

Erratum to Completed Iwahori-Hecke algebra and parahoric Hecke algebras for Kac-Moody groups over local fields

Ramla Abdellatif, Auguste Hébert

▶ To cite this version:

Ramla Abdellatif, Auguste Hébert. Erratum to Completed Iwahori-Hecke algebra and parahoric Hecke algebras for Kac-Moody groups over local fields. Journal de l'École polytechnique - Mathématiques, 2022, 9, pp.1293-1304. 10.5802/jep.206 . hal-04206162

HAL Id: hal-04206162 https://hal.science/hal-04206162

Submitted on 13 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Erratum to Completed Iwahori-Hecke algebra and parahoric Hecke algebras for Kac-Moody groups over local fields

Ramla ABDELLATIF LAMFA – UPJV UMR 7352 CNRS 80 039 AMIENS Cedex 1, France ramla.abdellatif@u-picardie.fr Auguste HÉBERT Université de Lorraine Institut Élie Cartan de Lorraine UMR 7502 CNRS auguste.hebert@univ-lorraine.fr

Abstract

We modify the definition of the completed Iwahori-Hecke algebra given in our previous article (J. Ec. Polytechnique 6, 79–118) and explain why the construction we gave earlier is not correct as such.

The goal of this erratum is to fix the construction of the *completed Iwahori-Hecke algebra* given in [AH19, Section 4.4], as the definition given there does not always provide an actual algebra (see Section 2 below). We define here an algebra that must be used instead. Consequently, the following modifications should be operated in [AH19] : the paragraph before [AH19, Proposition 4.33] is wrong and must be replaced by Section 3 below; Theorem 4.21, Corollary 4.23 and Theorem 4.30 of [AH19] are wrong as stated there and must respectively be replaced by Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.

1 Introduction

Let G be a split Kac-Moody group (as defined by Tits in [Tit87]) over a non-Archimedean local field \mathcal{K} . Given a ring \mathscr{R} containing \mathbb{Z} and satisfying light technical conditions (as in [AH19, Remark 4.1]), Braverman-Kazdhan-Patnaik [BK11, BKP16] and Bardy–Panse-Gaussent-Rousseau [GR14, BPGR16] associated to G a spherical Hecke algebra \mathcal{H}_s and an Iwahori-Hecke algebra \mathcal{H} , both defined over \mathscr{R} . Fixing a maximal split torus T of G, and letting Y (resp. Y^+) be the cocharacter lattice (resp. its intersection with the Tits cone) and W^v be the Weyl group of (G, T), then these authors moreover proved the existence of a Satake isomorphism from \mathcal{H}_s to $\mathscr{R}[Y]^{W^v}$, where $\mathscr{R}[Y]$ is the Looijenga algebra, which is a completion of the group algebra $\mathscr{R}[Y]$ of Y over \mathscr{R} (see [AH19, Definition 4.6] for its definition). A striking difference with the classical case of reductive groups is that for G non-reductive, this spherical Hecke algebra is not isomorphic to the center of the Iwahori-Hecke algebra \mathcal{H} .

In [AH19], we announced the definition of a completed Iwahori-Hecke algebra $\widehat{\mathcal{H}}$ that contains \mathcal{H} and whose center is isomorphic to $\mathscr{R}\llbracket Y \rrbracket^{W^v}$, hence to \mathcal{H}_s [AH19, Theorem 4.30]. Nevertheless, the construction of $\widehat{\mathcal{H}}$ we gave in [AH19, Section 4.4, pages 94-100] is not correct as stated, since as such, $\widehat{\mathcal{H}}$ is actually not stable in general under the convolution product. Indeed, the product of two elements of $\widehat{\mathcal{H}}$ can lead to infinite coefficients, as will be seen below in Sections 2.1 and 2.2. This erratum corrects this mistake by defining a slightly different algebra $\widehat{\mathcal{H}}$, contained in the vector space $\widehat{\mathcal{H}}$, for a suitable convolution product (see Corollary 3.13). The main point is to use the correct notion of almost-finiteness in the definition of the support of the elements of the completed algebra. We check here that $\widetilde{\mathcal{H}}$ contains \mathcal{H} and that the center of $\widetilde{\mathcal{H}}$ is isomorphic to $\mathscr{R}\llbracket Y \rrbracket^{W^v}$, hence to \mathcal{H}_s (see Theorem 3.14), as aimed at first. Moreover note that this modified definition of the completed Iwahori-Hecke algebra suppresses the aforementioned gap between the reductive and non-reductive cases, as for G reductive, $\widetilde{\mathcal{H}}$ is actually isomorphic to the classical Iwahori-Hecke algebra (see Proposition 3.17).

This erratum is organized as follows. In Section 2, we give two counter-examples to [AH19, Theorem 4.21]: one in the reductive case and one in the non-reductive case. Then we introduce the required modifications in the definition of the completed algebra to build $\tilde{\mathcal{H}}$ in Section 3. In particular, we explain in Section 3.3 how to adapt the content of [AH19, Section 4.4, pages 94-100] to prove that the center of $\tilde{\mathcal{H}}$ is isomorphic to $\mathscr{R}[\![Y]\!]^{W^v}$.

Acknowledgements We thank the referees for their valuable comments and suggestions.

2 Two counter-examples to [AH19, Theorem 4.21]

We keep the notation of [AH19, section 2]. Let us briefly recall that, as in [AH19, Section 2.1], given a root generating system $\mathcal{S} = (A, X, Y, (\alpha)_{i \in I}, (\alpha_i^{\vee})_{i \in I})$, we set $\mathbb{A} := Y \otimes \mathbb{R}$, let W^v denote the Weyl group of \mathcal{S} , $Q^{\vee} := \bigoplus_{i \in I} \mathbb{Z} \alpha_i^{\vee}$ denote its coroot lattice, $Q_{\mathbb{R},+}^{\vee} := \bigoplus_{i \in I} \mathbb{R}_+ \alpha_i^{\vee}$ and $Q_{\mathbb{R}}^{\vee} = \bigoplus_{i \in I} \mathbb{R} \alpha_i^{\vee}$.

We then have $\mathcal{H} = \bigoplus_{\lambda \in Y^+, w \in W^v} \mathscr{R}Z^{\lambda}H_w$, where Z^{λ} and H_w are symbols that satisfy relations (BL1) to (BL4) of [AH19, Section 4.1, page 91]. The notion of support is defined in [AH19, Definition 4.11]. In this section, we give two examples of elements $(a_j)_{j \in J}, (b_k)_{k \in K}$ in \mathcal{H} that are summable in $\widehat{\mathcal{H}}$ in the sense of [AH19, Definition 4.20], but such that $(a_j * b_k)_{j \in J, k \in K}$ is not summable: one is a reductive case (Section 2.1), the other one is an affine Kac-Moody case (Section 2.2). This prevents [AH19, Theorem 4.21] from being true as stated, and we explain in Section 2.3 why the problem stands in the notion of almost-finiteness initially used in the definition of $\widehat{\mathcal{H}}$, and how to modify it to get a correct analogue of [AH19, Theorem 4.21].

2.1 A counter-exemple in the reductive case

Assume that the standard apartment \mathbb{A} is associated with a Cartan matrix. Fix $\lambda \in Y = Y^+$ and $i \in I$. For $j \in \mathbb{N}$, set $a_j = Z^{\lambda - j\alpha_i^{\vee}} H_i$ and $b_j = Z^{\lambda - j\alpha_i^{\vee}}$. As $\lambda - \mathbb{N}\alpha_i^{\vee}$ is almost finite, $(a_j)_{j \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ are summable in $\widehat{\mathcal{H}}$. Now let $j, k \in \mathbb{N}$. By (BL4) (see [AH19, Section 4.1 page 91]), there exists $c_{j,k} \in \mathscr{R}[Y]$ such that

$$a_i * b_k = Z^{\lambda + r_i(\lambda) + (k-j)\alpha_i^{\vee}} H_i + c_{i,k} .$$

This implies that $(\lambda + r_i(\lambda) + (k - j)\alpha_i^{\vee}, r_i) \in \operatorname{supp}(a_j * b_k)$, hence $(a_j * b_k)_{(j,k) \in \mathbb{N}^2}$ satisfies none of the conditions of [AH19, Definition 4.20], so it is not summable in $\widehat{\mathcal{H}}$.

2.2 A counter-example in the non-reductive case

Assume now that \mathbb{A} is associated with an indecomposable affine Kac-Moody matrix A. Let $\delta : \mathbb{A} \to \mathbb{R}$ be the smallest positive imaginary root associated with A. Fix $\lambda \in Y^+$ such that $\delta(\lambda) > 0$ and $i \in I$: then [AH19, Section 4.2.2, pages 92-93] ensures that $\lambda - \mathbb{N}\alpha_i^{\vee}$ is an almost finite subset of Y^+ . For $j \in \mathbb{N}$, set $a_j = Z^{\lambda - j\alpha_i^{\vee}} H_i$ and $b_j = Z^{\lambda - j\alpha_i^{\vee}}$: then the same process as in the reductive case (Section 2.1 above) shows that $(a_j * b_k)_{(j,k) \in \mathbb{N}^2}$ satisfies none of the conditions of [AH19, Definition 4.20], hence is not summable in $\widehat{\mathcal{H}}$, although both (a_i) and (b_k) are summable in $\widehat{\mathcal{H}}$.

2.3 Position of the problem and modifications required

The definition of $\widehat{\mathcal{H}}$ given in [AH19, page 95] crucially relies on the notion of almost finiteness defined in [AH19, Definition 4.12]. The problem is that almost finiteness is not preserved by the action of W^v on Y^+ : there can exist (depending on the Kac-Moody matrix A) an almost finite set E such that w.E is not almost finite, for some $w \in W^v$. To fix this problem, we introduce in the next section a refined notion of almost-finiteness, namely the notion of W^v -almost finiteness. Using this new notion, we define an algebra $\widetilde{\mathcal{H}}$ through an analogous construction to the one done for $\widehat{\mathcal{H}}$ in [AH19, Section 4.4, pages 94-100]. We then explain why the results and proofs stated for $\widehat{\mathcal{H}}$ in [AH19] are now valid for $\widetilde{\mathcal{H}}$.

Before going further, let us list precisely what modifications are actually done in this erratum.

- * The notion of almost finiteness defined in [AH19, Definition 4.12] must be replaced by the notion of W^v -almost finiteness introduced in Definition 3.1 below to define $\tilde{\mathcal{H}}$ as we defined \hat{H} but with the aforementioned replacement.
- * The statement and proof of [AH19, Theorem 4.21, Corollary 4.23 and Theorem 4.30] must be respectively replaced by the statement and proof of Theorem 3.12, Corollary 3.13 and Theorem 3.14 below.
- ★ The content of the paragraph before [AH19, Proposition 4.33], which explains what happens in the reductive case, must be replaced by Section 3.4 below.

3 The completed Iwahori-Hecke algebra \mathcal{H}

The goal of this section is to build an algebra $\widetilde{\mathcal{H}}$ that appears to be smaller than $\widehat{\mathcal{H}}$ (that is NOT always an algebra) whose center is (still) isomorphic to $\mathscr{R}[\![Y]\!]^{W^v}$ and that (still) contains \mathcal{H} as the subalgebra of finitely

supported elements. It actually boils down to defining the right notion of almost-finiteness and checking that what we did in [AH19, Section 4] transposes in this setting to define an actual algebra $\widetilde{\mathcal{H}}$ with the required properties.

3.1 W^{v} -almost finiteness and definition of $\widetilde{\mathcal{H}}$

The idea behind the use of the following refined notion of almost finiteness is that it is preserved by the action of W^{v} .

Definition 3.1. Let $u \in W^v$.

- A subset E of Y^+ is *u*-almost finite if u.E is almost finite in the sense of [AH19, Definition 4.3].
- A subset E of $Y^+ \times W^v$ is called *u-almost finite* if its projection on W^v is finite and if its projection on Y^+ is *u*-almost finite (as a subset of Y^+).
- A subset of Y^+ or of $Y^+ \times W^v$ is W^v -almost finite if it is u-almost finite for any $u \in W^v$.

As in [AH19, page 95], we set $\mathscr{B} = \prod_{\lambda \in Y^+, w \in W^v} \mathscr{R}$ and for $(\lambda, w) \in Y^+ \times W^v$, we let $Z^{\lambda}H_w$ denote the element whose coefficients are all equal to 0 apart from the coefficient indexed by (λ, w) , which is equal to 1. This allows us to write $a = (a_{\lambda,w})_{(\lambda,w) \in Y^+ \times W^v} \in \mathscr{B}$ as the formal linear combination $a = \sum_{\lambda \in Y^+, w \in W^v} a_{\lambda,w} Z^{\lambda}H_w$. Also recall that any $(\nu, u) \in Y^+ \times W^v$ is associated to a projection map $\pi_{\nu,u} : \mathscr{B} \to \mathscr{R}$

$$\pi_{\nu,u} \left(\sum_{u' \in W^{\nu}, \nu' \in Y^+} c_{\nu',u'} Z^{\nu'} H_{u'} \right) := c_{\nu,u}$$

for any $\sum c_{\nu',u'} Z^{\nu'} H_{u'} \in \mathscr{B}.$

defined by

We can now define $\widetilde{\mathcal{H}}$ as the set of elements of \mathscr{B} with W^v -almost finite support. To prove that $\widetilde{\mathcal{H}}$ can be endowed with a convolution product * that turns it into an associative algebra containing \mathcal{H} , we will basically follow the same steps as in [AH19, Section 4.4], replacing the almost finiteness condition by the W^v -almost finiteness condition.

We let $\operatorname{conv}_{\mathbb{R}}(F)$ denote the convex hull of any part F of \mathbb{A} , and we set $\operatorname{conv}(E) := \operatorname{conv}_{\mathbb{R}}(E) \cap Y$ for any subset E of Y. Following [AH19, page 95], recall that for any part E of Y and any $i \in I$, we let $R_i(E) = \operatorname{conv}(E \cup r_i(E)) \subset E + Q^{\vee}$ and that, for any pair $(\lambda, w) \in Y^+ \times W^v$, we set

$$R_w(\lambda) := \bigcup R_{i_1} \left(R_{i_2} \left(\dots \left(R_{i_k}(\{\lambda\}) \dots \right) \right) \right) ,$$

where the union is taken over all the reduced writings $r_{i_1}r_{i_2} \ldots r_{i_k}$ of w. The next two results replace [AH19, Remark 4.13] and act as preparation for the proof of Lemma 3.4 below, which replaces [AH19, Lemma 4.15].

Lemma 3.2. For any $(\lambda, w) \in Y^+ \times W^v$, we have

$$R_w(\lambda) \subset \operatorname{conv}(\{u.\lambda \mid u \in [1, w]\}),$$

where $[1, w] := \{u \in W^v \mid u \leq w\}$ is defined as in [AH19, bottom of page 94].

Proof. We prove this result by induction on $\ell(w)$. If $\ell(w) = 0$, there is nothing to prove, so let $w \in W^v$ be an element of length $\ell(w) \ge 1$ and assume by induction that the lemma holds for any element $w' \in W^v$ such that $\ell(w') < \ell(w)$. Let $\mu \in R_w(\lambda)$, then there exists $i \in I$ such that $w' := r_i w$ satisfies w' < w and $\mu \in R_i(R_{w'}(\lambda)) = \operatorname{conv}(R_{w'}(\lambda), r_i R_{w'}(\lambda))$. As $\ell(w') < \ell(w)$, we have $R_{w'}(\lambda) \subset \operatorname{conv}(\{u.\lambda \mid u \in [1, w']\})$ by induction hypothesis. Since [Kum02, Corollary 1.3.19] ensures that $\{1, r_i\} \cdot [1, w'] \subset [1, w]$, we obtain that

$$\operatorname{conv}\left(\left\{u.\lambda \mid u \in [1, w']\right\}\right) \cup r_i.\operatorname{conv}\left(\left\{u.\lambda \mid u \in [1, w']\right\}\right) \subset \operatorname{conv}\left(\left\{u.\lambda \mid u \in [1, w]\right\}\right)$$

Consequently, we get that

$$\mu \in \operatorname{conv}\left(R_{w'}(E), r_i \cdot R_{w'}(E)\right) \subset \operatorname{conv}\left(\operatorname{conv}\left(\left\{u \cdot \lambda \mid u \in [1, w']\right\}\right) \cup r_i \cdot \operatorname{conv}\left(\left\{u \cdot \lambda \mid u \in [1, w']\right\}\right)\right) \\ \subset \operatorname{conv}\left(\operatorname{conv}\left(\left\{u \cdot \lambda \mid u \in [1, w]\right\}\right)\right) = \operatorname{conv}\left(\left\{u \cdot \lambda \mid u \in [1, w]\right\}\right) \ .$$

This proves that $R_w(\lambda)$ is contained in conv $(\{u, \lambda \mid u \in [1, w]\})$, hence the lemma.

Lemma 3.3. Let E be a W^v -almost finite subset of Y^+ . Then, for any pair $(\nu, w) \in Y^+ \times W^v$, the set $\{\mu \in E \mid \nu \in R_w(\mu)\}$ is finite.

Proof. Let $E \subset Y^+$ and $(\nu, w) \in Y^+ \times W^v$ be as in the statement. Applying the definition of almost finiteness [AH19, Definition 4.3] to u.E for any $u \in [1, w]$ provides a finite set $F \subset Y^+$ such that :

$$\forall \ u \in [1,w], \ \forall \ \mu \in u.E, \ \exists \ \lambda \in F \ | \ \mu \leq_{Q^{\vee}} \lambda \ .$$

Set $\mathcal{X} := \{\mu \in E \mid \nu \in R_w(\mu)\}$ and pick some $\mu \in \mathcal{X}$. As ν belongs to $R_w(\mu)$, Lemma 3.2 implies the existence of $(t_u)_{u \in [1,w]} \in [0,1]^{[1,u]}$ such that

$$\sum_{u \in [1,w]} t_u = 1 \text{ and } \nu = \sum_{u \in [1,w]} t_u u . \mu .$$

For any $u \in [1, w]$, choose $\lambda(u) \in F$ such that $u \cdot \mu \leq_{Q^{\vee}} \lambda(u)$ and write $\lambda(u) - u \cdot \mu$ as $\sum_{i \in I} n_i(u) \alpha_i^{\vee}$ with $n_i(u) \in \mathbb{N}$ for all $i \in I$. Then we have:

$$\nu = \sum_{u \in [1,w]} t_u u \cdot \mu = \sum_{u \in [1,w]} t_u \lambda(u) - \sum_{u \in [1,w], i \in I} t_u n_i(u) \alpha_i^{\vee}$$

Set $a(\mu) := \sum_{u \in [1,w]} t_u \lambda(u) \in \operatorname{conv}_{\mathbb{R}}(F)$ and $q(\mu) := \sum_{u \in [1,w], i \in I} t_u n_i(u) \alpha_i^{\vee} \in Q_{\mathbb{R},+}^{\vee}$. Since F is finite, $\operatorname{conv}_{\mathbb{R}}(F)$

is bounded. As $q(\mu) = a(\mu) - \nu$ lies in $\operatorname{conv}_{\mathbb{R}}(F) - \nu$, the set $\{q(\mu'), \mu' \in \mathcal{X}\}$ is bounded too. Moreover, as $\sum_{u \in [1,w]} t_u = 1$, there exists $u' \in [1,w]$ such that $t_{u'} \geq \frac{1}{|[1,w]|}$. Letting $f_j(x)$ denote the *j*-th coordinate of $x \in Q_{\mathbb{R}}^{\vee}$ in the basis $(\alpha_i^{\vee})_{i \in I}$ for all $j \in J$, we have:

$$\forall i \in I, f_i(q(\mu)) = \sum_{u \in [1,w]} t_u n_i(u) \ge t_{u'} n_i(u') \ge 0.$$

We hence obtain that:

$$\forall i \in I, \ 0 \le n_i(u') \le \frac{\sup_{\mu' \in \mathcal{X}} f_i(q(\mu'))}{t_{u'}} \le \frac{\sup_{\mu' \in \mathcal{X}} f_i(q(\mu'))}{|[1, u]|}$$

Consequently, if we set $N := \max_{i \in I} \frac{\sup_{\mu' \in \mathcal{X}} f_i(q(\mu'))}{|[1, u]|}$, then we have:

$$u'.\mu \in \lambda(u') - \sum_{i \in I} \llbracket 0,N \rrbracket \alpha_i^{\vee} \subset F - \sum_{i \in I} \llbracket 0,N \rrbracket \alpha_i^{\vee}$$

This proves that μ lies in $\bigcup_{u \in [1,w]} u^{-1} \cdot (F - \sum_{i \in I} \llbracket 0, N \rrbracket \alpha_i^{\vee})$, hence that \mathcal{X} is contained in the finite set $\bigcup_{u \in [1,w]} u^{-1} \cdot (F - \sum_{i \in I} \llbracket 0, N \rrbracket \alpha_i^{\vee})$, which proves that \mathcal{X} is finite too, as claimed.

The next lemma replaces [AH19, Lemma 4.15]: the only modification consists in replacing [1, w]w' by [1, w']w in the aforementioned statement. In particular, the proof follows the exact same lines as [AH19, page 96], hence we do not rewrite it here.

Lemma 3.4. For all
$$w, w' \in W^v$$
 and all $\lambda \in Y$, $H_{w'} * Z^{\lambda}H_w$ is in $\bigoplus_{(\nu,t)\in R_{w'}(\lambda)\times[1,w'].w} \mathscr{R}.Z^{\nu}H_t$.

Using the definitions of $\operatorname{supp}_{W^v}$ and supp_Y given by [AH19, Definition 4.11], one can straightforward deduce from Lemma 3.4 the following inclusions.

Lemma 3.5. For all $a, b \in \mathcal{H}$, we have:

(1)
$$\operatorname{supp}_{Y}(a * b) \subset \operatorname{supp}_{Y}(a) + \bigcup_{w \in \operatorname{supp}_{W^{v}}(a), \lambda \in \operatorname{supp}_{Y}(b)} R_{w}(\lambda);$$

(2) $\operatorname{supp}_{W^v}(a * b) \subset \bigcup_{v \in \operatorname{supp}_{W^v}(a), w \in \operatorname{supp}_{W^v}(b)} [1, v].w.$

Before we give the definition of summable families in \mathcal{H} , we prove two more statements related to W^{v} -almost finiteness in Y^{+} .

Lemma 3.6. For any almost finite set $E \subset Y^+$, conv(E) is also almost finite.

Proof. Let $E \subset Y^+$ be an almost finite set and let F be a finite set such that:

$$\forall \ \lambda \in E, \ \exists \ \mu \in F \mid \lambda \leq_{Q^{\vee}} \mu$$

Given $\lambda \in \operatorname{conv}(E)$, there exist $n \in \mathbb{N}, t_1, \ldots, t_n \in [0, 1]$ and $\lambda_1, \ldots, \lambda_n \in E$ such that

$$\sum_{i=1}^{n} t_i = 1 \text{ and } \sum_{i=1}^{n} t_i \lambda_i = \lambda.$$

For each index $i \in [\![1,n]\!]$, choose $\kappa_i \in F$ such that $\lambda_i \leq_{Q^{\vee}} \kappa_i$: then $\sum_{i=1}^n t_i \kappa_i - \sum_{i=1}^n t_i \lambda_i \in \bigoplus_{i \in I} \mathbb{R}_+ \alpha_i^{\vee}$. We can

hence write $\sum_{i=1}^{n} t_i \kappa_i - \sum_{i=1}^{n} t_i \lambda_i = \sum_{i \in I} x_i \alpha_i^{\vee}$ for some nonnegative real numbers $(x_i)_{i \in I}$. Now let $(x'_i) \in [0, 1]^I$ be such that $x_i + x'_i$ lies in \mathbb{N} for all $i \in I$ and set $\nu := \sum_{i \in I} (x_i + x'_i)\alpha_i^{\vee} + \lambda$. Then we have $\nu \ge_{Q^{\vee}} \lambda$ and $\nu \in (\operatorname{conv}_{\mathbb{R}}(F) + \bigoplus_{i \in I} [0, 1]\alpha_i^{\vee}) \cap Y$. Since F is finite, $(\operatorname{conv}_{\mathbb{R}}(F) + \bigoplus_{i \in I} [0, 1]\alpha_i^{\vee}) \cap Y$ is a finite set that can be taken as J in [AH19, Definition 4.3] for $\operatorname{conv}(E)$, and the lemma is proven.

Lemma 3.7. Let E be a W^v-almost finite subset of Y⁺. Then, for any $w \in W^v$, the set $\bigcup_{\lambda \in E} R_w(\lambda)$ is W^v-almost finite.

Proof. Let $w \in W^v$. By Lemma 3.2, we have

$$\bigcup_{\lambda \in E} R_w(\lambda) \subset \bigcup_{\lambda \in E} \operatorname{conv}(\{u.\lambda, \ u \in [1,w]\}) \subset \operatorname{conv}\left(\bigcup_{u \in [1,w]} u.E\right) \ .$$

Let $v \in W^v$. Since [1, w] is finite, the set v. $\bigcup_{u \in [1, w]} u \cdot E$ is almost finite, hence Lemma 3.6 implies that $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right) = \operatorname{conv} \left(v \cdot \begin{array}{c|c} & u \cdot E \end{array} \right)$ is almost finite. This proves that $\operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is almost finite. This proves that $\operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$ is $v \cdot \operatorname{conv} \left(\begin{array}{c|c} & u \cdot E \end{array} \right)$

 $v.\operatorname{conv}\left(\bigcup_{u\in[1,w]}u.E\right) = \operatorname{conv}\left(v.\bigcup_{u\in[1,w]}u.E\right) \text{ is almost finite. This proves that } \operatorname{conv}\left(\bigcup_{u\in[1,w]}u.E\right) \text{ is } v-almost finite for any } v \in W^v, \text{ and the lemma is proven.} \qquad \Box$

3.2 $\widetilde{\mathcal{H}}$ is an associative algebra

This subsection contains the main modification of the paper, as it aims to prove that \mathcal{H} is actually an associative algebra. To do this, we first need to introduce the correct definition of summable families, which is the counterpart of [AH19, Definition 4.20] for W^v -almost finite sets.

Definition 3.8. A family $(a_j)_{j \in J} \in \left(\widetilde{\mathcal{H}}\right)^J$ is summable in $\widetilde{\mathcal{H}}$ when the two following properties hold:

(i) for any $\lambda \in Y^+$, the set $\{j \in J \mid \exists w \in W^v, \ \pi_{\lambda,w}(a_j) \neq 0\}$ is finite;

(*ii*) the set $\bigcup_{j \in J} \operatorname{supp}(a_j) := \bigcup_{j \in J} \{ (\lambda, w) \in Y^+ \times W^v | \pi_{\lambda, w}(a_j) \neq 0 \}$ is W^v -almost finite.

Given a summable family $(a_j)_{j \in J} \in \left(\widetilde{\mathcal{H}}\right)^J$, we define $\sum_{j \in J} a_j \in \widetilde{\mathcal{H}}$ by the following formula:

$$\sum_{j \in J} a_j := \sum_{(\lambda, w) \in Y^+ \times W^v} a_{\lambda, w} Z^\lambda H_w \text{, with } a_{\lambda, w} := \sum_{j \in J} \pi_{\lambda, w}(a_j) \text{ for all } (\lambda, w) \in Y^+ \times W^v \text{.}$$

Lemma 3.9. For any almost finite subsets E, E' of Y^+ and any $\rho \in Y^+$, the set

$$E'' := \{ \lambda \in E \mid \exists \ \nu \in E', \ \lambda + \nu = \rho \}$$

is finite.

Proof. By definition, there exists a finite set $F \subset Y^+$ such that: $\forall \mu \in E \cup E', \exists \kappa \in F \mid \mu \leq_{Q^{\vee}} \kappa$. Now let $\lambda \in E''$ and $\nu \in E'$ be such that $\lambda + \nu = \rho$. Then we have:

$$\exists (\kappa, \kappa') \in F^2 \mid \rho - \kappa' \leq_{Q^{\vee}} \lambda \leq_{Q^{\vee}} \kappa$$

and the lemma follows.

The next lemma is the cornerstone that ensures that the convolution product on \mathcal{H} is well-defined.

Lemma 3.10. Let $(a_{\lambda}), (b_{\mu}) \in \mathscr{R}^{Y^+}$ be such that $\{\lambda \in Y^+ | a_{\lambda} \neq 0 \text{ or } b_{\lambda} \neq 0\}$ is W^v -almost finite. Then, for any $w \in W^v$, $(a_{\lambda}b_{\mu}Z^{\lambda}H_w * Z_{\mu})_{(\lambda,\mu)\in (Y^+)^2}$ is summable in $\widetilde{\mathcal{H}}$. Said differently, if $\sum_{\lambda \in Y^+} a_{\lambda}Z^{\lambda}H_w, \sum_{\mu \in Y^+} b_{\mu}Z^{\mu} \in \mathcal{H}^{Y^+}$

 $\widetilde{\mathcal{H}}, then$

$$\left(\sum_{\lambda \in Y^+} a_{\lambda} Z^{\lambda} H_w\right) * \left(\sum_{\mu \in Y^+} b_{\mu} Z^{\mu}\right) := \sum_{\lambda, \mu \in Y^+} a_{\lambda} b_{\mu} Z^{\lambda} H_w * Z^{\mu}$$

is a well-defined element of $\widetilde{\mathcal{H}}$.

Proof. Set $S_a := \{\lambda \in Y^+ \mid a_\lambda \neq 0\}$, $S_b = \{\mu \in Y^+ \mid b_\mu \neq 0\}$ and $E = \bigcup_{\mu \in S_b} R_w(\mu)$. Note that E is almost finite by Lemma 3.7.

Given $\mu \in Y^+$, Lemma 3.4 ensures the existence of $(z_{\mu}^{v,\nu})_{\nu \in R_w(\mu), v \in [1,w]} \in \mathscr{R}^{R_w(\mu) \times [1,w]}$ such that

$$H_w * Z^{\mu} = \sum_{\nu \in R_w(\mu), v \in [1,w]} z^{\nu,\nu}_{\mu} Z^{\nu} H_v .$$

Let us fix $v \in [1, w]$. Given any $\lambda, \rho \in Y^+$, we have

$$\pi_{\rho,\nu}(a_{\lambda}Z^{\lambda}H_w * b_{\mu}Z^{\mu}) = \sum_{\nu \in R_w(\mu) \mid \lambda + \nu = \rho} a_{\lambda}b_{\mu}z_{\mu}^{\nu,\nu} .$$

$$\tag{1}$$

Set $F_1 := \{\lambda \in Y^+ \mid \exists \mu \in Y^+, \pi_{\rho,v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) \neq 0\} \subset S_a$ and let $\lambda \in F_1$. By equality (1), there exists $\nu \in E$ such that $\lambda + \nu = \rho$. Since λ lies in S_a , applying Lemma 3.9 to E and S_a implies that F_1 is finite. Fix now $\lambda \in F_1$ and set $F_2(\lambda) = \{\mu \in Y^+ \mid \pi_{\rho,v}(a_\lambda Z^\lambda H_w * b_\mu Z^\mu) \neq 0\} \subset S_b$. Given $\mu \in F_2(\lambda)$, we know from equality (1) that $\rho - \lambda \in R_w(\mu)$. As S_b is W^v -almost finite, Lemma 3.3 yields the finiteness of $F_2(\lambda)$, hence the finiteness of

$$F_{v} := \{ (\lambda, \mu) \in (Y^{+})^{2} \mid \pi_{\rho, v}(a_{\lambda}b_{\mu}Z^{\lambda}H_{w} * Z^{\mu}) \neq 0 \} = \bigcup_{\lambda \in F_{1}} F_{2}(\lambda) .$$

Finally, we obtain that

$$\{(\lambda,\mu)\in (Y^+)^2\mid \exists v\in W^v, \pi_{\rho,v}(a_\lambda b_\mu Z^\lambda H_w*Z^\mu)\neq 0\}=\bigcup_{v\in[1,w]}F_v$$

is finite, which proves that $(a_{\lambda}b_{\mu}Z^{\lambda}H_w * Z^{\mu})_{(\lambda,\mu)\in (Y^+)^2}$ satisfies condition (i) of Definition 3.8. Now let $\lambda, \mu \in Y^+$. Then equality (1) ensures that:

$$\operatorname{supp}(a_{\lambda}b_{\mu}Z^{\lambda}H_{w}*Z^{\mu})\subset(\lambda+R_{w}(\mu))\times[1,w]\subset(\lambda+E)\times[1,w]$$

In particular, we have

$$\bigcup_{(\lambda,\mu)\in (Y^+)^2} \operatorname{supp}(a_{\lambda}b_{\mu}Z^{\lambda}H_w * Z^{\mu}) \subset (S_a + E) \times [1,w].$$

As E is W^{v} -almost finite, and as the sum of two W^{v} -almost finite sets is W^{v} -almost finite, we obtain that $(a_{\lambda}b_{\mu}Z^{\lambda}H_{w}*Z^{\mu})_{(\lambda,\mu)\in (Y^{+})^{2}}$ satisfies condition (ii) of Definition 3.8, which ends the proof.

Lemma 3.11. For any summable family $(a_i)_{i \in J} \in (\widetilde{\mathcal{H}})^J$ in $\widetilde{\mathcal{H}}$ and any $i \in I$, the family $(a_i * H_i)_{i \in J}$ is summable in \mathcal{H} .

Proof. By definition, there exist a W^{v} -almost finite subset E of Y^{+} and a finite subset F of W^{v} such that: $\forall i \in J$, supp $(a_i) \subset E \times F$. By (BL2) of [AH19, page 91], we get that

$$\forall j \in J, \text{ supp}(a_j * H_i) \subset E \times (F \cup F.r_i)$$

which proves that $\bigcup \operatorname{supp}(a_j * H_i)$ is W^v -almost finite. Now let $\lambda \in Y^+$, $w \in W^v$ and $j \in J$. By (BL2) again, we know that $\pi_{\lambda,w}(a_j * H_i) \neq 0$ implies $\pi_{\lambda,w}(a_j) \neq 0$ or $\pi_{\lambda,wr_i}(a_j) \neq 0$. Since there are finitely many such $j \in J$, this completes the proof of the lemma.

The next statement replaces [AH19, Theorem 4.21] and its proof basically follows the same lines as the proof of [AH19, Theorem 4.21], replacing almost finiteness by W^{v} -almost finiteness. Recall that the elements of \mathcal{H} correspond to the elements of $\widetilde{\mathcal{H}}$ with finite support.

Theorem 3.12. Let $(a_j)_{j\in J} \in (\mathcal{H})^J$ and $(b_k)_{k\in J} \in (\mathcal{H})^K$ be two families that are both summable in $\widetilde{\mathcal{H}}$. Then $(a_j * b_k)_{(j,k)\in J\times K}$ is summable in $\widetilde{\mathcal{H}}$ and $\sum_{(j,k)\in J\times K} a_j * b_k$ only depends on the two elements $\sum_{j\in J} a_j$ and $\sum_{k\in K} b_k$

of
$$\mathcal{H}$$
.

Proof. For $j \in J$ and $k \in K$, write $a_j := \sum_{v \in W^v} a_{v,j} * H_v$ and $b_k := \sum_{w \in W^v} b_{w,k} * H_w$, with $(a_{v,j})_{j \in J} \in \mathscr{R}\llbracket Y \rrbracket^J$ and $(b_{w,k})_{k\in K} \in \mathscr{R}[Y]^K$ for any $v, w \in W^v$. Given $v, w \in W^v$, Lemma 3.10 ensures that $(a_{v,j} * H_v * b_{w,k})_{(j,k)\in J\times K}$ is summable in $\widetilde{\mathcal{H}}$. By induction on $\ell(w)$ and using Lemma 3.11, we get that $(a_{v,j} * H_v * b_{w,k} * H_w)_{(j,k) \in J \times K}$ is summable in $\widetilde{\mathcal{H}}$. Moreover, as (a_i) and (b_k) are summable in $\widetilde{\mathcal{H}}$, there are at most finitely many $v, w \in W^v$ satisfying $(a_{v,j})_{j\in J} \neq 0$ and $(b_{w,k})_{k\in K} \neq 0$. Consequently, the family $(a_j * b_k)_{(j,k)\in J\times K}$ is summable in $\widetilde{\mathcal{H}}$. Now, given any triple $(u, v, \mu) \in W^v \times W^v \times Y^+$, applying Lemma 3.4 to $H_u * Z^\mu H_v$ gives a family $(z_{\nu,t}^{u,v,\mu})_{(\nu,t)\in R_u(\mu)\times[1,u],v}$ of scalars that satisfy $H_u * Z^\mu H_v = \sum_{(\nu,t)\in R_u(\mu)\times[1,u],v} z_{\nu,t}^{u,\nu,\mu} Z^\nu H_t$.

For any pair $(\rho, s) \in Y^+ \times W^v$, we have

$$\pi_{\rho,s}\left(\sum_{(j,k)\in J\times K}a_j*b_k\right) = \sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\sum_{\nu\in R_u(\mu)}\sum_{|\lambda+\nu=\rho}\sum_{(j,k)\in J\times K}a_{j,\lambda,u}b_{k,\mu,\nu}z_{\nu,s}^{u,\nu,\mu}$$
$$= \sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v}\sum_{\nu\in R_u(\mu)}\sum_{|\lambda+\nu=\rho}a_{\lambda,u}b_{\mu,\nu}z_{\nu,s}^{u,\nu,\mu},$$

where we set $\sum_{j \in J} a_j = \sum_{(\lambda, u) \in Y^+ \times W^v} a_{\lambda, u} Z^{\lambda} H_u$ and $\sum_{k \in K} b_k = \sum_{(\mu, v) \in Y^+ \times W^v} b_{\mu, v} Z^{\mu} H_v$, hence the theorem is proven.

The mistake done in the former proof of [AH19, Theorem 4.20] is to implicitely assume that S_Y = $\bigcup \operatorname{supp}_Y(a_j) \cup \bigcup \operatorname{supp}_Y(b_k) \text{ is such that } \{\lambda^{++}, \lambda \in S_Y\} \text{ is almost finite, which is not true in general,}$ $j \in J$ $k \in K$ as shown by the counter-examples given in Section 2. As spotted by the referee, the same kind of subtlety underlies a mistake made by Looijenga in his seminal 1980 work [Loo80, (4.1), end of the first paragraph].

For $a = \sum_{(\lambda,v)\in Y^+\times W^v} a_{\lambda,v} Z^{\lambda} H_v \in \widetilde{\mathcal{H}}$ and $b = \sum_{(\mu,w)\in Y^+\times W^v} b_{\mu,w} Z^{\mu} H_w \in \widetilde{\mathcal{H}}$, we set

$$a * b = \sum_{(v,\lambda),(w,\mu) \in Y^+ \times W^v} a_{\lambda,v} b_{\mu,w} Z^\lambda H_v * Z^\mu H_w,$$

which is well-defined by Theorem 3.12. We can now formulate the statement that replaces [AH19, Corollary 4.23], providing the required structure on \mathcal{H} . Its proof is the same as [AH19, Corollary 4.23], replacing [AH19, Theorem 4.21] by Theorem 3.12 above.

Corollary 3.13. The convolution product * equips $\widetilde{\mathcal{H}}$ with the structure of an associative algebra over \mathscr{R} that contains \mathcal{H} as subspace of finitely supported elements.

The center of $\widetilde{\mathcal{H}}$ is isomorphic to \mathcal{H}_s 3.3

Recall that the definition of the Looijenga algebra $\mathscr{R}[\![Y]\!]$ and its variants $\mathscr{R}[\![Y]\!]^{W^v}$ and $\mathscr{R}[\![Y^+]\!]$ is given by [AH19, Definition 4.6]. Also, we proved in [AH19, Proposition 4.9] that $\mathscr{R}[\![Y]\!]^{W^v}$ is a subspace of $\mathscr{R}[\![Y^+]\!]$. Now note that the latter can be seen as a subspace of \mathscr{B} , so it makes sense to compare these algebras with the algebra \mathcal{H} we built earlier.

Given $a \in \mathscr{R}\llbracket Y \rrbracket^{W^v}$, we have $\operatorname{supp}_V(a) \times \{1\}$, with $\operatorname{supp}_V(a)$ being W^v -invariant and almost finite, hence $\operatorname{supp}(a)$ is W^v -almost finite. In particular, this implies that $\mathscr{R}\llbracket Y \rrbracket^{W^v}$ is contained in $\widetilde{\mathcal{H}}$. However, note that in general, $\mathscr{R}[Y]$ may not be entirely contained in $\widetilde{\mathcal{H}}$, as can be seen for instance in Example 2.2.

Replacing $\hat{\mathcal{H}}$ by $\hat{\mathcal{H}}$ in the proof of [AH19, Theorem 4.30] provides the following theorem, which replaces [AH19, Theorem 4.30].

Theorem 3.14. The center of the algebra $\widetilde{\mathcal{H}}$ is $\mathscr{Z}(\widetilde{\mathcal{H}}) = \mathscr{R}[\![Y]\!]^{W^v}$, hence is isomorphic to \mathcal{H}_s via the Satake isomorphism.

Remark 3.15. By [AH19, Lemma 4.5], we know that if A is associated with an indefinite size 2 Kac-Moody matrix, then any subset of Y^+ is almost finite, hence any subset of Y^+ is W^{ν} -almost finite.

3.4The reductive case

Lemma 3.16. Assume that A is associated with a Cartan matrix A. Then a subset of $Y^+ = Y$ is W^v -almost finite if, and only if, it is finite.

Proof. Thanks to [AH19, Lemma 5.17], we may assume that $\bigcap_{i \in I} \ker \alpha_i = \{0\}$. Let A_1, \ldots, A_r denote the indecomposable components of A: then $\mathbb{A} = \bigoplus_{i=1}^{r} \mathbb{A}_i$, where \mathbb{A}_i is a realization of A_i (as defined in [AH19,

5.4.1]) for all $i \in [\![1, r]\!]$. For $i \in [\![1, r]\!]$, we denote by W_i^v the Weyl group of \mathbb{A}_i , by Q_i^{\vee} its coroot lattice and

by Y_i its cocharacter lattice, so that we have $W^v = W_1^v \times \ldots \times W_r^v$, $Q^{\vee} = \bigoplus_{i=1}^r Q_i^{\vee}$ and $Y = \bigoplus_{i=1}^r Y_i$. Now let E be a W^v -almost finite subset of Y. For $w \in W^v$, set $E_w := E \cap w.\overline{C_f^v}$. Since E is w^{-1} -almost finite, there exists a finite set $F \subset Y$ such that: $\forall \lambda \in E, \exists \mu \in F \mid \lambda \leq_{Q^{\vee}\mu}$. For $i \in [\![1, r]\!]$, set $Y_i^{++} := Y_i \cap \overline{C_{f,i}^v}$, where $C_{f,i}^v$ denotes the fundamental chamber of \mathbb{A}_i : then we know from [Kac94, Theorem 4.3] that $Y_i^{++} \subset Q_{i,+}^{\vee}$. Given $\lambda = (\lambda_1, \dots, \lambda_r) \in E_w$, let $\mu = (\mu_1, \dots, \mu_r) \in F$ be such that $\lambda \leq_{Q^{\vee}} \mu$. Then we have $0 \leq_{Q_i^{\vee}} \lambda_i \leq_{Q_i^{\vee}} \mu_i$ for all $i \in [\![1, r]\!]$, hence E_w must be finite. As W^v is finite, we get that E is finite too, and the lemma is proven as the converse statement is straightforward.

Since \mathcal{H} corresponds to the subspace of elements of \mathcal{H} with finite support, we directly obtain the following result from Lemma 3.16, which states that \mathcal{H} is just the usual Iwahori-Hecke algebra in the reductive case. This replaces the first paragraph of [AH19, Section 4.6.1, page 105].

Proposition 3.17. If \mathbb{A} is associated with a Cartan matrix, then $\widetilde{\mathcal{H}} = \mathcal{H}$.

References

- Ramla Abdellatif and Auguste Hébert. Completed Iwahori-Hecke algebras and parahoric Hecke [AH19] algebras for Kac-Moody groups over local fields. Journal de l'École polytechnique — Mathématiques, 6:79–118, 2019.
- [BK11] Alexander Braverman and David Kazhdan. The spherical Hecke algebra for affine Kac-Moody groups I. Ann. of Math. (2), 174(3):1603–1642, 2011.
- [BKP16] Alexander Braverman, David Kazhdan, and Manish M. Patnaik. Iwahori-Hecke algebras for p-adic loop groups. Invent. Math., 204(2):347-442, 2016.
- [BPGR16] Nicole Bardy-Panse, Stéphane Gaussent, and Guy Rousseau. Iwahori-Hecke algebras for Kac-Moody groups over local fields. Pacific J. Math., 285(1):1–61, 2016.
- [GR14] Stéphane Gaussent and Guy Rousseau. Spherical Hecke algebras for Kac-Moody groups over local fields. Annals of Mathematics, 180(3):1051–1087, 2014.
- Victor G Kac. Infinite-dimensional Lie algebras, volume 44. Cambridge university press, 1994. [Kac94]

- [Kum02] Shrawan Kumar. Kac-Moody groups, their flag varieties and representation theory, volume 204 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2002.
- [Loo80] Eduard Looijenga. Invariant theory for generalized root systems. *Inventiones mathematicae*, 61(1):1–32, 1980.
- [Tit87] Jacques Tits. Uniqueness and presentation of Kac-Moody groups over fields. J. Algebra, 105(2):542–573, 1987.