ANALYTIC FUNCTION SPACES AND THEIR APPLICATIONS : MINI-COURSE ON TRUNCATED TOEPLITZ OPERATORS.

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Contents

1. Introduction

This is a set of lecture notes to accompany a series of talks given as part of the Fields Institute session on *Truncated Toeplitz Operators* from July–December 2021. To prepare these notes, we intensely use two recent surveys on truncated Toeplitz operators [17, 32], as well as [35, Chapter 13] and the founding paper [51].

Truncated Toeplitz operators on model spaces have been formally introduced by Sarason in [51], although some special cases have long ago appeared in literature, most notably as model operators for completely nonunitary contractions with defect numbers one and for their commutant. This new area of study has been recently very active and many open questions posed by Sarason in [51] have now been solved. See [6, 7, 9, 8, 16, 18, 31, 32, 33, 53, 54]. Nevertheless, there are still basic and interesting questions which remain mysterious.

The truncated Toeplitz operators live on the model spaces K_Θ, which are the closed invariant subspaces for the backward shift operator S^* acting on the Hardy space H^2 (see Section 2 for precise definitions). Given a model space K_Θ and a function $\phi \in L^2 = L^2(\mathbb{T})$, the truncated Toeplitz operator A_ϕ^Θ (or simply A_ϕ if there is no ambiguity regarding the model space) is defined on a dense subspace of K_Θ as the compression to K_Θ of multiplication by ϕ. The function ϕ is then called a symbol of the operator. Note that the symbol is never uniquely defined by the operator. From this and other points of view the truncated Toeplitz operators have much more in common with Hankel Operators than with Toeplitz operators.

We intend to give a short introduction to this fascinating area of research. Our objective is not to be exhaustive but rather to make discover the different techniques and the beauty of this theory through some key results.

The structure of the paper is the following. After a preliminary section with generalities about Hardy spaces, model spaces, one-component inner functions (which will play a special role here), Toeplitz and Hankel operators, and Carleson measures, first for the whole H^2 and then for model spaces, truncated Toeplitz operators are introduced in Section 3. Then, in Section 4, we discuss why it is worth studying truncated Toeplitz operators. We explain how they appear in several natural problems in operator theory and complex function theory. In Section 5, we describe the class of symbols, because one difficulty with truncated Toeplitz operators, compared to the classical Toeplitz operator, is that the symbol is never unique. In Sections 6 and 7, we give several useful characterization of truncated Toeplitz operators and discuss some interesting connections with another important class of operators, the so-called complex symmetric operators. In Section 8, we give some estimates on the norm of a truncated Toeplitz operator, while in Section 9, we obtain a complete description of the spectrum of a truncated Toeplitz operator associated

to a symbol in H^∞. In Section 9 and 10, we discuss the class of finite rank and compact truncated Toeplitz operators. Finally, in the last section, we discuss the important problem of the existence of a bounded symbol.

2. **Preliminaries**

For the content of this section, [26, 43] are classical references for general facts about Hardy spaces, while [46] can be used for Toeplitz and Hankel operators and [35] for model spaces. We recall the main definitions and properties but we assume that the reader is a little familiar with the theory of Hardy spaces and Toeplitz operators

2.1. **Function spaces, multiplication operators and their cognates.** Recall that for $1 \leq p < +\infty$, the *Hardy space* H^p of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the space of analytic functions f on $\mathbb{D}$ satisfying $\|f\|_p < +\infty$, where

$$\|f\|_p = \sup_{0\leq r<1} \left(\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right)^{1/p}.$$

The algebra of bounded analytic functions on $\mathbb{D}$, equipped with the sup norm, is denoted by H^∞. We denote also $H^p_0 = zH^p$ and $H^p_- = \overline{zH^p}$. Alternatively, H^p can be identified (via radial limits) to the subspace of functions $f \in L^p = L^p(\mathbb{T})$ for which $\hat{f}(n) = 0$ for all $n < 0$. Here $\mathbb{T}$ denotes the unit circle with normalized Lebesgue measure m and $\hat{f}(n)$ denotes the n-th Fourier coefficient of f. We also denote by $\mathbf{C}(\mathbb{T})$ the space of continuous functions on $\mathbb{T}$.

In the case $p = 2$, H^2 becomes a Hilbert space with respect to the scalar product inherited from L^2 and given by

$$\langle f, g\rangle_2 = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}\, dm(\zeta), \qquad f, g \in L^2.$$

The orthogonal projection from L^2 to H^2 will be denoted by P_+. The space H^2_- is precisely the orthogonal of H^2, and the corresponding orthogonal projection is $P_- = I - P_+$.

The *Poisson transform* of a function $\phi \in L^1$ is

$$(\mathfrak{P}\phi)(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\varphi(\zeta)\, dm(\zeta), \tag{2.1}$$

A classical property of Poisson integral (see [49, Chapter 11]) says that if $\phi \in L^1$ and ϕ is continuous on an open arc $I \subset \mathbb{T}$, then for every $\zeta \in I$, we have

$$\lim_{z\to\zeta} (\mathfrak{P}\phi)(z) = \phi(\zeta). \tag{2.2}$$

For $\phi \in L^\infty$, we denote by $M_\phi f = \phi f$ the *multiplication operator* on L^2; we have $\|M_\phi\| = \|\phi\|_\infty$. The *Toeplitz operator* $T_\phi : H^2 \longrightarrow H^2$ and the *Hankel operator* $H_\phi : H^2 \longrightarrow H^2_- = L^2 \ominus H^2$ are given by the formulae

$$T_\phi = P_+ M_\phi, \qquad H_\phi = P_- M_\phi.$$

We have $T_\phi^* = T_{\overline{\phi}}$ and $H_\phi^* = P_+ M_{\overline{\phi}} P_-$. In the case where ϕ is analytic, T_ϕ is just the restriction of M_ϕ to H^2.

Example 2.1. A particular important case is when $\phi(z) = z$. The corresponding Toeplitz operator $S = T_z$ is called the *shift operator* and plays a central role in analytic function spaces. Its adjoint $S^* = T_{\overline{z}}$ is called *the backward shift operator*. One can easily check that

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}, \qquad f \in H^2.$$

See [46] for a comprehensive study of the shift operator.

A well-known result of Brown–Halmos says that it $T : H^2 \longrightarrow H^2$ is a bounded linear operator on H^2, then T is a Toeplitz operator if and only if $T = STS^*$. We have an analogue for Hankel operator : if $H : H^2 \longrightarrow H^2_-$ is a bounded linear operator from H^2 into H^2_-, then H is a Hankel operator if and only if $HS = P_- ZH$, where $Z : L^2 \longrightarrow L^2$ is the shift operator on L^2 (i.e. the multiplication by the independent variable z).

It should be noted that, while the symbols of M_ϕ and T_ϕ are uniquely defined by the operators, this is not the case with H_ϕ. Indeed, it is easy to check that $H_\phi = H_\psi$ if and only if $\phi - \psi \in H^\infty$. So statements about Hankel operators often imply only the existence of a symbol with corresponding properties.

The definition of M_ϕ, T_ϕ and H_ϕ can be extended to the case when the symbol ϕ is only in L^2 instead of L^∞, obtaining (possibly unbounded) densily defined operators. Then M_ϕ and T_ϕ are bounded if and only if $\phi \in L^\infty$ (and $\|M_\phi\| = \|T_\phi\| = \|\phi\|_\infty$). The situation is more complicated for H_ϕ. Namely, H_ϕ is bounded if and only if there exists $\psi \in L^\infty$ with $H_\phi = H_\psi$, and

$$\|H_\phi\| = \inf\{\|\psi\|_\infty : H_\phi = H_\psi\}$$

This is known as Nehari's Theorem; see, for instance, [45, p. 182]. Moreover (but we will not pursue this in the sequel) an equivalent condition is $P_-\phi \in BMO$ (the space of bounded mean oscillation on $\mathbb{T}$), and $\|H_\phi\|$ is then a norm equivalent to $\|P_-\phi\|_{BMO}$.

Related results are known for compactness. The operators M_ϕ and T_ϕ are never compact except in the trivial case $\phi \equiv 0$. Hartman's Theorem states that H_ϕ is compact if and only if there exists $\psi \in \mathbf{C}(\mathbb{T})$ with $H_\phi = H_\psi$; or, equivalently, $P_-\phi \in VMO$ (the space of vanishing mean oscillation on $\mathbb{T}$). If we know that ϕ is bounded, then H_ϕ is compact if and only if $\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$.

We end this subsection by recalling some standard notation. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{L}(\mathcal{H})$ denotes the algebra of linear and bounded operator on $\mathcal{H}$. For $x, y \in \mathcal{H}$, the rank-one operator $x \otimes y$ is defined on $\mathcal{H}$ by the formula

$$(x \otimes y)(h) = \langle h, y\rangle x, \qquad h \in \mathcal{H},$$

and for any $A, B \in \mathcal{L}(\mathcal{H})$, we have

$$A(x \otimes y)B = Ax \otimes B^* y.$$

2.2. **Model spaces and one component inner functions.** Suppose now Θ is an inner function, that is a function in H^∞ whose radial limits are of modulus one almost everywhere on $\mathbb{T}$. Its *spectrum* is defined by

$$\mathfrak{s}(\Theta) := \{\zeta \in \mathbb{D}^- : \liminf_{\lambda\in\mathbb{D},\lambda\to\zeta} |\Theta(\lambda)| = 0\}, \tag{2.3}$$

where $\mathbb{D}^-$ denotes the closed unit disc. Equivalently, if $\Theta = BS$ is the decomposition of Θ into a Blaschke product and a singular inner function, then $\mathfrak{s}(\Theta)$ is the union between the closure of the limit points of the zeros of B and the support of the singular measure associated to S. We will also define

$$\sigma(\Theta) = \mathfrak{s}(\Theta) \cap \mathbb{T}.$$

We define the corresponding *shift-coinvariant subspace* generated by Θ (also called *model space*) by the formula

$$K^p_\Theta = H^p \cap \Theta\overline{H^p_0} = H^p \cap \Theta H^p_-,$$

where $1 \leq p < +\infty$. We will be especially interested in the Hilbert case, that is when $p = 2$. In this case, we also denote by $K_\Theta = K^2_\Theta$ and it is easy to see that K_Θ is also given by the following

$$K_\Theta = H^2 \ominus \Theta H^2 = \left\{ f \in H^2 : \langle f, g\rangle = 0, \forall g \in H^2 \right\}.$$

The orthogonal projection of L^2 onto K_Θ is denoted by P_Θ. It is well known (see [46, page 34]) that $P_\Theta = P_+ - \Theta P_+ \bar{\Theta}$. Since P_+ acts boundedly on L^p, $1 < p < \infty$, this formula shows that P_Θ can also be regarded as a bounded operator from L^p into K^p_Θ, $1 < p < \infty$. The famous Beurling's Theorem asserts that all non-trivial closed subspaces of H^2 which are invariant with respect to S^* are of the form K_Θ for some inner function Θ.

The spaces H^2 and K_Θ are reproducing kernel spaces over the unit disc $\mathbb{D}$. The respective *reproducing kernels* are, for $\lambda \in \mathbb{D}$,

$$k_\lambda(z) = \frac{1}{1 - \bar{\lambda} z},$$
$$k^\Theta_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda} z}.$$

In other words, for every $\lambda \in \mathbb{D}$, we have

$$f(\lambda) = \langle f, k_\lambda \rangle_2, \qquad f \in H^2,$$

and

$$f(\lambda) = \langle f, k_\lambda^\Theta \rangle_2, \qquad f \in K_\Theta.$$

Contrary to the situation of H^2, evaluations at certain points $\zeta \in \mathbb{T}$ may also be bounded in K_Θ; this happens precisely when Θ has an *angular derivative in the sense of Caratheodory* at ζ [1]. Recall that means that Θ and Θ' have a non-tangential limit at ζ and $|\Theta(\zeta)| = 1$. Then it is known [1] that evaluation at ζ is continuous on K_Θ, and the function k_ζ^Θ, defined by

$$k_\zeta^\Theta(z) := \frac{1 - \overline{\Theta(\zeta)}\Theta(z)}{1 - \bar{\zeta} z}, \qquad (z \in \mathbb{D}), \tag{2.4}$$

belongs to K_Θ and is the corresponding reproducing kernel. In other words, for every such points ζ and every $f \in K_\Theta$, we have

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta) = \langle f, k_\zeta^\Theta \rangle_2.$$

Moreover we have $\|k_\zeta^\Theta\|_2 = |\Theta'(\zeta)|^{1/2}$. We denote by $E(\Theta)$ the set of points $\zeta \in \mathbb{T}$ where Θ has an angular derivative in the sense of Carathéodory.

Of course, when $\lambda \in \mathbb{D}$, the function k_λ^Θ belongs to H^∞, and thus to every L^p, for $p \geq 1$. In [1] and [25] precise conditions are given for the inclusion of k_ζ^Θ into L^p (for $1 < p < \infty$); namely, if (a_k) are the zeros of Θ in $\mathbb{D}$ and σ is the singular measure on $\mathbb{T}$ corresponding to the singular part of Θ, then $k_\zeta^\Theta \in L^p$ if and only if

$$\sum_k \frac{1 - |a_k|^2}{|\zeta - a_k|^p} + \int_{\mathbb{T}} \frac{d\sigma(\tau)}{|\zeta - \tau|^p} < \infty. \tag{2.5}$$

Moreover, if (2.5) is satisfied for $p = 2$, then $\zeta \in E(\Theta)$ and in particular, $|\Theta(\zeta)| = 1$.

The following result will be used in Section 12.

Lemma 2.2 ([7])**.** *Let $1 < p < +\infty$ and let Θ be an inner function. Then we have:*

(a) $E(\Theta^2) = E(\Theta)$.

(b) $\inf_{\lambda \in \mathbb{D} \cup E(\Theta)} \|k_\lambda^\Theta\|_2 > 0$.

(c) *For $\lambda \in \mathbb{D}$, we have*

$$C\|k_\lambda^\Theta\|_p \leq \|k_\lambda^{\Theta^2}\|_p \leq 2\|k_\lambda^\Theta\|_p, \tag{2.6}$$

where $C = \|P_\Theta\|_{L^p \to L^p}^{-1}$ is a constant which depends only on Θ and p. Also, if $\zeta \in E(\Theta)$, then $k_\zeta^{\Theta^2} \in L^p$ if and only if $k_\zeta^\Theta \in L^p$ and (2.6) holds for $\lambda = \zeta$.

Proof. The proof of (a) is immediate using definition. For the proof of (b), note that, for $\lambda \in \mathbb{D} \cup E(\Theta)$, we have

$$1 - |\Theta(0)| \leq |1 - \overline{\Theta(0)}\Theta(\lambda)| = |k_0^\Theta(\lambda)| \leq \|k_0^\Theta\|_2 \|k_\lambda^\Theta\|_2 = (1 - |\Theta(0)|^2)1/2\|k_\lambda^\Theta\|_2,$$

which implies

$$\|k_\lambda^\Theta\|_2 \geq \left(\frac{1 - |\Theta(0)|}{1 + |\Theta(0)|}\right)^{1/2}.$$

It remains to prove (c). Using (2.4), we have $k_\lambda^{\Theta^2} = (1 + \overline{\Theta(\lambda)}\Theta)k_\lambda^\Theta$, whence $P_\Theta k_\lambda^{\Theta^2} = k_\lambda^\Theta$. Thus the result follows from the fact that P_Θ is bounded on L^p and from the trivial estimate $|1 + \overline{\Theta(\lambda)}\Theta(z)| \leq 2$, $z \in \mathbb{T}$. It is easy to check that the above arguments also hold when $\lambda = \zeta \in E(\Theta)$. □

Lemma 2.3. *Let Θ be an inner function. Then*

(1) $K_{\Theta^2} = K_\Theta \oplus \Theta K_\Theta$.
(2) $K_\Theta \cdot K_\Theta \subset K^1_{\Theta^2}$. *In particular, if $f, g \in K_\Theta$ and f or g is also bounded, then $fg \in K_{\Theta^2}$.*

Proof. (1) Observe that

$$H^2 = K_\Theta \oplus \Theta H^2 = K_\Theta \oplus \Theta(K_\Theta \oplus \Theta H^2) = K_\Theta \oplus \Theta K_\Theta \oplus \Theta^2 H^2,$$

which implies that $K_{\Theta^2} = H^2 \ominus \Theta^2 H^2 = K_\Theta \oplus \Theta K_\Theta$.

(2) Let $f, g \in K_\Theta$, and write $f = \Theta\overline{f_1}$, $g = \Theta\overline{g_1}$, with $f_1, g_1 \in H^2_0$. Then $fg \in H^1$ and since $f_1 g_1 \in H^1_0$, we also have $fg = \Theta^2\overline{f_1 g_1}$. That means that $fg \in H^1 \cap \Theta^2\overline{H^1_0} = K^1_{\Theta^2}$. Finally, observe that if f or g is bounded, then f_1 or g_1 is also bounded and then $fg \in H^2 \cap \Theta^2 H^2 = K_{\Theta^2}$. □

It follows from Lemma 2.3 that for every $\lambda \in \mathbb{D}$, $(k_\Theta^\Theta)^2 \in K_{\Theta^2}$.

A useful tool in the theory of model spaces is the notion of conjugation. Recall that a map C from a Hilbert space $\mathcal{H}$ into itself is a *conjugation* on $\mathcal{H}$ if C is anti-linear, isometric and involutive, meaning that the following properties are satisfied :

(1) for every $x, y \in \mathcal{H}$ and every $\lambda \in \mathbb{C}$, $C(\alpha x + y) = \overline{\lambda} Cx + Cy$;
(2) for every $x, y \in \mathcal{H}$, $\langle Cx, Cy\rangle = \langle y, x\rangle$;
(3) $C^2 = I$.

It is easy to check that the map C_Θ defined on L^2 by

$$C_\Theta f = \Theta \bar{z} \bar{f}; \tag{2.7}$$

is a conjugation on L^2 which has the convenient supplementary property of mapping K_Θ precisely onto K_Θ. In other words, its restriction to K_Θ is a conjugation on K_Θ. When

convenient we shall write $\tilde{f}$ for $C_\Theta f$. It can be easily verified that

$$\widetilde{k_\lambda^\Theta}(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}, \qquad z, \lambda \in \mathbb{D}.$$

In particular, $\widetilde{k_0^\Theta} = S^*\Theta$. Moreover, when Θ has an angular derivative in the sense of Carathéodory at $\zeta \in \mathbb{T}$, we have

$$\widetilde{k_\zeta^\Theta}(z) = \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} = \bar{\zeta}\Theta(\zeta)k_\zeta^\Theta(z), \qquad z \in \mathbb{D}.$$

In view of their main role in the study of operators on model spaces, we end this subsection by a discussion on a particular class of inner functions. Fix a number $0 < \varepsilon < 1$, and define

$$\Omega(\Theta, \varepsilon) = \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}. \tag{2.8}$$

The function Θ is called *one-component* if there exists a value of ε for which $\Omega(\Theta, \varepsilon)$ is connected. (If this happens, then $\Omega(\Theta, \delta)$ is connected for every $\varepsilon < \delta < 1$.) One-component functions have been introduced by Cohn [23]. An extensive study of these functions appears in [5, 4]; all results quoted below appear in [4]. See also [19, 20] for more recent results on this interesting class.

The above definition is not very transparent. In fact, one-component functions are rather special: a first immediate reason is that they must satisfy $m(\sigma(\Theta)) = 0$. This condition, of course, is not sufficient, but it suggests examining some simple cases.

The set $\sigma(\Theta)$ is of course empty for finite Blaschke products, which are one-component. The next simplest case is when $\sigma(\Theta)$ consists of just one point. One can prove easily that the elementary singular inner functions $\Theta(z) = e^{\frac{z+\zeta}{z-\zeta}}$ (for $\zeta \in \mathbb{T}$) are indeed one-component.

Suppose then that Θ is a Blaschke product whose zeros a_n tend nontangentially to a single point $\zeta \in \mathbb{T}$. If

$$\inf_{n \geq 1} \frac{|\zeta - a_{n+1}|}{|\zeta - a_n|} > 0, \tag{2.9}$$

then Θ is one-component. So, in particular, if $0 < r < 1$ and Θ is the Blaschke product with zeros $1 - r^n$, $n \geq 1$, then Θ is one-component. If condition (2.9) is not satisfied, then usually Θ is not one-component. A detailed discussion of such Blaschke products is given in [4], including the determination of Carleson measures for such model spaces (see Subsection 2.3).

2.3. **Carleson measures for the Hardy spaces and for the model spaces.** Let us discuss first some objects related to the Hardy space; we will afterwards see what analogous facts are true for the case of model spaces.

A finite measure μ on $\mathbb{D}$ is called a *Carleson measure* if $H^2 \subset L^2(|\mu|)$ (such an inclusion, if it exists, is automatically continuous). It is known that this is equivalent to $H^p \subset L^p(|\mu|)$ for all $1 \leq p < \infty$. Carleson measures can also be characterized by a geometrical condition, as follows. For an arc $I \subset \mathbb{T}$ such that $|I| := m(I) < 1$ we define its associated *Carleson window*

$$S(I) = \{z \in \mathbb{D} : 1 - |I| < |z| < 1 \text{ and } z/|z| \in I\}.$$

Then μ is a Carleson measure if and only if

$$\sup_I \frac{|\mu|(S(I))}{|I|} < \infty. \tag{2.10}$$

Condition (2.10) is called the *Carleson condition*. Note that Carleson measures cannot have mass on the unit circle (intervals containing a Lebesgue point of the corresponding density would contradict the condition (2.10)). These measures and condition (2.10) appeared in the famous work of Carleson on H^∞ interpolating sequences [13, 14].

Analogous results may be proved concerning compactness. In this case the relevant notion is that of *vanishing Carleson measure*, which is defined by the property

$$\lim_{|I|\to 0} \frac{|\mu|(S(I))}{|I|} = 0. \tag{2.11}$$

Then the embedding $H^p \subset L^p(|\mu|)$ is compact if and only if μ is a vanishing Carleson measure.

Similar questions for model spaces have been developed starting with the papers [23, 24] and [55]; however, the results in this case are less complete. Let us introduce first some notations. For $1 \leq p < \infty$, define

$$\mathcal{C}_p(\Theta) = \{\mu \text{ finite measure on } \mathbb{T} : K^p_\Theta \hookrightarrow L^p(|\mu|) \text{ is bounded}\},$$

It is clear that $\mathcal{C}_p(\Theta)$ is a complex vectorial subspace of the complex measures on the unit circle. Using the relations $K_{\Theta^2} = K_\Theta \oplus \Theta K_\Theta$ and $K_\Theta \cdot K_\Theta \subset K^1_{\Theta^2}$, it is easy to see that $\mathcal{C}_2(\Theta^2) = \mathcal{C}_2(\Theta)$ and $\mathcal{C}_1(\Theta^2) \subset \mathcal{C}_2(\Theta)$.

Example 2.4. Let $\zeta \in E(\Theta)$. Then the Dirac measure δ_ζ at point ζ belongs to $C_2(\Theta)$. Note that this is in contrast with Carleson measures for the Hardy spaces.

It is natural to look for geometric conditions to characterize these classes. Things are, however, more complicated, and the results are only partial. We start by fixing a number $0 < \varepsilon < 1$; then the (Θ, ε)-*Carleson condition* asserts that

$$\sup_I \frac{|\mu|(S(I))}{|I|} < \infty, \tag{2.12}$$

where the supremum is taken only over the intervals $|I|$ such that $S(I) \cap \Omega(\Theta, \varepsilon) \neq \emptyset$. (Remember that $\Omega(\Theta, \varepsilon)$ is given by (2.8).)

It is then proved in [55] that if μ satisfies the (Θ, ε)-Carleson condition, then the embedding $K^p_\Theta \subset L^p(|\mu|)$ is continuous. The converse is true if Θ is *one-component*; in which case the embedding condition does not depend on p, while fulfilling of the (Θ, ε)-Carleson condition does not depend on $0 < \varepsilon < 1$ (see Theorem 2.5 below).

As concerns the general case, it is shown by Aleksandrov [4] that if the converse is true for some $1 \leq p < \infty$, then Θ is one-component. Also, Θ is one-component if and only if the embedding condition does not depend on p. More precisely, the next theorem is proved in [4] (note that a version of this result for $p \in (1, \infty)$ already appears in [55]).

Theorem 2.5. *The following are equivalent for an inner function Θ:*

1. *Θ is one-component.*
2. *For some $0 < p < \infty$ and $0 < \varepsilon < 1$, $\mathcal{C}_p(\Theta)$ concides with the class of measures that satisfy the (Θ, ε)-Carleson condition.*
3. *For all $0 < p < \infty$ and $0 < \varepsilon < 1$, $\mathcal{C}_p(\Theta)$ concides with the class of measures that satisfy the (Θ, ε)-Carleson condition.*
4. *The class $\mathcal{C}_p(\Theta)$ does not depend on $p \in (0, \infty)$.*

In particular, if Θ is one component, then so is Θ^2, whence $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta^2) = \mathcal{C}_2(\Theta)$.

Observe that the (Θ, ε)-Carleson condition is less rigid than the Carleson condition (2.10) because we only need to test condition (2.10) on a subclass of Carleson windows. This can be explained by the fact that functions in K_Θ are in general more regular than an arbitrary H^2 function; in particular, they can be analytically continued in a neighborhood of $\mathbb{T} \setminus \sigma(\Theta)$, and $\overline{\Omega(\Theta, \varepsilon)} \cap \mathbb{T} \subset \sigma(\Theta)$. Note that a general characterization of $\mathcal{C}_2(\Theta)$ has recently been obtained in [41]; however, the geometric content of this result is not easy to see.

3. Truncated Toeplitz operators

3.1. **Definition of truncated Toeplitz operators.** Let Θ be a (non constant) inner function and $\phi \in L^2$. The *truncated Toeplitz operator* $A_\phi = A^\Theta_\phi$, introduced by Sarason in [51], is a densely defined, possibly unbounded operator on K_Θ. Its domain contains $K_\Theta \cap H^\infty$, on which it acts by the formula

$$A_\phi f = P_\Theta(\phi f), \qquad f \in K_\Theta \cap H^\infty. \tag{3.1}$$

Note that $K_\Theta \cap H^\infty$ is dense in K_Θ because it contains the reproducing kernels k^Θ_λ, since we trivially have the following estimate :

$$|k^\Theta_\lambda(z)| \leq \frac{2}{1 - |\lambda|}, \qquad z, \lambda \in \mathbb{D}.$$

If A_ϕ thus defined extends to a bounded operator, that operator is called a TTO. The class of all TTOs on K_Θ is denoted by $\mathcal{T}(\Theta)$. In other words,

$$\mathcal{T}(\Theta) = \{A_\phi : \phi \in L^2 \text{ and } A_\phi \in \mathcal{L}(K_\Theta)\}.$$

Example 3.1. When $\phi \in L^\infty$, then $\phi f \in L^2$ for all $f \in K_\Theta$ and so

$$\|P_\Theta(\phi f)\|_2 \leq \|\phi f\|_2 \leq \|\phi\|_\infty \|f\|_2.$$

In particular, A_ϕ is a TTO and $\|A_\phi\| \leq \|\phi\|_\infty$.

Example 3.2. The case when $\phi(z) = z$ will play a special role. It will be denoted by S_Θ and is called the *model operator* on K_Θ (see Section 4.1). In other words,

$$S_\Theta(f) = P_\Theta(zf), \qquad f \in K_\Theta.$$

It is well-known that, in general, Toeplitz operators do not commute. This is also the case for TTO but in a special case, TTO commute.

Proposition 3.3. *Let $\phi, \psi \in H^\infty$. Then $A_\phi A_\psi = A_\psi A_\phi = A_{\phi\psi}$.*

Proof. Let $\phi, \psi \in H^\infty$. Since $\phi\Theta H^2 \subset \Theta H^2$, for every $f \in K_\Theta$, we have

$$A_\phi A_\psi f = P_\Theta(\phi P_\Theta \psi f) = P_\Theta(\phi\psi f) = A_{\phi\psi} f.$$

Therefore $A_\phi A_\psi = A_{\phi\psi}$. The second relation $A_\psi A_\phi = A_{\phi\psi}$ immediately follows by changing the role of ϕ and ψ. □

In particular, according to Proposition 3.3, if $\phi \in H^\infty$, then

$$A_\phi^\Theta S_\Theta = S_\Theta A_\phi^\Theta.$$

In other words, A_ϕ^Θ is in the commutant of S_Θ when $\phi \in H^\infty$. An important result says that the converse is true (see Section 4.2).

Example 3.4. When $\Theta(z) = z^n$, for some $n \geq 1$, the space K_Θ consists of polynomials of degree less or equal to $n-1$ and the set $\{1, z, \ldots, z^{n-1}\}$ forms an orthonormal basis for K_Θ. Moreover, any truncated Toeplitz operator A_ϕ, when represented with respect to this basis, yields a Toeplitz matrix. Indeed, for $0 \leq j, k \leq n-1$, we have

$$\begin{aligned}
\langle A_\phi z^j, z^k\rangle_2 &= \langle P_\Theta(\phi z^j), z^k\rangle_2 \\
&= \langle \phi z^j, z^k\rangle_2 \\
&= \langle \phi, z^{k-j}\rangle_2 \\
&= \widehat{\phi}(k-j),
\end{aligned}$$

and so the matrix representation of A_ϕ with respect to the basis $\{1, z, \dots, z^{n-1}\}$ is the Toeplitz matrix

$$\begin{pmatrix} \hat{\phi}(0) & \hat{\phi}(-1) & \hat{\phi}(-2) & \dots & \dots & \hat{\phi}(-n+1) \\ \hat{\phi}(1) & \hat{\phi}(0) & \hat{\phi}(-1) & \ddots & & \vdots \\ \hat{\phi}(2) & \hat{\phi}(1) & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \hat{\phi}(-1) & \hat{\phi}(-2) \\ \vdots & & \ddots & \hat{\phi}(1) & \hat{\phi}(0) & \hat{\phi}(-1) \\ \hat{\phi}(-n+1) & \dots & \dots & \hat{\phi}(2) & \hat{\phi}(1) & \hat{\phi}(0) \end{pmatrix}$$

It is worth pointing out in this particular case that since K_Θ is a finite dimensional space, then $A_\phi \in \mathcal{T}(\Theta)$ for all $\phi \in L^2$. Conversely, any $n \times n$ Toeplitz matrix gives rise to a truncated Toeplitz operator on K_Θ.

3.2. **An equivalent definition and some basic properties.** The operator A_ϕ can alternatively be understood as follows. Note that the orthogonal projection P_Θ can be seen as an integral operator on L^2 by writing that, for $g \in L^2$, we have

$$(P_\Theta g)(\lambda) = \langle P_\Theta g, k_\lambda^\Theta \rangle_2 = \langle g, k_\lambda^\Theta \rangle_2 = \int_{\mathbb{T}} g\overline{k_\lambda^\Theta}\, dm, \qquad \lambda \in \mathbb{D}.$$

Now, since for each compact subset K of $\mathbb{D}$, there exists a constant C_K such that

$$\sup_{\lambda \in K} |k_\lambda^\Theta(\zeta)| \leq C_K,$$

and since for fixed $\zeta \in \mathbb{T}$, the function $\lambda \longmapsto \overline{k_\lambda^\Theta(\zeta)}$ is analytic on $\mathbb{D}$, the preceding integral formula still makes sense, and defines an analytic function on $\mathbb{D}$, even when g belongs to the larger space L^1. Moreover, for any $\phi \in L^2$, we have $\phi f \in L^1$ for all $f \in K_\Theta$. Thus we can define the linear transformation

$$A_\phi : K_\Theta \longmapsto \mathcal{O}(\mathbb{D})$$

(here $\mathcal{O}(\mathbb{D})$ denotes the analytic function on $\mathbb{D}$) by the integral formula

$$(A_\phi f)(\lambda) = \int_{\mathbb{T}} \phi f \overline{k_\lambda^\Theta}\, dm, \qquad f \in K_\Theta, \lambda \in \mathbb{D}. \tag{3.2}$$

Moreover, the Cauchy–Schwarz inequality yields

$$|A_\phi f)(\lambda)| \leq \|f\|_2 \|\phi\|_2 \|k_\lambda^\Theta\|_\infty, \qquad \lambda \in \mathbb{D}. \tag{3.3}$$

Proposition 3.5. *Let Θ be inner, let $\phi \in L^2$ and let A_ϕ defined by* (3.2)*. If $A_\phi f \in K_\Theta$ for every $f \in K_\Theta$, then A_ϕ is bounded on K_Θ. Furthermore, in that case, the definitions in* (3.1) *and* (3.2) *for A_ϕ coincide.*

Proof. To show that the operator defined by (3.2) is bounded, we will use the closed graph theorem. Suppose that $(f_n)_{n\geq 1}$ is a sequence in K_Θ satisfying $f_n \to f$ in K_Θ and $A_\phi f_n \to g$ in K_Θ. First, it follows from (3.3) that, for every $\lambda \in \mathbb{D}$, we have

$$(A_\phi f_n)(\lambda) \to (A_\phi f)(\lambda).$$

Second, since $A_\phi f_n \to g$ in K_Θ, we also have that $(A_\phi f_n)(\lambda) \to g(\lambda)$ for every $\lambda \in \mathbb{D}$. Thus $A_\phi f = g$ and by the closed graph theorem, we conclude that A_ϕ is bounded on K_Θ. For the proof of the second part of the proposition, observe that the bounded operators defined by the formulas in (3.1) and (3.2) agree on the dense set $K_\Theta \cap H^\infty$ and therefore must be equal. □

A basic computation shows that the adjoint of a TTO is itself a TTO.

Proposition 3.6. *If $\phi \in L^2$ such that $A_\phi \in \mathcal{T}(\Theta)$, then $A_\phi^* \in \mathcal{T}(\Theta)$ and*

$$A_\phi^* = A_{\overline{\phi}}.$$

Proof. Let $f, g \in K_\Theta \cap H^\infty$. We have

$$\begin{aligned}\langle f, A_\phi^* g\rangle_2 &= \langle P_\Theta(\phi f), g\rangle_2 = \langle \phi f, g\rangle_2 \\ &= \langle f, \overline{\phi} g\rangle_2 = \langle f, P_\Theta(\overline{\phi} g)\rangle_2 \\ &= \langle f, A_{\overline{\phi}} g\rangle_2.\end{aligned}$$

That proves that $A_\phi^* = A_{\overline{\phi}}$, and in particular $A_\phi^* \in \mathcal{T}(\Theta)$. □

Example 3.7. When ϕ is in H^∞, it turns out that ΘH^2 is obviously invariant with respect to $T_\phi = M_\phi$, and so K_Θ is invariant with respect to $T_\phi^* = T_{\overline{\phi}}$. It then follows that

$$A_\phi^* = T_{\overline{\phi}} | K_\Theta.$$

In particular,

$$S_\Theta^* = S^* | K_\Theta.$$

Remark 3.8. The space $\mathcal{T}(\Theta)$ forms a linear space since we easily have

$$\alpha A_\phi + \beta A_\psi = A_{\alpha\phi+\beta\psi}, \qquad \alpha, \beta \in \mathbb{C}, \phi, \psi \in L^2.$$

However, the product of two truncated Toeplitz operators is not always a truncated Toeplitz operator. See Example 10.4.

For the class of symbols in H^∞, the TTO enable to define a functional calculus for S_Θ.

Proposition 3.9 (The H^∞ functional calculus of Sz.-Nagy–Foias)**.** *Let $T : H^\infty \longrightarrow \mathcal{L}(K_\Theta)$ defined by $T(\phi) = A_\phi$, $\phi \in H^\infty$. Then T is a contractive morphism of Banach algebra which extends the polynomial calculus for S_Θ. In other words, for every $\phi, \psi \in H^\infty$, for every $\alpha, \beta \in \mathbb{C}$ and for every $p \in \mathbb{C}[X]$, we have*

(1) $T(\alpha\phi + \beta\psi) = \alpha T(\phi) + \beta T(\psi)$;
(2) $T(\phi\psi) = T(\phi)T(\psi)$;
(3) $T(p) = p(S_\Theta)$;
(4) $\|T(\phi)\| \leq \|\phi\|_\infty$.

Proof. (1) follows from Remark 3.8, (2) follows from Proposition 3.3 and (4) follows from Example 3.1. So the only point to check is (3). Let p be a polynomial, $p(z) = \sum_{k=0}^{N} a_k z^k$. Using (1), we have

$$T(p) = \sum_{k=0}^{N} a_k T(z^k),$$

and by (2), $T(z^k) = T(z)^k$. But $T(z) = A_z = S_\Theta$. Hence

$$T(p) = \sum_{k=0}^{N} a_k S_\Theta^k = p(S_\Theta).$$

□

In Section 9, we will see that this functional calculus satisfies a spectral mapping theorem for the subclass of symbols in $H^\infty \cap \mathbf{C}(\mathbb{T})$.

As we have just seen, the class of bounded analytic symbols already generates a rich class of operators. Thus, the reader might wonder why we bother defining A_ϕ for $\phi \in L^2$ (and possibly unbounded) when one can define A_ϕ everywhere on K_Θ if $\phi \in L^\infty$. We did not go through all this trouble when defining Toeplitz operators on H^2. Indeed for Toeplitz operators, the symbol is unique in that $T_{\phi_1} = T_{\phi_2}$ if and only if $\phi_1 = \phi_2$ almost everywhere. Furthermore, one can show that for a symbol ϕ in L^2, the densely defined operator $T_\phi f = P_+(\phi f)$, $f \in H^\infty$, has a bounded extension to H^2 if and only if $\phi \in L^\infty$. For truncated Toeplitz operators, the symbol is never unique (see Theorem 5.1). Moreover, as will see in Section 12, there are A_ϕ with $\phi \in L^2$ that extend to bounded operators on K_Θ but for which there is no bounded symbol that represents A_ϕ. Moreover, it is well known that the set of bounded Toeplitz operators on H^2 forms a weakly closed linear space in $\mathcal{L}(H^2)$. To have a similar result for truncated Toeplitz operators (see Corollary 6.4), we need to include such unusual operators with symbols in L^2.

4. **Why studying truncated Toeplitz operators?**

Interest in truncated Toeplitz operators has been very strong over the past 15 years. Besides the fascinating structure of these operators, they can be seen as a natural generalization of Toeplitz matrices (see Example 3.4). On the other hand, they naturally appear in several questions of operator theory and functions theory.

4.1. **The Sz.-Nagy–Foias model.** One of the main reasons that model spaces and truncated Toeplitz operators are worthy of study in their own rights stems from the so-called *model theory* developed by Sz.-Nagy–Foias, which shows that a wide range of Hilbert space contractions can be realized concretely as $S_\Theta = A_z^\Theta$ on some model space K_Θ.

Theorem 4.1 (Sz.-Nagy–Foias)**.** *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a Hilbert space contraction. Assume that*

$$\lim_{n\to\infty} \|T^n x\|_{\mathcal{H}} = 0, \qquad (x \in \mathcal{H})$$

and that the operators $I - T^*T$ and $I - TT^*$ are of rank one. Then there exists an inner function Θ such that T is unitarily equivalent to $S_\Theta^* = S^*|K_\Theta$.

Proof. Let D_T be the unique positive square root of the positive operator $I - T^*T$, i.e. $D_T = (I - T^*T)^{1/2}$. The key point of the proof is the following simple observation. Let x be any vector in $\mathcal{H}$. Then we have

$$\begin{aligned} \|D_T T^n x\|_{\mathcal{H}}^2 &= \langle D_T^2 T^n x, T^n x\rangle_{\mathcal{H}} \\ &= \langle (I - T^*T) T^n x, T^n x\rangle_{\mathcal{H}} \\ &= \|T^n x\|_{\mathcal{H}}^2 - \|T^{n+1} x\|_{\mathcal{H}}^2, \end{aligned}$$

which, using the fact that $\|T^n x\| \longrightarrow 0$, implies that the sequence $(D_T T^n x)_{n\geq 0}$ is in ℓ^2 and its ℓ^2-norm is equal to $\|x\|$. Note that $D_T T^n x$ belongs to the range of D_T, which is of dimension one and thus can be identified by $\mathbb{C}$.

Now, define a linear map U from $\mathcal{H}$ into H^2 by

$$(Ux)(z) = \sum_{n=0}^{\infty} (D_T T^n x) z^n, \qquad (z \in \mathbb{D}).$$

Since the sequence $(D_T T^n x)_{n\geq 0}$ is in ℓ^2 and of norm $\|x\|$, the map U is well-defined and isometric from $\mathcal{H}$ onto its range E, which is a closed subspace of H^2. Then we have

$$\begin{aligned} (UTx)(z) &= \sum_{n=0}^{\infty} (D_T T^{n+1} x) z^n \\ &= S^* \left(D_T T x + \sum_{n=0}^{\infty} (D_T T^{n+1} x) z^{n+1} \right) \\ &= S^* \left(\sum_{n=0}^{\infty} (D_T T^n x) z^n \right) \\ &= (S^* U x)(z), \qquad (z \in \mathbb{D}), \end{aligned}$$

which reveals that $UT = S^*U$. This relations implies two things. Firstly, E is a closed S^*-invariant subspace of H^2 and, secondly, T is unitarily equivalent to $S^*|E$. Note that E

cannot be equal to H^2. Indeed, if $E = H^2$, then we would obtain that the rank of $I - TT^*$ would be the rank of $I - S^*S$ which is zero (because S is a contraction on H^2), which contradicts the assumption that $I - TT^*$ is of rank one. Therefore, by Beurling's theorem, there exists an inner function Θ such that $E = K_\Theta$ and this proves that T is unitarily equivalent to $S^*|K_\Theta = S^*_\Theta$. □

We can restate the Sz.-Nagy–Foias theorem as follows.

Corollary 4.2 (Sz.-Nagy–Foias)**.** *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a Hilbert space contraction. Assume that*

$$\lim_{n\to\infty} \|T^{*n}x\|_{\mathcal{H}} = 0, \qquad (x \in \mathcal{H})$$

and that the operators $I - T^*T$ and $I - TT^*$ are of rank one. Then there exists an inner function Θ such that T is unitarily equivalent to S_Θ.

4.2. **The commutant of S_Θ.** We discuss now the commutant of S_Θ, that is the set

$$\{S_\Theta\}' = \{A \in \mathcal{L}(K_\Theta) : AS_\Theta = S_\Theta A\}.$$

The description of this commutant was obtained by Sarason [50]. His work was motivated by the study of interpolation problems for bounded analytic functions on the open unit disc. The description of $\{S_\Theta\}'$ is a deep result known as the *commutant lifting theorem* for S_Θ. The term "commutant lifting" stems from the following phenomenon. To find solutions to the operator equation

$$AS_\Theta = S_\Theta A \tag{4.1}$$

for an $A \in \mathcal{L}(K_\Theta)$, we "lift it" to the operator equation

$$BS = SB, \tag{4.2}$$

where S is the shift operator on the larger space H^2 and $B \in \mathcal{L}(H^2)$. Then, we apply a theorem of Brown–Halmos saying that B must be a Toeplitz operator T_ϕ with $\phi \in H^\infty$. We then return to (4.1) to prove that $A = A^\Theta_\phi$. The main difficulty is to lift equation (4.1) into equation (4.2). This can be proved using dilation theory, further developed by Sz.-Nagy–Foias in an abstract context. In this note, following [45, Section 3.1.9], we will present another approach, based on Nehari's theorem, which uses a link between the commutant of M_Θ and Hankel operators. This link is precisely stated in the following lemma.

Lemma 4.3. *Let Θ be an inner function, and let $A \in \mathcal{L}(K_\Theta)$. Define*

$$\begin{array}{rccc} A_* : & H^2 & \longrightarrow & H^2_- \\ & f & \longmapsto & \overline{\Theta} A P_\Theta f. \end{array}$$

Then the following assertions are equivalent.

(i) $AS_\Theta = S_\Theta A$.
(ii) A_* *is a Hankel operator.*

Proof. First, let us verify that A_* is a well-defined operator from H^2 into H^2_-. Indeed, since $K_\Theta = H^2 \cap \Theta\, H^2_-$, we have $AP_\Theta H^2 \subset \Theta\, H^2_-$, which implies that $\overline{\Theta} AP_\Theta H^2 \subset H^2_-$.

Moreover,

$$\begin{aligned} AS_\Theta = S_\Theta A &\iff AP_\Theta S = P_\Theta SA, \qquad (\text{on } K_\Theta), \\ &\iff AP_\Theta SP_\Theta = P_\Theta SAP_\Theta, \qquad (\text{on } H^2). \end{aligned}$$

Since $z\Theta H^2 \subset \Theta H^2$, we have $P_\Theta S(I - P_\Theta) = 0$, whence $P_\Theta SP_\Theta = P_\Theta S$ and

$$AP_\Theta SP_\Theta = AP_\Theta S, \qquad (\text{on } H^2).$$

Recall that $P_\Theta = \Theta P_- \overline{\Theta}$ (on H^2), where $P_- = I - P_+$. Thus

$$\begin{aligned} AS_\Theta = S_\Theta A &\iff AP_\Theta S = P_\Theta SAP_\Theta \quad (\text{on } H^2), \\ &\iff AP_\Theta S = \Theta P_- \overline{\Theta} SAP_\Theta \\ &\iff \overline{\Theta} AP_\Theta S = P_- Z \overline{\Theta} AP_\Theta \\ &\iff A_* S = P_- Z A_*. \end{aligned}$$

Here Z is the shift operator (i.e. the multiplication by the independent variable z) on $L^2(\mathbb{T})$. The latter condition exactly means that A^* is a Hankel operator (see Subsection 2.1). □

As we will see in Section 5, the symbol of a truncated Toeplitz operator is never unique. In the proof of the commutant lifting theorem, we will need a special case of this.

Lemma 4.4. *Let $\phi \in H^\infty$. Then $A^\Theta_\phi = 0$ if and only if $\phi \in \Theta H^\infty$.*

Proof. Assume that $\phi = \Theta g$ for some $g \in H^\infty$. Then, for each $f \in K_\Theta$, we have

$$A^\Theta_\phi f = P_\Theta(\phi f) = P_\Theta(\Theta f g) = 0,$$

because $\Theta f g \in \Theta H^2 = (K_\Theta)^\perp$. Conversely, assume that $A^\Theta_\phi = 0$. Since for each $f \in H^2$, $\phi P_{\Theta H^2} f \in \Theta H^2$, we get

$$P_\Theta(\phi f) = P_\Theta(\phi P_\Theta f) + P_\Theta(\phi P_{\Theta H^2} f) = A^\Theta_\phi P_\Theta f = 0,$$

which implies that $\phi H^2 \subset \Theta H^2$. In particular, $\phi = g\Theta$, for some $g \in H^2$. But, taking the absolute values of both sides shows that $|\phi| = |g|$ a.e. on $\mathbb{T}$. Hence $g \in H^\infty$, and finally $\phi \in \Theta H^\infty$. □

We are now ready for the commutant lifting theorem for the operator S_Θ.

Theorem 4.5 (Sarason, [50]). *Let Θ be an inner function, and let $A \in \{S_\Theta\}'$, i.e. $A \in \mathcal{L}(K_\Theta)$ and $AS_\Theta = S_\Theta A$. Then there exists $\phi \in H^\infty$ such that $A = A_\phi^\Theta$.*

Moreover, the following assertions hold.

(i) *For any representation $A = A_\phi^\Theta$, with $\phi \in H^\infty$, we have*

$$\|A\| = \operatorname{dist}(\phi, \Theta H^\infty).$$

(ii)

$$\|A\| = \inf\{\|\phi\|_\infty : A = A_\phi^\Theta \text{ with } \phi \in H^\infty\}.$$

(iii) *There exists a particular choice $\phi \in H^\infty$ such that*

$$A = A_\phi^\Theta \quad \text{and} \quad \|A\| = \|\phi\|_\infty.$$

Proof. According to Lemma 4.3, the operator $A_* = \overline{\Theta} A P_\Theta$ is a Hankel operator. Since A is bounded, we have

$$\|A_* f\|_2 = \|\overline{\Theta} A P_\Theta f\|_2 = \|A P_\Theta f\|_2 \leq \|A\| \, \|f\|_2, \qquad (f \in H^2),$$

which implies that A_* is bounded and $\|A_*\| \leq \|A\|$. Furthermore,

$$\|Af\|_2 = \|AP_\Theta f\|_2 = \|\overline{\Theta} A P_\Theta f\|_2 = \|A_* f\|_2 \leq \|A_*\| \, \|f\|_2, \qquad (f \in K_\Theta),$$

whence $\|A\| \leq \|A_*\|$. Therefore,

$$\|A\|_{\mathcal{L}(K_\Theta)} = \|A_*\|_{\mathcal{L}(H^2, H^2_-)}.$$

Nehari's theorem implies that there exists a function $\eta \in L^\infty(\mathbf{T})$ such that $A_* = H_\eta$ and $\|A_*\| = \|\eta\|_\infty = \operatorname{dist}(\eta, H^\infty)$. Hence,

$$\|A\| = \|A_*\| = \|\eta\|_\infty = \operatorname{dist}(\eta, H^\infty). \tag{4.3}$$

Since $\Theta\eta \in L^2(\mathbf{T}) = H^2 \oplus H^2_-$, we can write

$$\Theta\eta = \phi + \psi,$$

where $\phi \in H^2$ and $\psi \in H^2_-$. But,

$$P_- \Theta\eta = H_\eta \Theta = A_* \Theta = \overline{\Theta} A P_\Theta \Theta = 0,$$

which means $\psi = P_- \Theta\eta = 0$. Hence,

$$\Theta\eta = \phi \in H^2 \cap L^\infty = H^\infty.$$

Rewrite the last identity as $\eta = \overline{\Theta}\phi$, with $\phi \in H^\infty$. This is a rewarding representation. In fact, for $f \in K_\Theta$, we have

$$\begin{aligned} Af &= \Theta(\overline{\Theta} A P_\Theta f) \\ &= \Theta A_* f \\ &= \Theta H_{\overline{\Theta}\phi} f \\ &= \Theta P_- \overline{\Theta}\phi f \\ &= P_\Theta \phi f \\ &= A^\Theta_\phi f, \end{aligned}$$

whence $A = A^\Theta_\phi$. Moreover, according to (4.3) and using the fact that $|\Theta| = 1$ a.e. on $\mathbb{T}$, we have

$$\|A\| = \|\phi\|_\infty = \operatorname{dist}(\phi, \Theta H^\infty). \tag{4.4}$$

Since $A^\Theta_\phi = A$, we have

$$\|A\| = \|\phi\|_\infty \geq \inf\{\|h\|_\infty : A = A^\Theta_h, \text{ with } h \in H^\infty\}.$$

If $h \in H^\infty$ is such that $A^\Theta_h = A = A^\Theta_\phi$, then according to Lemma 4.4, $h - \phi \in \Theta H^\infty$. In other words, there exists $g \in H^\infty$ such that $h = \phi + \Theta g$. Thus,

$$\operatorname{dist}(h, \Theta H^\infty) = \operatorname{dist}(\phi, \Theta H^\infty) = \|A\|,$$

and

$$\|h\|_\infty = \|\phi + \Theta g\| \geq \operatorname{dist}(\phi, \Theta H^\infty) = \|A\|.$$

Hence,

$$\inf\{\|h\|_\infty : A = A^\Theta_h, \text{ with } h \in H^\infty\} \geq \|A\|,$$

which proves that

$$\|A\| = \inf\{\|h\|_\infty : A = A^\Theta_h, \text{ with } h \in H^\infty\} = \operatorname{dist}(\phi, \Theta H^\infty).$$

Finally, (4.4) shows that the infimum is attained, which ends the proof.

□

4.3. **The Nevanlinna–Pick interpolation problem.** We will now explain how we can use the commutant lifting theorem to solve the Nevanlinna-Pick interpolation problem.

Given n points $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the open unit disc $\mathbb{D}$, and n points $\omega_1, \omega_2, \ldots, \omega_n$ in the complex plane, we would like to know if there exists a function f in the closed unit ball of H^∞ interpolating the points λ_i to the points ω_i, that is

$$f(\lambda_i) = \omega_i, \qquad (1 \leq i \leq n).$$

Before giving the answer to this question, we need an additional property of truncated Toeplitz operators with symbols in $\overline{H^\infty}$.

Lemma 4.6. *Let $\phi \in H^\infty$ and $\lambda \in \mathbb{D}$ such that $\Theta(\lambda) = 0$. Then*

$$A_{\overline{\phi}} k_\lambda = \overline{\phi(\lambda)} k_\lambda.$$

Proof. Observe that $k_\lambda = k_\lambda^\Theta$ since $\Theta(\lambda) = 0$. In particular, $k_\lambda \in K_\Theta$. Using Proposition 3.6, we have, for every $f \in K_\Theta$,

$$\begin{aligned}\langle A_{\overline{\phi}} k_\lambda, f\rangle_2 =& \langle k_\lambda, A_\phi f\rangle_2 \\ =& \langle k_\lambda, \phi f\rangle_2 \\ =& \overline{\phi(\lambda) f(\lambda)} \\ =& \langle \overline{\phi(\lambda)} k_\lambda, f\rangle_2.\end{aligned}$$

Since this is true for every $f \in K_\Theta$, we conclude that $A_{\overline{\phi}} k_\lambda = \overline{\phi(\lambda)} k_\lambda$. □

The following result answers the question of Nevanlinna–Pick interpolation.

Theorem 4.7 (Nevanlinna–Pick)**.** *Let $(\lambda_i)_{1\le i\le n}$ be n distinct points in $\mathbb{D}$, and let $(\omega_i)_{1\le i\le n}$ be complex numbers. Then the following are equivalent.*

(i) *There exists a function f in H^∞ such that*

$$f(\lambda_i) = \omega_i, \qquad (1 \le i \le n),$$

and, moreover, $\|f\|_\infty \le 1$.

(ii) *The matrix $Q = (Q_{j,k})_{1\le j,k\le n}$, where*

$$Q_{j,k} = \frac{1 - \overline{\omega_j}\omega_k}{1 - \overline{\lambda_j}\lambda_k}, \qquad (j,k = 1,\dots,n),$$

is nonnegative.

Proof. We do not follow the original proof but the one based on the commutant lifting theorem of Sarason. We start by introducing the operator T which is exploited in the proof of equivalence. Let B be the (finite) Blaschke product associated to the sequence $(\lambda_i)_{1\le i\le n}$. It is not difficult to check that the sequence of reproducing kernels $(k_{\lambda_i})_{1\le i\le n}$ forms a basis of K_B. Note that K_B is finite dimensional. Hence we can consider the linear bounded operator T from K_B into itself defined by

$$Tk_{\lambda_i} = \overline{\omega_i}\, k_{\lambda_i}, \qquad (1 \le i \le n). \tag{4.5}$$

Using Lemma 4.6 for $S_B^* = A_{\overline{z}}^B$, we have

$$TS_B^* k_{\lambda_i} = \overline{\lambda_i}\,\overline{\omega_i}\, k_{\lambda_i} = S_B^* T k_{\lambda_i}, \qquad (1 \le i \le n).$$

Since $(k_{\lambda_i})_{1\le i\le n}$ is a basis of K_B, we deduce that $TS_B^* = S_B^*T$. Hence, Theorem 4.5 ensures that there exists $f \in H^\infty$ such that $T = A_{\bar f}^B$ and

$$\|T\| = \min\{\|g\|_\infty : T = A_{\bar g}^B,\ g \in H^\infty\}. \tag{4.6}$$

For any function g which fulfills $T = A_{\bar g}^B$, we have

$$Tk_{\lambda_i} = A_{\bar g}^B k_{\lambda_i} = \overline{g(\lambda_i)}k_{\lambda_i}, \qquad (1 \le i \le n).$$

Therefore,

$$g(\lambda_i) = \omega_i \qquad (1 \le i \le n). \tag{4.7}$$

To establish the connection between Q and T, let $a_i \in \mathbb{C}$, $1 \le i \le n$. Then, using the operator T, we can write

$$\begin{aligned}\sum_{1\le i,j\le n} a_i\overline{a_j}\frac{1-\overline{\omega_i}\omega_j}{1-\overline{\lambda_i}\lambda_j} &= \sum_{1\le i,j\le n} a_i\overline{a_j}\left(\langle k_{\lambda_i}, k_{\lambda_j}\rangle_2 - \langle Tk_{\lambda_i}, Tk_{\lambda_j}\rangle_2\right)\\ &= \sum_{1\le i,j\le n} a_i\overline{a_j}\langle (I-T^*T)k_{\lambda_i}, k_{\lambda_j}\rangle_2\\ &= \langle (I-T^*T)h, h\rangle_2,\end{aligned}$$

where $h = \sum_{i=1}^n a_i k_{\lambda_i}$. Hence, the matrix Q is nonnegative if and only if the operator $I - T^*T$ is positive, which is equivalent to say that T is a contraction.

$(i) \implies (ii)$: Assume that there exists $f \in H^\infty$, $\|f\|_\infty \le 1$, such that $f(\lambda_i) = \omega_i$, $1 \le i \le n$. Hence, using (4.5) and Lemma 4.6, we have

$$Tk_{\lambda_i} = \overline{\omega_i}k_{\lambda_i} = \overline{f(\lambda_i)}k_{\lambda_i} = A_{\bar f}^B k_{\lambda_i}, \qquad (1 \le i \le n).$$

Thus we get $T = A_{\bar f}^B$. Now, (4.6) implies that

$$\|T\| \le \|f\|_\infty \le 1,$$

i.e. T is a contraction, and thus Q is nonnegative.

$(ii) \implies (i)$: Assume that Q is nonnegative. Thus, T is a contraction. By (4.6), we know that there exists a function $f \in H^\infty$, $\|f\|_\infty \le 1$ such that $T = A_{\bar f}^B$. Thus, by (4.7), $f(\lambda_i) = \omega_i$, $1 \le i \le n$, which gives (i). □

4.4. **A link with truncated Wiener-Hopf operators.** It turns out that truncated Toeplitz operators on the model space K_Θ are closed connected with another class of operators, the truncated Wiener–Hopf operators, which arise naturally in projection methods to solve certain convolution equations. See [40]. Let us explain the link between these two important classes of operators.

Let $\varphi \in L^1(\mathbb{R})$, $a > 0$, and let

$$(W_\varphi f)(x) = \int_0^{2a} \varphi(x-t) f(t)\, dt, \qquad x \in (0, 2a),$$

for $f \in L^\infty(0, 2a)$. It is easy to see that $W_\varphi f \in L^2(0, 2a)$, and then W_φ is a densely defined operator on $L^2(0, 2a)$. If W_φ extends to a bounded operator on $L^2(0, 2a)$, then it is called a *truncated Wiener–Hopf operator*.

To explain the link between a truncated Wiener–Hopf operator and a truncated Toeplitz operator, it will be easier to go into the upper half-plane. We denote by $H^2(\mathbb{C}_+)$ the Hardy space of the upper-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im\mathrm{m}(z) > 0\}$ and if Θ is an inner function in $\mathbb{C}_+$, the associated model space K_Θ is defined as $K_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$. Finally, we recall that if $\mathcal{F}$ is the Fourier transform, the Paley–Wiener Theorem says that $\mathcal{F}^{-1} L^2(\mathbb{R}_+) = H^2(\mathbb{C}_+)$.

Proposition 4.8. *Let $\Theta_a(z) = e^{iaz}$, $a > 0$, and let φ in the Schwartz class $\mathcal{S}(\mathbb{R})$. Then,*

$$\mathcal{F} A_\varphi^{\Theta_{2a}} \mathcal{F}^{-1} = W_{\widehat{\varphi}}.$$

Proof. Denote by $\mathcal{U}_a$ the (unitary) operator of translation by a on $L^2(\mathbb{R})$, i.e. $(\mathcal{U}_a f)(x) = f(x+a)$, $f \in L^2(\mathbb{R})$. Write

$$\begin{aligned} \mathcal{U}_a L^2(\mathbb{R}_+) =& L^2(-a, +\infty) = L^2(-a, a) \oplus L^2(a, +\infty) \\ =& L^2(-a, a) \oplus \mathcal{U}_{-a} L^2(\mathbb{R}_+). \end{aligned}$$

Apply the inverse of the Fourier transform and use the Paley–Wiener Theorem and the well-known facts that $\mathcal{F}^{-1}\mathcal{U}_a = \overline{\Theta_a}\mathcal{F}^{-1}$ and $\mathcal{F}^{-1}\mathcal{U}_{-a} = \Theta_a \mathcal{F}^{-1}$ to get

$$\overline{\Theta_a} H^2(\mathbb{C}_+) = \mathcal{F}^{-1} L^2(-a, a) \oplus \Theta_a H^2(\mathbb{C}_+).$$

Multiply this equation by Θ_a gives

$$K_{\Theta_{2a}} = \Theta_a \mathcal{F}^{-1} L^2(-a, a) = \Theta_a \mathcal{F}^{-1} \chi_{(-a,a)} L^2(\mathbb{R}).$$

It is easy to check that the operator $T = \Theta_a \mathcal{F}^{-1} \chi_{(-a,a)}$ is a partial isometry on $L^2(\mathbb{R})$. Hence TT^* is the orthogonal projection onto $\mathrm{Im}(T) = K_{\Theta_{2a}}$ (see [27, Theorem 7.22] for this result on partial isometries). We deduce a nice formula for the orthogonal projection onto $K_{\Theta_{2a}}$:

$$P_{\Theta_{2a}} = \Theta_a \mathcal{F}^{-1} \chi_{(-a,a)} \mathcal{F} \overline{\Theta_a}.$$

Let $f \in L^2(0,2a)$. Then, we have

$$\begin{aligned}\mathcal{F}A_\varphi^{\Theta_{2a}}\mathcal{F}^{-1}f =& \mathcal{F}P_{\Theta_{2a}}\varphi\mathcal{F}^{-1}f\\ =& \mathcal{F}\Theta_a\mathcal{F}^{-1}\chi_{(-a,a)}\mathcal{F}(\overline{\Theta_a}\varphi\mathcal{F}^{-1}f)\\ =& \mathcal{U}_{-a}\chi_{(-a,a)}\mathcal{F}(\varphi\mathcal{F}^{-1}\mathcal{U}_a f)\\ =& \mathcal{U}_{-a}\chi_{(-a,a)}\widehat{\varphi} * \mathcal{U}_a f.\end{aligned}$$

Let $x \in \mathbb{R}$. We get

$$\begin{aligned}(\mathcal{F}A_\varphi^{\Theta_{2a}}\mathcal{F}^{-1}f)(x) =& \chi_{(-a,a)}(x-a)(\widehat{\varphi} * \mathcal{U}_a f)(x-a)\\ =& \chi_{(0,2a)}(x)\int_{-\infty}^{\infty}\widehat{\varphi}(x-a-y)f(y+a)\,dy\\ =& \chi_{(0,2a)}(x)\int_0^{2a}\widehat{\varphi}(x-t)f(t)\,dt.\end{aligned}$$

In other words, $\mathcal{F}A_\varphi^{\Theta_{2a}}\mathcal{F}^{-1} = W_{\widehat{\varphi}}$. □

Proposition 4.8 says that the class of truncated Wiener–Hopf operators (at least for a class of symbols in the Schwartz class) is unitary equivalent to a particular class of truncated Toeplitz operators (associated to the model space with inner function Θ_{2a}). Therefore, truncated Toeplitz operators can be viewed as a generalization of truncated Wiener–Hopf operators.

5. The class of symbols for a truncated Toeplitz operator

We know that the symbol ϕ of a Toeplitz operator T_ϕ is unique. As we already noticed (see Lemma 4.4), the story for truncated Toeplitz operators is much different.

Theorem 5.1 (Sarason, [51])**.** *Let Θ be an inner function. A truncated Toeplitz operator A_ϕ is identically zero if and only if $\phi \in \Theta H^2 + \overline{\Theta H^2}$. Consequently,*

$$A_{\phi_1} = A_{\phi_2} \iff \phi_1 - \phi_2 \in \Theta H^2 + \overline{\Theta H^2}.$$

Proof. Suppose that $\phi = \Theta h_1 + \overline{\Theta h_2}$, where $h_1, h_2 \in H^2$. Then, for $f \in K_\Theta \cap H^\infty$, we have

$$\phi f = \Theta f h_1 + \overline{\Theta h_2} f.$$

The first term belongs to ΘH^2 and so $P_\Theta(\Theta f h_1) = 0$. Moreover, since $K_\Theta = H^2 \cap \Theta\overline{zH^2} \subset \Theta\overline{zH^2}$, the second term belongs to $\overline{zH^2}$, and thus, we also have $P_\Theta(\Theta f h_2) = 0$. Hence

$$A_\phi(f) = P_\Theta(\phi f) = P_\Theta(\Theta f h_1) + P_\Theta(\Theta f h_2) = 0.$$

Conversely, suppose that $A_\phi(f) = 0$ for every $f \in K_\Theta \cap H^\infty$. Write the symbol $\phi \in L^2$ as $\phi = \psi + \overline{\chi}$, where $\psi, \chi \in H^2$. In particular, we have $A_\psi = -A_{\overline{\chi}}$. Now, using the fact that

ψ and χ are in H^2, it is not difficult to check the following identities on $K_\Theta \cap H^\infty$

$$A_\psi S_\Theta = S_\Theta A_\psi \quad \text{and} \quad A_{\overline{\chi}} S_\Theta^* = S_\Theta^* A_{\overline{\psi}}.$$

In particular, we deduce that A_ψ and $A_{\overline{\chi}}$ commute with both S_Θ and S_Θ^*. We now need the following well-known identity

$$I - S_\Theta S_\Theta^* = k_0^\Theta \otimes k_0^\Theta. \tag{5.1}$$

To check (5.1), note that for $f \in K_\Theta$, we have

$$\begin{aligned}(I - S_\Theta S_\Theta^*)f =& f - P_\Theta(SP_\Theta(S^* f)) \\ =& f - P_\Theta(SS^* f) \\ =& f - P_\Theta(f - f(0)) \\ =& f(0)P_\Theta(1) = f(0)k_0^\Theta \\ =& (k_0^\Theta \otimes k_0^\Theta)f.\end{aligned}$$

Now use (5.1) to deduce

$$A_\psi(k_0^\Theta \otimes k_0^\Theta) = (k_0^\Theta \otimes k_0^\Theta)A_\psi.$$

If we evaluate this operator identity at k_0^Θ, we get

$$\|k_0^\Theta\|_2^2 A_\psi k_0^\Theta = \langle A_\psi k_0^\Theta, k_0^\Theta\rangle_2 k_0^\Theta.$$

In particular, we have $A_\psi k_0^\Theta = c k_0^\Theta$ for some constant $c \in \mathbb{C}$. From here we can write

$$\begin{aligned}0 =& (A_\psi - cI)k_0^\Theta \\ =& P_\Theta(\psi - c)(1 - \overline{\Theta(0)}\Theta) \\ =& P_\Theta(\psi - c) - \overline{\Theta(0)}P_\Theta((\psi - c)\Theta) \\ =& P_\Theta(\psi - c),\end{aligned}$$

the last equality following from the fact that $(\psi - c)\Theta \in \Theta H^2$. Thus $\psi - c \in \Theta H^2$. In particular, if we apply the first part of the proof, we deduce $A_\psi = cI$. But since $A_{\overline{\chi}} = -A_\psi = -cI$, repeating the same arguments yields to $\chi + \overline{c} \in \Theta H^2$, and so $\overline{\chi} + c \in \overline{\Theta H^2}$. Therefore

$$\phi = \psi + \overline{\chi} = (\psi - c) + (\overline{\chi} + c) \in \Theta H^2 + \overline{\Theta H^2},$$

which completes the proof. □

In particular, the preceding result tells us that there are always infinitely many symbols (many of them unbounded) which represent the same truncated Toeplitz operator. On the other hand, if we denote by $\mathfrak{S}_\Theta = L^2 \ominus (\Theta H^2 + \overline{\Theta H^2})$, it follows from Theorem 5.1 that every truncated Toeplitz operator has a unique symbol $\phi \in \mathfrak{S}_\Theta$. This space is sometimes called the *space of standard symbols*. In the following result, we gather from [7, Lemma

3.1] some interesting properties of this space. Some of them will be used in Section 12 for the construction of a bounded truncated Toeplitz operator that has no bounded symbols.

Lemma 5.2 ([7], Lemma 3.1)**.** *Let Θ be an inner function and let Q_Θ (respectively $P_{\mathfrak{S}_\Theta}$) the orthogonal projection onto $K_\Theta \oplus \overline{zK_\Theta}$ (respectively onto $\mathfrak{S}_\Theta$). Then:*

(1) $Q_\Theta(\overline{\Theta}) = \overline{\Theta} - \overline{\Theta(0)}^2\Theta$;

(2) *we have*

$$K_\Theta \oplus \overline{zK_\Theta} = \mathfrak{S}_\Theta \oplus \mathbb{C}q_\Theta,$$

where $q_\Theta = \|Q_\Theta(\overline{\Theta})\|_2^{-1} Q_\Theta(\overline{\Theta})$;

(3) *Q_Θ and $P_{\mathfrak{S}_\Theta}$ are bounded on L^p for $1 < p < \infty$. In particular, $q_\Theta \in L^p$.*

Proof. (a) Note that

$$\overline{z}\overline{K_\Theta} = \overline{z}\overline{H^2 \cap \Theta\overline{zH^2}} = \overline{zH^2} \cap \overline{\Theta}H^2 = \overline{\Theta}(H^2 \cap \Theta\overline{zH^2}) = \overline{\Theta}K_\Theta.$$

Therefore, we have

$$K_\Theta \oplus \overline{zK_\Theta} = K_\Theta \oplus \overline{\Theta}K_\Theta,$$

and then, $Q_\Theta = P_\Theta + M_{\overline{\Theta}}P_\Theta M_\Theta$. In particular, we have

$$\begin{aligned} Q_\Theta(\overline{\Theta}) =& P_\Theta(\overline{\Theta}) + M_{\overline{\Theta}}P_\Theta M_\Theta(\overline{\Theta}) = P_\Theta(\overline{\Theta(0)}) + M_{\overline{\Theta}}P_\Theta 1 \\ =& (\overline{\Theta(0)} + \overline{\Theta})(1 - \overline{\Theta(0)}\Theta) = \overline{\Theta} - \overline{\Theta(0)}^2\Theta, \end{aligned}$$

and (a) is proved.

(b) Since $L^2 = \Theta H^2 \oplus \overline{\Theta H_0^2} \oplus K_\Theta \oplus \overline{zK_\Theta}$, if follows that $\mathfrak{S}_\Theta \subset K_\Theta \oplus \overline{zK_\Theta}$, and thus

$$K_\Theta \oplus \overline{zK_\Theta} = Q_\Theta(\mathfrak{S}_\Theta + \Theta H^2 + \overline{\Theta H_0^2} + \mathbb{C}\overline{\Theta}) = \mathfrak{S}_\Theta \oplus \mathbb{C}Q_\Theta(\overline{\Theta}), \tag{5.2}$$

which proves (b). Note that according to (a), one easily see that $Q_\Theta(\overline{\Theta}) \not\equiv 0$.

(c) If follows from the identity $Q_\Theta = P_\Theta + M_{\overline{\Theta}}P_\Theta M_\Theta$, that Q_Θ is bounded on L^p for all $1 < p < \infty$. Further, according to (b), we have

$$P_{\mathfrak{S}_\Theta} = Q_\Theta - q_\Theta \otimes q_\Theta, \tag{5.3}$$

and since q_Θ belongs to L^∞, we deduce that $P_{\mathfrak{S}_\Theta}$ is also bounded on L^p for $1 < p < \infty$. □

Another interesting result observed by Sarason [51] is the following.

Corollary 5.3. *Let $\phi \in L^2$. Then there exists a pair of functions $\phi_1, \phi_2 \in K_\Theta$ such that $A_\phi = A_{\phi_1 + \overline{\phi_2}}$. Moreover, if ϕ_1, ϕ_2 is one such pair, the most general such pair equals $\phi_1 + ck_0^\Theta$, $\phi_2 - \bar{c}k_0^\Theta$, where c is a complex number. In particular, ϕ_1 and ϕ_2 are uniquely determined if we fix the value of one of them at the origin.*

Proof. Decompose the symbol ϕ as $\phi = \psi_1 + \overline{\psi_2}$, with $\psi_1, \psi_2 \in H^2$. Define $\phi_\ell = P_\Theta \psi_\ell$, $\ell = 1, 2$. Observe that $\psi_1 - \phi_1$ and $\psi_2 - \phi_2$ both belong to ΘH^2. Hence

$$\phi - (\phi_1 + \overline{\phi_2}) = (\psi_1 - \phi_1) + (\overline{\psi_2 - \phi_2}) \in \Theta H^2 + \overline{\Theta H^2}.$$

Theorem 5.1 now ensures that $A_\phi = A_{\phi_1 + \overline{\phi_2}}$.

Moreover, suppose that φ_1, φ_2 are in K_Θ and $A_\phi = A_{\varphi_1 + \overline{\varphi_2}}$. Then, according to Theorem 5.1, we get that $(\varphi_1 - \phi_1) + (\overline{\varphi_2} - \overline{\phi_2})$ is in $\Theta H^2 + \overline{\Theta H^2}$. In particular, $\varphi_1 - \phi_1 \in \Theta H^2 + \overline{H^2}$. Observe now that the orthogonal projection P_Θ maps $\Theta H^2 + \overline{H^2}$ onto $\mathbb{C} k_0^\Theta$, which gives that $\varphi_1 = \phi_1 + c k_0^\Theta$ for some complex number c. Similarly, there exists a constant $d \in \mathbb{C}$ such that $\varphi_2 = \phi_2 + d k_0^\Theta$. We deduce that $c k_0^\Theta + \overline{d} \overline{k_0^\Theta}$ is in $\Theta H^2 + \overline{\Theta H^2}$. But observe that

$$c k_0^\Theta + \overline{d} \overline{k_0^\Theta} = c + \bar{d} - c\overline{\Theta(0)}\Theta - \bar{d}\Theta(0)\overline{\Theta},$$

which yields $c + \bar{d} \in \Theta H^2 + \overline{\Theta H^2}$. Take now any $f \in K_\Theta$, $f \neq 0$. According to Theorem 5.1, we get

$$0 = A_{c+\bar{d}}(f) = P_\Theta((c + \bar{d})f) = (c + \bar{d})f,$$

which implies that $c + \bar{d} = 0$, or $d = -\bar{c}$. Therefore, $\varphi_1 = \phi_1 + c k_0^\Theta$ and $\varphi_2 = \phi_2 - \bar{c} k_0^\Theta$. □

Example 5.4. Let $\Theta(z) = z^3$ and note that $K_\Theta = \bigvee\{1, z, z^2\}$. As already mentioned (see Example 3.4), in that case, the truncated Toeplitz operator on K_Θ can be viewed as operators whose matrix with respect to the orthonormal basis $\{1, z, z^2\}$ is a Toeplitz matrix. Suppose that A is a generic 3×3 Toeplitz matrix

$$\begin{pmatrix} a & d & e \\ b & a & d \\ c & b & a \end{pmatrix}.$$

Then consider $\varphi(z) = e\bar{z}^2 + d\bar{z} + a + bz + cz^2$. An easy computation shows that $A = A_\varphi$. The function φ is the standard symbol for A and we have $\varphi = \phi + \overline{\psi}$, where $\phi, \psi \in K_\Theta$ are defined as $\phi(z) = \frac{a}{2} + bz + cz^2$ and $\psi(z) = \frac{\bar{a}}{2} + \bar{d}z + \bar{e}z^2$.

6. **Algebraic characterization of truncated Toeplitz operators**

A well known result of Brown–Halmos says that $T \in \mathcal{L}(H^2)$ is a Toeplitz operator if and only if $T = STS^*$, where S is the shift operator on H^2. Sarason obtained an analogue of this result for truncated Toeplitz operators, where the compressed shift (or the model operator) S_Θ plays the role of S.

Theorem 6.1 (Sarason, [51])**.** *A bounded operator A on K_Θ belongs to $\mathcal{T}(\Theta)$ if and only if there are functions $\phi, \psi \in K_\Theta$ such that*

$$A = S_\Theta A S_\Theta^* + \phi \otimes k_0^\Theta + k_0^\Theta \otimes \psi.$$

In that case, we have $A = A_{\phi+\overline{\psi}}$.

The proof of this result is based on two lemmas.

Lemma 6.2. *For* $\phi, \psi \in K_\Theta$, *we have*

$$A_{\phi+\overline{\psi}} - S_\Theta A_{\phi+\overline{\psi}} S_\Theta^* = \phi \otimes k_0^\Theta + k_0^\Theta \otimes \psi.$$

Proof. Since S_Θ commutes with A_ϕ and S_Θ^* commutes with $A_{\overline{\psi}}$ (see Proposition 3.3), we have

$$A_{\phi+\overline{\psi}} - S_\Theta A_{\phi+\overline{\psi}} S_\Theta^* = A_\phi(I - S_\Theta S_\Theta^*) + (I - S_\Theta S_\Theta^*)A_{\overline{\psi}}.$$

By (5.1), we have $I - S_\Theta S_\Theta^* = k_0^\Theta \otimes k_0^\Theta$, which yields

$$A_{\phi+\overline{\psi}} - S_\Theta A_{\phi+\overline{\psi}} S_\Theta^* = A_\phi k_0^\Theta \otimes k_0^\Theta + k_0^\Theta \otimes A_{\overline{\psi}}^* k_0^\Theta$$

Observe now that, since $\phi \in K_\Theta$, we have $A_\phi k_0^\Theta = P_\Theta(\phi - \overline{\Theta(0)}\phi\Theta) = \phi$. According to Proposition 3.6, we have $A_{\overline{\psi}}^* = A_\psi$, and so $A_{\overline{\psi}}^* k_0^\Theta = A_\psi k_0^\Theta = \psi$, which gives the result. □

Lemma 6.3. *Let* $\phi, \psi \in K_\Theta$. *For* $f, g \in K_\Theta \cap H^\infty$, *we have*

$$\langle A_{\phi+\overline{\psi}} f, g\rangle_2 = \sum_{n=0}^{\infty} \left(\langle f, S_\Theta^n k_0^\Theta\rangle_2 \langle S_\Theta^n \phi, g, \rangle_2 + \langle f, S_\Theta^n \psi\rangle_2 \langle S_\Theta^n k_0^\Theta, g\rangle_2 \right).$$

Proof. By Lemma 6.2,

$$A_{\phi+\overline{\psi}} - S_\Theta A_{\phi+\overline{\psi}} S_\Theta^* = \phi \otimes k_0^\Theta + k_0^\Theta \otimes \psi.$$

Thus, for any integer $n \geq 0$,

$$S_\Theta^n A_{\phi+\overline{\psi}} {S_\Theta^*}^n - S_\Theta^{n+1} A_{\phi+\overline{\psi}} {S_\Theta^*}^{n+1} = S_\Theta^n \phi \otimes S_\Theta^n k_0^\Theta + S_\Theta^n k_0^\Theta \otimes S_\Theta^n \psi.$$

Now sum both sides of the equation above from $n = 0$ to $n = N$ to get

$$A_{\phi+\overline{\psi}} = \sum_{n=0}^{N} \left(S_\Theta^n \phi \otimes S_\Theta^n k_0^\Theta + S_\Theta^n k_0^\Theta \otimes S_\Theta^n \psi \right) + S_\Theta^{N+1} A_{\phi+\overline{\psi}} {S_\Theta^*}^{N+1}.$$

Thus, for $f, g \in K_\Theta \cap H^\infty$,

$$\begin{aligned}\langle A_{\phi+\overline{\psi}} f, g\rangle_2 = &\sum_{n=0}^{N} \left(\langle f, S_\Theta^n k_0^\Theta\rangle_2 \langle S_\Theta^n \phi, g, \rangle_2 + \langle f, S_\Theta^n \psi\rangle_2 \langle S_\Theta^n k_0^\Theta, g\rangle_2 \right) \\ &+ \langle A_{\phi+\overline{\psi}} {S_\Theta^*}^{N+1} f, {S_\Theta^*}^{N+1} g\rangle_2.\end{aligned}$$

It remains to show that the last summand on the right tends to 0 as $N \to \infty$. Using the fac that $A_{\overline{\psi}}$ and $A_{\overline{\phi}}$ commute with S_Θ^*, observe that

$$\langle A_{\phi+\overline{\psi}} {S_\Theta^*}^{N+1} f, {S_\Theta^*}^{N+1} g\rangle_2 = \langle {S_\Theta^*}^{N+1} f, {S_\Theta^*}^{N+1} A_{\overline{\phi}} g\rangle_2 + \langle {S_\Theta^*}^{N+1} A_{\overline{\psi}} f, {S_\Theta^*}^{N+1} g\rangle_2.$$

The desired conclusion now follows because $S^{*N} \to 0$, as $N \to \infty$, in the strong operator topology (remind that for any $h(z) = \sum_{n=0}^{\infty} a_n z^n$ in H^2, we have

$$\|S^{*N}h\|_2^2 = \left\| \sum_{n=N}^{\infty} a_n z^{n-1} \right\|_2^2 = \sum_{n=N}^{\infty} |a_n|^2 \to 0, \quad \text{as } N \to \infty.)$$

□

Proof of Theorem 6.1. It follows from Lemma 6.2 and Corollary 5.3 that every operator in $\mathcal{T}(\Theta)$ satisfies the condition of the theorem. Suppose, conversely, that A is a bounded operator on K_Θ that satisfies the condition

$$A - S_\Theta A S_\Theta^* = \phi \otimes k_0^\Theta + k_0^\Theta \otimes \psi,$$

with $\phi, \psi \in K_\Theta$. Arguing as in the proof of Lemma 6.3, we deduce that, for every integer $N \geq 0$,

$$A = \sum_{n=0}^{N} \left(S_\Theta^n \phi \otimes S_\Theta^n k_0^\Theta + S_\Theta^n k_0^\Theta \otimes S_\Theta^n \psi \right) + S_\Theta^{N+1} A {S_\Theta^*}^{N+1}.$$

Again use the fact that ${S_\Theta^*}^N \to 0$, $N \to \infty$, in the strong operator topology to see that

$$A = \sum_{n=0}^{\infty} \left(S_\Theta^n \phi \otimes S_\Theta^n k_0^\Theta + S_\Theta^n k_0^\Theta \otimes S_\Theta^n \psi \right),$$

where the series converges in the strong operator topology. Finally, by Lemma 6.3, we can conclude that $A = A_{\phi + \overline{\psi}}$, and in particular, $A \in \mathcal{T}(\Theta)$.

□

Theorem 6.1 admits an interesting corollary.

Corollary 6.4 (Sarason, [51])**.** *The set $\mathcal{T}(\Theta)$ is closed in the weak operator topology.*

Proof. Suppose the net $(A^{(\alpha)})$ in $\mathcal{T}(\Theta)$ converges weakly to the bounded operator A. By Theorem 6.1, for each index α, there are functions ϕ_α, ψ_α in K_Θ such that

$$A^{(\alpha)} - S_\Theta A^{(\alpha)} S_\Theta^* = \phi_\alpha \otimes k_0^\Theta + k_0^\Theta \otimes \psi_\alpha. \tag{6.1}$$

Moreover, by Corollary 5.3, the function ψ_α can be taken to satisfy $\psi_\alpha(0) = 0$. Then we have

$$A^{(\alpha)} k_0^\Theta - S_\Theta A^{(\alpha)} S_\Theta^* k_0^\Theta = \|k_0^\Theta\|_2^2 \phi_\alpha - \langle k_0^\Theta, \psi_\alpha \rangle_2 k_0^\Theta = \|k_0^\Theta\|_2^2 \phi_\alpha.$$

It then follows that the net (ϕ_α) converges weakly, say to a function ϕ in K_Θ. The net $(\phi_\alpha \otimes k_0^\Theta)$ thus converges in the weak operator topology, and so by (6.1), the net $(k_0^\Theta \otimes \psi_\alpha)$ also converges in the weak operator topology, implying that the net (ψ_α) converges weakly, say to a function ψ in K_Θ. Passing to the limit in (6.1), we obtain

$$A - S_\Theta A S_\Theta^* = \phi \otimes k_0^\Theta + k_0^\Theta \otimes \psi,$$

and it follows by Theorem 6.1 that $A = A_{\phi+\overline{\psi}}$ is in $\mathcal{T}(\Theta)$. □

Remark 6.5. When K_Θ is finite dimensional (which means that Θ is a finite Blaschke product), one can get more specific results using matrix representations. It has already been noticed (see Examples 3.4 and 5.4) that if $\Theta(z) = z^N$, then the corresponding truncated Toeplitz operators are the Toeplitz matrices (with respect to the standard orthonormal basis $\{1, z, \dots, z^{N-1}\}$ of K_Θ). For more general finite Blaschke products, we have the following result from [22].

Theorem 6.6 (Cima–Ross–Wogen)**.** *Let Θ be a finite Blaschke product of degree n with distincts zeros $\lambda_1, \lambda_2, \dots, \lambda_n$ and let A be any operator on K_Θ. If $M_A = (r_{i,j})_{1\le i,j\le n}$ is the matrix representation of A with respect to the basis $\{k_{\lambda_1}, k_{\lambda_2}, \dots, k_{\lambda_n}\}$, then $A \in \mathcal{T}(\Theta)$ if and only if*

$$r_{i,j} = \overline{\frac{\Theta'(\lambda_1)}{\Theta'(\lambda_i)}} \left(\frac{r_{1,i}\overline{(\lambda_1 - \lambda_i)} + r_{i,j}\overline{(\lambda_j - \lambda_1)}}{\overline{\lambda_j - \lambda_i}} \right),$$

for $1 \le i, j \le n$ and $i \ne j$.

7. Complex symmetric operators

7.1. **Truncated Toeplitz operators are complex symmetric.** A *conjugation* on a complex Hilbert space $\mathcal{H}$ is a map $C : \mathcal{H} \to \mathcal{H}$ that is conjugate-linear, involutive ($C^2 = I$) and isometric ($\langle Cx, Cy\rangle = \langle y, x\rangle$ for every $x, y \in \mathcal{H}$). We now say that $T \in \mathcal{L}(\mathcal{H})$ is *C-symmetric* if $T = CT^*C$ and *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ with respect to which T is C-symmetric. This class of operators received a lot of attention during the past few years, in particular due to the works of Garcia–Putinar [36, 37]. Among important examples of complex symmetric operators, we have the Volterra operator, normal operators, Toeplitz matrices,....

However, note that Toeplitz operators cannot be complex symmetric in general. Indeed, if an operator T is complex symmetric, then, in particular, it should satisfy

$$\dim\ker T = \dim\ker T^*.$$

But, if for instance $T = T_\varphi$, with $\varphi \in H^\infty$, $\varphi \not\equiv 0$, then $\ker T_\varphi = \{0\}$ and $\ker T_\varphi^* \ne \{0\}$ (because $T_\varphi^* k_\lambda = T_{\overline{\varphi}} k_\lambda = \overline{\varphi(\lambda)} k_\lambda$). Thus, for every $\varphi \in H^\infty$, $\varphi \not\equiv 0$, T_φ is not complex symmetric. However, the situation is dramatically different for TTO. It turns out that truncated Toeplitz operators on K_Θ are C_Θ-symmetric, where C_Θ is the natural conjugation on K_Θ introduced in (2.7).

Theorem 7.1 (Sarason, [51])**.** *Let Θ be an inner function and $A_\phi \in \mathcal{T}(\Theta)$. Then we have $A_\phi = C_\Theta A_\phi^* C_\Theta$.*

Proof. According to Proposition 3.6, for every $f, g \in K_\Theta \cap H^\infty$, we have

$$\begin{aligned}\langle C_\Theta A_\phi^* C_\Theta f, g\rangle_2 =& \langle C_\Theta g, A_\phi^* C_\Theta f\rangle_2 = \langle C_\Theta g, P_\Theta(\overline{\phi} C_\Theta f)\rangle_2\\ =& \langle C_\Theta g, \overline{\phi} C_\Theta f\rangle_2 = \langle \Theta\overline{zg}, \overline{\phi}\Theta\overline{zf}\rangle_2\\ =& \langle \overline{g}, \overline{\phi f}\rangle_2 = \langle \phi f, g\rangle_2 = \langle P_\Theta(\phi f), g\rangle_2\\ =& \langle A_\phi(f), g\rangle_2.\end{aligned}$$

Hence $A_\phi = C_\Theta A_\phi^* C_\Theta$. □

Using the conjugation C_Θ, we can give an analogue of the characterization of $\mathcal{T}(\Theta)$ given in Theorem 6.1.

Corollary 7.2. *Let Θ be an inner function and let $A \in \mathcal{L}(K_\Theta)$. Then $A \in \mathcal{T}(\Theta)$ if and only if there are functions $\phi, \psi \in K_\Theta$ such that*

$$A - S_\Theta^* A S_\Theta = \phi \otimes S^*\Theta + S^*\Theta \otimes \psi. \tag{7.1}$$

Moreover, in that case, we have $A = A_{C_\Theta\psi + \overline{C_\Theta\phi}}$.

Proof. Assume that A satisfies (7.1). If we apply on the left and on the right C_Θ to both sides of (7.1), we get

$$C_\Theta A C_\Theta - C_\Theta S_\Theta^* A S_\Theta C_\Theta = C_\Theta\phi \otimes C_\Theta S^*\Theta + C_\Theta S^*\Theta \otimes C_\Theta\psi.$$

According to Theorem 7.1 applied to S_Θ^*, we have $C_\Theta S_\Theta^* = S_\Theta C_\Theta$, which gives

$$C_\Theta A C_\Theta - S_\Theta C_\Theta A C_\Theta S_\Theta^* = C_\Theta\phi \otimes C_\Theta S^*\Theta + C_\Theta S^*\Theta \otimes C_\Theta\psi.$$

But on $\mathbb{T}$, we have

$$C_\Theta S^*\Theta = \Theta\bar{z}\overline{(\bar{z}(\Theta - \Theta(0)))} = \Theta(\overline{\Theta} - \overline{\Theta(0)}) = 1 - \overline{\Theta(0)}\Theta = k_0^\Theta.$$

Thus, if $B = C_\Theta A C_\Theta$, we obtain

$$B - S_\Theta B S_\Theta^* = C_\Theta\phi \otimes k_0^\Theta + k_0^\Theta \otimes C_\Theta\psi. \tag{7.2}$$

Then it follows from Theorem 6.1 that $B \in \mathcal{T}(\Theta)$. In particular, there exists $\chi \in L^2$ such that $B = A_\chi$. Now Theorem 7.1 implies that

$$A = C_\Theta B C_\Theta = C_\Theta A_\chi C_\Theta = A_\chi^* = A_{\overline{\chi}}.$$

In other words, $A \in \mathcal{T}(\Theta)$.

Conversely, assume that $A \in \mathcal{T}(\Theta)$. Then $B = C_\Theta A C_\Theta \in \mathcal{T}(\Theta)$ and the arguments above can be reversed showing that A must satisfy (7.1).

Moreover, according to (7.2) and Theorem 6.1, we have $C_\Theta A C_\Theta = B = A_{C_\Theta\phi + \overline{C_\Theta\psi}}$. It follows that $A = C_\Theta A_{C_\Theta\phi + \overline{C_\Theta\psi}} C_\Theta$, and we get from Theorem 7.2 that

$$A = A^*_{C_\Theta\phi + \overline{C_\Theta\psi}} = A_{C_\Theta\psi + \overline{C_\Theta\phi}}.$$

□

It should be noted that the matrix representation of a truncated Toeplitz operators A_ϕ with respect to a modified Clark basis is complex symmetric (i.e. self-transpose). This fact was first observed in [36] and developed further in [34].

It is suspected that the truncated Toeplitz operators might serve as some sort of model operator for various classes of complex symmetric operators. An increasing long list of complex symmetric operators have been proven to be unitarily equivalent to truncated Toeplitz operators. For example, Sarason [52] proved that the Volterra operator on $L^2(0,1)$, a standard example of complex symmetric operator, is unitarily equivalent to a truncated Toeplitz operator acting on the space K_Θ corresponding to the atomic inner function $\Theta(z) = \exp\left(\frac{1+z}{1-z}\right)$. Among other first examples of complex symmetric operators being unitarily equivalent to truncated Toeplitz operators are : rank-one operators, 2×2 matrices, normal operators, for $k \in \mathbb{N} \cup \{\infty\}$ the k-fold inflation of a finite Toeplitz matrix. See [32] for details and further examples. All these results yield to the following still open questions:

Question 7.3. Is every complex symmetric operator unitarily equivalent to a truncated Toeplitz operator? If not, which ones are?

One can also explore when a bounded operator is similar to a truncated Toeplitz operator. In finite dimensions, it is proved in [18] that every operator on a finite dimensional space is similar to a co-analytic truncated Toeplitz operator.

Question 7.4. Which operators are similar to truncated Toeplitz operators in infinite dimensional spaces?

7.2. **Another characterization of truncated Toeplitz operators and new examples.** We now discuss another interesting characterization of truncated Toeplitz operators, also given by Sarason [51], which yields to new examples of truncated Toeplitz operators.

We first start with three simple lemmas. In the following, we should understand $(S^*\Theta)^\perp$ as the orthogonal complement of $\mathbb{C}S^*\Theta$ in K_Θ, that is $(S^*\Theta)^\perp = K_\Theta \ominus \mathbb{C}S^*\Theta$.

Lemma 7.5. *Let $f \in H^2$. Then*

$$(f \in K_\Theta \text{ and } Sf \in K_\Theta) \Longleftrightarrow f \in (S^*\Theta)^\perp.$$

Proof. First let us assume that f and Sf belong to K_Θ. We thus have

$$\langle f, S^*\Theta\rangle_2 = \langle Sf, \Theta\rangle_2 = 0.$$

Hence $f \perp \mathbb{C}S^*\Theta$.

Conversely, assume that $f \in K_\Theta \ominus \mathbb{C}S^*\Theta$. Remember that $C_\Theta(k_0^\Theta) = S^*\Theta$, and then $k_0^\Theta = C_\Theta(S^*\Theta)$. We thus have

$$0 = \langle f, S^*\Theta\rangle_2 = \langle C_\Theta(S^*\Theta), C_\Theta f\rangle_2 = \langle k_0^\Theta, C_\Theta f\rangle_2 = \overline{(C_\Theta f)(0)}.$$

In other words, $C_\Theta f = zg$, for some $g \in H^2$. But since $g = S^*(C_\Theta f)$ and $S^*K_\Theta \subset K_\Theta$, the function g belongs indeed to K_Θ. Then we get

$$zf = zC_\Theta(zg) = z\Theta\overline{z^2 g} = \Theta\overline{zg} = C_\Theta g.$$

In particular, we deduce that $zf \in K_\Theta$. □

The second result is a simple functional analysis result.

Lemma 7.6. *Let X be a Banach space, V a dense subspace of X, and let $\varphi : X \longrightarrow \mathbb{C}$ be a linear continuous functional on X, $\varphi \not\equiv 0$. Then*

(1) *there exists $v_0 \in V$ such that $\varphi(v_0) = 1$.*
(2) *$\ker\varphi \cap V$ is dense in $\ker\varphi$.*

Proof. (1) Since V is dense in X, $\varphi|V$ is not identically zero, because otherwise by continuity φ would be identically zero, which contradicts the hypothesis. Then, in particular, there is a vector $v \in V$ such that $\lambda = \varphi(v) \neq 0$. Now take $v_0 = v/\lambda$.

(2) Let $w \in \ker\varphi$. Then we can find a sequence $(w_n)_n$ in V such that $\|w_n - w\|_X \to 0$, as $n \to \infty$. Observe now that

$$|\varphi(w_n)| = |\varphi(w_n - w)| \leq \|\varphi\|\|w_n - w\|_X,$$

which implies that $\varphi(w_n) \to 0$, as $n \to \infty$. Applying (1), let $v_0 \in V$ such that $\varphi(v_0) = 1$, and define $u_n = w_n - \varphi(w_n)v_0$. We easily see that $u_n \in V \cap \ker\varphi$, and

$$\|u_n - v\|_X \leq \|w_n - v\|_X + |\varphi(w_n)|\|v_0\|_X \to 0, \text{ as } n \to \infty.$$

□

Lemma 7.7. *Let Θ be an inner function. Then*

(1) *$K_\Theta \cap H^\infty \cap (S^*\Theta)^\perp$ is dense in $(S^*\Theta)^\perp$.*
(2) *$K_\Theta \cap A(\mathbb{D}) \cap (S^*\Theta)^\perp$ is dense in $(S^*\Theta)^\perp$.*

Proof. Of course (2) implies (1) and so we only prove (2). Recall that by Aleksandrov's theorem (see [21, Section 8.5]), the space $A(\mathbb{D}) \cap K_\Theta$ is dense in K_Θ. It remains to apply Lemma 7.6 to $V = A(\mathbb{D}) \cap K_\Theta$ and to the linear continuous functional

$$\begin{array}{rccc} \varphi : & K_\Theta & \longrightarrow & \mathbb{C} \\ & h & \longmapsto & \langle h, S^*\Theta\rangle_2, \end{array}$$

for which $\ker\varphi = (S^*\Theta)^\perp$.

□

We can now give a new characterization of operators in $\mathcal{T}(\Theta)$.

Theorem 7.8 (Sarason, [51])**.** *Let $A \in \mathcal{L}(K_\Theta)$. Then $A \in \mathcal{T}(\Theta)$ if and only if it satisfies the relation*

$$f, Sf \in K_\Theta \implies \langle ASf, Sf\rangle_2 = \langle Af, f\rangle_2. \tag{7.3}$$

Proof. First, let us assume that $A \in \mathcal{T}(\Theta)$. According to Corollary 5.3, there are $\phi, \psi \in K_\Theta$ such that $A = A_{\phi+\bar{\psi}}$. To prove that A satisfies (7.3), it is sufficient, according to Lemma 7.5 and Lemma 7.7, to check that

$$\langle ASf, Sf\rangle_2 = \langle Af, f\rangle_2,$$

for every $f \in K_\Theta \cap H^\infty \cap (S^*\Theta)^\perp$. We have

$$\langle ASf, Sf\rangle_2 = \langle A_{\phi+\overline{\psi}}Sf, Sf\rangle_2 = \langle A_\phi Sf, Sf\rangle_2 + \langle Sf, A_\psi Sf\rangle_2.$$

Since $f \in (S^*\Theta)^\perp$, according to Lemma 7.5, $Sf \in K_\Theta$, and thus

$$\langle A_\phi Sf, Sf\rangle_2 = \langle P_\Theta(\phi zf), zf\rangle_2 = \langle \phi zf, zf\rangle_2 = \langle \phi f, f\rangle_2 = \langle A_\phi f, f\rangle_2.$$

Similarly, we have

$$\langle Sf, A_\psi Sf, \rangle_2 = \langle zf, P_\Theta(\psi zf)\rangle_2 = \langle zf, \psi zf\rangle_2 = \langle \overline{\psi} f, f\rangle_2 = \langle A_{\overline{\psi}} f, f\rangle_2.$$

Therefore, we deduce

$$\langle ASf, Sf\rangle_2 = \langle A_\phi(f), f\rangle_2 + \langle A_{\overline{\psi}} f, f\rangle_2 = \langle Af, f\rangle_2.$$

For the other direction, suppose that A satisfies (7.3). We shall prove that $A \in \mathcal{T}(\Theta)$ by showing that A satisfies (7.1). Let $B = A - S_\Theta^* A S_\Theta$. Note that for every function $f \in (S^*\Theta)^\perp$, we have $S_\Theta f = Sf$ (by Lemma 7.5), and then

$$\begin{aligned}\langle Bf, f\rangle_2 =& \langle Af, f\rangle_2 - \langle S_\Theta^* A S_\Theta f, f\rangle_2\\ =& \langle Af, f\rangle_2 - \langle AS_\Theta f, S_\Theta f\rangle_2\\ =& \langle Af, f\rangle_2 - \langle ASf, Sf\rangle_2\\ =& 0.\end{aligned}$$

By the polarization identity, we get

$$\langle Bf, g\rangle_2 = 0 \qquad \text{whenever } f, g \in (S^*\Theta)^\perp.$$

Note that the orthogonal projection in K_Θ with range $(S^*\Theta)^\perp$ equals $I - cS^*\Theta \otimes S^*\Theta$, where $c = \|S^*\Theta\|_2^{-2}$. We have then

$$(I - cS^*\Theta \otimes S^*\Theta)B(I - cS^*\Theta \otimes S^*\Theta) = 0,$$

implying

$$A - S_\Theta^* A S_\Theta = B = cBS^*\Theta \otimes S^*\Theta + cS^*\Theta \otimes B^*S^*\Theta - c^2\langle BS^*\Theta, S^*\Theta\rangle_2 S^*\Theta \otimes S^*\Theta.$$

Defining $\phi = cBS^*\Theta - c^2\langle BS^*\Theta, S^*\Theta\rangle_2 S^*\Theta$ and $\psi = \bar{c}B^*S^*\Theta$, we obtain

$$A - S_\Theta^* A S_\Theta = \phi \otimes S^*\Theta + S^*\Theta \otimes \psi.$$

Corollary 7.2 finally implies that $A \in \mathcal{T}(\Theta)$. □

Using the above characterization, we can give another way of building bounded truncated Toeplitz operators. For $\mu \in \mathcal{C}_2(\Theta)$ (see Section 2.3), denote by Γ_μ^Θ the sesquilinear functional on $K_\Theta \times K_\Theta$ defined by

$$\Gamma_\mu^\Theta(f,g) = \int_{\mathbb{T}} f\bar{g}\, d\mu.$$

Observe that

$$\left|\Gamma_\mu^\Theta(f,g)\right| \leq \left(\int_{\mathbb{T}} |f|^2\, d|\mu|\right)^{1/2} \left(\int_{\mathbb{T}} |g|^2\, d|\mu|\right)^{1/2} \leq c\|f\|_2\|g\|_2,$$

because $K_\Theta \hookrightarrow L^2(|\mu|)$. In particular, there exists a bounded operator $A_\mu = A_\mu^\Theta$ on K_Θ such that

$$\langle A_\mu^\Theta f, g\rangle_2 = \int_{\mathbb{T}} f\bar{g}\, d\mu.$$

The operator A_μ is another type of truncated Toeplitz operator.

Theorem 7.9 (Sarason, [51])**.** *Let $\mu \in \mathcal{C}_2(\Theta)$. Then $A_\mu \in \mathcal{T}(\Theta)$.*

Proof. According to Theorem 7.8 and Lemma 7.7, it is sufficient to show that for every $f \in A(\mathbb{D}) \cap K_\Theta \cap (S^*\Theta)^\perp$, we have

$$\langle A_\mu(Sf), Sf\rangle_2 = \langle A_\mu(f), f\rangle_2.$$

But since $|Sf| = |f|$ on $\mathbb{T}$, we get

$$\langle A_\mu(Sf), Sf\rangle_2 = \int_{\mathbb{T}} |Sf|^2\, d\mu = \int_{\mathbb{T}} |f|^2\, d\mu = \langle A_\mu(f), f\rangle_2.$$

That proves that $A_\mu \in \mathcal{T}(\Theta)$.

□

Example 7.10. Assume that $\zeta \in \mathbb{T}$ is an angular point of Carathéodory for Θ. Then, we know (see Subsection 2.2) that $f \longmapsto f(\zeta)$ is continuous on K_Θ. In other words, the Dirac measure δ_ζ, at point ζ, is in $C_2(\Theta)$. By Theorem 7.9, the operator A_{δ_ζ} is in $\mathcal{T}(\Theta)$. Observe that if $f, g \in K_\Theta$, then

$$\langle A_{\delta_\zeta} f, g\rangle_2 = \int_{\mathbb{T}} f\bar{g} d\delta_\zeta = f(\zeta)\overline{g(\zeta)} = \langle f, k_\zeta^\Theta\rangle_2 \langle k_\zeta^\Theta, g\rangle_2,$$

where k_ζ^Θ is the reproducing kernel of K_Θ at point ζ. In other words, $A_{\delta_\zeta} f = \langle f, k_\zeta^\Theta\rangle_2 k_\zeta^\Theta$, $f \in K_\Theta$, meaning that

$$A_{\delta_\zeta} = k_\zeta^\Theta \otimes k_\zeta^\Theta.$$

Thus, in particular, A_{δ_ζ} is a rank-one TTO. We will come back in Section 10 to this operator.

A natural question, posed by Sarason [51, page 513], is whether the converse of Theorem 7.9 holds. In other words, does every bounded truncated Toeplitz operators arise from a measure $\mu \in \mathcal{C}_2(\Theta)$? This question was settled in the affirmative in a beautiful paper by Baranov, Bessonov and Kapustin [6].

Theorem 7.11 (Baranov–Bessonov–Kapustin)**.** *Let $A \in \mathcal{L}(K_\Theta)$. Then $A \in \mathcal{T}(\Theta)$ if and only if $A = A_\mu$ for some $\mu \in \mathcal{C}_2(\Theta)$.*

In the proof of this result (and in many results of [6]), a key role is played by the following Banach space X defined by

$$X = \left\{ \sum_k x_k \overline{y_k} : \, x_k, y_k \in K_\Theta, \, \sum_k \|x_k\|_2 \|y_k\|_2 < \infty \right\}, \tag{7.4}$$

where the norm in X is defined as the infimum of $\sum_k \|x_k\|_2\|y_k\|_2$ over all representations of the elements in the form $\sum_k x_k\overline{y_k}$.

8. **Norm of a truncated Toeplitz operator**

For Toeplitz operators, a classical result of Brown–Halmos says that if $\phi \in L^\infty$, then T_ϕ is bounded and $\|T_\phi\| = \|\phi\|_\infty$. In contrast to this, we can say little more than

$$\|A_\phi\| \le \|\phi\|_\infty, \tag{8.1}$$

for general truncated Toeplitz operators with bounded symbols ϕ. For instance, observe that $A_\Theta = 0$ and so

$$\|A_\Theta\| < \|\Theta\|_\infty = 1.$$

Nevertheless using Theorem 5.1, we can easily improve the estimate (8.1) and obtain

$$\|A_\phi\| \le \inf\{\|\psi\|_\infty : \phi - \psi \in \Theta H^2 + \overline{\Theta H^2}\}.$$

The fact that there are many symbols that represent the same truncated Toeplitz operator makes the problem of computing the norm difficult. However, it is possible to give lower estimates of $\|A_\phi\|$ for general ϕ in L^2. Although a variety of lower bounds on $\|A_\phi\|$ are provided in [39], we focus here on perhaps the most useful of these. Remind the definition of the Poisson integral in (2.1).

Theorem 8.1 (Garcia–Ross, [39]). *Let Θ be an inner function and $\phi \in L^2$. Then*

$$\sup\{|(\mathfrak{P}\phi)(\lambda)| : \lambda \in \mathbb{D},\, \Theta(\lambda) = 0\} \leq \|A_\phi\|,$$

where the supremum above is regarded as 0 if Θ never vanishes on $\mathbb{D}$.

Proof. Let $\lambda \in \mathbb{D}$ such that $\Theta(\lambda) = 0$. Then $k_\lambda^\Theta = k_\lambda$ and we have

$$\begin{aligned}
\|A_\phi\| &\geq \left|\langle A_\phi\left(\frac{k_\lambda}{\|k_\lambda\|_2}\right), \frac{k_\lambda}{\|k_\lambda\|_2}\rangle_2\right| \\
&= (1-|\lambda|^2)|\langle P_\Theta(\phi k_\lambda), k_\lambda\rangle_2| \\
&= (1-|\lambda|^2)|\langle \phi k_\lambda, k_\lambda\rangle_2| \\
&= (1-|\lambda|^2)\left|\int_{\mathbb{T}} \frac{\phi(\xi)}{|\xi-\lambda|^2}\, dm(\xi)\right| \\
&= |(\mathfrak{P}\phi)(\lambda)|.
\end{aligned}$$

Taking the supremum over all λ in $\mathbb{D}$ such that $\Theta(\lambda) = 0$ gives the result. □

Corollary 8.2. *Let Θ be an inner function whose zeros accumulate almost everywhere on $\mathbb{T}$ and let $\phi \in \mathbf{C}(\mathbb{T})$. Then $\|A_\phi\| = \|\phi\|_\infty$.*

Proof. Let ζ be an accumulation point of $\Theta^{-1}(\{0\})$. Then there exists a sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$, $\Theta(\lambda_n) = 0$ such that $\lambda_n \to \zeta$, $n \to \infty$. According to Theorem 8.1, for every $n \geq 1$, we have

$$|(\mathfrak{P}\phi)(\lambda_n)| \leq \|A_\phi\|.$$

Now, since $\phi \in \mathbf{C}(\mathbb{T})$, letting n go to ∞, we get from (2.2)

$$|\phi(\zeta)| \leq \|A_\phi\|.$$

But, this is true for almost every $\zeta \in \mathbb{T}$, which yields to $\|\phi\|_\infty \leq \|A_\phi\|$. Now combining with (8.1) gives the result. □

Example 8.3. Let $(t_n)_{n\geq 1}$ be a dense sequence in $[0, 2\pi]$ and let $\lambda_n = \left(1 - \frac{1}{n^2}\right) e^{it_n}$, $n \geq 1$. We easily see that $(\lambda_n)_{n\geq 1}$ is a Blaschke sequence and if we denote by B its associated Blaschke product, then its boundary spectrum $\sigma(B)$ is equal to $\mathbb{T}$. Indeed, if $e^{it} \in \mathbb{T}$, $t \in [0, 2\pi]$, we have by classical estimates

$$|e^{it} - \lambda_n| \leq |e^{it} - e^{it_n}| + \frac{1}{n^2} \leq |t - t_n| + \frac{1}{n^2}.$$

Since $(t_n)_{n\geq 1}$ is dense sequence in $[0, 2\pi]$, there exists a subsequence $(t_{n_\ell})_{\ell\geq 1}$ such that $t_{n_\ell} \to t$, as $\ell \to \infty$. Hence $\lambda_{n_\ell} \to e^{it}$, as $\ell \to \infty$. Now according to Corollary 8.2, for every $\phi \in \mathbf{C}(\mathbb{T})$, we have $\|A_\phi^B\| = \|\phi\|_\infty$.

The norm of some truncated Toeplitz operators can be related to the norm of some Hankel operators and certain classical extremal problems from function theory. We refer the reader to the papers [38, 15] for more on this. Moreover, if Θ is a finite Blaschke product (so that the corresponding model space K_Θ is finite dimensional) and ϕ belongs to H^∞, then straightforward residue computations allow us to represent A_ϕ with respect to any of the classical orthonormal bases of K_Θ (the Takeneda basis, the Clark basis,...). In any case, one can readily compute the norm of A_ϕ by computing the norm of one of its matrix representations. This approach was undertaken in the paper [38].

9. **Spectral properties of A_ϕ^Θ**

The Livsic–Möeller Theorem identified the spectrum of the model operator S_Θ with the spectrum of the inner function Θ (see (2.3)). The following result of P. Fuhrmann [28, 29] generalizes this for the analytic truncated Toeplitz operators A_ϕ, where $\phi \in H^\infty$. As we will see, the proof depends crucially on the famous Corona Theorem of L. Carleson.

Theorem 9.1 (Fuhrmann)**.** *Let Θ be an inner function and $\phi \in H^\infty$. Then*

$$\sigma(A_\phi) = \{\lambda \in \mathbb{C} : \inf_{z\in\mathbb{D}}(|\Theta(z)| + |\phi(z) - \lambda|) = 0\}.$$

Proof. Let $\lambda \in \mathbb{C}$ satisfying $\inf_{z\in\mathbb{D}}(|\Theta(z)| + |\phi(z) - \lambda|) = 0$. Then there exists a sequence $(z_n)_{n\geq 1} \subset \mathbb{D}$ such that $|\Theta(z_n)| \to 0$ and $|\phi(z_n) - \lambda| \to 0$ as $n \to \infty$. Since $\phi \in H^\infty$, we have $A_\phi^* = T_{\overline{\phi}}|K_\Theta$ and so

$$\begin{aligned}\|(A_\phi^* - \overline{\lambda}I)k_{z_n}^\Theta\|_2 &= \|T_{\overline{\phi}-\overline{\lambda}}k_{z_n}^\Theta\|_2 \\ &= \|P_+((\overline{\phi} - \overline{\lambda})(1 - \overline{\Theta(z_n)}\Theta)k_{z_n})\|_2 \\ &\leq \|P_+((\overline{\phi} - \overline{\lambda})k_{z_n})\|_2 + |\Theta(z_n)|\|P_+((\overline{\phi} - \overline{\lambda})\Theta k_{z_n})\|_2\end{aligned}$$

Now recall that for every $\psi \in H^\infty$, we have $P_+(\overline{\psi}k_\lambda) = \overline{\psi(\lambda)}k_\lambda$. Then it follows

$$\begin{aligned}\|(A_\phi^* - \overline{\lambda}I)k_{z_n}^\Theta\|_2 &\leq \|\overline{\phi(z_n)} - \overline{\lambda})k_{z_n}\|_2 + (\|\phi\|_\infty + |\lambda|)\Theta(z_n)|\|k_{z_n}\|_2 \\ &\leq C(|\phi(z_n) - \lambda| + |\Theta(z_n)|)\|k_{z_n}\|_2,\end{aligned}$$

where $C = \|\phi\|_\infty + |\lambda| + 1$. Now dividing both terms by $\|k_{z_n}^\Theta\|_2$ gives

$$\left\|(A_\phi^* - \overline{\lambda}I)\frac{k_{z_n}^\Theta}{\|k_{z_n}^\Theta\|_2}\right\|_2 \leq C(|\phi(z_n) - \lambda| + |\Theta(z_n)|)\frac{\|k_{z_n}\|_2}{\|k_{z_n}^\Theta\|_2}.$$

Observe now that $\frac{\|k_{z_n}\|_2}{\|k_{z_n}^\Theta\|_2} = (1 - |\Theta(z_n)|^2)^{-1/2} \to 1$ as $n \to +\infty$, which finally gives that

$$\left\|(A_\phi^* - \overline{\lambda}I)\frac{k_{z_n}^\Theta}{\|k_{z_n}^\Theta\|_2}\right\|_2 \to 0,\ n \to \infty.$$

In particular,

$$\inf\{\|(A_\phi^* - \overline{\lambda}I)f\|_2 : f \in K_\Theta,\ \|f\|_2 = 1\} = 0.$$

Hence $A_\phi^* - \overline{\lambda}I$ is not bounded below and hence not invertible. Thus

$$\overline{\lambda} \in \sigma(A_\phi^*) = \overline{\sigma(A_\phi)},$$

and so

$$\{\lambda \in \mathbb{C} : \inf_{z\in\mathbb{D}}(|\Theta(z)| + |\phi(z) - \lambda|) = 0\} \subset \sigma(A_\phi).$$

To prove the reverse inclusion, we will now show that

$$\inf_{z\in\mathbb{D}}(|\Theta(z)| + |\phi(z) - \lambda|) > 0 \implies \lambda \notin \sigma(A_\phi).$$

According to Carleson Corona Theorem, when $\inf_{z\in\mathbb{D}}(|\Theta(z)| + |\phi(z) - \lambda|) > 0$, there exists $f, g \in H^\infty$ such that

$$f(z)\Theta(z) + g(z)(\phi(z) - \lambda) = 1, \qquad z \in \mathbb{D}.$$

Hence, by Proposition 3.3, we get

$$A_f A_\Theta + A_g(A_\phi - \lambda I) = I.$$

But $A_\Theta = 0$ (see Lemma 4.4). Hence

$$A_g(A_\phi - \lambda) = I.$$

Since A_g and A_ϕ commute because g and ϕ are in H^∞, we also have $(A_\phi - \lambda I)A_g = I$, and so $A_\phi - \lambda I$ is invertible, showing that $\lambda \notin \sigma(A_\phi)$. □

Remind from (2.3) the definition of $\mathfrak{s}(\Theta)$, the spectrum of an inner function Θ.

Corollary 9.2. *Let Θ be an inner function and $\phi \in H^\infty \cap \mathbf{C}(\mathbb{T})$. Then*

$$\sigma(A_\phi) = \phi(\mathfrak{s}(\Theta)).$$

Proof. Let $\lambda \in \mathbb{C}$. According to Theorem 9.1, $\lambda \in \sigma(A_\phi)$ if and only if there exists a sequence $(z_n)_{n\geq 1} \subset \mathbb{D}$ satisfying

$$\Theta(z_n) \to 0 \quad \text{and} \quad \phi(z_n) \to \lambda, \quad \text{as } n \to \infty.$$

Passing to a subsequence if necessary, we may assume that $(z_n)_{n\geq 1}$ converges to some point $z_0 \in \mathbb{D}^-$ (the closed unit disc). By definition, $z_0 \in \mathfrak{s}(\Theta)$ and since ϕ is continuous on $\mathbb{D}^-$, we have $\phi(z_n) \to \phi(z_0)$, $n \to \infty$. In particular, $\lambda = \phi(z_0) \in \phi(\mathfrak{s}(\Theta))$. Hence $\sigma(A_\phi) \subset \phi(\mathfrak{s}(\Theta))$.

Conversely, if $\lambda \in \phi(\mathfrak{s}(\Theta))$, then there is $z_0 \in \mathfrak{s}(\Theta)$, such that $\lambda = \phi(z_0)$. In particular, there is a sequence $(z_n)_{n\geq 1} \subset \mathbb{D}$ such that $z_n \to z_0$ and $\Theta(z_n) \to 0$, as $n \to \infty$. By continuity of ϕ, we have once more that $\phi(z_n) \to \phi(z_0) = \lambda$ and so $\lambda \in \sigma(A_\phi)$.

□

The natural polynomial functional calculus $p(S_\Theta) = A_p$ can be extended to H^∞ in such a way that the symbol map $\phi \longmapsto \phi(S_\Theta) := A_\phi$ is linear, contractive and multiplicative (see Proposition 3.9). The above corollary says that we have a spectral mapping theorem for symbols in $H^\infty \cap \mathbf{C}(\mathbb{T})$.

Note that in the case when $\phi \in H^\infty$, it is possible also to give an alternate description of $\sigma(A_\phi)$ in terms of cluster sets for ϕ. See [35, page 294-296] for details. We can also describe the point spectrum of A_ϕ and is adjoint. See [35, Corollaries 13.15, 13.17 and 13.18] or [27, Theorem 14.34 and Corollary 14.35].

Let us mention also another interesting new approach by Camara and Partington [12]. They give invertibility and Fredholmness criterion for truncated Toeplitz operators in model spaces K^p_Θ of the upper half-plane, $1 < p < \infty$ with essentially bounded symbols in a class including the algebra $\mathbf{C}(\mathbb{R}) + H^\infty$, as well as sums of analytic and anti-analytic functions satisfying a Θ-separation condition. Their approach is based on the equivalence after extension of truncated Toeplitz operators to Toeplitz operators with 2×2 matrix symbols.

Let us conclude this section by saying that in general determine the spectrum of a truncated Toeplitz operator is a problem highly non trivial. We only know the answer for particular class of symbols and a lot remain to do.

10. **Finite rank truncated Toeplitz operators**

Brown and Halmos [11] have showed that there are no nonzero compact Toeplitz operators on H^2. In contrast, there are many example of finite rank (hence compact) truncated Toeplitz operators. In fact, the rank-one truncated Toeplitz operators were first identified by Sarason [51, Theorem 5.1]. For the proof of this result, we need the following two lemmas. Remind from Subsection 2.2 the notation $\widetilde{f}$ for $f \in K_\Theta$.

Lemma 10.1. *Let Θ be an inner function.*

(a) *For $\lambda \in \mathbb{D}$, we have*

$$S^*_\Theta k^\Theta_\lambda = \overline{\lambda} k^\Theta_\lambda - \overline{\Theta(\lambda)}\widetilde{k^\Theta_0}$$

and

$$S_\Theta \widetilde{k^\Theta_\lambda} = \lambda \widetilde{k^\Theta_\lambda} - \Theta(\lambda) k^\Theta_0.$$

(b) *For $\lambda \in \mathbb{D} \setminus \{0\}$, we have*

$$S_\Theta k^\Theta_\lambda = \frac{1}{\overline{\lambda}} k^\Theta_\lambda - \frac{1}{\overline{\lambda}} k^\Theta_0$$

and

$$S_\Theta^* \widetilde{k_\lambda^\Theta} = \frac{1}{\lambda}\widetilde{k_\lambda^\Theta} - \frac{1}{\lambda}\widetilde{k_0^\Theta}.$$

Proof. (a) For the first equality, note that for every $f, g \in H^\infty$, we have $S^*(fg) = fS^*(g) + g(0)S^*f$, which yields to

$$S_\Theta^* k_\lambda^\Theta = S^*((1 - \overline{\Theta(\lambda)}\Theta)k_\lambda) = (1 - \overline{\Theta(\lambda}\Theta)S^*k_\lambda + k_\lambda(0)S^*(1 - \overline{\Theta(\lambda)}\Theta).$$

But $S^*k_\lambda = \overline{\lambda}k_\lambda$ and $S^*\Theta = C_\Theta k_0^\Theta$, and so

$$S_\Theta^* k_\lambda^\Theta = \overline{\lambda}(1 - \overline{\Theta(\lambda)}\Theta)k_\lambda - \overline{\Theta(\lambda)}C_\Theta k_0^\Theta = \overline{\lambda}k_\lambda^\Theta - \overline{\Theta(\lambda)}C_\Theta k_0^\Theta.$$

Since $S_\Theta C_\Theta = C_\Theta S_\Theta^*$ (see Theorem 7.1), we obtain the second equality by applying C_Θ to the first one:

$$\begin{aligned} S_\Theta C_\Theta k_\lambda^\Theta &= C_\Theta(\overline{\lambda}k_\lambda^\Theta - \overline{\Theta(\lambda)}C_\Theta k_0^\Theta) \\ &= \lambda C_\Theta k_\lambda^\Theta - \Theta(\lambda)k_0^\Theta. \end{aligned}$$

(b) For $\lambda \in \mathbb{D} \setminus \{0\}$, we have

$$S_\Theta k_\lambda^\Theta = P_\Theta(zk_\lambda^\Theta) = P_\Theta(z(1 - \overline{\Theta(\lambda)}\Theta)k_\lambda) = P_\Theta(zk_\lambda).$$

Since $(1 - \overline{\lambda}z)k_\lambda = 1$, we have

$$zk_\lambda = \frac{1}{\overline{\lambda}}(k_\lambda - 1),$$

and we obtain

$$S_\Theta k_\lambda^\Theta = \frac{1}{\overline{\lambda}}P_\Theta(k_\lambda - 1) = \frac{1}{\overline{\lambda}}(k_\lambda^\Theta - k_0^\Theta),$$

which is the first equality. As in (a), the second equality is obtained from the first through an application of C_Θ. □

Lemma 10.2. *Let Θ be an inner function. Assume that Θ has an angular derivative in the sense of Carathéodory at the point $\zeta \in \mathbb{T}$. Then the equalities of the preceding lemma hold with ζ in place of λ.*

Proof. Since Θ has an angular derivative in the sense of Carathéodory at the point $\zeta \in \mathbb{T}$, it is known that $k_\lambda^\Theta \to k_\zeta^\Theta$, as λ tends nontangentially to ζ, in the H^2 norm. Now, we obtain the conclusion by letting λ tend nontangentially to ζ in the equalities of Lemma 10.1.

□

Theorem 10.3 (Sarason, [51])**.** *Let Θ be an inner function. The operators*

(i) $k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta} = A_{\frac{\overline{\Theta}}{\overline{z}-\overline{\lambda}}}$ *for* $\lambda \in \mathbb{D}$,

(ii) $\widetilde{k_\lambda^\Theta} \otimes k_\lambda^\Theta = A_{\frac{\Theta}{z-\lambda}}$ *for* $\lambda \in \mathbb{D}$,

(iii) $k_\zeta^\Theta \otimes k_\zeta^\Theta = A_{k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1}$, *where Θ has an angular derivative in the sense of Carathéodory at $\zeta \in \mathbb{T}$*

are truncated Toeplitz operators having rank one.

Proof. (i) We consider first a point $\lambda \in \mathbb{D} \setminus \{0\}$ and apply the criterion of Theorem 6.1 to the operator $k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta}$. By Lemma 10.1,

$$\begin{aligned} S_\Theta(k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta})S_\Theta^* &= S_\Theta k_\lambda^\Theta \otimes S_\Theta \widetilde{k_\lambda^\Theta} \\ &= \left(\frac{1}{\overline{\lambda}} k_\lambda^\Theta - \frac{1}{\overline{\lambda}} k_0^\Theta\right) \otimes \left(\lambda \widetilde{k_\lambda^\Theta} - \Theta(\lambda) k_0^\Theta\right) \\ &= k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta} - k_0^\Theta \otimes \widetilde{k_\lambda^\Theta} - \frac{\overline{\Theta(\lambda)}}{\overline{\lambda}} k_\lambda^\Theta \otimes k_0^\Theta + \frac{\overline{\Theta(\lambda)}}{\overline{\lambda}} k_0^\Theta \otimes k_0^\Theta. \end{aligned}$$

Thus

$$k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta} - S_\Theta(k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta})S_\Theta^* = k_0^\Theta \otimes \widetilde{k_\lambda^\Theta} + \frac{\overline{\Theta(\lambda)}}{\overline{\lambda}}((k_\lambda^\Theta - k_0^\Theta) \otimes k_0^\Theta).$$

By theorem 6.1, $k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta} = A_\varphi$ with symbol

$$\varphi = \frac{\overline{\Theta(\lambda)}}{\overline{\lambda}}(k_\lambda^\Theta - k_0^\Theta) + \overline{\widetilde{k_\lambda^\Theta}}.$$

According to Theorem 5.1, we have

$$A_{k_0^\Theta} = A_{1-\overline{\Theta(0)}\Theta} = A_1 \quad \text{and} \quad A_{k_\lambda^\Theta} = A_{(1-\overline{\Theta(\lambda)}\Theta)k_\lambda} = A_{k_\lambda},$$

whence $k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta} = A_\psi$, with ψ, written as a function of the variable $z \in \mathbb{T}$, equals to

$$\begin{aligned} \psi(z) =& \frac{\overline{\Theta(\lambda)}}{\overline{\lambda}}(k_\lambda(z) - 1) + \overline{\widetilde{k_\lambda^\Theta}(z)} \\ =& \frac{\overline{\Theta(\lambda)}}{\overline{\lambda}}\left(\frac{1}{1-\overline{\lambda}z} - 1\right) + \frac{\overline{\Theta(z)} - \overline{\Theta(\lambda)}}{\overline{z} - \overline{\lambda}} \\ =& \overline{\Theta(\lambda)}\frac{z}{1-\overline{\lambda}z} + \frac{z(\overline{\Theta(z)} - \overline{\Theta(\lambda)})}{1-\overline{\lambda}z} \\ =& \frac{z\overline{\Theta(z)}}{1-\overline{\lambda}z} = \frac{\overline{\Theta(z)}}{\overline{z} - \overline{\lambda}} \end{aligned}$$

Taking the limit as $\lambda \to 0$, we find that $k_0^\Theta \otimes \widetilde{k_0^\Theta}$ is the truncated Toeplitz operator with symbol $\frac{\overline{\Theta(z)}}{\overline{z}}$.

(ii) Conjugating the identity proved in (a) conclusion, we find that for every $\lambda \in \mathbb{D}$

$$\widetilde{k_\lambda^\Theta} \otimes k_\lambda^\Theta = (k_\lambda^\Theta \otimes \widetilde{k_\lambda^\Theta})^* = A_\psi^* = A_{\overline{\psi}},$$

where $\overline{\psi(z)} = \frac{\Theta(z)}{z-\lambda}$.

(iii) Let ζ be a point of $\mathbb{T}$ at which Θ has an angular derivative in the sense of Carathéodory. By Lemma 10.2, the first part of the proof of (i) can be repeated with ζ in place of λ to show that $k_\zeta \otimes \widetilde{k_\zeta^\Theta}$ is the truncated Toeplitz operator with symbols

$$\frac{\overline{\Theta(\zeta)}}{\bar{\zeta}}(k_\zeta^\Theta - k_0^\Theta) + \widetilde{\overline{k_\zeta^\Theta}}.$$

Using that $\widetilde{k_\zeta^\Theta} = \bar{\zeta}\Theta(\zeta)k_\zeta^\Theta$, we see that $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is the truncated Toeplitz operator with symbol $k_\zeta^\Theta + \overline{k_\zeta^\Theta} - k_0^\Theta$. As above in the proof of (i), we can replace k_0^Θ by 1, obtaining the symbol $k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1$. □

It turns out that Sarason proved [51, Theorem 5.1] that Theorem 10.3 gives all the possible rank-one operators in $\mathcal{T}(\Theta)$. In other words, any truncated Toeplitz operators of rank one is a nonzero scalar multiple of one of the above. We should also mention the somewhat more involved results of Sarason [51, Theorems 6.1 and 6.2] which identify a variety of natural finite-rank truncated Toeplitz operators. Finally, the full classification of the finite-rank truncated Toeplitz operators was given by Bessonov in [8].

We would like to finish this section by giving an example of a rank-two operator $A \in \mathcal{T}(\Theta)$ such that $A^2 \notin \mathcal{T}(\Theta)$.

Example 10.4. Let Θ be an inner function such that $\dim(K_\Theta) \geq 3$. Consider $A = k_0^\Theta \otimes \widetilde{k_0^\Theta} + \widetilde{k_0^\Theta} \otimes k_0^\Theta$. Then $A \in \mathcal{T}(\Theta)$ but $A^2 \notin \mathcal{T}(\Theta)$.

Indeed, the fact that $A \in \mathcal{T}(\Theta)$ follows directly from Theorem 10.3. Let us now check that A^2 is not a truncated Toeplitz operator. Argue by absurd, and assume that $A^2 \in \mathcal{T}(\Theta)$. We have

$$\begin{aligned} A^2 = \langle k_0^\Theta, \widetilde{k_0^\Theta}\rangle_2 (k_0^\Theta \otimes \widetilde{k_0^\Theta}) + \langle \widetilde{k_0^\Theta}, k_0^\Theta\rangle_2 (\widetilde{k_0^\Theta} \otimes k_0^\Theta) \\ + \|\widetilde{k_0^\Theta}\|_2^2 (k_0^\Theta \otimes k_0^\Theta) + \|k_0^\Theta\|_2^2 (\widetilde{k_0^\Theta} \otimes \widetilde{k_0^\Theta}). \end{aligned}$$

As $k_0^\Theta \otimes \widetilde{k_0^\Theta}$ and $\widetilde{k_0^\Theta} \otimes k_0^\Theta$ are in $\mathcal{T}(\Theta)$ and $\|k_0^\Theta\|_2 = \|\widetilde{k_\Theta^\Theta}\|_2$ (because C_Θ is an isometry), it follows that the operator $B := k_0^\Theta \otimes k_0^\Theta + \widetilde{k_0^\Theta} \otimes \widetilde{k_0^\Theta}$ is in $\mathcal{T}(\Theta)$. It now follows from Theorem 6.1 that the operator $B - S_\Theta B S_\Theta^*$ has rank one or two. By Lemma 10.1

$$\begin{aligned} S_\Theta(\widetilde{k_0^\Theta} \otimes \widetilde{k_0^\Theta})S_\Theta^* &= S_\Theta \widetilde{k_0^\Theta} \otimes S_\Theta \widetilde{k_0^\Theta} \\ &= (-\Theta(0)k_0^\Theta) \otimes (-\Theta(0)k_0^\Theta) = |\Theta(0)|^2 k_0^\Theta \otimes k_0^\Theta, \end{aligned}$$

and so

$$B - S_\Theta B S_\Theta^* = \widetilde{k_0^\Theta} \otimes \widetilde{k_0^\Theta} + (1 - |\Theta(0)|^2) k_0^\Theta \otimes k_0^\Theta - S_\Theta k_0^\Theta \otimes S_\Theta k_0^\Theta.$$

Observe that k_0^Θ is a cyclic vector for the operator S_Θ. Indeed, let $f \in K_\Theta$ and assume that $f \perp S_\Theta^n k_0^\Theta$ for all non negative integers n. Then, we have

$$0 = \langle f, P_\Theta(z^n k_0^\Theta)\rangle_2 = \langle f, z^n(1 - \overline{\Theta(0)}\Theta)\rangle_2 = \langle f, z^n\rangle_2$$

which says that all the Taylor coefficients of f at the origin vanish, hence that $f = 0$.

In particular, we get that k_0^Θ and $S_\Theta k_0^\Theta$ are linearly independent. If $\widetilde{k_0^\Theta}$ is not a linear combination of k_0^Θ and $S_\Theta k_0^\Theta$, then the operator $B - S_\Theta B S_\Theta^*$ would have rank three which is not possible. Hence there are scalar a and b such that $\widetilde{k_0^\Theta} = ak_0^\Theta + bS_\Theta k_0^\Theta$. Applying S_Θ to the last equality and using Lemma 10.1 again, we get

$$-\Theta(0)k_0^\Theta = aS_\Theta k_0^\Theta + bS_\Theta^2 k_0^\Theta.$$

Since k_0^Θ and $S_\Theta k_0^\Theta$ are linearly independent, $b \neq 0$. Hence $S_\Theta^2 k_0^\Theta$ is a linear combination of k_0^Θ and $S_\Theta k_0^\Theta$. By the cyclicity of k_0^Θ for S_Θ, we obtain that $\dim(K_\Theta) = 2$, a contradiction. Therefore, A^2 is not in $\mathcal{T}(\Theta)$.

11. **Compact truncated Toeplitz operators**

Surprisingly enough, the first result about compactness of truncated Toeplitz operators dates from 1970. In [1, Section 5] Ahern–Clark introduced what are, in our terminology, truncated Toeplitz operators with continuous symbol, and they proved the following result.

Theorem 11.1 (Ahern–Clark)**.** . *Let Θ be an inner function and $\phi \in \mathbf{C}(\mathbb{T})$. Then A_ϕ is compact if and only if $\phi|\sigma(\Theta) \equiv 0$.*

This result has been rediscovered more recently in [33]; see also [32]. We follow the proof given in [33].

Proof. Suppose that $\phi|\sigma(\Theta) \equiv 0$. Let $\varepsilon > 0$ and pick $\psi \in \mathbf{C}(\mathbb{T})$ such that ψ vanishes on an open set containing $\sigma(\Theta)$ and $\|\phi - \psi\|_\infty \leq \varepsilon$. Since $\|A_\phi - A_\psi\| \leq \|\phi - \psi\|_\infty \leq \varepsilon$, and the set of compact operators is norm closed in $\mathcal{L}(K_\Theta)$, it suffices to show that A_ψ is compact. Take a sequence $(f_n)_{n\geq 1}$ in K_Θ that tends weakly to zero, and let us check that $A_\psi f_n \to 0$ in norm.

To this end, let K denote the closure of $\psi^{-1}(\mathbb{C} \setminus \{0\})$, and observe that $K \subset \mathbb{T} \setminus \sigma(\Theta)$. Then, we know that for every $\zeta \in K$, the function

$$k_\zeta^\Theta(z) = \frac{1 - \overline{\Theta(\zeta)}\Theta(z)}{1 - \bar{\zeta}z},$$

belongs to K_Θ and for every $f \in K_\Theta$, we have

$$f(\zeta) = \langle f, k_\zeta^\Theta \rangle_2$$

and

$$\|k_\zeta^\Theta\|_2^2 = \frac{1 - |\Theta(\zeta)|^2}{1 - |\zeta|^2} = |\Theta'(\zeta)|.$$

In particular, since $(f_n)_n$ converges weakly to zero in K_Θ, we have $f_n(\zeta) = \langle f_n, k_\zeta^\Theta\rangle_2 \to 0$ as $n \to \infty$ and there exists a constant $C > 0$ such that for every $n \geq 1$, $\|f_n\|_2 \leq C$. Since Θ is analytic on a neighborhood of K, we obtain

$$|f_n(\zeta)| = |\langle f_n, k_\zeta^\Theta\rangle_2| \leq \|f_n\|_2 \, \|k_\zeta^\Theta\|_2 \leq C \sup_{\zeta\in K} \sqrt{|\Theta'(\zeta)|} < \infty, \tag{11.1}$$

for every $\zeta \in K$. By the Dominated Convergence Theorem, it follows that

$$\|A_\psi f_n\|_2^2 = \|P_\Theta(\psi f_n)\|_2^2 \leq \|\psi f_n\|_2^2 = \int_K |\psi|^2 |f_n|^2 \, dm \to 0,$$

as $n \to \infty$, whence $A_\psi f_n$ tends to zero in norm, as desired.

Conversely, suppose that $\phi \in \mathbf{C}(\mathbb{T})$ and A_ϕ is compact. Let

$$\kappa_\lambda = \frac{k_\lambda^\Theta}{\|k_\lambda^\Theta\|_2}$$

be the normalized reproducing kernel for K_Θ at point λ, and define

$$F_\lambda(z) = \frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2} \left| \frac{1-\overline{\Theta(\lambda)}\Theta(z)}{1-\bar\lambda z} \right|^2,$$

Observe that $F_\lambda(z) \geq 0$ and

$$\frac{1}{2\pi} \int_0^{2\pi} F_\lambda(e^{it}) \, dt = 1.$$

Suppose that $\xi = e^{i\alpha} \in \sigma(\Theta)$. By definition, there is a sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ such that $\lambda_n \to \xi$ and $|\Theta(\lambda_n)| \to 0$. If $|t-\alpha| \geq \delta > 0$, then

$$F_{\lambda_n}(e^{it}) \leq C_\delta \frac{1-|\lambda_n|^2}{1-|\Theta(\lambda_n)|^2},$$

for some positive absolute constant C_δ. In particular, since $|\Theta(\lambda_n)| \to 0$, as $n \to \infty$, we get

$$\sup_{|t-\alpha|\geq\delta} F_{\lambda_n}(e^{it}) \to 0, \quad \text{as } n \to \infty. \tag{11.2}$$

First we show that

$$\lim_{n\to\infty} \left| \phi(\xi) - \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{it}) F_{\lambda_n}(e^{it}) \, dt \right| = 0. \tag{11.3}$$

To do this, note that

$$\begin{aligned}\left|\phi(\xi) - \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it})F_{\lambda_n}(e^{it})\,dt\right| =& \frac{1}{2\pi}\left|\int_0^{2\pi}(\phi(\xi)-\phi(e^{it}))F_{\lambda_n}(e^{it})\,dt\right| \\ &\le \frac{1}{2\pi}\int_{|t-\alpha|\le\delta}|\phi(\xi)-\phi(e^{it})|F_{\lambda_n}(e^{it})\,dt+ \\ &+\frac{1}{2\pi}\int_{|t-\alpha|\ge\delta}|\phi(\xi)-\phi(e^{it})|F_{\lambda_n}(e^{it})\,dt \\ &\le \frac{1}{2\pi}\int_{|t-\alpha|\le\delta}|\phi(\xi)-\phi(e^{it})|F_{\lambda_n}(e^{it})\,dt+ \\ &+2\|\phi\|_\infty \sup_{|t-\alpha|\ge\delta} F_{\lambda_n}(e^{it}).\end{aligned}$$

The first integral can be made small by the absolute continuity of ϕ (choosing an appropriate δ) and the fact that F_{λ_n} always integrate to one. Once $\delta > 0$ is fixed, the second term goes to zero by (11.2). This verifies (11.3).

Next, we show that

$$\lim_{n\to\infty}\int_{\mathbb{T}} \phi F_{\lambda_n}\,dm = 0. \tag{11.4}$$

We need the fact that $\kappa_{\lambda_n} \to 0$ weakly in K_Θ. To prove this, note that, for every $z \in \mathbb{D}$, we have

$$\kappa_{\lambda_n}(z) = \frac{(1-|\lambda_n|^2)^{1/2}}{(1-|\Theta(\lambda_n)|^2)^{1/2}}\frac{1-\overline{\Theta(\lambda_n)}\Theta(z)}{1-\overline{\lambda_n}z},$$

and since $\lambda_n \to \zeta \in \mathbb{T}$ and $\Theta(\lambda_n) \to 0$, $n \to \infty$, we get that

$$\kappa_{\lambda_n}(z) \to 0, \qquad \text{as } n \to \infty.$$

On the other hand, since $(\kappa_{\lambda_n})_{n\ge1}$ is norm bounded (it is of norm 1 for every n), we deduce that $\kappa_{\lambda_n} \to 0$ weakly in H^2, whence in K_Θ.

Now to verify (11.4), observe that

$$\left|\int_{\mathbb{T}} \phi F_{\lambda_n}\,dm\right| = |\langle \phi\kappa_{\lambda_n}, \kappa_{\lambda_n}\rangle_2| = |\langle P_\Theta(\phi\kappa_{\lambda_n}), \kappa_{\lambda_n}\rangle_2| = |\langle A_\phi\kappa_{\lambda_n}, \kappa_{\lambda_n}\rangle_2|.$$

Then, an application of Cauchy-Schwartz inequality yields

$$\left|\int_{\mathbb{T}} \phi F_{\lambda_n}\,dm\right| \le \|A_\phi\kappa_{\lambda_n}\|_2\|\kappa_{\lambda_n}\|_2 = \|A_\phi\kappa_{\lambda_n}\|_2.$$

Now use the fact that A_ϕ is compact and $\kappa_{\lambda_n} \to 0$ weakly as $n \to \infty$ to conclude that $\|A_\phi\kappa_{\lambda_n}\|_2 \to 0$. This proves (11.4).

Combining (11.3) with (11.4) shows that $\phi(\xi) = 0$ for every $\xi \in \sigma(\Theta)$, and completes the proof the theorem. □

Remark 11.2. Note that Theorem 11.1 admits a more precise version: if $\phi \in \mathbf{C}(\mathbb{T})$, then the essential spectrum of A_ϕ is

$$\sigma_e(A_\phi) = \phi(\sigma(\Theta), \tag{11.5}$$

and

$$\|A_\phi\|_e = \sup_{\zeta \in \sigma(\Theta)} |\phi(\zeta)|.$$

This can be proved using the explicit triangularization theory developed by Ahern and Clark in [1] (see also the exposition in [46, Lecture V]). Recently, Garcia–Ross–Wogen gave an algebraic proof of these formulae using the unital C^*-algebra generated by S_Θ.

Thinking of Hartmann's theorem, it seems plausible to believe that continuous symbols play for compact TTOs the role played by bounded symbols for general TTOs. However, as we will see in the next section, there exist inner functions Θ for which even rank-one operators might not have bounded symbols (not to speak about continuous). So we have to consider only certain classes of inner functions, suggested by the boundedness results in the previous section. In this sense one has the following result proved by Bessonov [9].

Theorem 11.3 (Bessonov)**.** *Let Θ be an inner function and let $\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$. Then*

$$\sigma_e(A_\varphi) = \phi(\sigma(\Theta)).$$

Here we should understand $\phi(\sigma(\Theta) \cap \mathbb{T})$ as

$$\phi(\sigma(\Theta)) = \left\{ \zeta \in \mathbb{C} : \liminf_{z \in \mathbb{D}, |z| \to 1} (|(\mathfrak{P}\phi)(z) - \zeta| + |\Theta(z)| = 0 \right\},$$

where $\mathfrak{P}\phi$ is the Poisson extension of ϕ, see (2.1)*.*

The main step in the proof of Theorem 11.3 is an application of a corona Theorem for the algebra $H^\infty + \mathbf{C}(\mathbb{T})$ obtained by Mortini and Wick [44].

If we impose some conditions on the inner function, one can get some characterization of compact truncated Toeplitz operators. Remind here, from Subsection 2.3, the classes of Carleson measures $\mathcal{C}_p(\Theta)$, $p \geq 1$.

Theorem 11.4 (Bessonov)**.** *Let Θ be an inner function such that $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta)$, and let A be a truncated Toeplitz operator on K_Θ. Then the following are equivalent:*

1. *A is compact.*
2. *$A = A_{\phi\Theta}$ for some $\phi \in \mathbf{C}(\mathbb{T})$.*

In particular, this characterization of compact TTO applies when Θ is one component.

One can see that instead of $\mathbf{C}(\mathbb{T})$ the main role is played by $\Theta\mathbf{C}(\mathbb{T})$. We give below some ideas about the connection between these two classes.

Theorem 11.5 (Chalendar-Fricain-Timotin, [17]). *Let Θ be an inner function such that $\mathcal{C}_2(\Theta) = \mathcal{C}_1(\Theta^2)$ and $m(\sigma(\Theta)) = 0$, and let $A \in \mathcal{T}(\Theta)$. The following are equivalent:*

(i) *A is compact.*

(ii) *$A = A_\phi$ for some $\phi \in \mathbf{C}(\mathbb{T})$ with $\phi|\sigma(\Theta) = 0$.*

Proof. (ii) $\Longrightarrow$ (i) is proved in Theorem 11.1.

Suppose now (i) is true. By Theorem 11.4, there is a $\psi \in \mathbf{C}(\mathbb{T})$ such that $A = A_{\Theta\psi}$. By the Rudin–Carleson interpolation theorem (see, for instance, [30, Theorem II.12.6]), there exists a function $\psi_1 \in \mathbf{C}(\mathbb{T}) \cap H^\infty$ (that is, in the disk algebra) such that $\psi|\sigma(\Theta) = \psi_1|\sigma(\Theta)$. Then one checks easily that $\phi = \Theta(\psi - \psi_1)$ is continuous on $\mathbb{T}$, $\phi|\sigma(\Theta) = 0$, and $A_\phi = A_{\Theta\psi}$ (since $A_{\Theta\psi_1} = 0$). □

In particular, Theorem 11.5 applies to the case when Θ one-component, since for such functions we have $\mathcal{C}_2(\Theta) = \mathcal{C}_1(\Theta^2)$ and $m(\rho(\Theta)) = 0$ [2, Theorem 6.4].

We also have the following result which is contained in [9, Proposition 2.1]; it was revisited in [17].

Proposition 11.6 (Bessonov). *Let Θ be an inner function.*

(i) *If $\phi \in \Theta\mathbf{C}(\mathbb{T}) + \Theta H^\infty$, then A_ϕ is compact.*

(ii) *If $\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$, then the converse is also true.*

Proof. (i) Assume that $\phi \in \Theta\mathbf{C}(\mathbb{T}) + \Theta H^\infty$. Hence $\overline{\Theta}\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$, which implies by Hartmann's theorem that $H_{\overline{\Theta}\phi}$ is compact. On the other hand, since $\mathbf{C}(\mathbb{T}) + H^\infty$ is an algebra and $\Theta \in H^\infty \subset \mathbf{C}(\mathbb{T}) + H^\infty$, we deduce that the function $\phi = \Theta\overline{\Theta}\phi$ also belongs to $\mathbf{C}(\mathbb{T}) + H^\infty$. Another application of Hartmann's theorem gives that the operator H_ϕ is also compact. The fact that A_ϕ is compact follows now from the relation

$$A_\phi = (\Theta H_{\overline{\Theta}\phi} - H_\phi)|K_\Theta. \tag{11.6}$$

To prove (11.6), observe that for every $f \in K_\Theta$, we have

$$\begin{aligned}(\Theta H_{\overline{\Theta}\phi} - H_\phi)f =& \Theta P_-(\overline{\Theta}\phi f) - P_-(\phi f)\\ =& \Theta(\overline{\Theta}\phi f - P_+(\overline{\Theta}\phi f)) - (\phi f - P_+(\phi f)\\ =& P_+(\phi f) - \Theta P_+(\overline{\Theta}\phi f)\\ =& P_\Theta(\phi f)\\ =& A_\phi(f).\end{aligned}$$

(ii) Assume now that $\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$ and also that A_ϕ is compact. Then, by (11.6), the operator $\Theta H_{\overline{\Theta}\phi}|K_\Theta = A_\phi + H_\phi|K_\Theta$ is also compact, whence $H_{\overline{\Theta}\phi}|K_\Theta$ is compact. Let us now remark that

$$H_{\overline{\Theta}\phi} = H_\phi\overline{\Theta}P_{\Theta H^2} + H_{\overline{\Theta}\phi}P_\Theta|H^2, \tag{11.7}$$

where $P_{\Theta H^2}$ denotes the orthogonal projection from H^2 onto ΘH^2. Indeed, let $f \in H^2$, and decompose f as $f = \Theta g_1 + g_2$, where $g_1 \in H^2$ and $g_2 \in K_\Theta$. Then

$$\begin{aligned} H_{\overline{\Theta}\phi} f =& P_-(\overline{\Theta}\phi\Theta g_1) + P_-(\overline{\Theta}\phi g_2) \\ =& P_-(\phi g_1) + P_-(\overline{\Theta}\phi g_2) \\ =& H_\phi g_1 + H_{\overline{\Theta}\phi} g_2, \end{aligned}$$

and it remains to observe that $g_1 = \overline{\Theta} P_{\Theta H^2} f$ and $g_2 = P_\Theta f$. Now it follows from (11.7) that $H_{\overline{\Theta}\phi}$ is compact, and Hartmann's theorem implies that $\overline{\Theta}\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$. □

Finally, let us mention that the problem of deciding when certain truncated Toeplitz operators are in Schatten–von-Neumann classes $\mathcal{S}_p$ has no clear solution, yet, even in the usually simple case of the Hilbert–Schmidt ideal. In [42], Lopatto–Rochberg give criteria for particular cases. For the case of one-component inner functions Θ, a conjecture is proposed in [9] for the characterization of Schatten–von Neumann TTOs in K_Θ. It states essentially that a truncated Toeplitz operator is in S_p if and only if it has at least one symbol ϕ in the Besov space $B_{pp}^{1/p}$ (note that this would not necessarily be the standard symbol). This last space admits several equivalent characterizations; for instance, if we define, for $\tau \in \mathbb{T}$, $\Delta_\tau f(z) = f(\tau z) - f(z)$, then

$$B_{p,p}^{1/p} = \left\{ f \in L^p : \int_{\mathbb{T}} \frac{\|\Delta_\tau f\|_p^p}{|1-\tau|^2}\, dm(\tau) < \infty \right\}.$$

The conjecture is suggested by the similar result in the case of Hankel operators [47, Chapter 6]. It is true if $\Theta(z) = e^{\frac{z+1}{z-1}}$, as shown by Bessonov in [10]. Bessonov also proposes some alternate characterizations in terms of Clark measures.

12. **Problem of the existence of a bounded symbol**

Recall that a famous result of Brown–Halmos says that a Toeplitz operator T_ϕ on H^2 is bounded if and only if its symbol ϕ is in L^∞. Moreover, we have $\|T_\phi\| = \|\phi\|_\infty$. In other words, the map $\phi \mapsto T_\phi$ is isometric from L^∞ **onto** the space of bounded Toeplitz operator on H^2. In the case of truncated Toeplitz operators, the map $\phi \mapsto A_\phi^\Theta$ is again contractive from L^∞ **into** $\mathcal{T}(\Theta)$. It is then natural to ask whether it is onto, that is whether any bounded truncated Toeplitz operator on K_Θ possesses an L^∞ symbol. This question was addressed by Sarason in [51]. As one may expect, the answer will depend on the inner function Θ. In [7], Baranov, Chalendar, Fricain, Mashreghi and Timotin give an answer to this question by constructing an example of a rank-one truncated Toeplitz operator that has no bounded symbols. The construction is based on the following crucial lemma.

Lemma 12.1 ([7], Lemma 5.2)**.** *Let Θ be an inner function and $1 < p < \infty$. There exists a constant C depending only on Θ and p such that if $\phi, \psi \in L^2$ are two symbols for the same truncated Toeplitz operator, with $\phi \in K_\Theta \oplus \bar{z}\overline{K_\Theta}$, then*

$$\|\phi\|_p \leq C(\|\psi\|_p + \|\phi\|_2).$$

In particular, if $\psi \in L^p$, then $\phi \in L^p$.

Proof. Remind that $\mathfrak{S}_\Theta = L^2 \ominus (\Theta H^2 + \overline{\Theta H^2})$. By hypothesis and Theorem 5.1, we have $P_{\mathfrak{S}_\Theta}\phi = P_{\mathfrak{S}_\Theta}\psi$; therefore, using (5.3),

$$\phi = Q_\Theta \phi = P_{\mathfrak{S}_\Theta}\phi + \langle \phi, q_\Theta\rangle_2 q_\Theta = P_{\mathfrak{S}_\Theta}\psi + \langle \phi, q_\Theta\rangle_2 q_\Theta.$$

By Lemma 5.2, we have

$$\|P_{\mathfrak{S}_\Theta}\psi\|_p \leq C_1\|\psi\|_p,$$

while

$$\|\langle \phi, q_\Theta\rangle_2 q_\Theta\|_p \leq \|\phi\|_2 \|q_\Theta\|_p,$$

whence the lemma follows. □

In Theorem 10.3, it is proved that if $\zeta \in E(\Theta)$ (remind that it means that Θ has an angular derivative in the sense of Carathéodory at ζ), then $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a rank-one truncated Toeplitz operator with symbol $k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1$. Lemma 12.1 suggests to look for the symbol in $K_\Theta \oplus \overline{zK_\Theta}$.

Lemma 12.2. *Let $\zeta \in E(\Theta)$ and define $\phi_\zeta = \zeta\overline{\Theta(\zeta)}\Theta\overline{zk_\zeta^{\Theta^2}}$. Then ϕ_ζ belongs to $K_\Theta \oplus \overline{zK_\Theta}$ and $A_{\phi_\zeta} = k_\zeta^\Theta \otimes k_\zeta^\Theta$.*

Proof. First note that if $\zeta \in E(\Theta)$, then by Lemma 2.2, $\zeta \in E(\Theta^2)$ and $k_\zeta^{\Theta^2} \in K_{\Theta^2}$. Now, using that $K_{\Theta^2} = K_\Theta \oplus \Theta K_\Theta$, we can write $k_\zeta^{\Theta^2} = f + \Theta g$, with $f, g \in K_\Theta$. Then

$$\Theta\overline{zk_\zeta^{\Theta^2}} = \Theta(\overline{zf} + \overline{\Theta zg}) = \Theta\overline{zf} + \overline{zg} = C_\Theta(f) + \overline{zg}.$$

Since $C_\Theta(f) \in K_\Theta$, we deduce that $\phi_\zeta \in K_\Theta \oplus \overline{zK_\Theta}$.

It remains to prove that ϕ_ζ is a symbol for $k_\zeta^\Theta \otimes k_\zeta^\Theta$. According to Theorem 5.1, this is equivalent to $\phi_\zeta - (k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1) \in \Theta H^2 + \overline{\Theta H^2}$. First note that for almost all points $z, \zeta \in \mathbb{T}$, we have

$$\frac{1}{1-\overline{\zeta}z} + \frac{1}{1-\overline{z}\zeta} = \frac{\zeta\overline{z}}{\zeta\overline{z}-1} + \frac{1}{1-\overline{z}\zeta} = 1.$$

Hence

$$\begin{aligned}\phi_\zeta - (k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1) =& \zeta\overline{\Theta(\zeta)}\Theta(z)\overline{z}\frac{1-\Theta^2(\zeta)\overline{\Theta^2(z)}}{1-\zeta\overline{z}} - \left(\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\overline{\zeta}z} + \frac{1-\Theta(\zeta)\overline{\Theta(z)}}{1-\zeta\overline{z}} - 1\right)\\ =& \zeta\overline{\Theta(\zeta)}\Theta(z)\overline{z}\frac{1-\Theta^2(\zeta)\overline{\Theta^2(z)}}{1-\zeta\overline{z}} + \frac{\overline{\Theta(\zeta)}\Theta(z)}{1-\overline{\zeta}z} + \frac{\Theta(\zeta)\overline{\Theta(z)}}{1-\zeta\overline{z}}\\ =& \frac{\overline{\Theta(\zeta)}\Theta(z)}{\overline{\zeta}z-1} - \zeta\overline{z}\frac{\Theta(\zeta)\overline{\Theta(z)}}{1-\zeta\overline{z}} + \frac{\overline{\Theta(\zeta)}\Theta(z)}{1-\overline{\zeta}z} + \frac{\Theta(\zeta)\overline{\Theta(z)}}{1-\zeta\overline{z}}\\ =& \frac{\Theta(\zeta)\overline{\Theta(z)}}{1-\zeta\overline{z}}(1-\zeta\overline{z})\\ =& \Theta(\zeta)\overline{\Theta(z)}.\end{aligned}$$

Therefore $\phi_\zeta - (k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1) \in \overline{\Theta H^2}$ and then $A_{\phi_\zeta} = A_{k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1} = k_\zeta^\Theta \otimes k_\zeta^\Theta$.

□

We are now ready to answer negatively to the question of Sarason.

Theorem 12.3 (Baranov-Chalendar-Fricain-Mashreghi-Timotin, [7])**.** *Suppose that Θ is an inner function which has an angular derivative in the sense of Carathéodory at $\zeta \in \mathbb{T}$. Let $p \in (2, \infty)$. Then, the following are equivalent:*

(i) *the bounded truncated Toeplitz operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ has a symbol $\psi \in L^p$;*
(ii) *$k_\zeta^\Theta \in L^p$.*

In particular, if $k_\zeta^\Theta \notin L^p$, for some $p \in (2, \infty)$, then $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a bounded truncated Toeplitz operator with no bounded symbol.

Proof. According to Lemma 12.2, a symbol for the operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is $\phi_\zeta = \overline{\Theta(\zeta)}\zeta\Theta\overline{zk_\zeta^{\Theta^2}}$. Since by Lemma 2.2, $\phi \in L^p$ if and only if $k_\zeta^\Theta \in L^p$, we obtain that (ii) implies (i). Conversely, assume that the bounded truncated Toeplitz operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ has a symbol $\psi \in L^p$. Since ϕ_ζ is a symbol in $K_\Theta \oplus \overline{zK_\Theta}$, we may then apply Lemma 12.1 and obtain that $\phi \in L^p$. Once again according to Lemma 2.2, we get that $k_\zeta^\Theta \in L^p$, which proves that (i) implies (ii). □

To obtain a bounded truncated Toeplitz operator with no bounded symbol, it is now sufficient to have a point $\zeta \in \mathbb{T}$ such that (2.5) is true for $p = 2$ but not for some strictly larger value of p. It is easy to give concrete examples, as, for instance:

(1) a Blaschke product with zeros a_k accumulating to the point 1, and such that

$$\sum_k \frac{1-|a_k|^2}{|1-a_k|^2} < \infty, \quad \sum_k \frac{1-|a_k|^2}{|1-a_k|^p} = \infty \quad \text{for some } p > 2;$$

(2) a singular function $\sigma = \sum_k c_k \delta_{\zeta_k}$ with $\sum_k c_k < \infty$, $\zeta_k \to 1$, and

$$\sum_k \frac{c_k}{|1-\zeta_k|^2} < \infty, \quad \sum_k \frac{c_k}{|1-\zeta_k|^p} = \infty \quad \text{for some } p > 2.$$

As we have just seen, one can construct some inner functions Θ for which there exists some bounded truncated Toeplitz operators with no bounded symbols. On the contrary, it is also proved in [7] that for the singular inner function $\Theta_1(z) = \exp\left(\frac{z+1}{z-1}\right)$, every bounded truncated Toeplitz operator has a bounded symbol. This follows from results obtained by Rochberg [48] on the Paley–Wiener space. That of course yields the following question: can we characterize inner functions Θ for which every bounded truncated Toeplitz operator on K_Θ has a bounded symbol? In [6], Baranov–Bessonov–Kapustin gave interesting characterizations of such inner functions in terms of set $C_p(\Theta)$, and also some factorization properties. We will now detail a little bit some of their results.

Theorem 12.4 (Baranov–Bessonov–Kapustin, [6])**.** *Let $A \in \mathcal{T}(\Theta)$. Then A admits a bounded symbol if and only if $A = A_\mu$ for some $\mu \in C_1(\Theta^2)$.*

This result should be compared with Theorem 7.11.

Proof. Let us assume that $A = A_\phi$, where $\phi \in L^\infty$. Then $A = A_\mu$ with $d\mu := \phi\, dm$. Notice now that for every $f \in K^1_{\Theta^2}$, we have

$$\int_{\mathbb{T}} |f|\, d|\mu| = \int_{\mathbb{T}} |f||\phi|\, dm \le \|\phi\|_\infty \|f\|_1,$$

whence $K^1_{\Theta^2} \hookrightarrow L^1(|\mu|)$, proving that $\mu \in C_1(\Theta^2)$.

Conversely, assume that $A = A_\mu$, where $\mu \in C_1(\Theta^2)$. Consider $\mathcal{E}$ the subspace of L^1 formed by functions which are finite sums of functions of the forms $x_k\overline{y_k}$, with $x_k, y_k \in K_\Theta$, and define the functional ℓ on $\mathcal{E}$ by

$$\ell : f \longmapsto \int_{\mathbb{T}} f\, d\mu,$$

Let us check that ℓ is well defined and continuous (if $\mathcal{E}$ is equipped with the L^1-norm). Note that for every $x_k, y_k \in K_\Theta$, we have $x_k\overline{y_k} = x_k\overline{\Theta} z\widetilde{y_k}$, where $\widetilde{y_k} = C_\Theta(y_k) \in K_\Theta \subset H^2$. In particular, $\Theta\bar{z}x_k\overline{y_k} = x_k\widetilde{y_k} \in H^1$. On the other hand, writing $x_k = \Theta\overline{z\widetilde{x_k}}$, we also have

$$\Theta\bar{z}x_k\overline{y_k} = \Theta^2\overline{z^2\widetilde{x_k}y_k} \in \Theta^2\overline{zH^1}.$$

Thus $\Theta\bar{z}x_k\overline{y_k} \in H^1 \cap \Theta^2\overline{zH^1} = K^1_{\Theta^2}$. Moreover, since $\mu \in C_1(\Theta^2) \subset C_2(\Theta)$, it can be proved (see [6, Proposition 3.2]) that $|\Theta| = 1$, $|\mu|$-almost everywhere. Thus, for any $f \in \mathcal{E}$, we have $\Theta\bar{z}f \in K^1_{\Theta^2}$, and we obtain

$$\int_{\mathbb{T}} |f|\, d|\mu| = \int_{\mathbb{T}} |\Theta\bar{z}f|\, d|\mu| \le C\|f\|_1.$$

In particular, ℓ is well defined and continuous, and so, by Hahn-Banach Theorem, it can be continuously extended to L^1. Hence, there exists a function $\phi \in L^\infty$ such that

$$\int_{\mathbb{T}} f\, d\mu = \ell(f) = \int_{\mathbb{T}} f\phi\, dm, \qquad \forall f \in \mathcal{E}.$$

It follows that, for every $x, y \in K_\Theta$,

$$\langle A_\mu x, y\rangle_2 = \int_{\mathbb{T}} x\bar{y}\, d\mu = \ell(x\bar{y}) = \int_{\mathbb{T}} x\bar{y}\phi\, dm = \langle\, A_\phi x, y\rangle_2,$$

and thus $A_\mu = A_\phi$, with $\phi \in L^\infty$. □

It follows immediately from Theorem 12.4 and Theorem 7.11 that if $C_1(\Theta^2) = C_2(\Theta)$, then every bounded truncated Toeplitz operator admits a bounded symbol. Baranov–Bessonov–Kapustin have showed that the converse is also true and they also make an interesting connection with a factorization problem involving the space X (see (7.4)).

Theorem 12.5 (Baranov–Bessonov-Kapustin, [6])**.** *Let Θ be an inner function. The following assertions are equivalent:*

(i) *every bounded truncated Toeplitz operator on K_Θ admits a bounded symbol;*
(ii) $C_1(\Theta^2) = C_2(\Theta^2)$*;*
(iii) *for any $f \in H^1 \cap \Theta^2\overline{z^2 H^1}$, there exists $x_k, y_k \in K_\Theta$ with*

$$f = \sum_k x_k y_k \quad \text{and} \quad \sum_k \|x_k\|_2 \|y_k\|_2 < \infty.$$

If Θ is a one-component inner function, then all classes $C_p(\Theta)$ coincide (see [4, Theorem 1.4]). Moreover, if Θ is a one-component inner function, the Θ^2 is, too, hence $C_1(\Theta^2) = C_2(\Theta^2)$. An immediate consequence of Theorem 12.5 is then the following.

Corollary 12.6. *Let Θ be a one-component inner function. Then every bounded truncated Toeplitz operator on K_Θ admits a bounded symbol.*

Question 12.7. Does the converse is true? That means: assume that every bounded truncated Toeplitz operator on K_Θ admits a bounded symbol. Does it follows that Θ is one component?

Let us notice that it is shown in [3, Theorem 8] that the condition $C_1(\Theta^2) = C_2(\Theta^2)$ implies that all the Clark measures σ_α, $\alpha \in \mathbb{T}$, (associated to Θ) are discrete. It implies immediately another class of counterexamples to the existence of a bounded symbol.

Corollary 12.8. *Let Θ be an inner function and assume that for some $\alpha \in \mathbb{T}$, the Clark measure σ_α is not discrete. Then there exist a bounded truncated Toeplitz operator on K_Θ that do not admit a bounded symbol.*

Let μ be a continuous singular measure. It is well known (see [35, Chapter 11]) that there exists an inner function Θ such that

$$\frac{1-|\Theta(z)^2|}{|1-\Theta(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2}\, d\mu(\zeta).$$

In particular, we get that μ is the Clark measure σ_1 associated to Θ at point 1. By Corollary 12.8, we know that there exist a bounded truncated Toeplitz operator on K_Θ that does not admit a bounded symbol.

References

[1] P. R. Ahern and D. N. Clark. On functions orthogonal to invariant subspaces. *Acta Math.*, 124:191–204, 1970.

[2] A. B. Aleksandrov. Inner functions and related spaces of pseudocontinuable functions. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 170(Issled. Linein. Oper. Teorii Funktsii. 17):7–33, 321, 1989. English translation in J. Soviet Math. 63 (1993), no. 2, 115–129.

[3] A. B. Aleksandrov. On the existence of angular boundary values of pseudocontinuable functions. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 222(Issled. po Linein. Oper. i Teor. Funktsii. 23):5–17, 307, 1995.

[4] A. B. Aleksandrov. Embedding theorems for coinvariant subspaces of the shift operator. II. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 262(Issled. po Linein. Oper. i Teor. Funkts. 27):5–48, 231, 1999. English translation in J. Math. Sci. (New York) 110 (2002), no. 5, 2907–2929.

[5] A.B. Aleksandrov. On embedding theorems for coinvariant subspaces of the shift operator I. In *Complex analysis, operators and related topics*, volume 113 of *Oper. Theory Adv. Appl.*, pages 45–64. Birkhäuser, Basel, 2000.

[6] A. Baranov, R. Bessonov, and V. Kapustin. Symbols of truncated Toeplitz operators. *J. Funct. Anal.*, 261(12):3437–3456, 2011.

[7] A. Baranov, I. Chalendar, E. Fricain, J. Mashreghi, and D. Timotin. Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators. *J. Funct. Anal.*, 259(10):2673–2701, 2010.

[8] R. V. Bessonov. Truncated Toeplitz operators of finite rank. *Proc. Amer. Math. Soc.*, 142(4):1301–1313, 2014.

[9] R. V. Bessonov. Fredholmness and compactness of truncated Toeplitz and Hankel operators. *Integral Equations Operator Theory*, 82(4):451–467, 2015.

[10] R. V. Bessonov. Schatten properties of Toeplitz operators on the Paley-Wiener space. *Ann. Inst. Fourier (Grenoble)*, 68(1):195–215, 2018.

[11] Arlen Brown and P. R. Halmos. Algebraic properties of Toeplitz operators. *J. Reine Angew. Math.*, 213:89–102, 1963/64.

[12] M. Cristina Câmara and Jonathan R. Partington. Spectral properties of truncated Toeplitz operators by equivalence after extension. *J. Math. Anal. Appl.*, 433(2):762–784, 2016.

[13] Lennart Carleson. An interpolation problem for bounded analytic functions. *Amer. J. Math.*, 80:921–930, 1958.

[14] Lennart Carleson. Interpolations by bounded analytic functions and the corona problem. *Ann. of Math. (2)*, 76:547–559, 1962.

[15] I. Chalendar, E. Fricain, and D. Timotin. On an extremal problem of Garcia and Ross. *Oper. Matrices*, 3(4):541–546, 2009.

[16] I. Chalendar and D. Timotin. Commutation relations for truncated Toeplitz operators. *Oper. Matrices*, 8(3):877–888, 2014.

[17] Isabelle Chalendar, Emmanuel Fricain, and Dan Timotin. A survey of some recent results on truncated Toeplitz operators. In *Recent progress on operator theory and approximation in spaces of analytic functions*, volume 679 of *Contemp. Math.*, pages 59–77. Amer. Math. Soc., Providence, RI, 2016.

[18] J. A. Cima, S. R. Garcia, W. T. Ross, and W. R. Wogen. Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity. *Indiana Univ. Math. J.*, 59(2):595–620, 2010.

[19] Joseph Cima and Raymond Mortini. One-component inner functions. *Complex Anal. Synerg.*, 3(1):Paper No. 2, 15, 2017.

[20] Joseph Cima and Raymond Mortini. One-component inner functions II. In *Advancements in complex analysis—from theory to practice*, pages 39–49. Springer, Cham, [2020] ©2020.

[21] Joseph A. Cima, Alec L. Matheson, and William T. Ross. *The Cauchy transform*, volume 125 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[22] Joseph A. Cima, William T. Ross, and Warren R. Wogen. Truncated Toeplitz operators on finite dimensional spaces. *Oper. Matrices*, 2(3):357–369, 2008.

[23] B. Cohn. Carleson measures for functions orthogonal to invariant subspaces. *Pacific J. Math.*, 103(2):347–364, 1982.

[24] W. S. Cohn. Carleson measures and operators on star-invariant subspaces. *J. Operator Theory*, 15(1):181–202, 1986.

[25] William S. Cohn. Radial limits and star invariant subspaces of bounded mean oscillation. *Amer. J. Math.*, 108(3):719–749, 1986.

[26] P. L. Duren. *Theory of H^p spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.

[27] Emmanuel Fricain and Javad Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, volume 20 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2016.

[28] Paul A. Fuhrmann. On the corona theorem and its application to spectral problems in Hilbert space. *Trans. Amer. Math. Soc.*, 132:55–66, 1968.

[29] Paul A. Fuhrmann. Exact controllability and observability and realization theory in Hilbert space. *J. Math. Anal. Appl.*, 53(2):377–392, 1976.

[30] Theodore W. Gamelin. *Uniform algebras*. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.

[31] S. R. Garcia, D. E. Poore, and W. T. Ross. Unitary equivalence to a truncated Toeplitz operator: analytic symbols. *Proc. Amer. Math. Soc.*, 140(4):1281–1295, 2012.

[32] S. R. Garcia and W. T. Ross. Recent progress on truncated Toeplitz operators. In *Blaschke products and their applications*, volume 65 of *Fields Inst. Commun.*, pages 275–319. Springer, New York, 2013.

[33] S. R. Garcia, W. T. Ross, and W. R. Wogen. C^*-algebras generated by truncated Toeplitz operators. In *Concrete operators, spectral theory, operators in harmonic analysis and approximation*, volume 236 of *Oper. Theory Adv. Appl.*, pages 181–192. Birkhäuser/Springer, Basel, 2014.

[34] Stephan Ramon Garcia. Conjugation and Clark operators. In *Recent advances in operator-related function theory*, volume 393 of *Contemp. Math.*, pages 67–111. Amer. Math. Soc., Providence, RI, 2006.

[35] Stephan Ramon Garcia, Javad Mashreghi, and William T. Ross. *Introduction to model spaces and their operators*, volume 148 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[36] Stephan Ramon Garcia and Mihai Putinar. Complex symmetric operators and applications. *Trans. Amer. Math. Soc.*, 358(3):1285–1315, 2006.

[37] Stephan Ramon Garcia and Mihai Putinar. Complex symmetric operators and applications. II. *Trans. Amer. Math. Soc.*, 359(8):3913–3931, 2007.

[38] Stephan Ramon Garcia and William T. Ross. A non-linear extremal problem on the Hardy space. *Comput. Methods Funct. Theory*, 9(2):485–524, 2009.

[39] Stephan Ramon Garcia and William T. Ross. The norm of a truncated Toeplitz operator. In *Hilbert spaces of analytic functions*, volume 51 of *CRM Proc. Lecture Notes*, pages 59–64. Amer. Math. Soc., Providence, RI, 2010.

[40] I. C. Gohberg and I. A. Fel′ dman. *Convolution equations and projection methods for their solution*. Translations of Mathematical Monographs, Vol. 41. American Mathematical Society, Providence, R.I., 1974. Translated from the Russian by F. M. Goldware.

[41] M.T. Lacey, E.T. Sawyer, C.-Y. Shen, I. Uriarte-Tuero, and B.D. Wick. Two weight inequalities for the Cauchy transform from $\mathbb{R}$ to $\mathbb{C}_+$. Preprint arXiv:1310.4820.

[42] Patrick Lopatto and Richard Rochberg. Schatten-class truncated Toeplitz operators. *Proc. Amer. Math. Soc.*, 144(2):637–649, 2016.

[43] Javad Mashreghi. *Representation theorems in Hardy spaces*, volume 74 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2009.

[44] Raymond Mortini, Amol Sasane, and Brett D. Wick. The corona theorem and stable rank for the algebra $\mathbb{C} + BH^\infty$. *Houston J. Math.*, 36(1):289–302, 2010.

[45] N. K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann.

[46] N. K. Nikol′skiĭ. *Treatise on the shift operator*, volume 273 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1986. Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.

[47] Vladimir V. Peller. *Hankel operators and their applications*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[48] R. Rochberg. Toeplitz and Hankel operators on the Paley-Wiener space. *Integral Equations Operator Theory*, 10(2):187–235, 1987.

[49] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[50] D. Sarason. Generalized interpolation in H^∞. *Trans. Amer. Math. Soc.*, 127:179–203, 1967.

[51] D. Sarason. Algebraic properties of truncated Toeplitz operators. *Oper. Matrices*, 1(4):491–526, 2007.

[52] Donald Sarason. A remark on the Volterra operator. *J. Math. Anal. Appl.*, 12:244–246, 1965.

[53] N. A. Sedlock. Algebras of truncated Toeplitz operators. *Oper. Matrices*, 5(2):309–326, 2011.

[54] E. Strouse, D. Timotin, and M. Zarrabi. Unitary equivalence to truncated Toeplitz operators. *Indiana Univ. Math. J.*, 61(2):525–538, 2012.

[55] A. L. Vol′berg and S. R. Treil′. Embedding theorems for invariant subspaces of the inverse shift operator. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 149(Issled. Linein. Teor. Funktsii. XV):38–51, 186–187, 1986.

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