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Erratum to Completed Iwahori-Hecke algebra and parahoric Hecke algebras for Kac-Moody groups over local fields

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Abstract

We modify the definition of the completed Iwahori-Hecke algebra given in our previous article (J. Ec. Polytechnique **6**, 79–118) and explain why the construction we gave earlier is not correct as such.

The goal of this erratum is to fix the construction of the *completed Iwahori-Hecke algebra* given in [AH19, Section 4.4], as the definition given there does not always provide an actual algebra (see Section 2 below). We define here an algebra that must be used instead. Consequently, the following modifications should be operated in [AH19] : the paragraph before [AH19, Proposition 4.33] is wrong and must be replaced by Section 3 below; Theorem 4.21, Corollary 4.23 and Theorem 4.30 of [AH19] are wrong as stated there and must respectively be replaced by Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.

1 Introduction

Let G be a split Kac-Moody group (as defined by Tits in [Tit87]) over a non-Archimedean local field \mathcal{K} . Given a ring \mathcal{R} containing \mathbb{Z} and satisfying light technical conditions (as in [AH19, Remark 4.1]), Braverman-Kazhdan-Patnaik [BK11, BKP16] and Bardy-Panse-Gaussent-Rousseau [GR14, BPGR16] associated to G a spherical Hecke algebra \mathcal{H}_s and an Iwahori-Hecke algebra \mathcal{H} , both defined over \mathcal{R} . Fixing a maximal split torus T of G , and letting Y (resp. Y^+) be the cocharacter lattice (resp. its intersection with the Tits cone) and W^v be the Weyl group of (G, T) , then these authors moreover proved the existence of a Satake isomorphism from \mathcal{H}_s to $\mathcal{R}[Y]^{W^v}$, where $\mathcal{R}[Y]$ is the Looijenga algebra, which is a completion of the group algebra $\mathcal{R}[Y]$ of Y over \mathcal{R} (see [AH19, Definition 4.6] for its definition). A striking difference with the classical case of reductive groups is that for G non-reductive, this spherical Hecke algebra is not isomorphic to the center of the Iwahori-Hecke algebra \mathcal{H} .

In [AH19], we announced the definition of a completed Iwahori-Hecke algebra $\widehat{\mathcal{H}}$ that contains \mathcal{H} and whose center is isomorphic to $\mathcal{R}[Y]^{W^v}$, hence to \mathcal{H}_s [AH19, Theorem 4.30]. Nevertheless, the construction of $\widehat{\mathcal{H}}$ we gave in [AH19, Section 4.4, pages 94-100] is not correct as stated, since as such, $\widehat{\mathcal{H}}$ is actually not stable in general under the convolution product. Indeed, the product of two elements of $\widehat{\mathcal{H}}$ can lead to infinite coefficients, as will be seen below in Sections 2.1 and 2.2. This erratum corrects this mistake by defining a slightly different algebra $\widetilde{\mathcal{H}}$, contained in the vector space $\widehat{\mathcal{H}}$, for a suitable convolution product (see Corollary 3.13). The main point is to use the correct notion of almost-finiteness in the definition of the support of the elements of the completed algebra. We check here that $\widetilde{\mathcal{H}}$ contains \mathcal{H} and that the center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathcal{R}[Y]^{W^v}$, hence to \mathcal{H}_s (see Theorem 3.14), as aimed at first. Moreover note that this modified definition of the completed Iwahori-Hecke algebra suppresses the aforementioned gap between the reductive and non-reductive cases, as for G reductive, $\widetilde{\mathcal{H}}$ is actually isomorphic to the classical Iwahori-Hecke algebra (see Proposition 3.17).

This erratum is organized as follows. In Section 2, we give two counter-examples to [AH19, Theorem 4.21]: one in the reductive case and one in the non-reductive case. Then we introduce the required modifications in the definition of the completed algebra to build $\widetilde{\mathcal{H}}$ in Section 3. In particular, we explain in Section 3.3 how to adapt the content of [AH19, Section 4.4, pages 94-100] to prove that the center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathcal{R}[Y]^{W^v}$.

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2 Two counter-examples to [AH19, Theorem 4.21]

We keep the notation of [AH19, section 2]. Let us briefly recall that, as in [AH19, Section 2.1], given a root generating system $\mathcal{S} = (A, X, Y, (\alpha)_{i \in I}, (\alpha_i^\vee)_{i \in I})$, we set $\mathbb{A} := Y \otimes \mathbb{R}$, let W^v denote the Weyl group of \mathcal{S} , $Q^\vee := \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$ denote its coroot lattice, $Q_{\mathbb{R},+}^\vee := \bigoplus_{i \in I} \mathbb{R}_+ \alpha_i^\vee$ and $Q_{\mathbb{R}}^\vee = \bigoplus_{i \in I} \mathbb{R} \alpha_i^\vee$.

We then have $\mathcal{H} = \bigoplus_{\lambda \in Y^+, w \in W^v} \mathcal{R} Z^\lambda H_w$, where Z^λ and H_w are symbols that satisfy relations (BL1) to (BL4) of [AH19, Section 4.1, page 91]. The notion of support is defined in [AH19, Definition 4.11]. In this section, we give two examples of elements $(a_j)_{j \in J}, (b_k)_{k \in K}$ in \mathcal{H} that are summable in $\widehat{\mathcal{H}}$ in the sense of [AH19, Definition 4.20], but such that $(a_j * b_k)_{j \in J, k \in K}$ is not summable: one is a reductive case (Section 2.1), the other one is an affine Kac-Moody case (Section 2.2). This prevents [AH19, Theorem 4.21] from being true as stated, and we explain in Section 2.3 why the problem stands in the notion of almost-finiteness initially used in the definition of $\widehat{\mathcal{H}}$, and how to modify it to get a correct analogue of [AH19, Theorem 4.21].

2.1 A counter-exemple in the reductive case

Assume that the standard apartment \mathbb{A} is associated with a Cartan matrix. Fix $\lambda \in Y = Y^+$ and $i \in I$. For $j \in \mathbb{N}$, set $a_j = Z^{\lambda - j \alpha_i^\vee} H_i$ and $b_j = Z^{\lambda - j \alpha_i^\vee}$. As $\lambda - \mathbb{N} \alpha_i^\vee$ is almost finite, $(a_j)_{j \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ are summable in $\widehat{\mathcal{H}}$. Now let $j, k \in \mathbb{N}$. By (BL4) (see [AH19, Section 4.1 page 91]), there exists $c_{j,k} \in \mathcal{R}[Y]$ such that

$$a_j * b_k = Z^{\lambda + r_i(\lambda) + (k-j)\alpha_i^\vee} H_i + c_{j,k} .$$

This implies that $(\lambda + r_i(\lambda) + (k-j)\alpha_i^\vee, r_i) \in \text{supp}(a_j * b_k)$, hence $(a_j * b_k)_{(j,k) \in \mathbb{N}^2}$ satisfies none of the conditions of [AH19, Definition 4.20], so it is not summable in $\widehat{\mathcal{H}}$.

2.2 A counter-example in the non-reductive case

Assume now that \mathbb{A} is associated with an indecomposable affine Kac-Moody matrix A . Let $\delta : \mathbb{A} \rightarrow \mathbb{R}$ be the smallest positive imaginary root associated with A . Fix $\lambda \in Y^+$ such that $\delta(\lambda) > 0$ and $i \in I$: then [AH19, Section 4.2.2, pages 92-93] ensures that $\lambda - \mathbb{N} \alpha_i^\vee$ is an almost finite subset of Y^+ . For $j \in \mathbb{N}$, set $a_j = Z^{\lambda - j \alpha_i^\vee} H_i$ and $b_j = Z^{\lambda - j \alpha_i^\vee}$: then the same process as in the reductive case (Section 2.1 above) shows that $(a_j * b_k)_{(j,k) \in \mathbb{N}^2}$ satisfies none of the conditions of [AH19, Definition 4.20], hence is not summable in $\widehat{\mathcal{H}}$, although both (a_j) and (b_k) are summable in $\widehat{\mathcal{H}}$.

2.3 Position of the problem and modifications required

The definition of $\widehat{\mathcal{H}}$ given in [AH19, page 95] crucially relies on the notion of almost finiteness defined in [AH19, Definition 4.12]. The problem is that almost finiteness is not preserved by the action of W^v on Y^+ : there can exist (depending on the Kac-Moody matrix A) an almost finite set E such that $w.E$ is not almost finite, for some $w \in W^v$. To fix this problem, we introduce in the next section a refined notion of almost-finiteness, namely the notion of W^v -almost finiteness. Using this new notion, we define an algebra $\widetilde{\mathcal{H}}$ through an analogous construction to the one done for $\widehat{\mathcal{H}}$ in [AH19, Section 4.4, pages 94-100]. We then explain why the results and proofs stated for $\widehat{\mathcal{H}}$ in [AH19] are now valid for $\widetilde{\mathcal{H}}$.

Before going further, let us list precisely what modifications are actually done in this erratum.

- ★ The notion of almost finiteness defined in [AH19, Definition 4.12] must be replaced by the notion of W^v -almost finiteness introduced in Definition 3.1 below to define $\widetilde{\mathcal{H}}$ as we defined $\widehat{\mathcal{H}}$ but with the aforementioned replacement.
- ★ The statement and proof of [AH19, Theorem 4.21, Corollary 4.23 and Theorem 4.30] must be respectively replaced by the statement and proof of Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.
- ★ The content of the paragraph before [AH19, Proposition 4.33], which explains what happens in the reductive case, must be replaced by Section 3.4 below.

3 The completed Iwahori-Hecke algebra $\widetilde{\mathcal{H}}$

The goal of this section is to build an algebra $\widetilde{\mathcal{H}}$ that appears to be smaller than $\widehat{\mathcal{H}}$ (that is NOT always an algebra) whose center is (still) isomorphic to $\mathcal{R}[Y]^{W^v}$ and that (still) contains \mathcal{H} as the subalgebra of finitely

supported elements. It actually boils down to defining the right notion of almost-finiteness and checking that what we did in [AH19, Section 4] transposes in this setting to define an actual algebra $\tilde{\mathcal{H}}$ with the required properties.

3.1 W^v -almost finiteness and definition of $\tilde{\mathcal{H}}$

The idea behind the use of the following refined notion of almost finiteness is that it is preserved by the action of W^v .

Definition 3.1. Let $u \in W^v$.

- A subset E of Y^+ is *u-almost finite* if $u.E$ is almost finite in the sense of [AH19, Definition 4.3].
- A subset E of $Y^+ \times W^v$ is called *u-almost finite* if its projection on W^v is finite and if its projection on Y^+ is *u-almost finite* (as a subset of Y^+).
- A subset of Y^+ or of $Y^+ \times W^v$ is *W^v -almost finite* if it is *u-almost finite* for any $u \in W^v$.

As in [AH19, page 95], we set $\mathcal{B} = \prod_{\lambda \in Y^+, w \in W^v} \mathcal{R}$ and for $(\lambda, w) \in Y^+ \times W^v$, we let $Z^\lambda H_w$ denote the element whose coefficients are all equal to 0 apart from the coefficient indexed by (λ, w) , which is equal to 1. This allows us to write $a = (a_{\lambda, w})_{(\lambda, w) \in Y^+ \times W^v} \in \mathcal{B}$ as the formal linear combination $a = \sum_{\lambda \in Y^+, w \in W^v} a_{\lambda, w} Z^\lambda H_w$. Also recall that any $(\nu, u) \in Y^+ \times W^v$ is associated to a projection map $\pi_{\nu, u} : \mathcal{B} \rightarrow \mathcal{R}$ defined by

$$\pi_{\nu, u} \left(\sum_{u' \in W^v, \nu' \in Y^+} c_{\nu', u'} Z^{\nu'} H_{u'} \right) := c_{\nu, u}$$

for any $\sum c_{\nu', u'} Z^{\nu'} H_{u'} \in \mathcal{B}$.

We can now define $\tilde{\mathcal{H}}$ as the set of elements of \mathcal{B} with W^v -almost finite support. To prove that $\tilde{\mathcal{H}}$ can be endowed with a convolution product $*$ that turns it into an associative algebra containing \mathcal{H} , we will basically follow the same steps as in [AH19, Section 4.4], replacing the almost finiteness condition by the W^v -almost finiteness condition.

We let $\text{conv}_{\mathbb{R}}(F)$ denote the convex hull of any part F of \mathbb{A} , and we set $\text{conv}(E) := \text{conv}_{\mathbb{R}}(E) \cap Y$ for any subset E of Y . Following [AH19, page 95], recall that for any part E of Y and any $i \in I$, we let $R_i(E) = \text{conv}(E \cup r_i(E)) \subset E + Q^\vee$ and that, for any pair $(\lambda, w) \in Y^+ \times W^v$, we set

$$R_w(\lambda) := \bigcup R_{i_1}(R_{i_2}(\dots(R_{i_k}(\{\lambda\})\dots)) ,$$

where the union is taken over all the reduced writings $r_{i_1} r_{i_2} \dots r_{i_k}$ of w . The next two results replace [AH19, Remark 4.13] and act as preparation for the proof of Lemma 3.4 below, which replaces [AH19, Lemma 4.15].

Lemma 3.2. For any $(\lambda, w) \in Y^+ \times W^v$, we have

$$R_w(\lambda) \subset \text{conv}(\{u.\lambda \mid u \in [1, w]\}),$$

where $[1, w] := \{u \in W^v \mid u \leq w\}$ is defined as in [AH19, bottom of page 94].

Proof. We prove this result by induction on $\ell(w)$. If $\ell(w) = 0$, there is nothing to prove, so let $w \in W^v$ be an element of length $\ell(w) \geq 1$ and assume by induction that the lemma holds for any element $w' \in W^v$ such that $\ell(w') < \ell(w)$. Let $\mu \in R_w(\lambda)$, then there exists $i \in I$ such that $w' := r_i w$ satisfies $w' < w$ and $\mu \in R_i(R_{w'}(\lambda)) = \text{conv}(R_{w'}(\lambda), r_i R_{w'}(\lambda))$. As $\ell(w') < \ell(w)$, we have $R_{w'}(\lambda) \subset \text{conv}(\{u.\lambda \mid u \in [1, w']\})$ by induction hypothesis. Since [Kum02, Corollary 1.3.19] ensures that $\{1, r_i\} \cdot [1, w'] \subset [1, w]$, we obtain that

$$\text{conv}(\{u.\lambda \mid u \in [1, w']\}) \cup r_i \cdot \text{conv}(\{u.\lambda \mid u \in [1, w']\}) \subset \text{conv}(\{u.\lambda \mid u \in [1, w]\}) .$$

Consequently, we get that

$$\begin{aligned} \mu \in \text{conv}(R_{w'}(E), r_i \cdot R_{w'}(E)) &\subset \text{conv}(\text{conv}(\{u.\lambda \mid u \in [1, w']\}) \cup r_i \cdot \text{conv}(\{u.\lambda \mid u \in [1, w']\})) \\ &\subset \text{conv}(\text{conv}(\{u.\lambda \mid u \in [1, w]\})) = \text{conv}(\{u.\lambda \mid u \in [1, w]\}) . \end{aligned}$$

This proves that $R_w(\lambda)$ is contained in $\text{conv}(\{u.\lambda \mid u \in [1, w]\})$, hence the lemma. \square

Lemma 3.3. *Let E be a W^v -almost finite subset of Y^+ .*

Then, for any pair $(\nu, w) \in Y^+ \times W^v$, the set $\{\mu \in E \mid \nu \in R_w(\mu)\}$ is finite.

Proof. Let $E \subset Y^+$ and $(\nu, w) \in Y^+ \times W^v$ be as in the statement. Applying the definition of almost finiteness [AH19, Definition 4.3] to $u.E$ for any $u \in [1, w]$ provides a finite set $F \subset Y^+$ such that :

$$\forall u \in [1, w], \forall \mu \in u.E, \exists \lambda \in F \mid \mu \leq_{Q^v} \lambda .$$

Set $\mathcal{X} := \{\mu \in E \mid \nu \in R_w(\mu)\}$ and pick some $\mu \in \mathcal{X}$. As ν belongs to $R_w(\mu)$, Lemma 3.2 implies the existence of $(t_u)_{u \in [1, w]} \in [0, 1]^{[1, w]}$ such that

$$\sum_{u \in [1, w]} t_u = 1 \text{ and } \nu = \sum_{u \in [1, w]} t_u u . \mu .$$

For any $u \in [1, w]$, choose $\lambda(u) \in F$ such that $u . \mu \leq_{Q^v} \lambda(u)$ and write $\lambda(u) - u . \mu$ as $\sum_{i \in I} n_i(u) \alpha_i^\vee$ with $n_i(u) \in \mathbb{N}$ for all $i \in I$. Then we have:

$$\nu = \sum_{u \in [1, w]} t_u u . \mu = \sum_{u \in [1, w]} t_u \lambda(u) - \sum_{u \in [1, w], i \in I} t_u n_i(u) \alpha_i^\vee .$$

Set $a(\mu) := \sum_{u \in [1, w]} t_u \lambda(u) \in \text{conv}_{\mathbb{R}}(F)$ and $q(\mu) := \sum_{u \in [1, w], i \in I} t_u n_i(u) \alpha_i^\vee \in Q_{\mathbb{R}, +}^\vee$. Since F is finite, $\text{conv}_{\mathbb{R}}(F)$ is bounded. As $q(\mu) = a(\mu) - \nu$ lies in $\text{conv}_{\mathbb{R}}(F) - \nu$, the set $\{q(\mu'), \mu' \in \mathcal{X}\}$ is bounded too. Moreover, as $\sum_{u \in [1, w]} t_u = 1$, there exists $u' \in [1, w]$ such that $t_{u'} \geq \frac{1}{|[1, w]|}$. Letting $f_j(x)$ denote the j -th coordinate of $x \in Q_{\mathbb{R}}^\vee$ in the basis $(\alpha_i^\vee)_{i \in I}$ for all $j \in J$, we have:

$$\forall i \in I, f_i(q(\mu)) = \sum_{u \in [1, w]} t_u n_i(u) \geq t_{u'} n_i(u') \geq 0 .$$

We hence obtain that:

$$\forall i \in I, 0 \leq n_i(u') \leq \frac{\sup_{\mu' \in \mathcal{X}} f_i(q(\mu'))}{t_{u'}} \leq \frac{\sup_{\mu' \in \mathcal{X}} f_i(q(\mu'))}{|[1, u]|} .$$

Consequently, if we set $N := \max_{i \in I} \frac{\sup_{\mu' \in \mathcal{X}} f_i(q(\mu'))}{|[1, u]|}$, then we have:

$$u' . \mu \in \lambda(u') - \sum_{i \in I} \llbracket 0, N \rrbracket \alpha_i^\vee \subset F - \sum_{i \in I} \llbracket 0, N \rrbracket \alpha_i^\vee .$$

This proves that μ lies in $\bigcup_{u \in [1, w]} u^{-1} . (F - \sum_{i \in I} \llbracket 0, N \rrbracket \alpha_i^\vee)$, hence that \mathcal{X} is contained in the finite set $\bigcup_{u \in [1, w]} u^{-1} . (F - \sum_{i \in I} \llbracket 0, N \rrbracket \alpha_i^\vee)$, which proves that \mathcal{X} is finite too, as claimed. \square

The next lemma replaces [AH19, Lemma 4.15]: the only modification consists in replacing $[1, w]w'$ by $[1, w']w$ in the aforementioned statement. In particular, the proof follows the exact same lines as [AH19, page 96], hence we do not rewrite it here.

Lemma 3.4. *For all $w, w' \in W^v$ and all $\lambda \in Y$, $H_{w'} * Z^\lambda H_w$ is in $\bigoplus_{(\nu, t) \in R_{w'}(\lambda) \times [1, w'] . w} \mathcal{R} . Z^\nu H_t$.*

Using the definitions of supp_{W^v} and supp_Y given by [AH19, Definition 4.11], one can straightforward deduce from Lemma 3.4 the following inclusions.

Lemma 3.5. *For all $a, b \in \mathcal{H}$, we have:*

- (1) $\text{supp}_Y(a * b) \subset \text{supp}_Y(a) + \bigcup_{w \in \text{supp}_{W^v}(a), \lambda \in \text{supp}_Y(b)} R_w(\lambda)$;
- (2) $\text{supp}_{W^v}(a * b) \subset \bigcup_{v \in \text{supp}_{W^v}(a), w \in \text{supp}_{W^v}(b)} [1, v] . w$.

Before we give the definition of summable families in $\tilde{\mathcal{H}}$, we prove two more statements related to W^v -almost finiteness in Y^+ .

Lemma 3.6. *For any almost finite set $E \subset Y^+$, $\text{conv}(E)$ is also almost finite.*

Proof. Let $E \subset Y^+$ be an almost finite set and let F be a finite set such that:

$$\forall \lambda \in E, \exists \mu \in F \mid \lambda \leq_{Q^v} \mu .$$

Given $\lambda \in \text{conv}(E)$, there exist $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, 1]$ and $\lambda_1, \dots, \lambda_n \in E$ such that

$$\sum_{i=1}^n t_i = 1 \quad \text{and} \quad \sum_{i=1}^n t_i \lambda_i = \lambda .$$

For each index $i \in \llbracket 1, n \rrbracket$, choose $\kappa_i \in F$ such that $\lambda_i \leq_{Q^v} \kappa_i$: then $\sum_{i=1}^n t_i \kappa_i - \sum_{i=1}^n t_i \lambda_i \in \bigoplus_{i \in I} \mathbb{R}_+ \alpha_i^\vee$. We can hence write $\sum_{i=1}^n t_i \kappa_i - \sum_{i=1}^n t_i \lambda_i = \sum_{i \in I} x_i \alpha_i^\vee$ for some nonnegative real numbers $(x_i)_{i \in I}$. Now let $(x'_i) \in [0, 1]^I$ be such that $x_i + x'_i$ lies in \mathbb{N} for all $i \in I$ and set $\nu := \sum_{i \in I} (x_i + x'_i) \alpha_i^\vee + \lambda$. Then we have $\nu \geq_{Q^v} \lambda$ and $\nu \in (\text{conv}_{\mathbb{R}}(F) + \bigoplus_{i \in I} [0, 1] \alpha_i^\vee) \cap Y$. Since F is finite, $(\text{conv}_{\mathbb{R}}(F) + \bigoplus_{i \in I} [0, 1] \alpha_i^\vee) \cap Y$ is a finite set that can be taken as J in [AH19, Definition 4.3] for $\text{conv}(E)$, and the lemma is proven. \square

Lemma 3.7. *Let E be a W^v -almost finite subset of Y^+ . Then, for any $w \in W^v$, the set $\bigcup_{\lambda \in E} R_w(\lambda)$ is W^v -almost finite.*

Proof. Let $w \in W^v$. By Lemma 3.2, we have

$$\bigcup_{\lambda \in E} R_w(\lambda) \subset \bigcup_{\lambda \in E} \text{conv}(\{u \cdot \lambda, u \in [1, w]\}) \subset \text{conv} \left(\bigcup_{u \in [1, w]} u \cdot E \right) .$$

Let $v \in W^v$. Since $[1, w]$ is finite, the set $v \cdot \bigcup_{u \in [1, w]} u \cdot E$ is almost finite, hence Lemma 3.6 implies that $v \cdot \text{conv} \left(\bigcup_{u \in [1, w]} u \cdot E \right) = \text{conv} \left(v \cdot \bigcup_{u \in [1, w]} u \cdot E \right)$ is almost finite. This proves that $\text{conv} \left(\bigcup_{u \in [1, w]} u \cdot E \right)$ is v -almost finite for any $v \in W^v$, and the lemma is proven. \square

3.2 $\tilde{\mathcal{H}}$ is an associative algebra

This subsection contains the main modification of the paper, as it aims to prove that $\tilde{\mathcal{H}}$ is actually an associative algebra. To do this, we first need to introduce the correct definition of summable families, which is the counterpart of [AH19, Definition 4.20] for W^v -almost finite sets.

Definition 3.8. A family $(a_j)_{j \in J} \in \left(\tilde{\mathcal{H}} \right)^J$ is *summable in $\tilde{\mathcal{H}}$* when the two following properties hold:

- (i) for any $\lambda \in Y^+$, the set $\{j \in J \mid \exists w \in W^v, \pi_{\lambda, w}(a_j) \neq 0\}$ is finite;
- (ii) the set $\bigcup_{j \in J} \text{supp}(a_j) := \bigcup_{j \in J} \{(\lambda, w) \in Y^+ \times W^v \mid \pi_{\lambda, w}(a_j) \neq 0\}$ is W^v -almost finite.

Given a summable family $(a_j)_{j \in J} \in \left(\tilde{\mathcal{H}} \right)^J$, we define $\sum_{j \in J} a_j \in \tilde{\mathcal{H}}$ by the following formula:

$$\sum_{j \in J} a_j := \sum_{(\lambda, w) \in Y^+ \times W^v} a_{\lambda, w} Z^\lambda H_w , \quad \text{with } a_{\lambda, w} := \sum_{j \in J} \pi_{\lambda, w}(a_j) \text{ for all } (\lambda, w) \in Y^+ \times W^v .$$

Lemma 3.9. For any almost finite subsets E, E' of Y^+ and any $\rho \in Y^+$, the set

$$E'' := \{\lambda \in E \mid \exists \nu \in E', \lambda + \nu = \rho\}$$

is finite.

Proof. By definition, there exists a finite set $F \subset Y^+$ such that: $\forall \mu \in E \cup E', \exists \kappa \in F \mid \mu \leq_{Q^v} \kappa$. Now let $\lambda \in E''$ and $\nu \in E'$ be such that $\lambda + \nu = \rho$. Then we have:

$$\exists (\kappa, \kappa') \in F^2 \mid \rho - \kappa' \leq_{Q^v} \lambda \leq_{Q^v} \kappa$$

and the lemma follows. \square

The next lemma is the cornerstone that ensures that the convolution product on $\tilde{\mathcal{H}}$ is well-defined.

Lemma 3.10. Let $(a_\lambda), (b_\mu) \in \mathcal{R}^{Y^+}$ be such that $\{\lambda \in Y^+ \mid a_\lambda \neq 0 \text{ or } b_\lambda \neq 0\}$ is W^v -almost finite. Then, for any $w \in W^v$, $(a_\lambda b_\mu Z^\lambda H_w * Z_\mu)_{(\lambda, \mu) \in (Y^+)^2}$ is summable in $\tilde{\mathcal{H}}$. Said differently, if $\sum_{\lambda \in Y^+} a_\lambda Z^\lambda H_w, \sum_{\mu \in Y^+} b_\mu Z^\mu \in \tilde{\mathcal{H}}$, then

$$\left(\sum_{\lambda \in Y^+} a_\lambda Z^\lambda H_w \right) * \left(\sum_{\mu \in Y^+} b_\mu Z^\mu \right) := \sum_{\lambda, \mu \in Y^+} a_\lambda b_\mu Z^\lambda H_w * Z^\mu$$

is a well-defined element of $\tilde{\mathcal{H}}$.

Proof. Set $S_a := \{\lambda \in Y^+ \mid a_\lambda \neq 0\}$, $S_b = \{\mu \in Y^+ \mid b_\mu \neq 0\}$ and $E = \bigcup_{\mu \in S_b} R_w(\mu)$. Note that E is almost finite by Lemma 3.7.

Given $\mu \in Y^+$, Lemma 3.4 ensures the existence of $(z_\mu^{v, \nu})_{\nu \in R_w(\mu), v \in [1, w]} \in \mathcal{R}^{R_w(\mu) \times [1, w]}$ such that

$$H_w * Z^\mu = \sum_{\nu \in R_w(\mu), v \in [1, w]} z_\mu^{v, \nu} Z^\nu H_v.$$

Let us fix $v \in [1, w]$. Given any $\lambda, \rho \in Y^+$, we have

$$\pi_{\rho, v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) = \sum_{\nu \in R_w(\mu) \mid \lambda + \nu = \rho} a_\lambda b_\mu z_\mu^{v, \nu}. \quad (1)$$

Set $F_1 := \{\lambda \in Y^+ \mid \exists \mu \in Y^+, \pi_{\rho, v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) \neq 0\} \subset S_a$ and let $\lambda \in F_1$. By equality (1), there exists $\nu \in E$ such that $\lambda + \nu = \rho$. Since λ lies in S_a , applying Lemma 3.9 to E and S_a implies that F_1 is finite.

Fix now $\lambda \in F_1$ and set $F_2(\lambda) = \{\mu \in Y^+ \mid \pi_{\rho, v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) \neq 0\} \subset S_b$. Given $\mu \in F_2(\lambda)$, we know from equality (1) that $\rho - \lambda \in R_w(\mu)$. As S_b is W^v -almost finite, Lemma 3.3 yields the finiteness of $F_2(\lambda)$, hence the finiteness of

$$F_v := \{(\lambda, \mu) \in (Y^+)^2 \mid \pi_{\rho, v}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu) \neq 0\} = \bigcup_{\lambda \in F_1} F_2(\lambda).$$

Finally, we obtain that

$$\{(\lambda, \mu) \in (Y^+)^2 \mid \exists v \in W^v, \pi_{\rho, v}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu) \neq 0\} = \bigcup_{v \in [1, w]} F_v$$

is finite, which proves that $(a_\lambda b_\mu Z^\lambda H_w * Z^\mu)_{(\lambda, \mu) \in (Y^+)^2}$ satisfies condition (i) of Definition 3.8.

Now let $\lambda, \mu \in Y^+$. Then equality (1) ensures that:

$$\text{supp}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu) \subset (\lambda + R_w(\mu)) \times [1, w] \subset (\lambda + E) \times [1, w].$$

In particular, we have

$$\bigcup_{(\lambda, \mu) \in (Y^+)^2} \text{supp}(a_\lambda b_\mu Z^\lambda H_w * Z^\mu) \subset (S_a + E) \times [1, w].$$

As E is W^v -almost finite, and as the sum of two W^v -almost finite sets is W^v -almost finite, we obtain that $(a_\lambda b_\mu Z^\lambda H_w * Z^\mu)_{(\lambda, \mu) \in (Y^+)^2}$ satisfies condition (ii) of Definition 3.8, which ends the proof. \square

Lemma 3.11. *For any summable family $(a_j)_{j \in J} \in (\tilde{\mathcal{H}})^J$ in $\tilde{\mathcal{H}}$ and any $i \in I$, the family $(a_j * H_i)_{j \in J}$ is summable in $\tilde{\mathcal{H}}$.*

Proof. By definition, there exist a W^v -almost finite subset E of Y^+ and a finite subset F of W^v such that: $\forall j \in J$, $\text{supp}(a_j) \subset E \times F$. By (BL2) of [AH19, page 91], we get that

$$\forall j \in J, \text{supp}(a_j * H_i) \subset E \times (F \cup F.r_i),$$

which proves that $\bigcup_{j \in J} \text{supp}(a_j * H_i)$ is W^v -almost finite. Now let $\lambda \in Y^+$, $w \in W^v$ and $j \in J$. By (BL2) again, we know that $\pi_{\lambda, w}(a_j * H_i) \neq 0$ implies $\pi_{\lambda, w}(a_j) \neq 0$ or $\pi_{\lambda, w.r_i}(a_j) \neq 0$. Since there are finitely many such $j \in J$, this completes the proof of the lemma. \square

The next statement replaces [AH19, Theorem 4.21] and its proof basically follows the same lines as the proof of [AH19, Theorem 4.21], replacing almost finiteness by W^v -almost finiteness. Recall that the elements of \mathcal{H} correspond to the elements of $\tilde{\mathcal{H}}$ with finite support.

Theorem 3.12. *Let $(a_j)_{j \in J} \in (\mathcal{H})^J$ and $(b_k)_{k \in K} \in (\mathcal{H})^K$ be two families that are both summable in $\tilde{\mathcal{H}}$. Then $(a_j * b_k)_{(j,k) \in J \times K}$ is summable in $\tilde{\mathcal{H}}$ and $\sum_{(j,k) \in J \times K} a_j * b_k$ only depends on the two elements $\sum_{j \in J} a_j$ and $\sum_{k \in K} b_k$ of $\tilde{\mathcal{H}}$.*

Proof. For $j \in J$ and $k \in K$, write $a_j := \sum_{v \in W^v} a_{v,j} * H_v$ and $b_k := \sum_{w \in W^v} b_{w,k} * H_w$, with $(a_{v,j})_{j \in J} \in \mathcal{R}[Y]^J$ and $(b_{w,k})_{k \in K} \in \mathcal{R}[Y]^K$ for any $v, w \in W^v$. Given $v, w \in W^v$, Lemma 3.10 ensures that $(a_{v,j} * H_v * b_{w,k})_{(j,k) \in J \times K}$ is summable in $\tilde{\mathcal{H}}$. By induction on $\ell(w)$ and using Lemma 3.11, we get that $(a_{v,j} * H_v * b_{w,k} * H_w)_{(j,k) \in J \times K}$ is summable in $\tilde{\mathcal{H}}$. Moreover, as (a_j) and (b_k) are summable in $\tilde{\mathcal{H}}$, there are at most finitely many $v, w \in W^v$ satisfying $(a_{v,j})_{j \in J} \neq 0$ and $(b_{w,k})_{k \in K} \neq 0$. Consequently, the family $(a_j * b_k)_{(j,k) \in J \times K}$ is summable in $\tilde{\mathcal{H}}$. Now, given any triple $(u, v, \mu) \in W^v \times W^v \times Y^+$, applying Lemma 3.4 to $H_u * Z^\mu H_v$ gives a family $(z_{\nu,t}^{u,v,\mu})_{(\nu,t) \in R_u(\mu) \times [1,u].v}$ of scalars that satisfy $H_u * Z^\mu H_v = \sum_{(\nu,t) \in R_u(\mu) \times [1,u].v} z_{\nu,t}^{u,v,\mu} Z^\nu H_t$.

For any pair $(\rho, s) \in Y^+ \times W^v$, we have

$$\begin{aligned} \pi_{\rho,s} \left(\sum_{(j,k) \in J \times K} a_j * b_k \right) &= \sum_{(\lambda,u), (\mu,v) \in Y^+ \times W^v} \sum_{\nu \in R_u(\mu) \mid \lambda + \nu = \rho} \sum_{(j,k) \in J \times K} a_{\lambda,\nu} b_{\mu,v} z_{\nu,s}^{u,v,\mu} \\ &= \sum_{(\lambda,u), (\mu,v) \in Y^+ \times W^v} \sum_{\nu \in R_u(\mu) \mid \lambda + \nu = \rho} a_{\lambda,\nu} b_{\mu,v} z_{\nu,s}^{u,v,\mu}, \end{aligned}$$

where we set $\sum_{j \in J} a_j = \sum_{(\lambda,u) \in Y^+ \times W^v} a_{\lambda,u} Z^\lambda H_u$ and $\sum_{k \in K} b_k = \sum_{(\mu,v) \in Y^+ \times W^v} b_{\mu,v} Z^\mu H_v$, hence the theorem is proven. \square

The mistake done in the former proof of [AH19, Theorem 4.20] is to implicitly assume that $S_Y = \bigcup_{j \in J} \text{supp}_Y(a_j) \cup \bigcup_{k \in K} \text{supp}_Y(b_k)$ is such that $\{\lambda^{++}, \lambda \in S_Y\}$ is almost finite, which is not true in general, as shown by the counter-examples given in Section 2. As spotted by the referee, the same kind of subtlety underlies a mistake made by Looijenga in his seminal 1980 work [Loo80, (4.1), end of the first paragraph].

For $a = \sum_{(\lambda,v) \in Y^+ \times W^v} a_{\lambda,v} Z^\lambda H_v \in \tilde{\mathcal{H}}$ and $b = \sum_{(\mu,w) \in Y^+ \times W^v} b_{\mu,w} Z^\mu H_w \in \tilde{\mathcal{H}}$, we set

$$a * b = \sum_{(v,\lambda), (w,\mu) \in Y^+ \times W^v} a_{\lambda,v} b_{\mu,w} Z^\lambda H_v * Z^\mu H_w,$$

which is well-defined by Theorem 3.12. We can now formulate the statement that replaces [AH19, Corollary 4.23], providing the required structure on $\tilde{\mathcal{H}}$. Its proof is the same as [AH19, Corollary 4.23], replacing [AH19, Theorem 4.21] by Theorem 3.12 above.

Corollary 3.13. *The convolution product $*$ equips $\tilde{\mathcal{H}}$ with the structure of an associative algebra over \mathcal{R} that contains \mathcal{H} as subspace of finitely supported elements.*

3.3 The center of $\tilde{\mathcal{H}}$ is isomorphic to \mathcal{H}_s

Recall that the definition of the Looijenga algebra $\mathcal{R}[Y]$ and its variants $\mathcal{R}[Y]^{W^v}$ and $\mathcal{R}[Y^+]$ is given by [AH19, Definition 4.6]. Also, we proved in [AH19, Proposition 4.9] that $\mathcal{R}[Y]^{W^v}$ is a subspace of $\mathcal{R}[Y^+]$. Now note that the latter can be seen as a subspace of \mathcal{B} , so it makes sense to compare these algebras with the algebra $\tilde{\mathcal{H}}$ we built earlier.

Given $a \in \mathcal{R}[Y]^{W^v}$, we have $\text{supp}(a) = \text{supp}_Y(a) \times \{1\}$, with $\text{supp}_Y(a)$ being W^v -invariant and almost finite, hence $\text{supp}(a)$ is W^v -almost finite. In particular, this implies that $\mathcal{R}[Y]^{W^v}$ is contained in $\tilde{\mathcal{H}}$. However, note that in general, $\mathcal{R}[Y]$ may not be entirely contained in $\tilde{\mathcal{H}}$, as can be seen for instance in Example 2.2.

Replacing $\hat{\mathcal{H}}$ by $\tilde{\mathcal{H}}$ in the proof of [AH19, Theorem 4.30] provides the following theorem, which replaces [AH19, Theorem 4.30].

Theorem 3.14. *The center of the algebra $\tilde{\mathcal{H}}$ is $\mathcal{Z}(\tilde{\mathcal{H}}) = \mathcal{R}[Y]^{W^v}$, hence is isomorphic to \mathcal{H}_s via the Satake isomorphism.*

Remark 3.15. By [AH19, Lemma 4.5], we know that if \mathbb{A} is associated with an indefinite size 2 Kac-Moody matrix, then any subset of Y^+ is almost finite, hence any subset of Y^+ is W^v -almost finite.

3.4 The reductive case

Lemma 3.16. *Assume that \mathbb{A} is associated with a Cartan matrix A . Then a subset of $Y^+ = Y$ is W^v -almost finite if, and only if, it is finite.*

Proof. Thanks to [AH19, Lemma 5.17], we may assume that $\bigcap_{i \in I} \ker \alpha_i = \{0\}$. Let A_1, \dots, A_r denote the indecomposable components of A : then $\mathbb{A} = \bigoplus_{i=1}^r \mathbb{A}_i$, where \mathbb{A}_i is a realization of A_i (as defined in [AH19, 5.4.1]) for all $i \in \llbracket 1, r \rrbracket$. For $i \in \llbracket 1, r \rrbracket$, we denote by W_i^v the Weyl group of \mathbb{A}_i , by Q_i^\vee its coroot lattice and by Y_i its cocharacter lattice, so that we have $W^v = W_1^v \times \dots \times W_r^v$, $Q^\vee = \bigoplus_{i=1}^r Q_i^\vee$ and $Y = \bigoplus_{i=1}^r Y_i$.

Now let E be a W^v -almost finite subset of Y . For $w \in W^v$, set $E_w := E \cap w \cdot \overline{C_f^v}$. Since E is w^{-1} -almost finite, there exists a finite set $F \subset Y$ such that: $\forall \lambda \in E, \exists \mu \in F \mid \lambda \leq_{Q^\vee} \mu$. For $i \in \llbracket 1, r \rrbracket$, set $Y_i^{++} := Y_i \cap \overline{C_{f,i}^v}$, where $C_{f,i}^v$ denotes the fundamental chamber of \mathbb{A}_i : then we know from [Kac94, Theorem 4.3] that $Y_i^{++} \subset Q_i^\vee$. Given $\lambda = (\lambda_1, \dots, \lambda_r) \in E_w$, let $\mu = (\mu_1, \dots, \mu_r) \in F$ be such that $\lambda \leq_{Q^\vee} \mu$. Then we have $0 \leq_{Q_i^\vee} \lambda_i \leq_{Q_i^\vee} \mu_i$ for all $i \in \llbracket 1, r \rrbracket$, hence E_w must be finite. As W^v is finite, we get that E is finite too, and the lemma is proven as the converse statement is straightforward. \square

Since \mathcal{H} corresponds to the subspace of elements of $\tilde{\mathcal{H}}$ with finite support, we directly obtain the following result from Lemma 3.16, which states that $\tilde{\mathcal{H}}$ is just the usual Iwahori-Hecke algebra in the reductive case. This replaces the first paragraph of [AH19, Section 4.6.1, page 105].

Proposition 3.17. *If \mathbb{A} is associated with a Cartan matrix, then $\tilde{\mathcal{H}} = \mathcal{H}$.*

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