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# Erratum to Completed Iwahori-Hecke algebra and parahoric Hecke algebras for Kac-Moody groups over local fields 

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#### Abstract

We modify the definition of the completed Iwahori-Hecke algebra given in our previous article (J. Ec. Polytechnique 6, 79-118) and explain why the construction we gave earlier is not correct as such.


The goal of this erratum is to fix the construction of the completed Iwahori-Hecke algebra given in [AH19, Section 4.4], as the definition given there does not always provide an actual algebra (see Section 2 below). We define here an algebra that must be used instead. Consequently, the following modifications should be operated in [AH19] : the paragraph before [AH19, Proposition 4.33] is wrong and must be replaced by Section 3 below; Theorem 4.21, Corollary 4.23 and Theorem 4.30 of [AH19] are wrong as stated there and must respectively be replaced by Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.

## 1 Introduction

Let $G$ be a split Kac-Moody group (as defined by Tits in [Tit87]) over a non-Archimedean local field $\mathcal{K}$. Given a ring $\mathscr{R}$ containing $\mathbb{Z}$ and satisfying light technical conditions (as in [AH19, Remark 4.1]), Braverman-Kazdhan-Patnaik [BK11, BKP16] and Bardy-Panse-Gaussent-Rousseau [GR14, BPGR16] associated to $G$ a spherical Hecke algebra $\mathcal{H}_{s}$ and an Iwahori-Hecke algebra $\mathcal{H}$, both defined over $\mathscr{R}$. Fixing a maximal split torus $T$ of $G$, and letting $Y$ (resp. $Y^{+}$) be the cocharacter lattice (resp. its intersection with the Tits cone) and $W^{v}$ be the Weyl group of ( $G, T$ ), then these authors moreover proved the existence of a Satake isomorphism from $\mathcal{H}_{s}$ to $\mathscr{R} \llbracket Y \rrbracket^{W^{v}}$, where $\mathscr{R} \llbracket Y \rrbracket$ is the Looijenga algebra, which is a completion of the group algebra $\mathscr{R}[Y]$ of $Y$ over $\mathscr{R}$ (see [AH19, Definition 4.6] for its definition). A striking difference with the classical case of reductive groups is that for $G$ non-reductive, this spherical Hecke algebra is not isomorphic to the center of the Iwahori-Hecke algebra $\mathcal{H}$.

In [AH19], we announced the definition of a completed Iwahori-Hecke algebra $\widehat{\mathcal{H}}$ that contains $\mathcal{H}$ and whose center is isomorphic to $\mathscr{R} \llbracket Y \rrbracket^{W^{v}}$, hence to $\mathcal{H}_{s}$ [AH19, Theorem 4.30]. Nevertheless, the construction of $\widehat{\mathcal{H}}$ we gave in [AH19, Section 4.4, pages $94-100]$ is not correct as stated, since as such, $\widehat{\mathcal{H}}$ is actually not stable in general under the convolution product. Indeed, the product of two elements of $\widehat{\mathcal{H}}$ can lead to infinite coefficients, as will be seen below in Sections 2.1 and 2.2. This erratum corrects this mistake by defining a slightly different algebra $\widetilde{\mathcal{H}}$, contained in the vector space $\widehat{\mathcal{H}}$, for a suitable convolution product (see Corollary 3.13). The main point is to use the correct notion of almost-finiteness in the definition of the support of the elements of the completed algebra. We check here that $\widetilde{\mathcal{H}}$ contains $\mathcal{H}$ and that the center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathscr{B} \llbracket Y \rrbracket^{W^{v}}$, hence to $\mathcal{H}_{s}$ (see Theorem 3.14), as aimed at first. Moreover note that this modified definition of the completed Iwahori-Hecke algebra suppresses the aforementioned gap between the reductive and non-reductive cases, as for $G$ reductive, $\mathcal{H}$ is actually isomorphic to the classical Iwahori-Hecke algebra (see Proposition 3.17).

This erratum is organized as follows. In Section 2, we give two counter-examples to [AH19, Theorem 4.21]: one in the reductive case and one in the non-reductive case. Then we introduce the required modifications in the definition of the completed algebra to build $\widetilde{\mathcal{H}}$ in Section 3. In particular, we explain in Section 3.3 how to adapt the content of [AH19, Section 4.4, pages 94-100] to prove that the center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathscr{R} \llbracket Y \rrbracket^{W^{v}}$.

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## 2 Two counter-examples to [AH19, Theorem 4.21]

We keep the notation of [AH19, section 2]. Let us briefly recall that, as in [AH19, Section 2.1], given a root generating system $\mathcal{S}=\left(A, X, Y,(\alpha)_{i \in I},\left(\alpha_{i}^{\vee}\right)_{i \in I}\right)$, we set $\mathbb{A}:=Y \otimes \mathbb{R}$, let $W^{v}$ denote the Weyl group of $\mathcal{S}$, $Q^{\vee}:=\bigoplus_{i \in I} \mathbb{Z} \alpha_{i}^{\vee}$ denote its coroot lattice, $Q_{\mathbb{R},+}^{\vee}:=\bigoplus_{i \in I} \mathbb{R}_{+} \alpha_{i}^{\vee}$ and $Q_{\mathbb{R}}^{\vee}=\bigoplus_{i \in I} \mathbb{R} \alpha_{i}^{\vee}$.

We then have $\mathcal{H}=\bigoplus_{\lambda \in Y^{+}, w \in W^{v}} \mathscr{R} Z^{\lambda} H_{w}$, where $Z^{\lambda}$ and $H_{w}$ are symbols that satisfy relations (BL1) to (BL4) of [AH19, Section 4.1, page 91]. The notion of support is defined in [AH19, Definition 4.11]. In this section, we give two examples of elements $\left(a_{j}\right)_{j \in J},\left(b_{k}\right)_{k \in K}$ in $\mathcal{H}$ that are summable in $\widehat{\mathcal{H}}$ in the sense of [AH19, Definition 4.20], but such that $\left(a_{j} * b_{k}\right)_{j \in J, k \in K}$ is not summable: one is a reductive case (Section 2.1), the other one is an affine Kac-Moody case (Section 2.2). This prevents [AH19, Theorem 4.21] from being true as stated, and we explain in Section 2.3 why the problem stands in the notion of almost-finiteness initially used in the definition of $\widehat{\mathcal{H}}$, and how to modify it to get a correct analogue of [AH19, Theorem 4.21].

### 2.1 A counter-exemple in the reductive case

Assume that the standard apartment $\mathbb{A}$ is associated with a Cartan matrix. Fix $\lambda \in Y=Y^{+}$and $i \in I$. For $j \in \mathbb{N}$, set $a_{j}=Z^{\lambda-j \alpha_{i}^{\vee}} H_{i}$ and $b_{j}=Z^{\lambda-j \alpha_{i}^{\vee}}$. As $\lambda-\mathbb{N} \alpha_{i}^{\vee}$ is almost finite, $\left(a_{j}\right)_{j \in \mathbb{N}}$ and $\left(b_{k}\right)_{k \in \mathbb{N}}$ are summable in $\widehat{\mathcal{H}}$. Now let $j, k \in \mathbb{N}$. By (BL4) (see [AH19, Section 4.1 page 91]), there exists $c_{j, k} \in \mathscr{R} \llbracket Y \rrbracket$ such that

$$
a_{j} * b_{k}=Z^{\lambda+r_{i}(\lambda)+(k-j) \alpha_{i}^{\vee}} H_{i}+c_{j, k} .
$$

This implies that $\left(\lambda+r_{i}(\lambda)+(k-j) \alpha_{i}^{\vee}, r_{i}\right) \in \operatorname{supp}\left(a_{j} * b_{k}\right)$, hence $\left(a_{j} * b_{k}\right)_{(j, k) \in \mathbb{N}^{2}}$ satisfies none of the conditions of [AH19, Definition 4.20], so it is not summable in $\widehat{\mathcal{H}}$.

### 2.2 A counter-example in the non-reductive case

Assume now that $\mathbb{A}$ is associated with an indecomposable affine Kac-Moody matrix $A$. Let $\delta: \mathbb{A} \rightarrow \mathbb{R}$ be the smallest positive imaginary root associated with $A$. Fix $\lambda \in Y^{+}$such that $\delta(\lambda)>0$ and $i \in I$ : then [AH19, Section 4.2.2, pages 92-93] ensures that $\lambda-\mathbb{N} \alpha_{i}^{\vee}$ is an almost finite subset of $Y^{+}$. For $j \in \mathbb{N}$, set $a_{j}=Z^{\lambda-j \alpha_{i}^{\curlyvee}} H_{i}$ and $b_{j}=Z^{\lambda-j \alpha_{i}^{\curlyvee}}$ : then the same process as in the reductive case (Section 2.1 above) shows that $\left(a_{j} * b_{k}\right)_{(j, k) \in \mathbb{N}^{2}}$ satisfies none of the conditions of [AH19, Definition 4.20], hence is not summable in $\widehat{\mathcal{H}}$, although both $\left(a_{j}\right)$ and $\left(b_{k}\right)$ are summable in $\widehat{\mathcal{H}}$.

### 2.3 Position of the problem and modifications required

The definition of $\widehat{\mathcal{H}}$ given in [AH19, page 95] crucially relies on the notion of almost finiteness defined in [AH19, Definition 4.12]. The problem is that almost finiteness is not preserved by the action of $W^{v}$ on $Y^{+}$: there can exist (depending on the Kac-Moody matrix $A$ ) an almost finite set $E$ such that $w . E$ is not almost finite, for some $w \in W^{v}$. To fix this problem, we introduce in the next section a refined notion of almost-finiteness, namely the notion of $W^{v}$-almost finiteness. Using this new notion, we define an algebra $\widetilde{\mathcal{H}}$ through an analogous construction to the one done for $\widehat{\mathcal{H}}$ in [AH19, Section 4.4, pages 94-100]. We then explain why the results and proofs stated for $\widehat{\mathcal{H}}$ in [AH19] are now valid for $\widetilde{\mathcal{H}}$.

Before going further, let us list precisely what modifications are actually done in this erratum.

* The notion of almost finiteness defined in [AH19, Definition 4.12] must be replaced by the notion of $W^{v}$-almost finiteness introduced in Definition 3.1 below to define $\widetilde{\mathcal{H}}$ as we defined $\widehat{H}$ but with the aforementioned replacement.
* The statement and proof of [AH19, Theorem 4.21, Corollary 4.23 and Theorem 4.30] must be respectively replaced by the statement and proof of Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.
* The content of the paragraph before [AH19, Proposition 4.33], which explains what happens in the reductive case, must be replaced by Section 3.4 below.


## 3 The completed Iwahori-Hecke algebra $\widetilde{\mathcal{H}}$

The goal of this section is to build an algebra $\widetilde{\mathcal{H}}$ that appears to be smaller than $\widehat{\mathcal{H}}$ (that is NOT always an algebra) whose center is (still) isomorphic to $\mathscr{R} \llbracket Y \rrbracket^{W^{v}}$ and that (still) contains $\mathcal{H}$ as the subalgebra of finitely
supported elements. It actually boils down to defining the right notion of almost-finiteness and checking that what we did in [AH19, Section 4] transposes in this setting to define an actual algebra $\widetilde{\mathcal{H}}$ with the required properties.

## $3.1 \quad W^{v}$-almost finiteness and definition of $\widetilde{\mathcal{H}}$

The idea behind the use of the following refined notion of almost finiteness is that it is preserved by the action of $W^{v}$.

Definition 3.1. Let $u \in W^{v}$.

- A subset $E$ of $Y^{+}$is $u$-almost finite if $u$. $E$ is almost finite in the sense of [AH19, Definition 4.3].
- A subset $E$ of $Y^{+} \times W^{v}$ is called $u$-almost finite if its projection on $W^{v}$ is finite and if its projection on $Y^{+}$is $u$-almost finite (as a subset of $Y^{+}$).
- A subset of $Y^{+}$or of $Y^{+} \times W^{v}$ is $W^{v}$-almost finite if it is $u$-almost finite for any $u \in W^{v}$.

As in [AH19, page 95], we set $\mathscr{B}=\prod_{\lambda \in Y^{+}, w \in W^{v}} \mathscr{R}$ and for $(\lambda, w) \in Y^{+} \times W^{v}$, we let $Z^{\lambda} H_{w}$ denote the element whose coefficients are all equal to 0 apart from the coefficient indexed by $(\lambda, w)$, which is equal to 1. This allows us to write $a=\left(a_{\lambda, w}\right)_{(\lambda, w) \in Y^{+} \times W^{v}} \in \mathscr{B}$ as the formal linear combination $a=$ $\sum_{\substack{\lambda \in Y^{+}, w \in W^{v} \\ \text { defined by }}} a_{\lambda, w} Z^{\lambda} H_{w}$. Also recall that any $(\nu, u) \in Y^{+} \times W^{v}$ is associated to a projection map $\pi_{\nu, u}: \mathscr{B} \rightarrow \mathscr{R}$ defined by

$$
\pi_{\nu, u}\left(\sum_{u^{\prime} \in W^{v}, \nu^{\prime} \in Y^{+}} c_{\nu^{\prime}, u^{\prime}} Z^{\nu^{\prime}} H_{u^{\prime}}\right):=c_{\nu, u}
$$

for any $\sum c_{\nu^{\prime}, u^{\prime}} Z^{\nu^{\prime}} H_{u^{\prime}} \in \mathscr{B}$.
We can now define $\widetilde{\mathcal{H}}$ as the set of elements of $\mathscr{B}$ with $W^{v}$-almost finite support. To prove that $\widetilde{\mathcal{H}}$ can be endowed with a convolution product $*$ that turns it into an associative algebra containing $\mathcal{H}$, we will basically follow the same steps as in [AH19, Section 4.4], replacing the almost finiteness condition by the $W^{v}$-almost finiteness condition.

We let $\operatorname{conv}_{\mathbb{R}}(F)$ denote the convex hull of any part $F$ of $\mathbb{A}$, and we set $\operatorname{conv}(E):=\operatorname{conv}_{\mathbb{R}}(E) \cap Y$ for any subset $E$ of $Y$. Following [AH19, page 95], recall that for any part $E$ of $Y$ and any $i \in I$, we let $R_{i}(E)=\operatorname{conv}\left(E \cup r_{i}(E)\right) \subset E+Q^{\vee}$ and that, for any pair $(\lambda, w) \in Y^{+} \times W^{v}$, we set

$$
R_{w}(\lambda):=\bigcup R_{i_{1}}\left(R_{i_{2}}\left(\ldots\left(R_{i_{k}}(\{\lambda\}) \ldots\right)\right)\right.
$$

where the union is taken over all the reduced writings $r_{i_{1}} r_{i_{2}} \ldots r_{i_{k}}$ of $w$. The next two results replace [AH19, Remark 4.13] and act as preparation for the proof of Lemma 3.4 below, which replaces [AH19, Lemma 4.15].

Lemma 3.2. For any $(\lambda, w) \in Y^{+} \times W^{v}$, we have

$$
R_{w}(\lambda) \subset \operatorname{conv}(\{u . \lambda \mid u \in[1, w]\}),
$$

where $[1, w]:=\left\{u \in W^{v} \mid u \leq w\right\}$ is defined as in [AH19, bottom of page 94].
Proof. We prove this result by induction on $\ell(w)$. If $\ell(w)=0$, there is nothing to prove, so let $w \in W^{v}$ be an element of length $\ell(w) \geq 1$ and assume by induction that the lemma holds for any element $w^{\prime} \in W^{v}$ such that $\ell\left(w^{\prime}\right)<\ell(w)$. Let $\mu \in R_{w}(\lambda)$, then there exists $i \in I$ such that $w^{\prime}:=r_{i} w$ satisfies $w^{\prime}<w$ and $\mu \in R_{i}\left(R_{w^{\prime}}(\lambda)\right)=\operatorname{conv}\left(R_{w^{\prime}}(\lambda), r_{i} R_{w^{\prime}}(\lambda)\right)$. As $\ell\left(w^{\prime}\right)<\ell(w)$, we have $R_{w^{\prime}}(\lambda) \subset \operatorname{conv}\left(\left\{u \cdot \lambda \mid u \in\left[1, w^{\prime}\right]\right\}\right)$ by induction hypothesis. Since [Kum02, Corollary 1.3.19] ensures that $\left\{1, r_{i}\right\} \cdot\left[1, w^{\prime}\right] \subset[1, w]$, we obtain that

$$
\operatorname{conv}\left(\left\{u . \lambda \mid u \in\left[1, w^{\prime}\right]\right\}\right) \cup r_{i} \cdot \operatorname{conv}\left(\left\{u . \lambda \mid u \in\left[1, w^{\prime}\right]\right\}\right) \subset \operatorname{conv}(\{u . \lambda \mid u \in[1, w]\}) .
$$

Consequently, we get that

$$
\begin{aligned}
\mu \in \operatorname{conv}\left(R_{w^{\prime}}(E), r_{i} \cdot R_{w^{\prime}}(E)\right) & \subset \operatorname{conv}\left(\operatorname{conv}\left(\left\{u \cdot \lambda \mid u \in\left[1, w^{\prime}\right]\right\}\right) \cup r_{i} \cdot \operatorname{conv}\left(\left\{u \cdot \lambda \mid u \in\left[1, w^{\prime}\right]\right\}\right)\right) \\
& \subset \operatorname{conv}(\operatorname{conv}(\{u \cdot \lambda \mid u \in[1, w]\}))=\operatorname{conv}(\{u \cdot \lambda \mid u \in[1, w]\}) .
\end{aligned}
$$

This proves that $R_{w}(\lambda)$ is contained in $\operatorname{conv}(\{u . \lambda \mid u \in[1, w]\})$, hence the lemma.

Lemma 3.3. Let $E$ be a $W^{v}$-almost finite subset of $Y^{+}$.
Then, for any pair $(\nu, w) \in Y^{+} \times W^{v}$, the set $\left\{\mu \in E \mid \nu \in R_{w}(\mu)\right\}$ is finite.
Proof. Let $E \subset Y^{+}$and $(\nu, w) \in Y^{+} \times W^{v}$ be as in the statement. Applying the definition of almost finiteness [AH19, Definition 4.3] to $u . E$ for any $u \in[1, w]$ provides a finite set $F \subset Y^{+}$such that :

$$
\forall u \in[1, w], \forall \mu \in u . E, \exists \lambda \in F \mid \mu \leq_{Q^{\vee}} \lambda
$$

Set $\mathcal{X}:=\left\{\mu \in E \mid \nu \in R_{w}(\mu)\right\}$ and pick some $\mu \in \mathcal{X}$. As $\nu$ belongs to $R_{w}(\mu)$, Lemma 3.2 implies the existence of $\left(t_{u}\right)_{u \in[1, w]} \in[0,1]^{[1, u]}$ such that

$$
\sum_{u \in[1, w]} t_{u}=1 \text { and } \nu=\sum_{u \in[1, w]} t_{u} u \cdot \mu .
$$

For any $u \in[1, w]$, choose $\lambda(u) \in F$ such that $u \cdot \mu \leq_{Q^{\vee}} \lambda(u)$ and write $\lambda(u)-u \cdot \mu$ as $\sum_{i \in I} n_{i}(u) \alpha_{i}^{\vee}$ with $n_{i}(u) \in \mathbb{N}$ for all $i \in I$. Then we have:

$$
\nu=\sum_{u \in[1, w]} t_{u} u \cdot \mu=\sum_{u \in[1, w]} t_{u} \lambda(u)-\sum_{u \in[1, w], i \in I} t_{u} n_{i}(u) \alpha_{i}^{\vee} .
$$

Set $a(\mu):=\sum_{u \in[1, w]} t_{u} \lambda(u) \in \operatorname{conv}_{\mathbb{R}}(F)$ and $q(\mu):=\sum_{u \in[1, w], i \in I} t_{u} n_{i}(u) \alpha_{i}^{\vee} \in Q_{\mathbb{R},+}^{\vee}$. Since $F$ is finite, $\operatorname{conv}_{\mathbb{R}}(F)$ is bounded. As $q(\mu)=a(\mu)-\nu$ lies in $\operatorname{conv}_{\mathbb{R}}(F)-\nu$, the set $\left\{q\left(\mu^{\prime}\right), \mu^{\prime} \in \mathcal{X}\right\}$ is bounded too. Moreover, as $\sum_{u \in[1, w]} t_{u}=1$, there exists $u^{\prime} \in[1, w]$ such that $t_{u^{\prime}} \geq \frac{1}{|[1, w]|}$. Letting $f_{j}(x)$ denote the $j$-th coordinate of $x \in Q_{\mathbb{R}}^{\vee}$ in the basis $\left(\alpha_{i}^{\vee}\right)_{i \in I}$ for all $j \in J$, we have:

$$
\forall i \in I, f_{i}(q(\mu))=\sum_{u \in[1, w]} t_{u} n_{i}(u) \geq t_{u^{\prime}} n_{i}\left(u^{\prime}\right) \geq 0
$$

We hence obtain that:

$$
\forall i \in I, 0 \leq n_{i}\left(u^{\prime}\right) \leq \frac{\sup _{\mu^{\prime} \in \mathcal{X}} f_{i}\left(q\left(\mu^{\prime}\right)\right)}{t_{u^{\prime}}} \leq \frac{\sup _{\mu^{\prime} \in \mathcal{X}} f_{i}\left(q\left(\mu^{\prime}\right)\right)}{|[1, u]|}
$$

Consequently, if we set $N:=\max _{i \in I} \frac{\sup _{\mu^{\prime} \in \mathcal{X}} f_{i}\left(q\left(\mu^{\prime}\right)\right)}{|[1, u]|}$, then we have:

$$
u^{\prime} \cdot \mu \in \lambda\left(u^{\prime}\right)-\sum_{i \in I} \llbracket 0, N \rrbracket \alpha_{i}^{\vee} \subset F-\sum_{i \in I} \llbracket 0, N \rrbracket \alpha_{i}^{\vee}
$$

This proves that $\mu$ lies in $\bigcup_{u \in[1, w]} u^{-1} \cdot\left(F-\sum_{i \in I} \llbracket 0, N \rrbracket \alpha_{i}^{\vee}\right)$, hence that $\mathcal{X}$ is contained in the finite set $\bigcup_{u \in[1, w]} u^{-1} \cdot(F-$ $\left.\sum_{i \in I} \llbracket 0, N \rrbracket \alpha_{i}^{\vee}\right)$, which proves that $\mathcal{X}$ is finite too, as claimed.

The next lemma replaces [AH19, Lemma 4.15]: the only modification consists in replacing [ $1, w] w^{\prime}$ by $\left[1, w^{\prime}\right] w$ in the aforementioned statement. In particular, the proof follows the exact same lines as [AH19, page 96], hence we do not rewrite it here.

Lemma 3.4. For all $w, w^{\prime} \in W^{v}$ and all $\lambda \in Y, H_{w^{\prime}} * Z^{\lambda} H_{w}$ is in $\bigoplus_{(\nu, t) \in R_{w^{\prime}}(\lambda) \times\left[1, w^{\prime}\right] \cdot w} \mathscr{R} \cdot Z^{\nu} H_{t}$.
Using the definitions of $\operatorname{supp}_{W^{v}}$ and $\operatorname{supp}_{Y}$ given by [AH19, Definition 4.11], one can straightforward deduce from Lemma 3.4 the following inclusions.

Lemma 3.5. For all $a, b \in \mathcal{H}$, we have:

$$
\begin{aligned}
& \text { (1) } \operatorname{supp}_{Y}(a * b) \subset \operatorname{supp}_{Y}(a)+\bigcup_{w \in \operatorname{supp}_{W^{v}}(a), \lambda \in \operatorname{supp}_{Y}(b)} R_{w}(\lambda) ; \\
& (2) \operatorname{supp}_{W^{v}}(a * b) \subset \bigcup_{v \in \operatorname{supp}_{W^{v}}(a), w \in \operatorname{supp}_{W^{v}}(b)}[1, v] \cdot w
\end{aligned}
$$

Before we give the definition of summable families in $\widetilde{\mathcal{H}}$, we prove two more statements related to $W^{v}{ }_{-}$ almost finiteness in $Y^{+}$.

Lemma 3.6. For any almost finite set $E \subset Y^{+}, \operatorname{conv}(E)$ is also almost finite.
Proof. Let $E \subset Y^{+}$be an almost finite set and let $F$ be a finite set such that:

$$
\forall \lambda \in E, \exists \mu \in F \mid \lambda \leq_{Q^{\vee}} \mu
$$

Given $\lambda \in \operatorname{conv}(E)$, there exist $n \in \mathbb{N}, t_{1}, \ldots, t_{n} \in[0,1]$ and $\lambda_{1}, \ldots, \lambda_{n} \in E$ such that

$$
\sum_{i=1}^{n} t_{i}=1 \quad \text { and } \quad \sum_{i=1}^{n} t_{i} \lambda_{i}=\lambda
$$

For each index $i \in \llbracket 1, n \rrbracket$, choose $\kappa_{i} \in F$ such that $\lambda_{i} \leq_{Q^{\vee}} \kappa_{i}$ : then $\sum_{i=1}^{n} t_{i} \kappa_{i}-\sum_{i=1}^{n} t_{i} \lambda_{i} \in \bigoplus_{i \in I} \mathbb{R}_{+} \alpha_{i}^{\vee}$. We can hence write $\sum_{i=1}^{n} t_{i} \kappa_{i}-\sum_{i=1}^{n} t_{i} \lambda_{i}=\sum_{i \in I} x_{i} \alpha_{i}^{\vee}$ for some nonnegative real numbers $\left(x_{i}\right)_{i \in I}$. Now let $\left(x_{i}^{\prime}\right) \in\left[0,1\left[I^{I}\right.\right.$ be such that $x_{i}+x_{i}^{\prime}$ lies in $\mathbb{N}$ for all $i \in I$ and set $\nu:=\sum_{i \in I}\left(x_{i}+x_{i}^{\prime}\right) \alpha_{i}^{\vee}+\lambda$. Then we have $\nu \geq_{Q^{\vee}} \lambda$ and $\nu \in\left(\operatorname{conv}_{\mathbb{R}}(F)+\bigoplus_{i \in I}[0,1] \alpha_{i}^{\vee}\right) \cap Y$. Since $F$ is finite, $\left(\operatorname{conv}_{\mathbb{R}}(F)+\bigoplus_{i \in I}[0,1] \alpha_{i}^{\vee}\right) \cap Y$ is a finite set that can be taken as $J$ in [AH19, Definition 4.3] for $\operatorname{conv}(E)$, and the lemma is proven.

Lemma 3.7. Let $E$ be a $W^{v}$-almost finite subset of $Y^{+}$. Then, for any $w \in W^{v}$, the set $\bigcup_{\lambda \in E} R_{w}(\lambda)$ is $W^{v}$-almost finite.

Proof. Let $w \in W^{v}$. By Lemma 3.2, we have

$$
\bigcup_{\lambda \in E} R_{w}(\lambda) \subset \bigcup_{\lambda \in E} \operatorname{conv}(\{u \cdot \lambda, u \in[1, w]\}) \subset \operatorname{conv}\left(\bigcup_{u \in[1, w]} u . E\right)
$$

Let $v \in W^{v}$. Since $[1, w]$ is finite, the set $v . \bigcup_{u \in[1, w]} u . E$ is almost finite, hence Lemma 3.6 implies that $v . \operatorname{conv}\left(\bigcup_{u \in[1, w]} u . E\right)=\operatorname{conv}\left(v . \bigcup_{u \in[1, w]} u . E\right)$ is almost finite. This proves that $\operatorname{conv}\left(\bigcup_{u \in[1, w]} u . E\right)$ is $v$ almost finite for any $v \in W^{v}$, and the lemma is proven.

## $3.2 \tilde{\mathcal{H}}$ is an associative algebra

This subsection contains the main modification of the paper, as it aims to prove that $\widetilde{\mathcal{H}}$ is actually an associative algebra. To do this, we first need to introduce the correct definition of summable families, which is the counterpart of [AH19, Definition 4.20] for $W^{v}$-almost finite sets.
Definition 3.8. A family $\left(a_{j}\right)_{j \in J} \in(\widetilde{\mathcal{H}})^{J}$ is summable in $\widetilde{\mathcal{H}}$ when the two following properties hold:
(i) for any $\lambda \in Y^{+}$, the set $\left\{j \in J \mid \exists w \in W^{v}, \pi_{\lambda, w}\left(a_{j}\right) \neq 0\right\}$ is finite;
(ii) the set $\bigcup_{j \in J} \operatorname{supp}\left(a_{j}\right):=\bigcup_{j \in J}\left\{(\lambda, w) \in Y^{+} \times W^{v} \mid \pi_{\lambda, w}\left(a_{j}\right) \neq 0\right\}$ is $W^{v}$-almost finite.

Given a summable family $\left(a_{j}\right)_{j \in J} \in(\widetilde{\mathcal{H}})^{J}$, we define $\sum_{j \in J} a_{j} \in \widetilde{\mathcal{H}}$ by the following formula:

$$
\sum_{j \in J} a_{j}:=\sum_{(\lambda, w) \in Y^{+} \times W^{v}} a_{\lambda, w} Z^{\lambda} H_{w}, \text { with } a_{\lambda, w}:=\sum_{j \in J} \pi_{\lambda, w}\left(a_{j}\right) \text { for all }(\lambda, w) \in Y^{+} \times W^{v}
$$

Lemma 3.9. For any almost finite subsets $E, E^{\prime}$ of $Y^{+}$and any $\rho \in Y^{+}$, the set

$$
E^{\prime \prime}:=\left\{\lambda \in E \mid \exists \nu \in E^{\prime}, \lambda+\nu=\rho\right\}
$$

is finite.
Proof. By definition, there exists a finite set $F \subset Y^{+}$such that: $\forall \mu \in E \cup E^{\prime}, \exists \kappa \in F \mid \mu \leq_{Q^{\vee}} \kappa$. Now let $\lambda \in E^{\prime \prime}$ and $\nu \in E^{\prime}$ be such that $\lambda+\nu=\rho$. Then we have:

$$
\exists\left(\kappa, \kappa^{\prime}\right) \in F^{2} \mid \rho-\kappa^{\prime} \leq_{Q^{\vee}} \lambda \leq_{Q^{\vee}} \kappa
$$

and the lemma follows.
The next lemma is the cornerstone that ensures that the convolution product on $\widetilde{\mathcal{H}}$ is well-defined.
Lemma 3.10. Let $\left(a_{\lambda}\right),\left(b_{\mu}\right) \in \mathscr{R}^{Y^{+}}$be such that $\left\{\lambda \in Y^{+} \mid a_{\lambda} \neq 0\right.$ or $\left.b_{\lambda} \neq 0\right\}$ is $W^{v}$-almost finite. Then, for any $w \in W^{v},\left(a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z_{\mu}\right)_{(\lambda, \mu) \in\left(Y^{+}\right)^{2}}$ is summable in $\widetilde{\mathcal{H}}$. Said differently, if $\sum_{\lambda \in Y^{+}} a_{\lambda} Z^{\lambda} H_{w}, \sum_{\mu \in Y^{+}} b_{\mu} Z^{\mu} \in$ $\widetilde{\mathcal{H}}$, then

$$
\left(\sum_{\lambda \in Y^{+}} a_{\lambda} Z^{\lambda} H_{w}\right) *\left(\sum_{\mu \in Y^{+}} b_{\mu} Z^{\mu}\right):=\sum_{\lambda, \mu \in Y^{+}} a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z^{\mu}
$$

is a well-defined element of $\widetilde{\mathcal{H}}$.
Proof. Set $S_{a}:=\left\{\lambda \in Y^{+} \mid a_{\lambda} \neq 0\right\}, S_{b}=\left\{\mu \in Y^{+} \mid b_{\mu} \neq 0\right\}$ and $E=\bigcup_{\mu \in S_{b}} R_{w}(\mu)$. Note that $E$ is almost finite by Lemma 3.7.
Given $\mu \in Y^{+}$, Lemma 3.4 ensures the existence of $\left(z_{\mu}^{v, \nu}\right)_{\nu \in R_{w}(\mu), v \in[1, w]} \in \mathscr{R}^{R_{w}(\mu) \times[1, w]}$ such that

$$
H_{w} * Z^{\mu}=\sum_{\nu \in R_{w}(\mu), v \in[1, w]} z_{\mu}^{v, \nu} Z^{\nu} H_{v}
$$

Let us fix $v \in[1, w]$. Given any $\lambda, \rho \in Y^{+}$, we have

$$
\begin{equation*}
\pi_{\rho, v}\left(a_{\lambda} Z^{\lambda} H_{w} * b_{\mu} Z^{\mu}\right)=\sum_{\nu \in R_{w}(\mu) \mid \lambda+\nu=\rho} a_{\lambda} b_{\mu} z_{\mu}^{v, \nu} \tag{1}
\end{equation*}
$$

Set $F_{1}:=\left\{\lambda \in Y^{+} \mid \exists \mu \in Y^{+}, \pi_{\rho, v}\left(a_{\lambda} Z^{\lambda} H_{w} * b_{\mu} Z^{\mu}\right) \neq 0\right\} \subset S_{a}$ and let $\lambda \in F_{1}$. By equality (1), there exists $\nu \in E$ such that $\lambda+\nu=\rho$. Since $\lambda$ lies in $S_{a}$, applying Lemma 3.9 to $E$ and $S_{a}$ implies that $F_{1}$ is finite. Fix now $\lambda \in F_{1}$ and set $F_{2}(\lambda)=\left\{\mu \in Y^{+} \mid \pi_{\rho, v}\left(a_{\lambda} Z^{\lambda} H_{w} * b_{\mu} Z^{\mu}\right) \neq 0\right\} \subset S_{b}$. Given $\mu \in F_{2}(\lambda)$, we know from equality (1) that $\rho-\lambda \in R_{w}(\mu)$. As $S_{b}$ is $W^{v}$-almost finite, Lemma 3.3 yields the finiteness of $F_{2}(\lambda)$, hence the finiteness of

$$
F_{v}:=\left\{(\lambda, \mu) \in\left(Y^{+}\right)^{2} \mid \pi_{\rho, v}\left(a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z^{\mu}\right) \neq 0\right\}=\bigcup_{\lambda \in F_{1}} F_{2}(\lambda)
$$

Finally, we obtain that

$$
\left\{(\lambda, \mu) \in\left(Y^{+}\right)^{2} \mid \exists v \in W^{v}, \pi_{\rho, v}\left(a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z^{\mu}\right) \neq 0\right\}=\bigcup_{v \in[1, w]} F_{v}
$$

is finite, which proves that $\left(a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z^{\mu}\right)_{(\lambda, \mu) \in\left(Y^{+}\right)^{2}}$ satisfies condition (i) of Definition 3.8.
Now let $\lambda, \mu \in Y^{+}$. Then equality (1) ensures that:

$$
\operatorname{supp}\left(a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z^{\mu}\right) \subset\left(\lambda+R_{w}(\mu)\right) \times[1, w] \subset(\lambda+E) \times[1, w]
$$

In particular, we have

$$
\bigcup_{(\lambda, \mu) \in\left(Y^{+}\right)^{2}} \operatorname{supp}\left(a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z^{\mu}\right) \subset\left(S_{a}+E\right) \times[1, w] .
$$

As $E$ is $W^{v}$-almost finite, and as the sum of two $W^{v}$-almost finite sets is $W^{v}$-almost finite, we obtain that $\left(a_{\lambda} b_{\mu} Z^{\lambda} H_{w} * Z^{\mu}\right)_{(\lambda, \mu) \in\left(Y^{+}\right)^{2}}$ satisfies condition (ii) of Definition 3.8, which ends the proof.

Lemma 3.11. For any summable family $\left(a_{j}\right)_{j \in J} \in(\widetilde{\mathcal{H}})^{J}$ in $\widetilde{\mathcal{H}}$ and any $i \in I$, the family $\left(a_{j} * H_{i}\right)_{j \in J}$ is summable in $\widetilde{\mathcal{H}}$.

Proof. By definition, there exist a $W^{v}$-almost finite subset $E$ of $Y^{+}$and a finite subset $F$ of $W^{v}$ such that: $\forall j \in J, \operatorname{supp}\left(a_{j}\right) \subset E \times F$. By (BL2) of [AH19, page 91], we get that

$$
\forall j \in J, \operatorname{supp}\left(a_{j} * H_{i}\right) \subset E \times\left(F \cup F \cdot r_{i}\right)
$$

which proves that $\bigcup_{j \in J} \operatorname{supp}\left(a_{j} * H_{i}\right)$ is $W^{v}$-almost finite. Now let $\lambda \in Y^{+}, w \in W^{v}$ and $j \in J$. By (BL2) again, we know that $\pi_{\lambda, w}\left(a_{j} * H_{i}\right) \neq 0$ implies $\pi_{\lambda, w}\left(a_{j}\right) \neq 0$ or $\pi_{\lambda, w r_{i}}\left(a_{j}\right) \neq 0$. Since there are finitely many such $j \in J$, this completes the proof of the lemma.

The next statement replaces [AH19, Theorem 4.21] and its proof basically follows the same lines as the proof of [AH19, Theorem 4.21], replacing almost finiteness by $W^{v}$-almost finiteness. Recall that the elements of $\mathcal{H}$ correspond to the elements of $\widetilde{\mathcal{H}}$ with finite support.
Theorem 3.12. Let $\left(a_{j}\right)_{j \in J} \in(\mathcal{H})^{J}$ and $\left(b_{k}\right)_{k \in J} \in(\mathcal{H})^{K}$ be two families that are both summable in $\widetilde{\mathcal{H}}$. Then $\left(a_{j} * b_{k}\right)_{(j, k) \in J \times K}$ is summable in $\widetilde{\mathcal{H}}$ and $\sum_{(j, k) \in J \times K} a_{j} * b_{k}$ only depends on the two elements $\sum_{j \in J} a_{j}$ and $\sum_{k \in K} b_{k}$ of $\widetilde{\mathcal{H}}$.

Proof. For $j \in J$ and $k \in K$, write $a_{j}:=\sum_{v \in W^{v}} a_{v, j} * H_{v}$ and $b_{k}:=\sum_{w \in W^{v}} b_{w, k} * H_{w}$, with $\left(a_{v, j}\right)_{j \in J} \in \mathscr{R} \llbracket Y \rrbracket^{J}$ and $\left(b_{w, k}\right)_{k \in K} \in \mathscr{R}\left[Y \rrbracket^{K}\right.$ for any $v, w \in W^{v}$. Given $v, w \in W^{v}$, Lemma 3.10 ensures that $\left(a_{v, j} * H_{v} * b_{w, k}\right)_{(j, k) \in J \times K}$ is summable in $\widetilde{\mathcal{H}}$. By induction on $\ell(w)$ and using Lemma 3.11, we get that $\left(a_{v, j} * H_{v} * b_{w, k} * H_{w}\right)_{(j, k) \in J \times K}$ is summable in $\widetilde{\mathcal{H}}$. Moreover, as $\left(a_{j}\right)$ and $\left(b_{k}\right)$ are summable in $\widetilde{\mathcal{H}}$, there are at most finitely many $v, w \in W^{v}$ satisfying $\left(a_{v, j}\right)_{j \in J} \neq 0$ and $\left(b_{w, k}\right)_{k \in K} \neq 0$. Consequently, the family $\left(a_{j} * b_{k}\right)_{(j, k) \in J \times K}$ is summable in $\widetilde{\mathcal{H}}$. Now, given any triple $(u, v, \mu) \in W^{v} \times W^{v} \times Y^{+}$, applying Lemma 3.4 to $H_{u} * Z^{\mu} H_{v}$ gives a family $\left(z_{\nu, t}^{u, v, \mu}\right)_{(\nu, t) \in R_{u}(\mu) \times[1, u] . v}$ of scalars that satisfy $H_{u} * Z^{\mu} H_{v}=\sum_{(\nu, t) \in R_{u}(\mu) \times[1, u] . v} z_{\nu, t}^{u, v, \mu} Z^{\nu} H_{t}$.
For any pair $(\rho, s) \in Y^{+} \times W^{v}$, we have

$$
\begin{aligned}
\pi_{\rho, s}\left(\sum_{(j, k) \in J \times K} a_{j} * b_{k}\right) & =\sum_{(\lambda, u),(\mu, v) \in Y^{+} \times W^{v}} \sum_{\nu \in R_{u}(\mu) \mid \lambda+\nu=\rho} \sum_{(j, k) \in J \times K} a_{j, \lambda, u} b_{k, \mu, v} z_{\nu, s}^{u, v, \mu} \\
& =\sum_{(\lambda, u),(\mu, v) \in Y^{+} \times W^{v}} \sum_{\nu \in R_{u}(\mu) \mid \lambda+\nu=\rho} a_{\lambda, u} b_{\mu, v} z_{v, s, s}^{u, v, \mu}
\end{aligned}
$$

where we set $\sum_{j \in J} a_{j}=\sum_{(\lambda, u) \in Y^{+} \times W^{v}} a_{\lambda, u} Z^{\lambda} H_{u}$ and $\sum_{k \in K} b_{k}=\sum_{(\mu, v) \in Y^{+} \times W^{v}} b_{\mu, v} Z^{\mu} H_{v}$, hence the theorem is proven.

The mistake done in the former proof of [AH19, Theorem 4.20] is to implicitely assume that $S_{Y}=$ $\bigcup_{j \in J} \operatorname{supp}_{Y}\left(a_{j}\right) \cup \bigcup_{k \in K} \operatorname{supp}_{Y}\left(b_{k}\right)$ is such that $\left\{\lambda^{++}, \lambda \in S_{Y}\right\}$ is almost finite, which is not true in general, as shown by the counter-examples given in Section 2. As spotted by the referee, the same kind of subtlety underlies a mistake made by Looijenga in his seminal 1980 work [Loo80, (4.1), end of the first paragraph].

For $a=\sum_{(\lambda, v) \in Y^{+} \times W^{v}} a_{\lambda, v} Z^{\lambda} H_{v} \in \widetilde{\mathcal{H}}$ and $b=\sum_{(\mu, w) \in Y^{+} \times W^{v}} b_{\mu, w} Z^{\mu} H_{w} \in \widetilde{\mathcal{H}}$, we set

$$
a * b=\sum_{(v, \lambda),(w, \mu) \in Y^{+} \times W^{v}} a_{\lambda, v} b_{\mu, w} Z^{\lambda} H_{v} * Z^{\mu} H_{w}
$$

which is well-defined by Theorem 3.12. We can now formulate the statement that replaces [AH19, Corollary 4.23], providing the required structure on $\widetilde{\mathcal{H}}$. Its proof is the same as [AH19, Corollary 4.23], replacing [AH19, Theorem 4.21] by Theorem 3.12 above.

Corollary 3.13. The convolution product $*$ equips $\widetilde{\mathcal{H}}$ with the structure of an associative algebra over $\mathscr{R}$ that contains $\mathcal{H}$ as subspace of finitely supported elements.

### 3.3 The center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathcal{H}_{s}$

Recall that the definition of the Looijenga algebra $\mathscr{R} \llbracket Y \rrbracket$ and its variants $\mathscr{R} \llbracket Y]^{W^{v}}$ and $\mathscr{R} \llbracket Y^{+} \rrbracket$ is given by [AH19, Definition 4.6]. Also, we proved in [AH19, Proposition 4.9] that $\mathscr{R} \llbracket Y \rrbracket^{W^{v}}$ is a subspace of $\mathscr{R} \llbracket Y^{+} \rrbracket$. Now note that the latter can be seen as a subspace of $\mathscr{B}$, so it makes sense to compare these algebras with the algebra $\widetilde{\mathcal{H}}$ we built earlier.

Given $a \in \mathscr{R} \llbracket Y \rrbracket^{W^{v}}$, we have $\operatorname{supp}(a)=\operatorname{supp}_{Y}(a) \times\{1\}$, with $\operatorname{supp}_{Y}(a)$ being $W^{v}$-invariant and almost finite, hence $\operatorname{supp}(a)$ is $W^{v}$-almost finite. In particular, this implies that $\mathscr{R} \llbracket Y \rrbracket^{W^{v}}$ is contained in $\widetilde{\mathcal{H}}$. However, note that in general, $\mathscr{R} \llbracket Y \rrbracket$ may not be entirely contained in $\widetilde{\mathcal{H}}$, as can be seen for instance in Example 2.2.

Replacing $\widehat{\mathcal{H}}$ by $\widetilde{\mathcal{H}}$ in the proof of [AH19, Theorem 4.30] provides the following theorem, which replaces [AH19, Theorem 4.30].

Theorem 3.14. The center of the algebra $\widetilde{\mathcal{H}}$ is $\mathscr{Z}(\widetilde{\mathcal{H}})=\mathscr{R} \llbracket Y \rrbracket^{W^{v}}$, hence is isomorphic to $\mathcal{H}_{s}$ via the Satake isomorphism.

Remark 3.15. By [AH19, Lemma 4.5], we know that if $\mathbb{A}$ is associated with an indefinite size 2 Kac-Moody matrix, then any subset of $Y^{+}$is almost finite, hence any subset of $Y^{+}$is $W^{v}$-almost finite.

### 3.4 The reductive case

Lemma 3.16. Assume that $\mathbb{A}$ is associated with a Cartan matrix $A$. Then a subset of $Y^{+}=Y$ is $W^{v}$-almost finite if, and only if, it is finite.

Proof. Thanks to [AH19, Lemma 5.17], we may assume that $\bigcap_{i \in I} \operatorname{ker} \alpha_{i}=\{0\}$. Let $A_{1}, \ldots, A_{r}$ denote the indecomposable components of $A$ : then $\mathbb{A}=\bigoplus_{i=1}^{r} \mathbb{A}_{i}$, where $\mathbb{A}_{i}$ is a realization of $A_{i}$ (as defined in [AH19, 5.4.1]) for all $i \in \llbracket 1, r \rrbracket$. For $i \in \llbracket 1, r \rrbracket$, we denote by $W_{i}^{v}$ the Weyl group of $\mathbb{A}_{i}$, by $Q_{i}^{\vee}$ its coroot lattice and by $Y_{i}$ its cocharacter lattice, so that we have $W^{v}=W_{1}^{v} \times \ldots \times W_{r}^{v}, Q^{\vee}=\bigoplus_{i=1}^{r} Q_{i}^{\vee}$ and $Y=\bigoplus_{i=1}^{r} Y_{i}$.

Now let $E$ be a $W^{v}$-almost finite subset of $Y$. For $w \in W^{v}$, set $E_{w}:=E \cap w . \overline{C_{f}^{v}}$. Since $E$ is $w^{-1}$ almost finite, there exists a finite set $F \subset Y$ such that: $\forall \lambda \in E, \exists \mu \in F \mid \lambda \leq_{Q^{\vee} \mu}$. For $i \in \llbracket 1, r \rrbracket$, set $Y_{i}^{++}:=Y_{i} \cap \overline{C_{f, i}^{v}}$, where $C_{f, i}^{v}$ denotes the fundamental chamber of $\mathbb{A}_{i}$ : then we know from [Kac94, Theorem 4.3] that $Y_{i}^{++} \subset Q_{i,+}^{\vee}$. Given $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right) \in E_{w}$, let $\mu=\left(\mu_{1}, \ldots, \mu_{r}\right) \in F$ be such that $\lambda \leq_{Q^{\vee}} \mu$. Then we have $0 \leq_{Q_{i}^{\vee}} \lambda_{i} \leq_{Q_{i}^{\vee}} \mu_{i}$ for all $i \in \llbracket 1, r \rrbracket$, hence $E_{w}$ must be finite. As $W^{v}$ is finite, we get that $E$ is finite too, and the lemma is proven as the converse statement is straightforward.

Since $\mathcal{H}$ corresponds to the subspace of elements of $\widetilde{\mathcal{H}}$ with finite support, we directly obtain the following result from Lemma 3.16, which states that $\widetilde{\mathcal{H}}$ is just the usual Iwahori-Hecke algebra in the reductive case. This replaces the first paragraph of [AH19, Section 4.6.1, page 105].
Proposition 3.17. If $\mathbb{A}$ is associated with a Cartan matrix, then $\widetilde{\mathcal{H}}=\mathcal{H}$.

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