ , we have

$$d_t F(\mu_t) = \sum_{i=0}^N \langle \bar{\nabla} F, V_{\phi_i(t, \cdot)} \rangle_{\bar{\mathbf{T}}_{\mu_t}} dB_t^i + \frac{1}{2} \sum_{i=1}^N (\bar{D}_{V_{\phi_i(t, \cdot)}}^2 F)(\mu_t) dt. \quad (3.5)$$

*Proof.* For two functionals  $F_1$  and  $F_2$  satisfying Formula (3.5), by Itô formula,

$$d_t(F_1 F_2)(\mu_t) = d_t F_1(\mu_t) F_2(\mu_t) + F_1(\mu_t) d_t F_2(\mu_t) + d_t F_1(\mu_t) \cdot d_t F_2(\mu_t).$$

Note that

$$\bar{D}_{V_{\phi_i(t, \cdot)}}^2 (F_1 F_2) = F_2 \bar{D}_{V_{\phi_i(t, \cdot)}}^2 F_1 + F_1 \bar{D}_{V_{\phi_i(t, \cdot)}}^2 F_2 + 2 \langle \bar{\nabla} F_1, V_{\phi_i(t, \cdot)} \rangle \cdot \langle \bar{\nabla} F_2, V_{\phi_i(t, \cdot)} \rangle,$$

and  $d_t F_1(\mu_t) \cdot d_t F_2(\mu_t) = \sum_{i=1}^N \langle \bar{\nabla} F_1, V_{\phi_i(t, \cdot)} \rangle \cdot \langle \bar{\nabla} F_2, V_{\phi_i(t, \cdot)} \rangle dt$ ; so Formula (3.5) holds for  $F_1 F_2$ .

A polynomial  $F$  on  $\mathbb{P}_2(M)$  is a finite sum of  $F_{\varphi_1} \cdots F_{\varphi_k}$ , Formula (3.5) remains true for  $F$ . We complete the proof.  $\square$

Now are going to see what happens with internal energy functional  $\mathcal{F}$ , which is not continuous. Note that if  $\rho = \frac{d\mu}{dx} > 0$ , then for almost all  $\omega$ ,  $\mu_t(\omega)$  has a positive density  $\rho_t$  with respect to  $dx$ .

**Proposition 3.3.** *The stochastic process  $\{\rho_t, t \geq 0\}$  satisfies the following SPDE*

$$d\rho_t = - \sum_{i=0}^N \operatorname{div}(\rho_t \nabla \phi_i(t)) dB_t^i + \frac{1}{2} \sum_{i=1}^N \operatorname{div}(\operatorname{div}(\rho_t \nabla \phi_i(t)) \nabla \phi_i(t)) dt. \quad (3.6)$$

*Proof.* We have

$$\begin{aligned} \int_M \langle \nabla \varphi, \nabla \phi_i(t) \rangle \mu_t(dx) &= \int_M \langle \nabla \varphi, \rho_t \nabla \phi_i(t) \rangle dx \\ &= - \int_M \varphi \operatorname{div}(\rho_t \nabla \phi_i(t)) dx. \end{aligned}$$

In the same way,

$$\int_M \mathcal{L}_{\nabla \phi_i(t)}^2 \varphi \mu_t(dx) = \int_M \varphi \operatorname{div}(\operatorname{div}(\rho_t \nabla \phi_i(t)) \nabla \phi_i(t)) dx.$$

Formula (3.6) follows by considering  $F_\varphi(\mu_t) = \int_M \varphi \rho_t dx$  for any  $\varphi \in C^2(M)$ , together with Formula (3.5).  $\square$

Let  $\chi$  be a  $C^2$  function defined on  $]0, +\infty[$ . According to (3.6), we have

$$\begin{aligned} d\chi(\rho_t) &= - \sum_{i=0}^N \chi'(\rho_t) \operatorname{div}(\rho_t \nabla \phi_i(t)) dB_t^i + \frac{1}{2} \sum_{i=1}^N \chi'(\rho_t) \operatorname{div}(\operatorname{div}(\rho_t \nabla \phi_i(t)) \nabla \phi_i(t)) dt \\ &\quad + \frac{1}{2} \sum_{i=1}^N \chi''(\rho_t) (\operatorname{div}(\rho_t \nabla \phi_i(t)))^2 dt. \end{aligned}$$

Remarking that  $\operatorname{div}(\rho_t \nabla \phi_i(t)) = \rho_t \Delta \phi_i(t) + \langle \nabla \rho_t, \nabla \phi_i(t) \rangle$ , that we will use several times in the sequel, we have

$$\int_M \chi'(\rho_t) \operatorname{div}(\rho_t \nabla \phi_i(t)) dx = \int_M \chi'(\rho_t) \rho_t \Delta \phi_i(t) dx + \int_M \chi'(\rho_t) \langle \nabla \rho_t, \nabla \phi_i(t) \rangle dx,$$

this last term is equal to

$$\int_M \langle \nabla(\chi(\rho_t)), \nabla \phi_i(t) \rangle dx = - \int_M \chi(\rho_t) \Delta \phi_i(t) dx.$$

Therefore

$$\int_M \chi'(\rho_t) \operatorname{div}(\rho_t \nabla \phi_i(t)) dx = \int_M (\chi'(\rho_t) \rho_t - \chi(\rho_t)) \Delta \phi_i(t) dx,$$

which is  $-(\bar{D}_{V_{\phi_i(t)}} \mathcal{F})(\mu_t)$  by Proposition 2.1. Now we are going to deal with drift terms. First of all, we compute

$$I_1 = \int_M \chi'(\rho_t) \operatorname{div}(\operatorname{div}(\rho_t \nabla \phi_i(t)) \nabla \phi_i(t)) dx = - \int_M \chi''(\rho_t) \langle \nabla \rho_t, \nabla \phi_i(t) \rangle \operatorname{div}(\rho_t \nabla \phi_i(t)) dx.$$

Again using  $\operatorname{div}(\rho_t \nabla \phi_i(t)) = \rho_t \Delta \phi_i(t) + \langle \nabla \rho_t, \nabla \phi_i(t) \rangle$ , we write down  $I_1$  as

$$I_1 = - \int_M \chi''(\rho_t) \rho_t \langle \nabla \rho_t, \nabla \phi_i(t) \rangle \Delta \phi_i(t) dx - \int_M \chi''(\rho_t) \langle \nabla \rho_t, \nabla \phi_i(t) \rangle^2 dx.$$



Put  $I_2 = \int_M \chi''(\rho_t) (\operatorname{div}(\rho_t \nabla \phi_i(t)))^2 dx$ . In the same way,

$$\begin{aligned} I_2 &= \int_M \chi''(\rho_t) \rho_t^2 (\Delta \phi_i(t))^2 dx + 2 \int_M \chi''(\rho_t) \rho_t \langle \nabla \rho_t, \nabla \phi_i(t) \rangle \Delta \phi_i(t) dx \\ &\quad + \int_M \chi''(\rho_t) \langle \nabla \rho_t, \nabla \phi_i(t) \rangle^2 dx. \end{aligned}$$

Combining above terms gives

$$I_1 + I_2 = \int_M \chi''(\rho_t) \rho_t^2 (\Delta \phi_i(t))^2 dx + \int_M \chi''(\rho_t) \rho_t \langle \nabla \rho_t, \nabla \phi_i(t) \rangle \Delta \phi_i(t) dx. \quad (3.7)$$

On the other hand, according to the expression for  $\bar{D}_{V_{\phi_i(t)}}^2 \mathcal{F}$  in Proposition 2.1, we set

$$I_3 = \int_M p'(\rho_t) (\Delta \phi_i(t))^2 \rho_t^2 dx.$$

Replacing  $\tilde{\chi}$  by its expression,  $I_3$  becomes

$$\begin{aligned} I_3 &= \int_M \chi''(\rho_t) (\Delta \phi_i(t))^2 \rho_t^2 dx - \int_M \chi'(\rho_t) (\Delta \phi_i(t))^2 \rho_t dx \\ &\quad + \int_M \chi(\rho_t) (\Delta \phi_i(t))^2 dx. \end{aligned}$$

We put

$$I_4 = \int_M \chi'(\rho_t) \langle \nabla \phi_i(t), \nabla \Delta \phi_i(t) \rangle \rho_t dx - \int_M \chi(\rho_t) \langle \nabla \phi_i(t), \nabla \Delta \phi_i(t) \rangle dx.$$

In order to more clearly see the relation between  $I_1 + I_2$  and  $I_3 - I_4$ , we now deal with the second term in  $I_1 + I_2$ , that is,

$$J = \int_M \chi''(\rho_t) \rho_t \langle \nabla \rho_t, \nabla \phi_i(t) \rangle \Delta \phi_i(t) dx = \int_M \langle \nabla(\chi'(\rho_t)), \nabla \phi_i(t) \rangle \rho_t \Delta \phi_i(t) dx$$

Via divergence operator  $\operatorname{div}$ , we get

$$J = - \int_M \chi'(\rho_t) \operatorname{div}(\rho_t \Delta \phi_i(t) \nabla \phi_i(t)) dx.$$

Note that  $\operatorname{div}(\rho_t \Delta \phi_i(t) \nabla \phi_i(t)) = \rho_t (\Delta \phi_i(t))^2 + \rho_t \langle \nabla \Delta \phi_i(t), \nabla \phi_i(t) \rangle + \langle \nabla \rho_t, \nabla \phi_i(t) \rangle \Delta \phi_i(t)$ , and

$$\begin{aligned} - \int_M \chi'(\rho_t) \langle \nabla \rho_t, \nabla \phi_i(t) \rangle \Delta \phi_i(t) dx &= \int_M \chi(\rho_t) \operatorname{div}(\Delta \phi_i(t) \nabla \phi_i(t)) dx \\ &= \int_M \chi(\rho_t) ((\Delta \phi_i(t))^2 + \langle \nabla \phi_i(t), \nabla \Delta \phi_i(t) \rangle) dx. \end{aligned}$$

Combining all of these terms, finally we prove that  $I_1 + I_2 = I_3 - I_4$ . In other words, we get the following result.

**Proposition 3.4.** *For any internal energy functional  $\mathcal{F}$  with  $\chi \in C^2(]0, +\infty[)$ , we have*

$$d_t \mathcal{F}(\mu_t) = \sum_{i=0}^N \langle \bar{\nabla} \mathcal{F}, V_{\phi_i(t)} \rangle_{\bar{\mathbf{T}}_{\mu_t}} dB_t^i + \frac{1}{2} \sum_{i=1}^N (\bar{D}_{V_{\phi_i(t)}}^2 \mathcal{F})(\mu_t) dt.$$

**Proposition 3.5.** *Itô's formula (3.5) also holds for interaction energy functional  $\mathcal{W}$ .*

*Proof.* Applying Itô formula to  $W(X_{t,0}(x), X_{t,0}(y))$  and proceeding as above yields the result.  $\square$

Now having these results in hand, we say now that the stochastic process  $\{\mu_t; t \geq 0\}$  solves the following SDE on  $\mathbb{P}_{2,\infty}(M)$ ,

$$\circ d_t^I \mu_t = \sum_{i=0}^N V_{\phi_i(t)}(\mu_t) \circ dB_t^i, \quad \mu_0 = \rho dx. \quad (3.8)$$

In what follows, we introduce quite general vector fields on  $\mathbb{P}_2(M)$ .

**Definition 3.6.** *We say that  $Z$  is a vector field on  $\mathbb{P}_2(M)$  if there exists a Borel map  $\Phi : M \times \mathbb{P}_2(M) \rightarrow \mathbb{R}$  such that, for any  $\mu \in \mathbb{P}_2(M)$ ,  $x \rightarrow \Phi(x, \mu)$  is  $C^1$  and  $Z(\mu) = V_{\Phi(\cdot, \mu)}$ .*

Recall that for two probability measures  $\mu, \nu \in \mathbb{P}_{2,\infty}(M)$ , there is a unique optimal transport map  $T_{\mu,\nu} : M \rightarrow M$ , which pushes  $\mu$  to  $\nu$  and has the expression (see [28, 34]):

$$T_{\mu,\nu}(x) = \exp_x(\nabla \xi(x)).$$

Let  $\xi_x^{\mu,\nu}(t) = \exp_x(t \nabla \xi(x))$  and let  $//_t^{\xi_x^{\mu,\nu}}$  be the parallel translation along  $\{\xi_x^{\mu,\nu}(t); t \in [0, 1]\}$ .

**Definition 3.7.** *We say that  $Z$  is Lipschitzian if there exists a constant  $\kappa > 0$  such that*

$$\int_M \left| //_1^{\xi_x^{\mu,\nu}} \nabla \Phi(x, \mu) - \nabla \Phi(T_{\mu,\nu}(x), \nu) \right|^2 \mu(dx) \leq \kappa^2 W_2^2(\mu, \nu), \quad (3.9)$$

for any couple of probability measures  $(\mu, \nu) \in \mathbb{P}_{2,\infty}(M) \times \mathbb{P}_{2,\infty}(M)$ .

It was proved in [10] that, under above Lipschitzian condition, the ODE on  $\mathbb{P}_{2,\infty}(M)$

$$\frac{d^I \nu_t}{dt} = Z(\nu_t); \quad \nu_0 = \rho dx \text{ given} \quad (3.10)$$

admits a unique solution. Condition (3.9) is satisfied [10] if (i)  $x \rightarrow \nabla^2 \Phi(x, \mu)$  exists such that

$$\Lambda_1 = \sup_{\mu \in \mathbb{P}_2(M)} \|\nabla^2 \Phi(\cdot, \mu)\|_\infty < +\infty, \quad (3.11)$$

and (ii) there exists a constant  $\Lambda_2$  such that

$$|\nabla \Phi(x, \mu) - \nabla \Phi(x, \nu)| \leq \Lambda_2 W_2(\mu, \nu), \quad \text{for any } x \in M. \quad (3.12)$$

**Remark 3.8.** *The function  $\Phi$  involved in the interaction energy functional  $\mathcal{W}$  satisfies Condition (3.11) and (3.12) if  $W \in C^2(M \times M)$ . This functional  $\mathcal{W}$  plays an important role in the work [23]. A definition of absolutely continuous vector field using parallel translations was given in Chapter 3 in [16].*

In the sequel, for simplicity, we suppose that  $\phi_0, \dots, \phi_N$  are time-independent. The following result was established by F. Y. Wang in [36] in a quite general setting. For the own interest, we give a construction of solutions.

**Proposition 3.9.** *Suppose that  $\Phi$  satisfies Conditions (3.11) and (3.12), then there is a unique solution  $(X_t, \mu_t)$  to the following McKean-Vlasov SDE:*

$$dX_t = \sum_{i=0}^N \nabla \phi_i(X_t) \circ dB_t^i + \nabla \Phi(X_t, \mu_t) dt, \quad \mu_t = (X_t)_{\#} \mu_0, \quad (3.13)$$

*Proof.* We will construct a solution to (3.13). Let  $(U_t)_{t \geq 0}$  be the stochastic flow associated to the following SDE

$$dU_t = \sum_{i=0}^N \nabla \phi_i(U_t) \circ dB_t^i.$$

Define the stochastic measure-dependent vector fields  $V_t(\omega, x, \mu)$  on  $M$  by

$$V_t(\omega, x, \mu) = (U_t^{-1}(\omega, \cdot))_* \nabla \Phi(x, (U_t)_{\#} \mu) = (U_t^{-1})'(\omega, U_t(x)) \nabla \Phi(U_t(x), (U_t)_{\#} \mu),$$

where the prime denotes the differential with respect to  $x$ . Since the manifold  $M$  is compact, we have, for  $\omega$  given,

$$|V_t(\omega, x, \mu) - V_t(\omega, x, \nu)| \leq \|(U_t^{-1})'\|_{\infty} |\nabla \Phi(U_t(x), (U_t)_{\#} \mu) - \nabla \Phi(U_t(x), (U_t)_{\#} \nu)|.$$

Note that

$$W_2((U_t)_{\#} \mu, (U_t)_{\#} \nu) \leq \|U_t'\|_{\infty} W_2(\mu, \nu);$$

then under Condition (3.12),  $\mu \rightarrow V_t(\omega, x, \mu)$  is Lipschitzian uniformly in  $(t, x) \in [0, 1] \times M$ ; moreover by Condition (3.11),  $x \rightarrow V_t(\omega, x, \mu)$  is Lipschitzian uniformly in  $(t, \mu) \in [0, 1] \times \mathbb{P}_2(M)$ . So there is a unique solution  $(Y_t, \nu_t)$  to the following McKean-Vlasov ODE on  $M$

$$\frac{d}{dt} Y_t = V_t(Y_t, \nu_t), \quad \nu_t = (Y_t)_{\#} \mu_0.$$

Let  $\tilde{X}_t = U_t(Y_t)$ . By Itô-Wentzell formula,

$$d\tilde{X}_t = \sum_{i=0}^N \nabla \phi_i(U_t(Y_t)) \circ dB_t^i + U_t'(Y_t) V_t(Y_t, \nu_t),$$

the last term in above equality is  $\nabla \Phi(\tilde{X}_t, (U_t)_{\#} \nu_t)$ . Note that  $(\tilde{X}_t)_{\#} \mu_0 = (U_t)_{\#} (Y_t)_{\#} \mu_0 = (U_t)_{\#} \nu_t$ ; therefore  $(\tilde{X}_t, (U_t)_{\#} \nu_t)$  is a solution to McKean-Vlasov SDE (3.13) on  $M$ .  $\square$

**Proposition 3.10.** *The stochastic process  $\{\mu_t; t \in [0, 1]\}$  obtained in Proposition 3.9 solves the following SDE on  $\mathbb{P}_2(M)$ :*

$$\circ d_t^I \mu_t = \sum_{i=0}^N V_{\phi_i}(\mu_t) \circ dB_t^i + Z(\mu_t) dt, \quad \mu|_{t=0} = \mu_0, \quad (3.14)$$

where  $Z(\mu) = V_{\Phi(\cdot, \mu)}$ , see Definition 3.6.

## 4 Regular curves and parallel translations on $\mathbb{P}_{2,\infty}(M)$

In this section, we only consider the space  $\mathbb{P}_{2,\infty}$ . According to J. Lott [24], the Levi-Civita covariant derivative defined by usual formula

$$\begin{aligned} 2\langle \bar{\nabla}_{V_{\psi_1}} V_{\psi_2}, V_{\psi_3} \rangle_{\bar{\mathbf{T}}_\mu} &= \bar{D}_{V_{\psi_1}} \langle V_{\psi_2}, V_{\psi_3} \rangle_{\bar{\mathbf{T}}_\mu} + \bar{D}_{V_{\psi_2}} \langle V_{\psi_3}, V_{\psi_1} \rangle_{\bar{\mathbf{T}}_\mu} - \bar{D}_{V_{\psi_3}} \langle V_{\psi_1}, V_{\psi_2} \rangle_{\bar{\mathbf{T}}_\mu} \\ &\quad + \langle V_{\psi_3}, [V_{\psi_1}, V_{\psi_2}] \rangle_{\bar{\mathbf{T}}_\mu} - \langle V_{\psi_2}, [V_{\psi_1}, V_{\psi_3}] \rangle_{\bar{\mathbf{T}}_\mu} - \langle V_{\psi_1}, [V_{\psi_2}, V_{\psi_3}] \rangle_{\bar{\mathbf{T}}_\mu}, \end{aligned}$$

admits the following expression

$$\langle \bar{\nabla}_{V_{\psi_1}} V_{\psi_2}, V_{\psi_3} \rangle_{\bar{\mathbf{T}}_\mu} = \int_M \langle \nabla^2 \psi_2, \nabla \psi_1 \otimes \nabla \psi_3 \rangle \mu(dx). \quad (4.1)$$

Also as usual, above bracket  $[V_{\psi_1}, V_{\psi_2}]$  denotes the Lie bracket of two vector fields on  $\mathbb{P}_{2,\infty}(M)$ . Note that  $[V_{\psi_1}, V_{\psi_2}]$  as well as  $\bar{\nabla}_{V_{\psi_1}} V_{\psi_2}$  are not constant vector fields. Let

$$\Pi_\mu : L^2(M, TM; \mu) \rightarrow \bar{\mathbf{T}}_\mu$$

be the orthogonal projection; then

$$\bar{\nabla}_{V_{\psi_1}} V_{\psi_2}(\mu) = \Pi_\mu(\nabla \nabla \psi_1 \nabla \psi_2). \quad (4.2)$$

For  $\mu \in \mathbb{P}_{2,\infty}(M)$  with  $d\mu = \rho dx$ ,  $\rho > 0$ , we denote by  $\Delta_\mu$  the Witten Laplacian:

$$\Delta_\mu = \Delta + \langle \nabla \log \rho, \nabla \cdot \rangle,$$

and  $\text{div}_\mu$  the divergence operator defined by  $\int_M \langle \nabla \varphi, Z \rangle \mu(dx) = - \int_M \varphi \text{div}_\mu(Z) \mu(dx)$  for any  $\varphi \in C^\infty(M)$ . We have the relation  $\text{div}_\mu(Z) = \text{div}(Z) + \langle \nabla \log \rho, Z \rangle$ . By Hodge decomposition [20], there is a function  $f$  and a vector field  $Y$  of  $\text{div}_\mu(Y) = 0$  such that  $Z = \nabla f + Y$ . Therefore  $\text{div}_\mu(Z) = \Delta_\mu f$  and  $f = \Delta_\mu^{-1}(\text{div}_\mu(Z))$ , or

$$\Pi_\mu(Z) = \nabla \Delta_\mu^{-1}(\text{div}_\mu(Z)). \quad (4.3)$$

Formula (4.3) first holds for smooth  $Z$ , then extends to the space  $L^2(M, TM; \mu)$ .

Now let  $\{c_t\}$  be a curve in  $\mathbb{P}_{2,\infty}(M)$  defined by a flow of diffeomorphisms  $X_{t,s}$  associate to ODE:

$$dX_{t,s} = \nabla \phi_t(X_{t,s}) dt, \quad t \geq s, \quad X_s(x) = x, \quad (4.4)$$

with  $c_t = (X_{t,0})_\#(\rho dx)$ .

The following derivative formula of  $\Pi_{c_t}$  was obtained in chapter 5 of [16]. For reader's convenience, we include a proof here.

**Theorem 4.1.** *For a smooth vector  $Z$  on  $M$ ,  $t \rightarrow \mathbb{P}_{c_t}(Z)$  is absolutely continuous and*

$$\frac{d}{dt} \Pi_{c_t}(Z) = -\Pi_{c_t} \left( \Delta_{c_t} \phi_t \Pi_{c_t}^\perp(Z) \right), \quad (4.5)$$

where  $\Pi_\mu^\perp = I - \Pi_\mu$ .

*Proof.* Denote for a moment  $\hat{\rho}_t$  the density of  $c_t$  with respect to  $c_0$ :  $\hat{\rho}_t = \frac{dc_t}{dc_0}$ , then  $\hat{\rho}_t$  satisfies the following equation

$$\frac{d\hat{\rho}_t}{dt} = -\operatorname{div}_{c_t}(\nabla\phi_t) \hat{\rho}_t = -(\Delta_{c_t}\phi_t) \hat{\rho}_t. \quad (4.6)$$

Let  $f \in C^\infty(M)$ , we have the relation

$$\int_M \langle \nabla f, Z \rangle c_t(dx) = \int_M \langle \nabla f, \Pi_{c_t}(Z) \rangle c_t(dx).$$

Using the density  $\hat{\rho}_t$  in the two hand sides, above equality becomes

$$\int_M \langle \nabla f, Z \rangle \hat{\rho}_t c_0(dx) = \int_M \langle \nabla f, \Pi_{c_t}(Z) \rangle \hat{\rho}_t c_0(dx).$$

Taking the derivative on the two hand sides with respect to  $t$ , and using (4.6), we get

$$\begin{aligned} - \int_M \langle \nabla f, Z \rangle (\Delta_{c_t}\phi_t) \hat{\rho}_t c_0(dx) &= \int_M \langle \nabla f, \frac{d}{dt}\Pi_{c_t}(Z) \rangle \hat{\rho}_t c_0(dx) \\ &\quad - \int_M \langle \nabla f, \Pi_{c_t}(Z) \rangle \Delta_{c_t}\phi_t \hat{\rho}_t c_0(dx), \end{aligned}$$

which implies the equality

$$\int_M \langle \nabla f, \frac{d}{dt}\Pi_{c_t}(Z) \rangle \hat{\rho}_t c_0(dx) = - \int_M \langle \nabla f, \Pi_{c_t}^\perp(Z) \rangle \Delta_{c_t}\phi_t \hat{\rho}_t c_0(dx).$$

By (4.3),  $\frac{d}{dt}\Pi_{c_t}(Z) \in \bar{\mathbf{T}}_{c_t}$ . Since  $f$  is arbitrary, we get the result (4.5) from above equality.  $\square$

Now let's briefly describe results obtained in the literature for parallel translations on  $\mathbb{P}_2(M)$ . Let  $\{Y_t; t \in [0, 1]\}$  be a family of vector fields along  $\{c_t; t \in [0, 1]\}$ , that is,  $Y_t \in \bar{\mathbf{T}}_{c_t}$ . Suppose there are smooth functions  $(t, x) \rightarrow \Phi_t(x)$  and  $(t, x) \rightarrow \Psi_t(x)$  such that

$$\frac{d^J c_t}{dt} = V_{\Phi_t}, \quad Y_t = V_{\Psi_t}.$$

J. Lott obtained in [24] that if  $\{Y_t; t \in [0, 1]\}$  is parallel along  $\{c_t; t \in [0, 1]\}$ , then  $\{\nabla\Psi_t; t \in [0, 1]\}$  is a solution to the following linear PDE

$$\frac{d}{dt}\nabla\Psi_t + \Pi_{c_t}(\nabla_{\nabla\Phi_t}\nabla\Psi_t) = 0. \quad (4.7)$$

Existence of weak solutions to (4.7) were established by L. Ambrosio and N. Gigli in [1]. By mimicing the section 5 of the paper [1], the first two authors obtained in [10] the following result

**Theorem 4.2.** *For any  $\nabla\Psi_0 \in L^2(c_0)$ , there is a unique weak solution  $\{\nabla\Psi_t, t \in [0, 1]\}$  in the sense that  $V_{\Psi_t} \in \bar{\mathbf{T}}_{c_t}$  and*

$$\Pi_{c_t} \left( \lim_{\varepsilon \downarrow 0} \frac{\tau_\varepsilon^{-1} \nabla\Psi_{t+\varepsilon}(X_{t+\varepsilon,t}) - \nabla\Psi_t}{\varepsilon} \right) = 0 \quad (4.8)$$

holds in  $L^2(c_t)$  for almost all  $t \in [0, 1]$ , where  $\tau_\varepsilon$  is the parallel translation along  $\{s \rightarrow X_{t+s,t}, s \in [0, \varepsilon]\}$ , that is equivalent to say that  $t \rightarrow \nabla\Psi_t$  is absolutely continuous and

$$\frac{d}{dt} \int_M \langle \nabla f, \nabla\Psi_t \rangle c_t(dx) = \int_M \langle \nabla^2 f, \nabla\phi_t \otimes \nabla\Psi_t \rangle c_t(dx), \quad f \in C^\infty(M). \quad (4.9)$$

**Proposition 4.3.** *Let  $\{\nabla\Psi_t; t \in [0, 1]\}$  be the parallel translation along  $\{c_t; t \in [0, 1]\}$  in Theorem 4.2, then for any smooth vector field  $Z$  on  $M$ ,*

$$\begin{aligned} \frac{d}{dt} \int_M \langle Z, \nabla\Psi_t \rangle c_t(dx) &= - \int_M \langle (\Delta_{c_t}\phi_t) \Pi_{c_t}^\perp(Z), \nabla\Psi_t \rangle c_t(dx) \\ &\quad + \int_M \langle \nabla_{\nabla\phi_t}(\Pi_{c_t}(Z)), \nabla\Psi_t \rangle c_t(dx). \end{aligned} \quad (4.10)$$

*Proof.* Let  $I_t = \int_M \langle \Pi_{c_t}(Z), \nabla\Psi_t \rangle c_t(dx)$ . For any  $\varepsilon > 0$ , we have

$$I_{t+\varepsilon} = \int_M \langle \Pi_{c_{t+\varepsilon}}(Z), \nabla\Psi_{t+\varepsilon} \rangle c_{t+\varepsilon}(dx) = \int_M \langle \tau_\varepsilon^{-1} \Pi_{c_{t+\varepsilon}}(Z), \tau_\varepsilon^{-1} \nabla\Psi_{t+\varepsilon} \rangle (X_{t+\varepsilon, t}) c_t(dx).$$

Then

$$\begin{aligned} I_{t+\varepsilon} - I_t &= \int_M \langle \tau_\varepsilon^{-1} \Pi_{c_{t+\varepsilon}}(Z)(X_{t+\varepsilon, t}) - \Pi_{c_t}(Z)(x), \tau_\varepsilon^{-1} \nabla\Psi_{t+\varepsilon}(X_{t+\varepsilon, t}) \rangle c_t(dx) \\ &\quad + \int_M \langle \Pi_{c_t}(Z), \tau_\varepsilon^{-1} \nabla\Psi_{t+\varepsilon}(X_{t+\varepsilon, t}) - \nabla\Psi_t(x) \rangle c_t(dx) = J_\varepsilon^1 + J_\varepsilon^2 \quad \text{respectively.} \end{aligned}$$

As  $\varepsilon \rightarrow 0$ ,  $\tau_\varepsilon^{-1} \nabla\Psi_{t+\varepsilon}(X_{t+\varepsilon, t})$  converges to  $\nabla\Psi_t(x)$  and  $J_\varepsilon^2/\varepsilon$  converges to 0 by (4.8). For  $J_\varepsilon^1$ , note that

$$\begin{aligned} \frac{1}{\varepsilon} \left( \tau_\varepsilon^{-1} \Pi_{c_{t+\varepsilon}}(Z)(X_{t+\varepsilon, t}) - \Pi_{c_t}(Z)(x) \right) &= \frac{1}{\varepsilon} \left( \tau_\varepsilon^{-1} \Pi_{c_{t+\varepsilon}}(Z)(X_{t+\varepsilon, t}) - \tau_\varepsilon^{-1} \Pi_{c_t}(Z)(X_{t+\varepsilon, t}) \right) \\ &\quad + \frac{1}{\varepsilon} \left( \tau_\varepsilon^{-1} \Pi_{c_t}(Z)(X_{t+\varepsilon, t}) - \Pi_{c_t}(Z)(x) \right) \end{aligned}$$

As  $\varepsilon \rightarrow 0$ , the last term in above equality converges to  $\nabla_{\nabla\phi_t} \Pi_{c_t}(Z)$ , while the first term on the right hand gives  $\frac{d}{dt} \Pi_{c_t}(Z)$ . Finally using (4.5), we get the result (4.10).  $\square$

Now we are going to establish the existence of strong solution to Equation (4.7) in the case of  $\mathbb{P}_{2,\infty}(\mathbb{T})$ , the base manifold  $M$  being the torus  $\mathbb{T}$ .

In this case, we can explicit the orthogonal projection  $\Pi_{c_t}$ . A function  $v$  on  $\mathbb{T}$  is the derivative of a function  $\phi$  if and only if  $\int_{\mathbb{T}} v(x) dx = 0$ . In order to make explicit dependence on  $x$ , the derivative of  $\phi$  on  $\mathbb{T}$  is denoted by  $\partial_x\phi$ . Let  $\mu \in \mathbb{P}_{2,\infty}(\mathbb{T})$  with  $\rho = \frac{d\mu}{dx} > 0$ . Let  $\partial_x\phi = \Pi_\mu(v)$ ; then for any function  $f$ ,

$$\int_{\mathbb{T}} \partial_x f v(x) \rho(x) dx = \int_{\mathbb{T}} \partial_x f \partial_x \phi \rho(x) dx.$$

This implies that  $\partial_x(v\rho) = \partial_x(\partial_x\phi \rho)$ , so that for a constant  $K$ ,

$$v\rho = \partial_x\phi \rho + K \quad \text{or} \quad v = \partial_x\phi + \frac{K}{\rho}.$$

Integrating the two hand sides over  $\mathbb{T}$  yields  $K = \frac{\int_{\mathbb{T}} v(x) dx}{\int_{\mathbb{T}} \frac{dx}{\rho}}$ . Therefore

$$\Pi_\mu(v) = v - \frac{\int_{\mathbb{T}} v(x) dx}{\int_{\mathbb{T}} \frac{dx}{\rho}} \cdot \frac{1}{\rho}.$$

For simplifying notations, we put

$$\hat{\rho} = \frac{1}{\left(\int_{\mathbb{T}} \frac{dx}{\rho}\right) \rho}. \quad (4.11)$$

It is obvious that  $\int_{\mathbb{T}} \hat{\rho} dx = 1$ . In the sequel, we use notation  $\Pi_{\rho}$  instead of  $\Pi_{\mu}$ . Then

$$\Pi_{\rho}(v) = v - \left(\int_{\mathbb{T}} v(x) dx\right) \hat{\rho}. \quad (4.12)$$

Let  $\phi_t \in C^{\infty}(\mathbb{T})$  and  $(X_t)$  be the flow associated to

$$\frac{dX_t}{dt} = \partial_x \phi_t(X_t).$$

Let  $c_t = (X_t)_{\#} \mu$  with  $d\mu = \rho dx$ . Set  $\rho_t = \frac{dc_t}{dx}$  the density of  $c_t$  with respect to  $dx$ . Let  $g_t \in C^2(\mathbb{T})$  such that  $\int_{\mathbb{T}} g_t(x) dx = 0$ . Then  $\{g_t; t \in [0, 1]\}$  is a solution to (4.7) if

$$\frac{dg_t}{dt} + \Pi_{\rho_t}(\partial_x g_t \partial_x \phi_t) = 0.$$

According to (4.12), we have

$$\frac{dg_t}{dt} = -\partial_x g_t \partial_x \phi_t + \left(\int_{\mathbb{T}} \partial_x g_t \partial_x \phi_t dx\right) \hat{\rho}_t. \quad (4.13)$$

From above equation, it is easy to see that  $\frac{d}{dt} \int_{\mathbb{T}} g_t(x) dx = 0$  since  $\int_{\mathbb{T}} \hat{\rho}_t(x) dx = 1$ . It follows that  $\int_{\mathbb{T}} g_t(x) dx = \int_{\mathbb{T}} g_0(x) dx = 0$ . In other words,  $g_t$  is derivative of a function on  $\mathbb{T}$  if the initial condition  $g_0$  does. Put  $f_t = g_t(X_t)$ . Then

$$\frac{df_t}{dt} = \left(\int_{\mathbb{T}} \partial_x g_t \partial_x \phi_t dx\right) \hat{\rho}_t(X_t).$$

Remark that

$$\int_{\mathbb{T}} \partial_x g_t \partial_x \phi_t dx = -\int_{\mathbb{T}} g_t \partial_x^2 \phi_t dx = -\int_{\mathbb{T}} \frac{g_t \partial_x^2 \phi_t}{\rho_t} \rho_t dx = \int_{\mathbb{T}} g_t(X_t) \left(\frac{\partial_x^2 \phi_t}{\rho_t}\right)(X_t) \rho dx.$$

Then  $f_t$  satisfies the following equation

$$\frac{df_t}{dt} = -\left(\int_{\mathbb{T}} f_t \frac{\partial_x^2 \phi_t}{\rho_t}(X_t) \rho dx\right) \hat{\rho}_t(X_t). \quad (4.14)$$

Define  $\Lambda(t, f) = -\left(\int_{\mathbb{T}} f \frac{\partial_x^2 \phi_t}{\rho_t}(X_t) \rho dx\right) \hat{\rho}_t(X_t)$ . Then above equation can be written in the form

$$\frac{df_t}{dt} = \Lambda(t, f_t).$$

**Lemma 4.4.** *We have*

$$\|\Lambda(t, f) - \Lambda(t, g)\|_{L^2(\rho dx)} \leq \left(\sup_{t \in [0, 1]} \|\partial_x^2 \phi_t\|_{\infty}\right) \|f - g\|_{L^2(\rho dx)}, \quad t \in [0, 1]. \quad (4.15)$$

*Proof.* Note that

$$\int_{\mathbb{T}} \left( \frac{\partial_x^2 \phi_t}{\rho_t} \right)^2 (X_t) \rho dx = \int_{\mathbb{T}} \frac{(\partial_x^2 \phi_t)^2}{\rho_t} dx \leq \|\partial_x^2 \phi_t\|_{\infty}^2 \int_{\mathbb{T}} \frac{dx}{\rho_t},$$

and  $\int_{\mathbb{T}} \hat{\rho}_t (X_t)^2 \rho dx = \left( \int_{\mathbb{T}} \frac{dx}{\rho_t} \right)^{-1}$ ; it follows that

$$\int_{\mathbb{T}} \left| \int_{\mathbb{T}} f \frac{\partial_x^2 \phi_t}{\rho_t} (X_t) \rho dx \right|^2 \hat{\rho}_t (X_t)^2 \rho dx \leq \|\partial_x^2 \phi_t\|_{\infty}^2 \|f\|_{L^2(\rho dx)}^2$$

and global Lipschitz condition (4.15) holds.  $\square$

By classical theory of ODE on Banach spaces, for  $f_0 \in L^2(\rho dx)$  given, there is a unique solution  $f_t$  to Equation (4.14). Now having this solution  $(f_t)_{t \in [0,1]}$  in hand, we set

$$g_t = f_t(X_t^{-1}).$$

We have

$$\int_{\mathbb{T}} |g_t|^2 \rho_t dx = \int_{\mathbb{T}} |f_t|^2 \frac{\rho_t}{\rho} (X_t) \tilde{\rho}_t dx,$$

where  $\tilde{\rho}_t = \frac{d(X_t^{-1})_{\#}(\rho dx)}{dx}$ . It is known that  $\frac{\rho_t}{\rho}(X_t) \tilde{\rho}_t = \rho$ . Hence

$$\int_{\mathbb{T}} |g_t|^2 \rho_t dx = \int_{\mathbb{T}} |f_t|^2 \rho dx. \quad (4.16)$$

Now by a quite standard computation, we prove that

$$\frac{dg_t}{dt} = - \left( \int_{\mathbb{T}} g_t \partial_x^2 \phi_t dx \right) \hat{\rho}_t - \partial_x g_t \partial_x \phi_t,$$

$g_t$  is a solution to Equation (4.13), therefore  $\int_{\mathbb{T}} g_t dx = 0$ .

**Proposition 4.5.** *The solution  $\{g_t; t \in [0, 1]\}$  to Equation (4.13) preserves norms, that is*

$$\int_{\mathbb{T}} |g_t|^2 \rho_t dx = \int_{\mathbb{T}} |g_0|^2 \rho dx, \quad t \in [0, 1]. \quad (4.17)$$

*Proof.* By (4.16), it is sufficient to check  $\int_{\mathbb{T}} f_t \frac{df_t}{dt} \rho dx = 0$ . But by (4.14), we compute

$$\int_{\mathbb{T}} \hat{\rho}_t (X_t) f_t \rho dx = \int_{\mathbb{T}} \hat{\rho}_t (X_t) g_t (X_t) \rho dx = \int_{\mathbb{T}} \hat{\rho}_t g_t \rho_t dx = \frac{\int_{\mathbb{T}} g_t dx}{\int_{\mathbb{T}} \frac{dx}{\rho_t}},$$

this last term is equal to 0.  $\square$

Finally we get the main result of this section.

**Theorem 4.6.** *For any  $g_0 \in \bar{\mathbf{T}}_{\rho dx}$  given, there is a unique solution  $g_t \in \bar{\mathbf{T}}_{\rho_t dx}$  to parallel translation equation (4.13) such that  $\int_{\mathbb{T}} |g_t|^2 \rho_t dx = \int_{\mathbb{T}} |g_0|^2 \rho dx$  for any  $t \in [0, 1]$ .*



## 5 Stochastic parallel translations

Let  $\{\phi_0, \phi_1, \dots, \phi_N\}$  be a finite family of smooth function on  $M$  and  $\{\mu_t; t \in [0, 1]\}$  a solution to SDE (3.8), which comes from SDE on  $M$

$$dX_t = \sum_{i=0}^N \nabla \phi_i(X_t) \circ dB_t^i, \quad \text{with } B_t^0 = t.$$

According to (4.7), if a stochastic process  $\{\nabla \Psi_t; t \in [0, 1]\}$  is parallel along  $\{\mu_t; t \in [0, 1]\}$ , Equation (4.7) would be replaced by the following formal Stratanovich SDE

$$\circ d_t(\nabla \Psi_t) = - \sum_{i=0}^N \Pi_{\mu_t} \left( \nabla_{\nabla \phi_i} \nabla \Psi_t \right) \circ dB_t^i. \quad (5.1)$$

Using notation (4.2), the above equation becomes

$$\circ d_t V_{\Psi_t} = - \sum_{i=0}^N (\bar{\nabla}_{V_{\phi_i}} V_{\Psi_t}) \circ dB_t^i. \quad (5.2)$$

If (5.2) has a smooth solution, it preserves norm. More precisely, we have the following proposition.

**Proposition 5.1.** *Suppose that  $\{\nabla \Psi_t; t \in [0, 1]\}$  is a solution to Equation (5.2), then for any  $t \in [0, 1]$ ,*

$$\int_M |\nabla \Psi_t(x)|^2 \mu_t(dx) = \int_M |\nabla \Psi_0(x)|^2 \mu(dx). \quad (5.3)$$

*Proof.* We give a heuristic proof of (5.3). Let  $\rho_t = \frac{d\mu_t}{d\mu}$  be the density with respect to the initial probability measure  $\mu$ , then  $\{\rho_t; t \in [0, 1]\}$  satisfies the following SPDE:

$$\circ d_t \rho_t = - \sum_{i=0}^N \left( \operatorname{div}_{\mu_t} (\nabla \phi_i) \rho_t \right) \circ dB_t^i.$$

Now

$$\circ d_t \int_M |\nabla \Psi_t|^2 \rho_t \mu(dx) = 2 \int_M \langle \nabla \Psi_t, \circ d_t \nabla \Psi_t \rangle \rho_t \mu(dx) + \int_M \langle \nabla \Psi_t, \nabla \Psi_t \rangle \circ d_t \rho_t \mu(dx).$$

Note that

$$\int_M \langle \nabla \Psi_t, \nabla \Psi_t \rangle \operatorname{div}_{\mu_t} (\nabla \phi_i) \rho_t \mu(dx) = 2 \int_M \langle \nabla \Psi_t, \nabla_{\nabla \phi_i} \nabla \Psi_t \rangle \mu_t(dx).$$

Combining these equalities yields

$$\circ d_t \int_M |\nabla \Psi_t|^2 \mu_t(dx) = 2 \int_M \langle \nabla \Psi_t, \circ d_t \nabla \Psi_t + \sum_{i=0}^N \nabla_{\nabla \phi_i} \nabla \Psi_t \circ dB_t^i \rangle \mu_t(dx) = 0.$$

We get (5.3). □

The weak form of (5.2), the stochastic counterpart of (4.9), would be

$$d_t \int_M \langle \nabla f, \nabla \Psi_t \rangle \mu_t(dx) = \sum_{i=0}^N \left( \int_M \langle \nabla_{\nabla \phi_i}(\nabla f), \nabla \Psi_t \rangle \mu_t(dx) \right) \circ dB_t^i.$$

Since  $\nabla_{\nabla \phi_i}(\nabla f)$  is not a vector field of gradient type, the last term in above equality really is

$$\sum_{i=0}^N \left( \int_M \langle \Pi_{\mu_t}(\nabla_{\nabla \phi_i}(\nabla f)), \nabla \Psi_t \rangle \mu_t(dx) \right) \circ dB_t^i.$$

**Proposition 5.2.** For any  $f \in C^3(M)$ , set

$$R_t^f = \sum_{i=1}^N \Pi_{\mu_t} \left( \nabla_{\nabla \phi_i} \Pi_{\mu_t}(\nabla_{\nabla \phi_i}(\nabla f)) \right), \quad (5.4)$$

$$S_t = \sum_{i=1}^N \Pi_{\mu_t} \left( (\Delta_{\mu_t} \phi_i) \Pi_{\mu_t}^\perp((\nabla_{\nabla \phi_i}(\nabla f))) \right). \quad (5.5)$$

Then the stochastic counterpart of (4.9) has the following form

$$\begin{aligned} \int_M \langle \nabla f, \nabla \Psi_t \rangle \mu_t(dx) &= \int_M \langle \nabla f, \nabla \Psi_0 \rangle \mu(dx) + \sum_{i=0}^N \int_0^t \left( \int_M \langle \nabla_{\nabla \phi_i}(\nabla f), \nabla \Psi_s \rangle \mu_s(dx) \right) dB_s^i \\ &\quad + \frac{1}{2} \int_0^t \left( \int_M \langle R_s^f - S_s^f, \nabla \Psi_s \rangle \mu_s(dx) \right) ds, \end{aligned}$$

or intrinsically

$$\begin{aligned} \langle V_f, V_{\Psi_t} \rangle_{\bar{\mathbb{T}}_t} &= \langle V_f, V_{\Psi_0} \rangle_{\bar{\mathbb{T}}_\mu} + \sum_{i=0}^N \int_0^t \langle \bar{\nabla}_{V_{\phi_i}} V_f, V_{\Psi_s} \rangle_{\bar{\mathbb{T}}_{\mu_s}} dB_s^i \\ &\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t \langle \bar{\nabla}_{V_{\phi_i}} \bar{\nabla}_{V_{\phi_i}} V_f, V_{\Psi_s} \rangle_{\bar{\mathbb{T}}_{\mu_s}} ds - \frac{1}{2} \int_0^t \langle S_s^f, V_{\Psi_s} \rangle_{\bar{\mathbb{T}}_{\mu_s}} ds. \end{aligned}$$

*Proof.* We only remark that the right hand of (4.10) is the sum of two terms, the first term involving variations of orthogonal projection along the time  $t$ , while the second one provides the term  $R_t^f$ .  $\square$

Now we are going to see what happens in the case of  $\mathbb{P}_{2,\infty}(\mathbb{T})$ . For simplicity, we consider SDE on  $\mathbb{T}$ ,

$$dX_t = \partial_x \phi_t(X_t) \circ dB_t, \quad (5.6)$$

where  $B_t$  is a one-dimensional Brownian motion.

Let  $d\mu = \rho dx$  and  $\mu_t = (X_t)_\# \mu$ ; set  $\rho_t = \frac{d\mu_t}{dx}$  the density with respect to  $dx$ . Suppose that  $\{\partial_x \Psi_t; t \in [0, 1]\}$  is a solution to the equation of parallel translations:

$$d_t \partial_x \Psi_t = -\Pi_{\rho_t}(\partial_x^2 \Psi_t \partial_x \phi_t) dB_t + \left( \frac{1}{2} R_t^{\Psi_t} + \frac{1}{2} S_t^{\Psi_t} \right) dt. \quad (5.7)$$

We first explicit  $R_t$  and  $S_t$  in this special case. Using the expression of  $\Pi_\rho$  (see (4.12)), we have

$$\begin{aligned} R_t^{\Psi_t} &= \partial_x(\partial_x^2 \Psi_t \partial_x \phi_t) \partial_x \phi_t - \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \partial_x \hat{\rho}_t \partial_x \phi_t \\ &\quad - \left( \int_{\mathbb{T}} \partial_x(\partial_x^2 \Psi_t \partial_x \phi_t) \partial_x \phi_t dx \right) \hat{\rho}_t + \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \left( \int_{\mathbb{T}} \partial_x \hat{\rho}_t \partial_x \phi_t dx \right) \hat{\rho}_t \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

respectively, and

$$\begin{aligned} S_t^{\Psi_t} &= (\partial_x^2 \phi_t + \partial_x \log(\rho_t) \partial_x \phi_t) \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \hat{\rho}_t \\ &\quad - \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \left( \int_{\mathbb{T}} \partial_x^2 \phi_t \hat{\rho}_t dx \right) \hat{\rho}_t \\ &\quad - \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \left( \int_{\mathbb{T}} \partial_x \log(\rho_t) \partial_x \phi_t \hat{\rho}_t dx \right) \hat{\rho}_t \\ &= J_1 + J_2 + J_3 + J_4 \end{aligned}$$

respectively. We have

$$I_2 + J_2 = -2 \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \partial_x \phi_t \partial_x \hat{\rho}_t,$$

and

$$I_4 + J_4 = 2 \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \left( \int_{\mathbb{T}} \partial_x \phi_t \partial_x \hat{\rho}_t dx \right) \hat{\rho}_t.$$

Therefore we get the following expression for  $R_t^{\Psi_t} + S_t^{\Psi_t}$ :

$$\begin{aligned} R_t^{\Psi_t} + S_t^{\Psi_t} &= \partial_x(\partial_x^2 \Psi_t \partial_x \phi_t) \partial_x \phi_t - \left( \int_{\mathbb{T}} \partial_x(\partial_x^2 \Psi_t \partial_x \phi_t) \partial_x \phi_t dx \right) \hat{\rho}_t \\ &\quad + \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \partial_x^2 \phi_t \hat{\rho}_t - \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \left( \int_{\mathbb{T}} \partial_x^2 \phi_t \hat{\rho}_t dx \right) \hat{\rho}_t \\ &\quad - 2 \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \partial_x \phi_t \partial_x \hat{\rho}_t + 2 \left( \int_{\mathbb{T}} \partial_x^2 \Psi_t \partial_x \phi_t dx \right) \left( \int_{\mathbb{T}} \partial_x \phi_t \partial_x \hat{\rho}_t dx \right) \hat{\rho}_t. \end{aligned} \tag{5.8}$$

Let  $f_t = \partial \Psi_t(X_t)$ . Then by Kunita-Itô-Wentzell formula, we get

$$\begin{aligned} d_t f_t &= - \left( \int_{\mathbb{T}} \partial_x \Psi_t \partial_x^2 \phi_t dx \right) \hat{\rho}_t(X_t) dB_t - \frac{1}{2} \left( \int_{\mathbb{T}} \partial_x \Psi_t \partial_x^2 \phi_t dx \right) (\partial_x^2 \phi_t)(X_t) \hat{\rho}_t(X_t) dt \\ &\quad - \frac{1}{2} \left( \int_{\mathbb{T}} \partial_x \Psi_t \partial_x(\partial_x^2 \phi_t \partial_x \phi_t) dx \right) \hat{\rho}_t(X_t) dt + \frac{3}{2} \left( \int_{\mathbb{T}} \partial_x \Psi_t \partial_x^2 \phi_t dx \right) \left( \int_{\mathbb{T}} \partial_x^2 \phi_t \hat{\rho}_t dx \right) \hat{\rho}_t(X_t) dt. \end{aligned}$$

As in Section 4, we remark that

$$\int_{\mathbb{T}} \partial_x \Psi_t \partial_x^2 \phi_t dx = \int_{\mathbb{T}} f_t \times \frac{\partial_x^2 \phi_t(X_t)}{\rho_t} \rho dx,$$

and also

$$\int_{\mathbb{T}} \partial_x \Psi_t \partial_x(\partial_x^2 \phi_t \partial_x \phi_t) dx = \int_{\mathbb{T}} f_t \times \frac{\partial_x(\partial_x^2 \phi_t \partial_x \phi_t)}{\rho_t}(X_t) \rho dx.$$

We introduce two notations

$$a_t = \frac{\partial_x^2 \phi_t}{\rho_t}(X_t), \quad b_t = \frac{\partial_x(\partial_x^2 \phi_t \partial_x \phi_t)}{\rho_t}(X_t). \quad (5.9)$$

Then  $\{f_t; t \in [0, 1]\}$  satisfies the following equation

$$\begin{aligned} d_t f_t = & - \left( \int_{\mathbb{T}} f_t a_t \rho dx \right) \hat{\rho}_t(X_t) dB_t - \frac{1}{2} \left( \int_{\mathbb{T}} f_t a_t \rho dx \right) (\hat{\rho}_t \partial_x^2 \phi_t)(X_t) dt \\ & - \frac{1}{2} \left( \int_{\mathbb{T}} f_t b_t \rho dx \right) \hat{\rho}_t(X_t) dt + \frac{3}{2} \left( \int_{\mathbb{T}} f_t a_t \rho dx \right) \left( \int_{\mathbb{T}} \partial_x^2 \phi_t \hat{\rho}_t dx \right) \hat{\rho}_t(X_t) dt. \end{aligned} \quad (5.10)$$

Let  $\Lambda(t, f) = - \left( \int_{\mathbb{T}} f a_t \rho dx \right) \hat{\rho}_t(X_t)$  and

$$\begin{aligned} \Theta(t, f) = & - \frac{1}{2} \left( \int_{\mathbb{T}} f_t a_t \rho dx \right) (\hat{\rho}_t \partial_x^2 \phi_t)(X_t) - \frac{1}{2} \left( \int_{\mathbb{T}} f_t b_t \rho dx \right) \hat{\rho}_t(X_t) \\ & + \frac{3}{2} \left( \int_{\mathbb{T}} f_t a_t \rho dx \right) \left( \int_{\mathbb{T}} \partial_x^2 \phi_t \hat{\rho}_t dx \right) \hat{\rho}_t(X_t). \end{aligned}$$

We put  $d_t f_t$  in the form

$$d_t f_t = \Lambda(t, f_t) dB_t + \Theta(t, f_t) dt.$$

**Lemma 5.3.** *We have, for any  $t \in [0, 1]$  and  $f \in L^2(\rho dx)$ ,*

$$\|\Lambda(t, f)\|_{L^2(\rho dx)} \leq \left( \sup_{t \in [0, 1]} \|\partial_x^2 \phi_t\|_{\infty} \right) \|f\|_{L^2(\rho dx)}, \quad (5.11)$$

$$\|\Theta(t, f)\|_{L^2(\rho dx)} \leq \left( 2 \sup_{t \in [0, 1]} \|\partial_x^2 \phi_t\|_{\infty}^2 + \sup_{t \in [0, 1]} \|\partial_x(\partial_x^2 \phi_t \partial_x \phi_t)\|_{\infty} \right) \|f\|_{L^2(\rho dx)}. \quad (5.12)$$

*Proof.* We proceed in the same way as in the proof of Lemma 4.4.  $\square$

By standard Picard iteration or by SDE on Hilbert spaces, finally we get the following result.

**Theorem 5.4.** *There is a unique solution  $\{f_t; t \in [0, 1]\}$  to Equation (5.10).*

Define  $g_t = f_t(X_t^{-1})$ . Contrary to ODE, we have no SDE directly expressing  $X_t^{-1}$ .

The following result will be used several times in the sequel.

**Lemma 5.5.** *Let  $d\mu = \rho dx$  be a probability measure on a compact Riemannian manifold  $M$  such that  $\rho > 0$  and  $\Phi : M \rightarrow M$  a diffeomorphism. Set*

$$\rho_{\Phi} = \frac{d\Phi_{\#}(\rho dx)}{dx}, \quad \tilde{K} = \frac{d(\Phi^{-1})_{\#}(dx)}{dx},$$

then  $\rho_{\Phi}(\Phi)\tilde{K} = \rho$ .

*Proof.* Let  $f \in C(M)$ , we have

$$\int_M f \rho dx = \int_M f(\Phi^{-1}(\Phi)) \rho dx = \int_M f(\Phi^{-1}) \rho_{\Phi} dx = \int_M f \rho_{\Phi}(\Phi) \tilde{K} dx,$$

the result follows.  $\square$

**Proposition 5.6.** *Suppose that  $\int_{\mathbb{T}} g_0(x) dx = 0$ , then for any  $t \in [0, 1]$ ,  $\int_{\mathbb{T}} g_t(x) dx = 0$ .*

*Proof.* Let  $\tilde{K}_t = \frac{d(X_t^{-1})_{\#}(dx)}{dx}$ ; then by Kunita [19] (see also [13]), we have the following explicit formula:

$$\tilde{K}_t = \exp\left(\int_0^t (\partial_x^2 \phi_s)(X_s) \circ dB_s\right).$$

Using  $\tilde{K}_t$ ,  $\int_{\mathbb{T}} g_t(x) dx = \int_{\mathbb{T}} f_t \tilde{K}_t dx$ . Remark that all of drift terms in  $f_t$  came from Itô's stochastic contraction; therefore in Stratanovich form

$$\circ dt f_t = -\left(\int_{\mathbb{T}} f_t a_t \rho dx\right) \hat{\rho}_t(X_t) \circ dB_t. \quad (5.13)$$

Now by Itô's formula,

$$\circ dt (f_t \tilde{K}_t) = -\left(\int_{\mathbb{T}} f_t a_t \rho dx\right) \hat{\rho}_t(X_t) \tilde{K}_t \circ dB_t + f_t \partial_x^2 \phi_t(X_t) \tilde{K}_t \circ dB_t. \quad (5.14)$$

Note that  $\int_{\mathbb{T}} \hat{\rho}_t(X_t) \tilde{K}_t dx = \int_{\mathbb{T}} \hat{\rho}_t(x) dx = 1$ . On the other hand,

$$\int_{\mathbb{T}} f_t \partial_x^2 \phi_t(X_t) \tilde{K}_t dx = \int_{\mathbb{T}} f_t \times \frac{\partial_x^2 \phi_t}{\rho_t}(X_t) \rho_t(X_t) \tilde{K}_t dx = \int_{\mathbb{T}} f_t a_t \rho dx,$$

the last equality being due to  $\rho_t(X_t) \tilde{K}_t = \rho$  by Lemma 5.5. Now we get  $\circ dt \int_{\mathbb{T}} f_t \tilde{K}_t dx = 0$  using (5.14). Therefore  $\int_{\mathbb{T}} g_t dx = \int_{\mathbb{T}} g_0 dx = 0$ .  $\square$

**Theorem 5.7.** *We have, for any  $t \in [0, 1]$ ,*

$$\int_{\mathbb{T}} |g_t(x)|^2 \rho_t(x) dx = \int_{\mathbb{T}} |g_0(x)|^2 \rho dx. \quad (5.15)$$

*Proof.* By (5.13),

$$\begin{aligned} \circ dt \int_{\mathbb{T}} f_t^2 \rho dx &= -2 \left[ \left( \int_{\mathbb{T}} f_t a_t \rho dx \right) \hat{\rho}_t(X_t) f_t \rho dx \right] \circ dB_t \\ &= -2 \left[ \left( \int_{\mathbb{T}} f_t a_t \rho dx \right) \left( \int_{\mathbb{T}} \hat{\rho}_t(X_t) f_t(x) dx \right) \right] \circ dB_t, \end{aligned}$$

But we have seen that

$$\int_{\mathbb{T}} \hat{\rho}_t(X_t) f_t(x) dx = \left( \int_{\mathbb{T}} g_t(x) dx \right) \left( \int_{\mathbb{T}} \frac{dx}{\rho_t} \right)^{-1}$$

which is equal to 0 by Proposition 5.6.  $\square$

Combining all above results, finally we get

**Theorem 5.8.** *Let  $\partial_x \Psi_t = g_t$ . Then for  $\mu = \rho dx$  and  $\mu_t = (X_t)_{\#}(\rho dx)$ ,  $\{\partial_x \Psi_t; t \in [0, 1]\}$  is the parallel translation along the stochastic regular curve  $\{\mu_t; t \in [0, 1]\}$ , that is,  $\partial_x \Psi_t \in \bar{\mathbf{T}}_{\mu_t}$  and*

$$\int_{\mathbb{T}} |\partial_x \Psi_t|^2 \mu_t(dx) = \int_{\mathbb{T}} |\partial_x \Psi_0|^2 \rho dx, \quad t \in [0, 1].$$

Actually, it is well-known that some quasi-invariant non-degenerated diffusion processes have been constructed in  $\mathbb{P}_2(\mathbb{T})$ , see for example [31, 36]. It seems that these diffusion processes do not charge the subspace  $\mathbb{P}_{2,\infty}(\mathbb{T})$ . In what follows, we construct a non-degenerated diffusion process  $\{\mu_t; t \in [0, 1]\}$  on  $\mathbb{P}_{2,\infty}(\mathbb{T})$  and parallel translations along it.

For  $k \in \mathbb{N}^*$ , set

$$\phi_{2k-1}(x) = \frac{\sin(kx)}{k}, \quad \phi_{2k}(x) = -\frac{\cos(kx)}{k}, \quad \text{and} \quad a_k = k^q.$$

For an integer  $N \geq 2$ , we consider the SDE on  $\mathbb{T}$ ,

$$dX_t^N = \sum_{k=1}^N \frac{1}{\alpha_k} \left( \partial_x \phi_{2k-1}(X_t^N) \circ dB_{2k-1}(t) + \partial_x \phi_{2k}(X_t^N) \circ dB_{2k}(t) \right).$$

Note that  $d_t \partial_x \phi_{2k-1}(X_t^N) \cdot dB_{2k-1} + d_t \partial_x \phi_{2k}(X_t^N) \cdot dB_{2k} = 0$ ; therefore the above Stratanovich SDE becomes the below Itô SDE:

$$dX_t^N = \sum_{k=1}^N \frac{1}{\alpha_k} \left( \partial_x \phi_{2k-1}(X_t^N) dB_{2k-1}(t) + \partial_x \phi_{2k}(X_t^N) dB_{2k}(t) \right). \quad (5.16)$$

It is well-known (see [11, 17, 19, 27], especially in [3, 12]) that for  $q > 2$ , almost surely, as  $N \rightarrow +\infty$ ,  $X_t^N$  converges in  $C([0, 1], \text{Diff}(\mathbb{T}))$  to  $X_t$ , which solves the following SDE:

$$dX_t = \sum_{k=1}^{\infty} \frac{1}{\alpha_k} \left( \partial_x \phi_{2k-1}(X_t) dB_{2k-1}(t) + \partial_x \phi_{2k}(X_t) dB_{2k}(t) \right). \quad (5.17)$$

Let  $d\mu = \rho dx \in \mathbb{P}_{2,\infty}(\mathbb{T})$  be given; for any  $N > 2$ , we denote by  $\rho_t^N$  the density of the measure  $(X_t^N)_\#(\mu)$  with respect to  $dx$ . It is obvious that almost surely,  $\rho_t^N$  converges to the density  $\rho_t$  of  $(X_t)_\#(\mu)$  uniformly in  $(t, x) \in [0, 1] \times \mathbb{T}$ . It is quite automatic that results in Theorem 5.8 remain valid for SDE (5.16). More precisely, for any  $\partial_x \Psi_0$  given in  $L^2(\rho dx)$ , there exists the parallel translation  $\{\partial_x \Psi_t^N; t \in [0, 1]\}$  along  $\{\mu_t^N; t \in [0, 1]\}$ , that is,  $\partial_x \Psi_t \in \mathbf{T}_{\mu_t^N}$  and

$$\int_{\mathbb{T}} |\partial_x \Psi_t^N|^2 \rho_t^N dx = \int_{\mathbb{T}} |\partial_x \Psi_0|^2 \rho dx.$$

For simplicity, we again use the notation  $g_t^N = \partial_x \Psi_t^N$  and  $f_t^N = g_t^N(X_t^N)$ . We introduce

$$a_k^N(t) = \left( \frac{\partial_x^2 \phi_k}{\rho_t^N} \right)(X_t^N), \quad b_k^N(t) = \left( \frac{\partial_x (\partial_x^2 \phi_k \partial_x \phi_k)}{\rho_t^N} \right)(X_t^N),$$

and

$$\begin{aligned} \Lambda_k^N(t, f) &= - \left( \int_{\mathbb{T}} f a_k^N(t) \rho dx \right) \hat{\rho}_t^N(X_t^N), \\ \Theta_k^N(t, f) &= - \left( \int_{\mathbb{T}} f a_k^N(t) \rho dx \right) (\hat{\rho}_t^N \partial_x^2 \phi_k)(X_t^N) - \left( \int_{\mathbb{T}} f b_k^N(t) \rho dx \right) \hat{\rho}_t^N(X_t^N) \\ &\quad + 3 \left( \int_{\mathbb{T}} f a_k^N(t) \rho dx \right) \left( \int_{\mathbb{T}} \partial_x^2 \phi_k \hat{\rho}_t^N dx \right) \hat{\rho}_t^N(X_t^N). \end{aligned}$$

Then by (5.10),  $f_t^N$  satisfies the following SDE

$$d_t f_t^N = \sum_{k=1}^{2N} \frac{1}{\alpha_k} \Lambda_k^N(t, f_t^N) dB_t^k + \sum_{k=1}^{2N} \frac{1}{2\alpha_k^2} \Theta_k^N(t, f_t^N) dt, \quad (5.18)$$

with  $\alpha_k = \left[\frac{k+1}{2}\right]^q$ . In the sequel, we will use the notation:  $\xi(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s}$  for  $s > 1$ .

**Theorem 5.9.** *Let  $q > \frac{5}{2}$ ; then almost surely, as  $N \rightarrow +\infty$ ,  $f^N$  converges in  $C([0, 1], L^2(\mathbb{T}, \rho dx))$ . More precisely, there exists  $f \in C([0, 1], L^2(\mathbb{T}, \rho dx))$  such that*

$$\lim_{N \rightarrow +\infty} \sup_{t \in [0, 1]} \int_{\mathbb{T}} |f_t^N - f_t|^2 \rho dx = 0. \quad (5.19)$$

*Proof.* We first remark that

$$\|\partial_x^2 \phi_k\|_{\infty} \leq k, \quad \|\partial_x(\partial_x^2 \phi_k \partial_x \phi_k)\|_{\infty} \leq k^2.$$

Then, by Lemma 5.3, for any  $N \geq 2$ ,

$$\|\Lambda_k^N(t, f)\|_{L^2(\rho dx)} \leq k \|f\|_{L^2(\rho dx)}, \quad \|\Theta_k^N(t, f)\|_{L^2(\rho dx)} \leq 3k^2 \|f\|_{L^2(\rho dx)}. \quad (5.20)$$

Using (5.20), we have,

$$\sum_{k=1}^{+\infty} \left\| \frac{1}{\alpha_k} \Lambda_k^N(t, f) \right\|_{L^2(\rho dx)}^2 \leq \xi(2q-2) \|f\|_{L^2(\rho dx)}^2,$$

as well as

$$\sum_{k=1}^{+\infty} \left\| \frac{1}{\alpha_k^2} \Theta_k^N(t, f) \right\|_{L^2(\rho dx)} \leq 3\xi(2q-2) \|f\|_{L^2(\rho dx)}.$$

Combining these estimates, together with SDE (5.18), we get that, for any  $p \geq 1$ ,

$$D_p = \sup_N \sup_{t \in [0, 1]} \mathbb{E} \left( \|f_t^N\|_{L^2(\rho dx)}^p \right) < +\infty. \quad (5.21)$$

Now let  $N' > N \geq 2$ , we have

$$\begin{aligned} d_t(f_t^{N'} - f_t^N) &= \sum_{k=1}^{2N} \frac{1}{\alpha_k} \left( \Lambda_k^{N'}(t, f_t^{N'}) - \Lambda_k^N(t, f_t^N) \right) dB_t^k \\ &\quad + \sum_{k=1}^{2N} \frac{1}{2\alpha_k^2} \left( \Theta_k^{N'}(t, f_t^{N'}) - \Theta_k^N(t, f_t^N) \right) dt \\ &\quad + \sum_{k=2N+1}^{2N'} \frac{1}{\alpha_k} \Lambda_k^{N'}(t, f_t^{N'}) dB_t^k + \sum_{k=2N+1}^{2N'} \frac{1}{2\alpha_k^2} \Theta_k^{N'}(t, f_t^{N'}) dt \\ &= dI_1(t) + dI_2(t) + dI_3(t) + dI_4(t) \quad \text{respectively.} \end{aligned}$$

By the Bürkhölder inequality,

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|I_3(s)\|_{L^2(\rho dx)}^2 \right) \leq 4 \sum_{k > 2N} \mathbb{E} \left( \int_0^t \frac{1}{\alpha_k^2} \|\Lambda_k^{N'}(s, f_s^{N'})\|_{L^2(\rho dx)}^2 ds \right),$$

which is dominated, using (5.20),

$$4 \sum_{k > 2N} \frac{k^2}{k^{2q}} \mathbb{E} \left( \int_0^t \|f_s^{N'}\|_{L^2(\rho dx)}^2 ds \right) \leq \frac{4}{(2N)^{2q-3}} \mathbb{E} \left( \int_0^1 \|f_s^{N'}\|_{L^2(\rho dx)}^2 ds \right).$$

According to (5.21), we get

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|I_3(s)\|_{L^2(\rho dx)}^2 \right) \leq \frac{4D_1}{(2N)^{2q-3}}. \quad (5.22)$$

Secondly, according to second estimate in (5.20), we have

$$\begin{aligned} \|I_4(t)\|_{L^2(\rho dx)} &\leq \sum_{k=2N+1}^{2N'} \frac{1}{2\alpha_k^2} \int_0^t \|\Theta_k^{N'}(s, f_s^{N'})\|_{L^2(\rho dx)} ds \\ &\leq \sum_{k=2N+1}^{2N'} \frac{3k^2}{2k^{2q}} \int_0^t \|f_s^{N'}\|_{L^2(\rho dx)} ds \leq \frac{6}{(2N)^{2q-3}} \int_0^t \|f_s^{N'}\|_{L^2(\rho dx)} ds. \end{aligned}$$

It follows that, for  $N$  big enough,

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|I_4(s)\|_{L^2(\rho dx)}^2 \right) \leq \left( \frac{6}{(2N)^{2q-3}} \right)^2 D_1 \leq \frac{4D_1}{(2N)^{2q-3}}. \quad (5.23)$$

Estimating  $I_1(t)$  and  $I_2(t)$  will make appear Gronwall type inequality for

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|f_s^{N'} - f_s^N\|_{L^2(\rho dx)}^2 \right),$$

which will yield desired result.

Again by B urkholder inequality, we have

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|I_1(s)\|_{L^2(\rho dx)}^2 \right) \leq \sum_{k=1}^{2N} \frac{4}{\alpha_k^2} \mathbb{E} \left( \int_0^t \|\Lambda_k^{N'}(s, f_s^{N'}) - \Lambda_k^N(s, f_s^N)\|_{L^2(\rho dx)}^2 ds \right). \quad (5.24)$$

By expression of  $\Lambda_k$ , we write down

$$\begin{aligned} \Lambda_k^{N'}(s, f_s^{N'}) - \Lambda_k^N(s, f_s^N) &= \left[ - \int_{\mathbb{T}} f_s^{N'} a_k^{N'}(s) \rho dx + \int_{\mathbb{T}} f_s^N a_k^N(s) \rho dx \right] \hat{\rho}_s^{N'}(X_s^{N'}) \\ &\quad + \left( \int_{\mathbb{T}} f_s^N a_k^N(s) \rho dx \right) \left( \hat{\rho}_s^N(X_s^N) - \hat{\rho}_s^{N'}(X_s^{N'}) \right) \\ &= J_1(s) + J_2(s) \quad \text{respectively.} \end{aligned} \quad (5.25)$$

As for getting estimates in (5.20), we have

$$\left\| \left( \int_{\mathbb{T}} (f_s^N - f_s^{N'}) a_k^{N'}(s) \rho dx \right) \hat{\rho}_s^{N'}(X_s^{N'}) \right\|_{L^2(\rho dx)} \leq k \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}. \quad (5.26)$$

Hence  $\|\Lambda_k^{N'}(s, f_s^{N'}) - \Lambda_k^N(s, f_s^N)\|_{L^2(\rho dx)}^2$  is dominated by the sum of two terms in the following way

$$2k^2 \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 + o_k(s, N, N'),$$

so that the right hand side of (5.24) has the following upper bound

$$\sum_{k=1}^{2N} \frac{8k^2}{k^{2q}} \int_0^t \mathbb{E} \left( \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 \right) ds + \sum_{k=1}^{2N} \frac{4}{k^{2q}} \int_0^t \mathbb{E} (o_k(s, N, N')) ds.$$



Therefore the right hand side of (5.24) is dominated by

$$8\xi(2q-2) \int_0^t \mathbb{E} \left( \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 \right) ds + o(N).$$

We have in fact the following inequality

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|I_1(s)\|_{L^2(\rho dx)}^2 \right) \leq 8\xi(2q-2) \int_0^t \mathbb{E} \left( \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 \right) ds + \frac{C}{N^{q-\frac{1}{2}}}. \quad (5.27)$$

Since the detail of the proof of (5.27) is lengthy, we will do it in the following proposition. In the same way, there is a constant  $C_2 > 0$  such that

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|I_2(s)\|_{L^2(\rho dx)}^2 \right) \leq C_2\xi(2q-2) \int_0^t \mathbb{E} \left( \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 \right) ds + \frac{C}{N^{q-\frac{1}{2}}}. \quad (5.28)$$

Also the proof of (5.28) is postponed in the following proposition. For  $q > \frac{5}{2}$ ,  $2q-3 > q-\frac{1}{2}$ , combining (5.22), (5.23), (5.27) and (5.28), we finally get

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 \right) \leq C_1 \int_0^t \mathbb{E} \left( \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 \right) ds + C_2 N^{-(q-\frac{1}{2})}.$$

By Gronwall's lemma, there is a constant  $C > 0$  such that, for any  $N' \geq N$ ,

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}^2 \right) \leq C N^{-(q-\frac{1}{2})} e^{C_1 t}.$$

It follows that  $\{f^N; N \geq 2\}$  is a Cauchy sequence in  $L^2(\Omega, C([0, 1], L^2(\rho dx)))$ ; then there exists  $f \in L^2(\Omega, C([0, 1], L^2(\rho dx)))$  such that

$$\mathbb{E} \left( \sup_{s \in [0, 1]} \|f_s^N - f_s\|_{L^2(\rho dx)}^2 \right) \leq C N^{-(q-\frac{1}{2})}. \quad (5.29)$$

This estimate is useful in order to obtain the almost surely convergence. Let  $\beta \in (0, 1/2)$ , we put

$$\Omega_N = \left\{ \omega; \sup_{t \in [0, 1]} \|f_t^N - f_t\|_{L^2(\rho dx)} \geq N^{-\beta} \right\}.$$

By (5.29),  $\mathbf{P}(\Omega_N) \leq C N^{-(q-(1/2)-2\beta)}$ . For  $q > \frac{5}{2}$  and  $\beta \in (0, 1/2)$ , the series  $\sum_{N>1} N^{-(q-(1/2)-2\beta)}$  converges; therefore the Borel-Cantelli lemma yields the almost sure convergence with a convergence rate  $N^{-\beta}$ .  $\square$

**Proposition 5.10.** *Above inequality (5.27) as well as inequality (5.28) hold true.*

*Proof.* Recall the expression of  $J_1(s)$  in (5.25):

$$J_1(s) = \left[ - \int_{\mathbb{T}} f_s^{N'} a_k^{N'}(s) \rho dx + \int_{\mathbb{T}} f_s^N a_k^N(s) \rho dx \right] \hat{\rho}_s^{N'}(X_s^{N'}).$$

Put

$$J_{11}(s) = \left[ \int_{\mathbb{T}} (f_s^N - f_s^{N'}) a_k^{N'}(s) \rho dx \right] \hat{\rho}_s^{N'}(X_s^{N'}),$$

and

$$J_{12}(s) = \left[ \int_{\mathbb{T}} f_s^N (a_k^N(s) - a_k^{N'}(s)) \rho dx \right] \hat{\rho}_s^{N'}(X_s^{N'}).$$

Then  $J_1(s) = J_{11}(s) + J_{12}(s)$ . Obviously

$$\|J_{11}(s)\|_{L^2(\rho dx)} \leq k \|f_s^N - f_s^{N'}\|_{L^2(\rho dx)}. \quad (5.30)$$

Remark that for any strictly positive probability density  $\sigma$ ,  $\int_{\mathbb{T}} \frac{dx}{\sigma} \geq 1$ , since

$$1 = \int_{\mathbb{T}} \sqrt{\sigma} \cdot \frac{1}{\sqrt{\sigma}} dx \leq \left( \int_{\mathbb{T}} \frac{dx}{\sigma} \right)^{1/2}.$$

Therefore

$$\int_{\mathbb{T}} (\hat{\rho}_s^N)^2(X_s^{N'}) \rho dx = 1 / \int_{\mathbb{T}} \frac{dx}{\rho_s^{N'}} \leq 1.$$

So by expression of  $J_{12}(s)$  and Cauchy-Schwarz inequality, we get

$$\int_{\mathbb{T}} |J_{12}(s)|^2 \rho dx \leq \|f_s^N\|_{L^2(\rho dx)}^2 \int_{\mathbb{T}} (a_k^N(s) - a_k^{N'}(s))^2 \rho dx. \quad (5.31)$$

Now

$$a_k^N(s) - a_k^{N'}(s) = \frac{\partial_x^2 \phi_k(X_s^N) - \partial_x^2 \phi_k(X_s^{N'})}{\rho_s^{N'}(X_s^{N'})} + \partial_x^2 \phi_k(X_s^N) \left( \frac{1}{\rho_s^N(X_s^N)} - \frac{1}{\rho_s^{N'}(X_s^{N'})} \right).$$

Remark that

$$\left| \frac{\partial_x^2 \phi_k(X_s^N) - \partial_x^2 \phi_k(X_s^{N'})}{\rho_s^{N'}(X_s^{N'})} \right| \leq k^2 \frac{|X_s^N - X_s^{N'}|}{\rho_s^{N'}(X_s^{N'})}.$$

Put  $V_k(s) = \int_{\mathbb{T}} (a_k^N(s) - a_k^{N'}(s))^2 \rho dx$ , then, according to above two relations,

$$V_k(s) \leq 2 \left[ k^4 \int_{\mathbb{T}} \frac{|X_s^N - X_s^{N'}|^2}{(\rho_s^{N'}(X_s^{N'}))^2} \rho dx + k^2 \int_{\mathbb{T}} \left( \frac{1}{\rho_s^N(X_s^N)} - \frac{1}{\rho_s^{N'}(X_s^{N'})} \right)^2 \rho dx \right].$$

Putting

$$V_{k1}(s) = \int_{\mathbb{T}} \frac{|X_s^N - X_s^{N'}|^2}{(\rho_s^{N'}(X_s^{N'}))^2} \rho dx, \quad V_{k2}(s) = \int_{\mathbb{T}} \left( \frac{1}{\rho_s^N(X_s^N)} - \frac{1}{\rho_s^{N'}(X_s^{N'})} \right)^2 \rho dx,$$

then  $V_k(s) \leq 2k^4 V_{k1}(s) + 2k^2 V_{k2}(s)$ . Using (5.31), we write down

$$\int_{\mathbb{T}} |J_{12}(s)|^2 \rho dx \leq 2k^4 \|f_s^N\|_{L^2(\rho dx)}^2 V_{k1}(s) + 2k^2 \|f_s^N\|_{L^2(\rho dx)}^2 V_{k2}(s). \quad (5.32)$$

Let  $r > 1$  and  $\tilde{r} > 1$  such that  $\frac{1}{r} + \frac{1}{\tilde{r}} = 1$ . By Hölder inequality, we have

$$\int_0^t \mathbb{E} \left( \|f_s^N\|_{L^2}^2 V_{k1}(s) \right) ds \leq \left[ \int_0^t \mathbb{E} \left( \|f_s^N\|_{L^2}^{2\tilde{r}} \right) ds \right]^{1/\tilde{r}} \left[ \int_0^t \mathbb{E} \left( \int_{\mathbb{T}} \frac{|X_s^N - X_s^{N'}|^{2r}}{(\rho_s^{N'}(X_s^{N'}))^{2r}} \rho dx \right) ds \right]^{1/r}.$$

Denote by  $o_N(t)^{1/r}$  the last term in the product in the right hand, again by Hölder inequality, we have

$$o_N(t) \leq \left[ \int_0^t \int_{\mathbb{T}} \mathbb{E} (|X_s^N - X_s^{N'}|^{2r^2}) \rho dx ds \right]^{1/r} \left[ \int_0^t \int_{\mathbb{T}} \mathbb{E} \left( \frac{1}{(\rho_s^{N'}(X_s^{N'}))^{2r\tilde{r}}} \right) \rho dx ds \right]^{1/\tilde{r}}. \quad (5.33)$$

Now using (6.3) in the next section and for  $N' \geq N$ , we have

$$\left( \mathbb{E} \left[ |X_s^N - X_s^{N'}|^{2r^2} \right] \right)^{1/2r^2} \leq 2 \left( \frac{C_{r^2}}{N^{2q-1}} \right)^{1/r^2};$$

therefore for some constant  $C > 0$ , independent of  $\omega, x, t$  such that

$$\left[ \int_0^t \int_{\mathbb{T}} \mathbb{E}(|X_s^N - X_s^{N'}|^{2r^2}) \rho dx ds \right]^{1/r^2} \leq \frac{C}{N^{(2q-1)/r^2}} = \frac{C}{N^{q-\frac{1}{2}}}, \quad (5.34)$$

if we take  $r^2 = 2$ . For estimating the second factor on the right hand of (5.33), we introduce  $\tilde{K}_s^N$  the density of  $(X_s^N)_{\#}^{-1}(dx)$  relative to  $dx$ . By Lemma 5.5,  $\rho_s^{N'}(X_s^{N'}) \tilde{K}_s^N = \rho$ , which implies that

$$\frac{1}{(\rho_s^{N'}(X_s^{N'}))^{2r\tilde{r}}} = \frac{(\tilde{K}_s^N)^{2r\tilde{r}}}{\rho^{2r\tilde{r}}}.$$

Since  $0 < \delta_1 \leq \rho \leq \delta_2$  and using (6.7) in Appendice, we get

$$\sup_N \sup_{s \in [0,1]} \mathbb{E} \left( \int_{\mathbb{T}} \frac{(\tilde{K}_s^N)^{2r\tilde{r}}}{\rho^{2r\tilde{r}}} \rho dx \right) < +\infty.$$

Now combining this result with (5.33), (5.34), we get

$$\int_0^t \mathbb{E} \left( \|f_s^N\|_{L^2}^2 V_{k1}(s) \right) ds \leq \frac{Ck^4}{N^{q-\frac{1}{2}}}. \quad (5.35)$$

Again by Lemma 5.5, we have  $V_{k2}(s) = \int_{\mathbb{T}} (\tilde{K}_s^N - \tilde{K}_s^{N'})^2 \rho^{-1} dx$ , so that

$$\int_0^t \mathbb{E} \left( \|f_s^N\|_{L^2}^2 V_{k2}(s) \right) ds \leq \left( \int_0^t \mathbb{E} \left[ \|f_s^N\|_{L^2}^{2\tilde{r}} \right] ds \right)^{1/\tilde{r}} \left( \int_0^t \int_{\mathbb{T}} \mathbb{E} \left[ |\tilde{K}_s^N - \tilde{K}_s^{N'}|^{2r} \right] \rho^{-r+1} dx ds \right)^{1/r}.$$

By Proposition 6.2 in Appendice, for some  $\delta > 0$  such that  $q - \frac{1}{2} < q - \delta < 2q - 3$ ,

$$\mathbb{E} \left[ |\tilde{K}_s^N - \tilde{K}_s^{N'}|^{2r} \right] \leq \frac{C}{N^{q-\delta}}.$$

For some  $r > 1$  close to 1, we have  $\frac{1}{N^{(q-\delta)/r}} = \frac{1}{N^{q-\frac{1}{2}}}$ . Finally we get

$$\int_0^t \mathbb{E} \left( \|f_s^N\|_{L^2}^2 V_{k2}(s) \right) ds \leq \frac{C}{N^{q-\frac{1}{2}}}. \quad (5.36)$$

Finally combining (5.32), (5.35) and (5.36), we obtain

$$\int_0^t \mathbb{E} \left[ \int_{\mathbb{T}} J_{12}(s)^2 \rho dx \right] ds \leq \frac{Ck^4}{N^{q-\frac{1}{2}}}. \quad (5.37)$$

For estimating  $J_2(s)$ , we remark that

$$\begin{aligned} \hat{\rho}_s^N(X_s^N) - \hat{\rho}_s^{N'}(X_s^{N'}) &= \left( \frac{1}{\rho_s^N(X_s^N)} - \frac{1}{\rho_s^{N'}(X_s^{N'})} \right) \frac{1}{\int_{\mathbb{T}} \frac{dx}{\rho_s^N}} + \frac{1}{\rho_s^{N'}(X_s^{N'})} \left( \frac{1}{\int_{\mathbb{T}} \frac{dx}{\rho_s^N}} - \frac{1}{\int_{\mathbb{T}} \frac{dx}{\rho_s^{N'}}} \right) \\ &= \frac{\tilde{K}_s^N - \tilde{K}_s^{N'}}{\rho \int_{\mathbb{T}} \frac{dx}{\rho_s^N}} + \frac{1}{\rho_s^{N'}(X_s^{N'})} \frac{\int_{\mathbb{T}} \frac{dx}{\rho_s^{N'}} - \int_{\mathbb{T}} \frac{dx}{\rho_s^N}}{\int_{\mathbb{T}} \frac{dx}{\rho_s^{N'}} \int_{\mathbb{T}} \frac{dx}{\rho_s^{N'}}}. \end{aligned}$$

Note that

$$\int_{\mathbb{T}} \frac{dx}{\rho_s^N} = \int_{\mathbb{T}} \frac{\rho dx}{(\rho_s^N(X_s^N))^2} = \int_{\mathbb{T}} \frac{(\tilde{K}_s^N)^2}{\rho} dx;$$

then

$$\int_{\mathbb{T}} \frac{dx}{\rho_s^{N'}} - \int_{\mathbb{T}} \frac{dx}{\rho_s^N} = \int_{\mathbb{T}} \frac{(\tilde{K}_s^{N'})^2 - (\tilde{K}_s^N)^2}{\rho} dx.$$

On the other hand,

$$\int_{\mathbb{T}} (a_k^N(s))^2 \rho dx \leq k^2 \int_{\mathbb{T}} \frac{\rho dx}{(\rho_s^N(X_s^N))^2} = k^2 \int_{\mathbb{T}} \frac{dx}{\rho_s^N},$$

so that

$$\left| \int_{\mathbb{T}} f_s^N a_k^N(s) \rho dx \right| \leq k \|f_s^N\|_{L^2} \left( \int_{\mathbb{T}} \frac{dx}{\rho_s^N} \right)^{1/2}.$$

Recall that

$$J_2(s) = \left( \int_{\mathbb{T}} f_s^N a_k^N(s) \rho dx \right) \left( \hat{\rho}_s^N(X_s^N) - \hat{\rho}_s^{N'}(X_s^{N'}) \right).$$

According to above calculation, we get

$$|J_2(s)| \leq k \|f_s^N\|_{L^2} \frac{|\tilde{K}_s^N - \tilde{K}_s^{N'}|}{\rho} + \frac{k \|f_s^N\|_{L^2}}{\rho_s^{N'}(X_s^{N'})} \left| \int_{\mathbb{T}} \frac{(\tilde{K}_s^{N'})^2 - (\tilde{K}_s^N)^2}{\rho} dx \right|.$$

Proceeding as for estimating  $J_1(s)$ , we get finally

$$\int_0^t \mathbb{E} \left[ |J_2(s)|^2 \rho dx \right] ds \leq \frac{Ck^2}{N^{q-\frac{1}{2}}}. \quad (5.38)$$

Now using (5.30), (5.37) and (5.38) and the fact that the series  $\sum_{k \geq 1} \frac{k^4}{k^{2q}}$  converges for  $q > \frac{5}{2}$ , we get (5.27). Inequality (5.28) can be proved in a similar way.  $\square$

**Definition 5.11.** We define  $g_t = f_t(X_t^{-1})$ .

It is obvious that  $\int_{\mathbb{T}} g_t^2 \rho_t dx < +\infty$ . In what follows, we justify that  $\{g_t; t \in [0, 1]\}$  is the stochastic parallel translation along the curve  $\{(X_t)_\#(\rho dx); t \in [0, 1]\}$ . We have no explicit SDE for  $g_t$ , but  $f_t = g_t(X_t)$  satisfies a SDE. More precisely,  $\{f_t; t \in [0, 1]\}$  is a solution to the following SDE, with  $q > \frac{5}{2}$ ,

$$df_t = \sum_{k=1}^{+\infty} \frac{1}{k^q} \Lambda_k(t, f_t) dB_t^k + \sum_{k=1}^{+\infty} \frac{1}{2k^{2q}} \Theta_k(t, f_t) dt, \quad (5.39)$$

where

$$\begin{aligned} \Lambda_k(t, f) &= - \left( \int_{\mathbb{T}} f a_k(t) \rho dx \right) \hat{\rho}_t(X_t), \\ \Theta_k(t, f) &= - \left( \int_{\mathbb{T}} f a_k(t) \rho dx \right) (\hat{\rho}_t \partial_x^2 \phi_k)(X_t) - \left( \int_{\mathbb{T}} f b_k(t) \rho dx \right) \hat{\rho}_t(X_t) \\ &\quad + 3 \left( \int_{\mathbb{T}} f a_k(t) \rho dx \right) \left( \int_{\mathbb{T}} \partial_x^2 \phi_k \hat{\rho}_t dx \right) \hat{\rho}_t(X_t), \end{aligned}$$

with

$$a_k(t) = \left( \frac{\partial_x^2 \phi_k}{\rho_t} \right)(X_t), \quad b_k(t) = \left( \frac{\partial_x (\partial_x^2 \phi_k \partial_x \phi_k)}{\rho_t} \right)(X_t).$$

**Corollary 5.12.** *Under hypothesis of theorem 5.9, almost surely,*

$$\lim_{N \rightarrow +\infty} \int_{\mathbb{T}} (g_t^N)^2 \rho_t^N dx = \int_{\mathbb{T}} g_t^2 \rho_t dx. \quad (5.40)$$

*Proof.* We have

$$\int_{\mathbb{T}} (g_t^N) \rho_t^N dx = \int_{\mathbb{T}} (f_t^N)^2 \rho dx$$

which converges, by (5.19), to

$$\int_{\mathbb{T}} f_t^2 \rho dx = \int_{\mathbb{T}} g_t^2 \rho_t dx.$$

□

An immediate consequence of above result is

**Corollary 5.13.** *For any  $t \in [0, 1]$ ,*

$$\int_{\mathbb{T}} g_t^2 \rho_t dx = \int_{\mathbb{T}} g_0^2 \rho dx. \quad (5.41)$$

**Theorem 5.14.** *Almost surely, for  $t \in [0, 1]$ ,*

$$\int_{\mathbb{T}} g_t(x) dx = 0. \quad (5.42)$$

*Proof.* Let  $\xi$  be a bounded Random variable, using the uniform (relative to  $N$ ) estimate of the density of  $(X_t^N)^{-1}(\rho dx)$ , it is straightforward (see [13]) to prove that

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left( \xi \int_{\mathbb{T}} f_t^N ((X_t^N)^{-1}) dx \right) = \mathbb{E} \left( \xi \int_{\mathbb{T}} f_t ((X_t)^{-1}) dx \right).$$

But by Proposition 5.6,  $\int_{\mathbb{T}} f_t^N ((X_t^N)^{-1}) dx = \int_{\mathbb{T}} g_t^N(x) dx = 0$ . Then for any bounded  $\xi$ ,

$$\mathbb{E} \left( \xi \int_{\mathbb{T}} f_t ((X_t)^{-1}) dx \right) = 0.$$

The result (5.42) follows. □

## 6 Appendice

First we recall the notations:  $\phi_{2k-1}(x) = \frac{\sin(kx)}{k}$ ,  $\phi_{2k}(x) = -\frac{\cos(kx)}{k}$  and  $\alpha_k = k^q$  for some  $q > 1$ . Let  $(X_t^N)$  and  $(X_t)$  be solutions to SDE:

$$dX_t^N = \sum_{k=1}^N \frac{1}{\alpha_k} \left( \partial_x \phi_{2k-1}(X_t^N) dB_{2k-1}(t) + \partial_x \phi_{2k}(X_t^N) dB_{2k}(t) \right), \quad (6.1)$$

$$dX_t = \sum_{k=1}^{\infty} \frac{1}{\alpha_k} \left( \partial_x \phi_{2k-1}(X_t) dB_{2k-1}(t) + \partial_x \phi_{2k}(X_t) dB_{2k}(t) \right) \quad (6.2)$$

respectively.

**Proposition 6.1.** *Let  $p \geq 1$  be an integer, there exists a constant  $C_p > 0$  independent of  $x$  such that*

$$\mathbb{E} \left[ \left( X_t^N(x) - X_t(x) \right)^{2p} \right] \leq \frac{C_p}{N^{2q-1}}. \quad (6.3)$$

*Proof.* Let  $\eta_t = X_t(x) - X_t^N(x)$ . Then

$$\begin{aligned} d\eta_t &= \sum_{k=1}^N \frac{1}{\alpha_k} \left[ (\cos(kX_t) - \cos(kX_t^N)) dB_{2k-1}(t) + (\sin(kX_t) - \sin(kX_t^N)) dB_{2k}(t) \right] \\ &\quad + \sum_{k>N} \frac{1}{\alpha_k} \left[ \cos(kX_t) dB_{2k-1}(t) + \sin(kX_t) dB_{2k}(t) \right]. \end{aligned}$$

Then Itô stochastic contraction  $d\eta_t \cdot d\eta_t$  admits the expression

$$\sum_{k=1}^N \frac{1}{\alpha_k^2} \left[ (\cos(kX_t) - \cos(kX_t^N))^2 + (\sin(kX_t) - \sin(kX_t^N))^2 \right] + \sum_{k>N} \frac{1}{\alpha_k^2},$$

which is equal to

$$\sum_{k=1}^N \frac{4}{k^{2q}} \sin^2 \left( k \frac{X_t - X_t^N}{2} \right) + \sum_{k>N} \frac{1}{k^{2q}};$$

the first above sum is dominated by

$$\sum_{k=1}^{+\infty} \frac{1}{k^{2q-2}} (X_t - X_t^N)^2,$$

while the second sum has the upper bound  $\frac{1}{N^{2q-1}}$ . Hence there is a constant  $C > 0$  independent of  $x$  such that

$$d\eta_t \cdot d\eta_t \leq \left( C \eta_t^2 + \frac{1}{N^{2q-1}} \right) dt. \quad (6.4)$$

By Itô's formula,  $d\eta_t^{2p} = 2p\eta_t^{2p-1} d\eta_t + 2p(2p-1)\eta_t^{2p-2} d\eta_t \cdot d\eta_t$ , which is dominated, according to (6.4), by

$$2p\eta_t^{2p-1} d\eta_t + 2p(2p-1)C \eta_t^{2p} dt + 2p(2p-1) \frac{\eta_t^{2p-2}}{N^{2q-1}} dt.$$

Note that  $2p(2p-1)\eta_t^{2p-2} \leq 2p(2p-1)(2\pi)^{2p-2}$  that is denoted by  $C'_p$ . Therefore

$$d\eta_t^{2p} \leq 2p\eta_t^{2p-1} d\eta_t + C'_p \eta_t^{2p} dt + \frac{C'_p}{N^{2q-1}} dt.$$

It follows that

$$\mathbb{E}(\eta_t^{2p}) \leq C'_p \int_0^T \mathbb{E}(\eta_s^{2p}) ds + \frac{C'_p t}{N^{2q-1}}.$$

Then Gronwall's lemma yields

$$\mathbb{E}(\eta_t^{2p}) \leq \frac{C'_p t}{N^{2q-1}} e^{C'_p t},$$

that is nothing but (6.3). □

Now we denote by  $\tilde{K}_t^N$  the density of  $(X_t^N)_\#^{-1}(dx)$  with respect to  $dx$  and  $\tilde{K}_t$  the density of  $(X_t^{-1})_\#(dx)$  with respect to  $dx$ . By Kunita formula [19], we have

$$\tilde{K}_t^N = \exp\left(\sum_{k=1}^{2N} \frac{1}{\alpha_k} \int_0^t \partial_x^2 \phi_k(X_s^N) \circ dB_s^k\right),$$

and

$$\tilde{K}_t = \exp\left(\sum_{k=1}^{+\infty} \frac{1}{\alpha_k} \int_0^t \partial_x^2 \phi_k(X_s) \circ dB_s^k\right).$$

First we consider

$$\hat{K}_t^N = \exp\left(\sum_{k=1}^{2N} \frac{1}{\alpha_k} \int_0^t \partial_x^2 \phi_k(X_s^N) dB_s^k\right),$$

and

$$\hat{K}_t = \exp\left(\sum_{k=1}^{+\infty} \frac{1}{\alpha_k} \int_0^t \partial_x^2 \phi_k(X_s) dB_s^k\right).$$

Then they are linked by

$$\tilde{K}_t^N = \hat{K}_t^N \exp\left(-\sum_{k=1}^{2N} \frac{k^2}{k^{2q}}\right). \quad (6.5)$$

To estimate  $\hat{K}_t^N$  in  $L^p$ , we use exponential martingale. More precisely, for a continuous martingale  $(M_t)$  with quadratic variation  $\langle M \rangle_t$ , we write down, for  $p \geq 1$ ,

$$e^{pM_t} = e^{pM_t - p^2 \langle M \rangle_t} e^{p^2 \langle M \rangle_t}.$$

By Cauchy-Schwarz inequality and the fact  $\mathbb{E}(e^{2pM_t - 2p^2 \langle M \rangle_t}) = 1$ , we get

$$\mathbb{E}(e^{pM_t}) \leq \left(\mathbb{E}(e^{2p^2 \langle M \rangle_t})\right)^{1/2}. \quad (6.6)$$

For  $M_t = \sum_{k=1}^{2N} \frac{1}{\alpha_k} \int_0^t \partial_x^2 \phi_k(X_s^N) dB_s^k$ , we have

$$\langle M \rangle_t = \sum_{k=1}^N \int_0^t \frac{1}{\alpha_k^2} (\partial_x^2 \phi_k(X_s^N))^2 ds \leq \sum_{k=1}^{+\infty} \frac{t}{k^{2q-2}} = \xi(2q-2)t.$$

Then by (6.6), we get

$$\mathbb{E}\left((\hat{K}_t^N)^p\right) \leq e^{p^2 \xi(2q-2)t}. \quad (6.7)$$

Obviously above estimate holds true for  $\tilde{K}_t$ , that is,

$$\mathbb{E}(\hat{K}_t^p) \leq e^{p^2 \xi(2q-2)t}. \quad (6.8)$$

**Proposition 6.2.** For  $q > \frac{5}{2}$ , choose  $\delta > 0$  such that  $q - \frac{1}{2} < q - \delta < 2q - 3$ ; then there is a constant  $C_p > 0$  such that

$$\mathbb{E}\left((\tilde{K}_t^N - \tilde{K}_t)^{2p}\right) \leq \frac{C}{N^{q-\delta}}. \quad (6.9)$$

*Proof.* Let  $\zeta_t = \hat{K}_t^N - \hat{K}_t$ . we have

$$d\hat{K}_t^N = \hat{K}_t^N \sum_{k=1}^{2N} \frac{1}{\alpha_k} \partial_x^2 \phi_k(X_t^N) dB_t^k,$$

and

$$d\hat{K}_t = \hat{K}_t \sum_{k=1}^{+\infty} \frac{1}{\alpha_k} \partial_x^2 \phi_k(X_t) dB_t^k.$$

So

$$\begin{aligned} d\zeta_t &= \sum_{k=1}^{2N} \frac{1}{\alpha_k} \left( \hat{K}_t^N \partial_x^2 \phi_k(X_t^N) - \hat{K}_t \partial_x^2 \phi_k(X_t) \right) dB_t^k \\ &\quad - \sum_{k>2N} \frac{1}{\alpha_k} \hat{K}_t \partial_x^2 \phi_k(X_t) dB_t^k. \end{aligned}$$

Now the Itô stochastic contraction  $d\zeta_t \cdot d\zeta_t$  admits the expression

$$\begin{aligned} d\zeta_t \cdot d\zeta_t &= \sum_{k=1}^{2N} \frac{1}{\alpha_k^2} \left( \hat{K}_t^N \partial_x^2 \phi_k(X_t^N) - \hat{K}_t \partial_x^2 \phi_k(X_t) \right)^2 dt \\ &\quad + \sum_{k>2N} \frac{1}{\alpha_k^2} \left( \hat{K}_t \partial_x^2 \phi_k(X_t) \right)^2 dt. \end{aligned}$$

Remark that

$$\hat{K}_t^N \partial_x^2 \phi_k(X_t^N) - \hat{K}_t \partial_x^2 \phi_k(X_t) = (\hat{K}_t^N - \hat{K}_t) \partial_x^2 \phi_k(X_s^N) + \hat{K}_t (\partial_x^2 \phi_k(X_s^N) - \partial_x^2 \phi_k(X_s)),$$

then

$$|\hat{K}_t^N \partial_x^2 \phi_k(X_t^N) - \hat{K}_t \partial_x^2 \phi_k(X_t)| \leq k |\hat{K}_t^N - \hat{K}_t| + k^2 \hat{K}_t |X_s^N - X_s|.$$

Therefore we get the following upper bound

$$d\zeta_t \cdot d\zeta_t \leq 2\xi(2q-2) |\hat{K}_t^N - \hat{K}_t|^2 + 2\xi(2q-4) \hat{K}_t^2 |X_t^N - X_t|^2 + \frac{\hat{K}_t^2}{N^{2q-3}}. \quad (6.10)$$

For  $p \geq 1$  an integer, by Itô's formula and above estimate, there are three constants  $C_1, C_2, C_3$  only dependent of  $p$  such that

$$\begin{aligned} d\zeta_t^{2p} &= 2p\zeta_t^{2p-1} d\zeta_t + 2p(2p-1)\zeta_t^{2p-2} d\zeta_t \cdot d\zeta_t \\ &\leq 2p\zeta_t^{2p-1} d\zeta_t + C_1(p)\zeta_t^{2p} dt + C_2(p)\zeta_t^{2p-2} \hat{K}_t^2 (X_t^N - X_t)^2 dt + C_3(p) \frac{\zeta_t^{2p-2} \hat{K}_t^2}{N^{2q-3}}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}(\zeta^{2p}) &\leq C_1(p) \int_0^t \mathbb{E}(\zeta_s^{2p}) ds + C_2(p) \int_0^t \mathbb{E} \left( \zeta_s^{2p-2} \hat{K}_s^2 (X_s^N - X_s)^2 \right) ds \\ &\quad + \frac{C_3(p)}{N^{2q-3}} \int_0^t \mathbb{E} \left( \zeta_s^{2p-2} \hat{K}_s^2 \right) ds. \end{aligned}$$



Let  $r > 1$  and  $\hat{r} > 1$  such that  $(1/r) + (1/\hat{r}) = 1$ . Now by (6.6) and (6.7), we have

$$C = \sup_{s \in [0,1]} \mathbb{E} \left( (\zeta_s^{2p-2} \hat{K}_s^2)^{\hat{r}} \right) < +\infty.$$

On the other hand, by (6.3),  $\mathbb{E}((X_s^N - X_s)^{2r}) \leq \frac{C_r}{N^{2q-1}}$ . Taking  $r = \frac{2q-1}{q-\delta}$  yields

$$\mathbb{E}(\zeta_t^{2p}) \leq C_1(p) \int_0^t \mathbb{E}(\zeta_s^{2p}) ds + \frac{C_2(p)}{N^{q-\delta}} t.$$

Gronwall lemma gives

$$\mathbb{E} \left( (\hat{K}_t^N - \hat{K}_t)^{2p} \right) \leq \frac{C}{N^{q-\delta}}.$$

Combing this with Relation (6.5), we obtain the desired estimate (6.9).  $\square$

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## References

- [1] L. Ambrosio and N. Gigli, Construction of the parallel transport in the Wasserstein space. *Methods Appl. Anal.* 15 (2008), no. 1, 1-29.
- [2] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lect. in Math., ETH Zürich, Birkhäuser Verlag, Basel, 2005.
- [3] H. Airault, J. Ren, Modulus of continuity of the canonic Brownian motion on the group of diffeomorphisms of the circle, *J. Funct. Anal.*, 196 (2002), 395-426.
- [4] D. Bakry and M. Emery, Diffusion hypercontractivities, *Sém. de Probab.*, XIX, Lect. Notes in Math., 1123 (1985), 177-206, Springer.
- [5] J.D. Benamou and Y. Brenier: A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.*, **84** (2000), 375-393.
- [6] Y. Brenier, Polar factorization and monotone rearrangement of vector valued functions, *Comm. Pure Appl. Math.*, 44 (1991), 375-417.
- [7] L.A. Caffarelli, Some regular properties of solutions of Monge-Ampère equations, *CPAM*, 44 (1991), 965-969.
- [8] A.B. Cruzeiro: Equations différentielles sur l'espace de Wiener et formules de Cameron-Martin non linéaires, *J. Funct. Analysis*, 54 (1983), 206-227.
- [9] Hao Ding, *Parallel translations, Newton flows and Q-Wiener processes on the Wasserstein space*, PhD thesis, tel-03716159, 2022

- [10] Hao Ding, Shizan Fang, Geometry on the Wasserstein space over a compact Riemannian manifold, *Acta Mathematica Scientia*, 41 (2021), 1959-1984.
- [11] K.D. Elworthy, *Stochastic flows on Riemannian manifolds*, in “Diffusion processes and related problems in Analysis”, Birkhäuser, Boston, 1992.
- [12] S. Fang, Canonical Brownian motion on the diffeomorphism group of the circle, *J. Funct. Anal.*, 196 (2002), 162-179.
- [13] Shizan Fang, Dejun Luo, A. Thalmaier, Stochastic differential equations with coefficients in Sobolev spaces. *J. Funct. Analysis*, 259 (2010), 1129–1168. .
- [14] S. Fang and J. Shao, Optimal transport maps for Monge-Kantorovich problem on loop groups, *J. Funct. Analysis*, 248 (2007), 225-257.
- [15] D. Feyel and A.S. Üstünel, Monge-Kantorovich measure transportation and Monge-Ampère equation on Wiener space, *Prob. Th. related Fields*, 128(2004), 347-385.
- [16] N. Gigli, *Second Order Analysis on  $(P_2(M), W_2)$* , Memoirs of the American Mathematical Society, 2012.
- [17] N. Ikeda, S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, Math. Library, 24, 1981.
- [18] R. Jordan, D. Kinderlehrer, and F. Otto. The Variational Formulation of the Fokker-Planck Equation. *SIAM Journal on Mathematical Analysis*, 29(1):1?17, 1998.
- [19] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, 1990.
- [20] Xiang-dong Li, On the strong  $L^p$ -Hodge decomposition over complete Riemannian manifolds, *J. Funct. Anal.*, 257 (2009), 3617-3646.
- [21] Songzi Li and Xiang-dong Li,  $W$ -entropy formulas and Langevin deformation of flows on the Wasserstein space over Riemannian manifolds, arXiv:1604.02596v1.
- [22] Songzi Li and Xiang-dong Li,  $W$ -entropy formulas on super Ricci flows and Langevin deformation on Wasserstein space over Riemannian manifolds *Sci. China Math.*, 61 (2018), 1385-1406.
- [23] Wei Liu, Liming Wu and Chaoen Zhang, Long-time behaviors of mean-field interacting particle systems related to McKean-Vlasov equations. *Comm. Math. Phys.* 387 (2021), 179-214.
- [24] J. Lott, Some geometric calculation on Wasserstein space, *Commun. Math. Phys.*, 277 (2008), 423-437.
- [25] J. Lott, An intrinsic parallel transport in Wasserstein space, *Proceedings of AMS*, 145 (2017), 5329-5340.
- [26] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann of Math.*, 169 (2009), 903-991.
- [27] P. Malliavin, *Stochastic analysis*, Grund. Math. Wissen., vol. 313, Springer, 1997.

- [28] R. McCann, Polar factorization of maps on Riemannian manifolds, *Geo. Funct. Anal.*, 11 (2001), 589-608.
- [29] F. Otto: The geometry of dissipative evolution equations: The porous medium equation, *Comm. partial Diff. equations*, **26** (2001), 101-174.
- [30] F. Otto and Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, *J. Funct. Anal.* 173(2000), 361-400.
- [31] M. K. Von Renesse, K. T. Sturm, Entropic measure and Wasserstein diffusion, *Ann. Prob.*, 37 (2009), 1114-1191.
- [32] M.K. Von Renesse, K.T. Sturm, Transport inequalities, gradient estimates, entropy and Ricci curvature, *Comm. Pures Appl. Math.*, 58 (2005), 923-940.
- [33] K. T. Sturm, On the geometry of metric measure spaces, *Acta Math.*, 196 (2006), 65-131.
- [34] C. Villani, *Optimal transport, Old and New*, vol. 338, *Grund. Math. Wiss.*, Springer-Verlag, Berlin, 2009.
- [35] C. Villani, *Topics in optimal transportation*, *Graduate Studies in Mathematics*, 58 (2003), AMS, Providence, Rhode Island.
- [36] Feng-Yu Wang, Image-dependent conditional McKean-Vlasov SDEs for measure-valued diffusion processes, *J. Evol. Equ.*, 21 (2021), 2009-2045.