USING STRUCTURED VARIANTS IN LATTICE-BASED CRYPTOGRAPHY

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Cryptography

Let's start with a simple example: you want to send a message to someone.

Two possibilities:

- Either you share a secret key (AES...),
- Either you don't $\Rightarrow$ public key cryptography.

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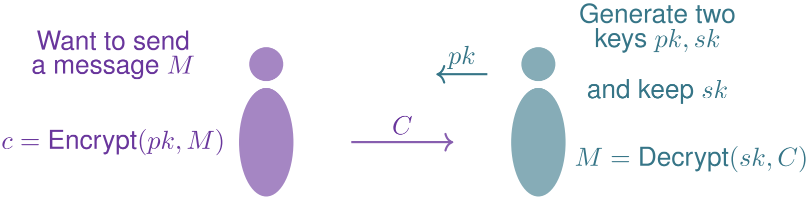
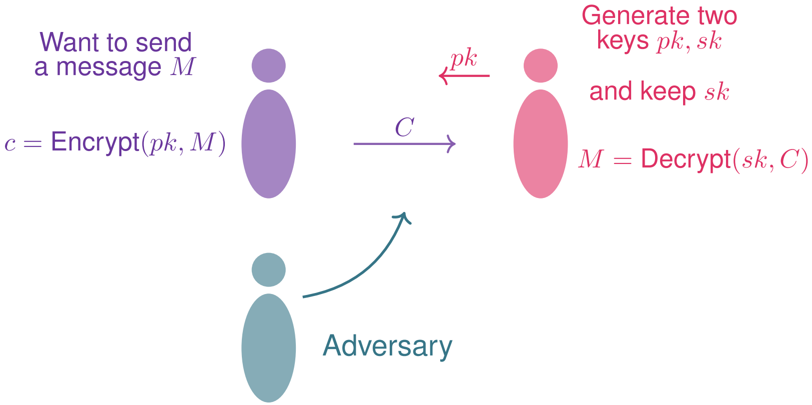


Public key cryptography



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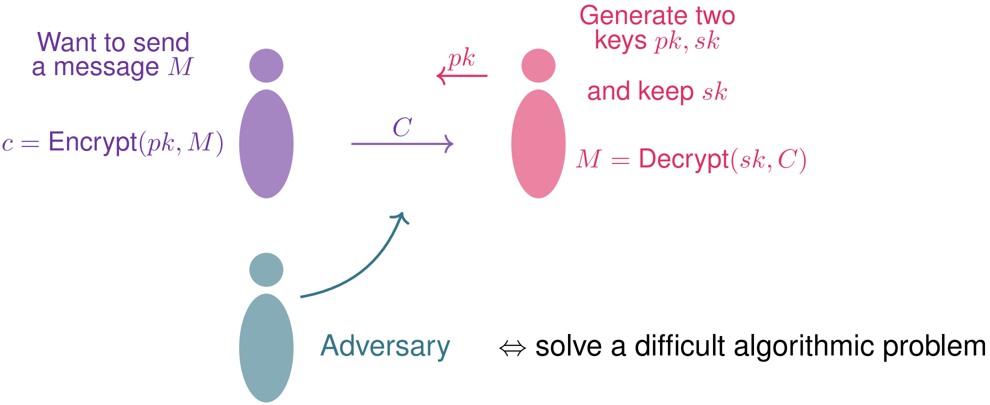

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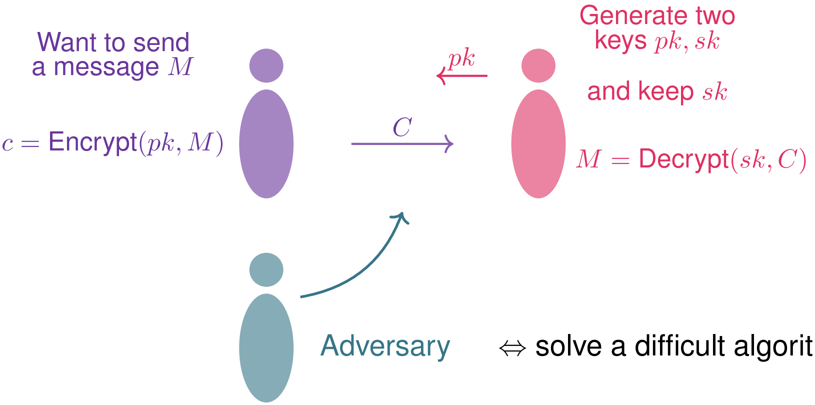
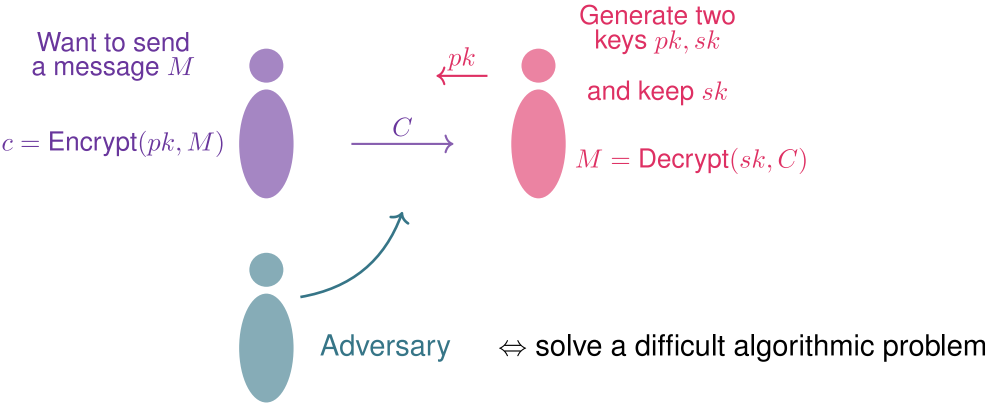


$\Leftrightarrow$ solve a difficult algorithmic problem

- Examples: factorisation (RSA), discrete log (El Gamal) ...
- Solving those problems needs an exponential complexity on a classical computer.

Public key cryptography



- Examples: factorisation (RSA), discrete log (El Gamal) ...
- Solving those problems needs an exponential complexity on a classical computer.
- Shor's algorithm (1997): **polynomial time on a quantum computer**.

Context

New goals in cryptography

- Resisting to quantum computers,
- Need of new functionalities,

$\rightarrow$ need alternatives

- Post-quantum secure,
- Efficient,
- New functionalities, different types of constructions.

NIST competition

- Lattice-based cryptography
- Code-based cryptography
- Multivariate, Isogenies, Hash based ...

NIST competition

From 2017 to 2024, NIST competition to develop new standards
on post-quantum cryptography

Total: 69 accepted submissions (round 1)

- Signature (5 lattice-based),
- Public key encryption / Key Encapsulation Mechanism (21 lattice-based)

Other candidates: 17 code-based PKE, 7 multivariate signatures, 3 hash-based signatures, 7 from ”other” assumptions (isogenies, PQ RSA ...) and 4 attacked + 5 withdrawn.

$\Rightarrow$ lattice-based constructions are very serious candidates

5 over 7 finalists are lattice-based

2022 first results: **3 over 4 new standards** are lattice-based

Why lattice-based cryptography?

- Likely to resist attacks from quantum computers,
- Strong security guarantees,
  from well-understood hard problems on lattices.

- Novel and powerful cryptographic functionalities,
  - Public key encryption and signature scheme (practical),
  - Advanced signature (group signature ...),
    and encryption scheme (IBE, ABE, ...),
  - Fully homomorphic encryption.

- Efficiency

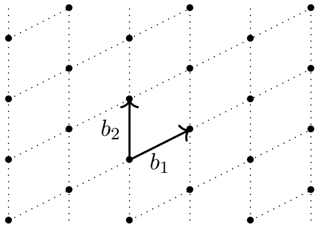
Today: an introduction to lattice-based cryptography

1. Lattices
   - Definition
   - Hard problem on lattices

2. The Learning With Errors problem
   - Definition
   - Difficulty
   - How to encrypt using LWE?

3. Practical scheme
   - Adding structure
   - Module-LWE
   - Kyber encryption scheme

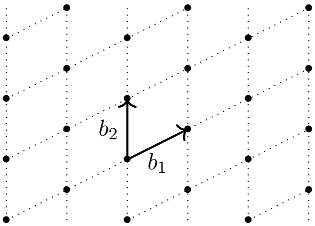
Lattices

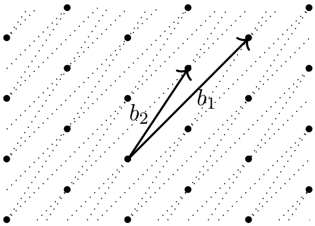


Lattice

$\mathcal{L}(\mathbf{B}) = \{\sum_{1=i}^{n} a_i \mathbf{b}_i, a_i \in \mathbb{Z}\}$, where the $(\mathbf{b}_i)_{1 \leq i \leq n}$'s, linearly independent vectors, are a basis of $\mathcal{L}(\mathbf{B})$.

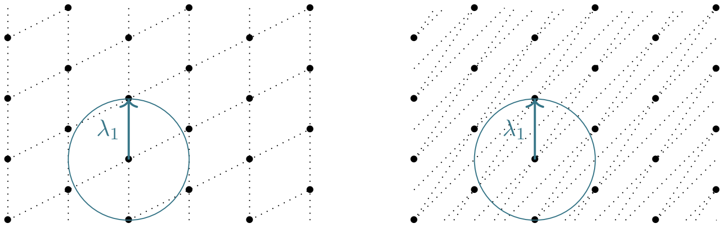
Lattices





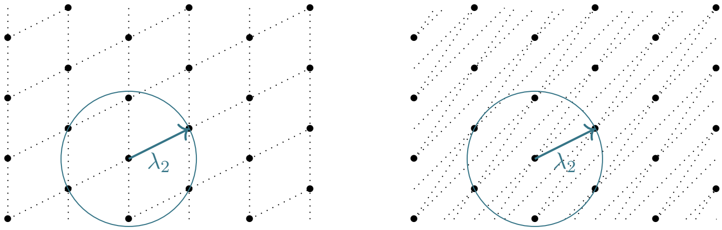
- Several basis define a lattice, some are better.

Lattices



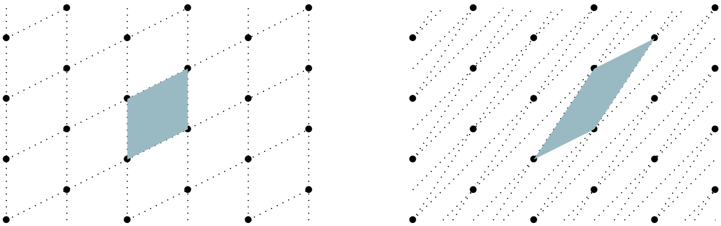
- Several basis define a lattice, some are better.
- The first minimum λ_1 is the norm of the smallest non-zero vector.

Lattices



- Several basis define a lattice, some are better.
- The first minimum λ_1 is the norm of the smallest non-zero vector.
- The n-th minima λ_n is the radius of a sphere which contains n linearly independent shortest vectors of the lattices.

Lattices



- Several basis define a lattice, some are better.
- The first minimum λ_1 is the norm of the smallest non-zero vector.
- The n-th minima λ_n is the radius of a sphere which contains n linearly independent shortest vectors of the lattices.
- The fundamental parallelepiped is defined by $\mathcal{P}(\mathbf{B}) = \{\sum_{i=1}^{n} c_i \mathbf{b}_i : c_i \in [0,1)\}$. Its volume defines the volume of the lattice: $\det(\Lambda) = |\det(\mathbf{B})|$.

Lattices

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- Minkowski Theorem:

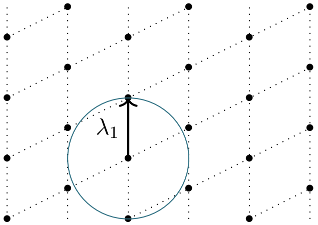
$$\lambda_1(\Lambda) \leq \sqrt{n} \cdot \det(\Lambda)^{1/n}$$

$$\left(\prod_{i=1}^{n} \lambda_i(\Lambda)\right)^{1/n} \leq \sqrt{n} \cdot \det(\Lambda)^{1/n}$$

Shortest Vector Problem (SVP)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n:

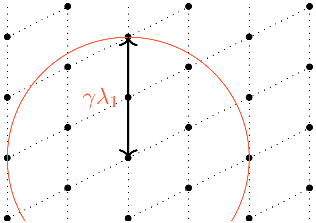
Output: find the shortest non-zero vector $\mathbf{x} \in \mathcal{L}(\mathbf{B})$.



Approx Shortest Vector Problem (Approx SVP$_\gamma$)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n:

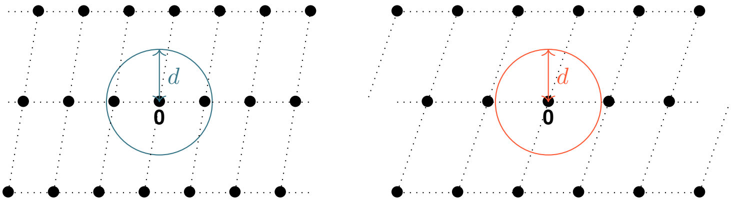
Output: find a non-zero vector $\mathbf{x} \in \mathcal{L}(\mathbf{B})$ such that $\|\mathbf{x}\| \leq \gamma\lambda_1(\mathcal{L}(\mathbf{B}))$



Gap Shortest Vector Problem (GapSVP)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n and $d > 0$:

Output:
- YES: there is $\mathbf{z} \in \mathcal{L}(\mathbf{B})$ non-zero such that $\|\mathbf{z}\| < d$,
- NO: for all non-zero vectors $\mathbf{z} \in \mathcal{L}(\mathbf{B})$: $\|\mathbf{z}\| \geq d$.



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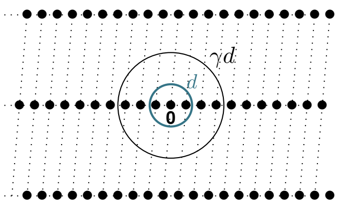
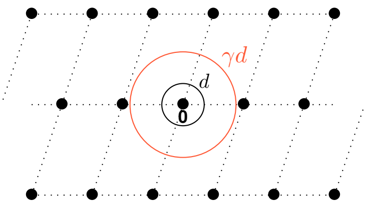


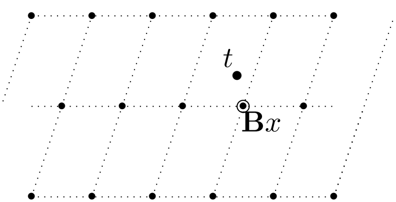


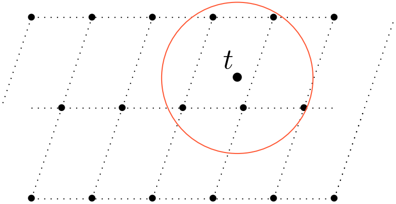
Closest Vector Problem

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n and $\mathbf{t} \in \mathbb{Z}^m$:

Output: find $\mathbf{x} \in \mathbb{Z}^n$ minimizing $||\mathbf{Bx} - \mathbf{t}||$.

Approx variant: find $\mathbf{x} \in \mathbb{Z}^n$ such that $||\mathbf{Bx} - \mathbf{t}|| \leq \gamma \cdot \text{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$.





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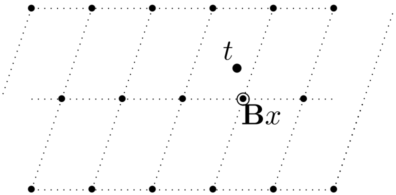
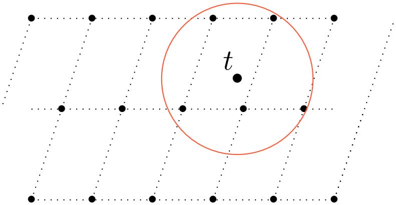
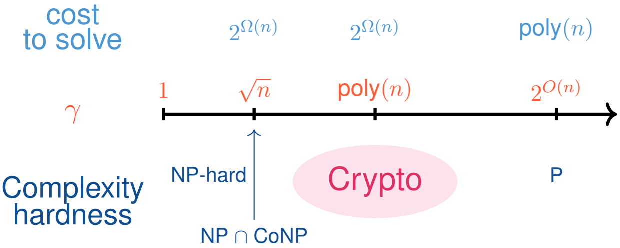




How hard is it to solve those problems?

Hardness of Approx SVP$_\gamma$



Conjecture

There is no polynomial time algorithm that approximates this lattice problem and its variants to within polynomial factors.

At the heart of lattice-based cryptography
the Learning With Errors problem

- Introduced by Regev in 2005

Problem: solve a linear system with m equations and n variables ($m \geq n$), with noise, and modulo an integer q.

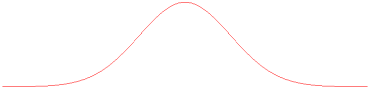
Find $(s_1, s_2, s_3, s_4, s_5)$ such that:

$$
\begin{array}{rcrl}
s_1 + 22s_2 + 17s_3 + 2s_4 + s_5 & \approx & 16 & \mod 23 \\
3s_1 + 2s_2 + 11s_3 + 7s_4 + 8s_5 & \approx & 17 & \mod 23 \\
15s_1 + 13s_2 + 10s_3 + 3s_4 + 5s_5 & \approx & 3 & \mod 23 \\
17s_1 + 11s_2 + 20s_3 + 9s_4 + 3s_5 & \approx & 8 & \mod 23 \\
2s_1 + 14s_2 + 13s_3 + 6s_4 + 7s_5 & \approx & 9 & \mod 23 \\
4s_1 + 21s_2 + 9s_3 + 5s_4 + s_5 & \approx & 18 & \mod 23 \\
11s_1 + 12s_2 + 5s_3 + s_4 + 9s_5 & \approx & 7 & \mod 23
\end{array}
$$

Gaussian distributions

Continuous Gaussian distribution of center c and parameter s:

$D_{s,c}(x) \sim \frac{1}{s} \exp\left(-\pi \frac{||x-c||^2}{s^2} \right)$
$\forall x \in \mathbb{R}$



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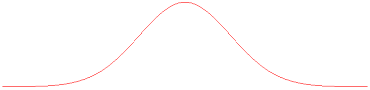
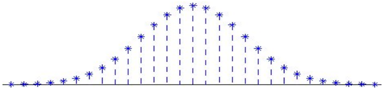


Gaussian distribution on $\mathbb{Z}$ of center c with parameter s:

$$D_{\mathbb{Z},s,c}(x) \sim \frac{1}{s} \exp\left(-\pi \frac{||x-c||^2}{s^2} \right)$$
$$\forall x \in \mathbb{Z}$$



- It is not the rounding of the continuous Gaussian.
- We now how to sample it efficiently.
- Almost all samples are in $[-t \cdot s, +t \cdot s]$ for a constant t, if s is not to small.

Discrete gaussian on lattices

Theorem (Gentry, Peikert, Vaikuntanathan 2008)

There exists a PPT algorithm which, given a basis $\mathbf{B}$ *of a lattice* $\Lambda(\mathbf{B})$ *of dimension* n*, a parameter* $s \geq \|\tilde{\mathbf{B}}\| \cdot \omega(\sqrt{\log n})$*, an a center* $c \in \mathbb{R}^n$*, outputs a sample from a distribution statistically close from* $D_{\Lambda,s,c}$*.*

Intuition: sampling on $\mathbb{Z}$ is quite easy, it is more complicated on a general lattice.

Important: Better is the basis (with short vectors), smaller is the parameter we can sample with, and then have short vectors.

Smoothing parameter

Definition

For all $\varepsilon > 0$, the *smoothing parameter* of a lattice Λ with parameter ε is the smallest s such that $\rho_{1/s}(\Lambda^* \setminus \{0\}) \leq \varepsilon$, we denote it by $\eta_\varepsilon(\Lambda)$.

When the Gaussian's parameter is bigger than smoothing parameter, the discrete gaussian distribution has the same properties than a continuous one. In particular:

- the discrete gaussian distribution $D_{\Lambda,s,c}$ is mainly concentrated in a sphere of radius $\sqrt{n}s$ around its center c.

  If $s > \eta_\varepsilon(\Lambda)$,

  $$\Pr_{x \hookleftarrow D_{\Lambda,s,c}} \left[\|x - c\| > \sqrt{n}s \right] \leq 2^{-n}.$$

- Addition: if $s, t > \eta_\varepsilon(\Lambda)$, we can also add two gaussian **on the same lattice** :

  $$D_{\Lambda,s} + D_{\Lambda,t} = D_{\Lambda,\sqrt{s^2+t^2}}.$$

Size of the smoothing parameter

The size of the smooting parameter can be compared to the size of the n-th minima.

- Micciancio, Regev 2004 and Regev 2005:
  For any lattice Λ and $\varepsilon > 0$

$$\sqrt{\frac{\ln(1/\varepsilon)}{\pi}} \cdot \frac{\lambda_n(\Lambda)}{n} \leq \eta_\varepsilon(\Lambda) \leq \sqrt{\frac{\ln(2n(1+1/\varepsilon))}{\pi}} \cdot \lambda_n(\Lambda).$$

The Learning With Errors problem [Regev 05]

Let $n > 1$, $q \geq 2$ and $\alpha \in]0, 1[$.
For any $\mathbf{s} \in \mathbb{Z}_q^n$, we define the distribution $\mathcal{D}_{n,q,\alpha}(\mathbf{s})$ by:

$$(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e)\ , \text{ with } \mathbf{a} \leftarrow U(\mathbb{Z}_q^n) \text{ and } e \leftarrow D_{\mathbb{Z},\alpha q}.$$

- Search LWE
  For any $\mathbf{s}$: find $\mathbf{s}$ given an arbitrary number of samples from $\mathcal{D}_{n,q,\alpha}(\mathbf{s})$.

- Decision LWE
  With non-negligible probability on $\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$: distinguish between the distributions $\mathcal{D}_{n,q,\alpha}(\mathbf{s})$ and $U(\mathbb{Z}_q^{n+1})$.

Decision version

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We consider an oracle $\mathcal{O}$ which produces independant samples, all from the same distribution being:

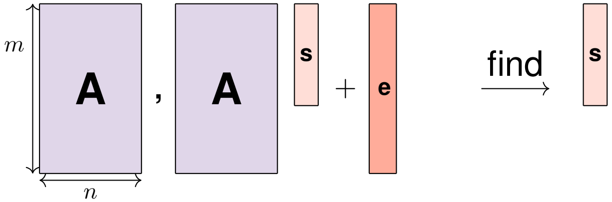
- either $\mathcal{D}_{n,q,\alpha}(\mathbf{s})$ for a fixed $\mathbf{s}$,
- either $U(\mathbb{Z}_q^{n+1})$.

The goal is to decide which one with a non-negligeable advantage.

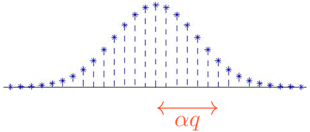
The Learning With Errors problem

$\mathsf{LWE}^n_{\alpha,q}$

Given



- $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$,
- $\mathbf{s} \leftarrow U(\mathbb{Z}_q^{n})$,
- $\mathbf{e} \leftarrow D_{\mathbb{Z}^m,\alpha q}$, small compared to q.



Discrete Gaussian error $D_{\mathbb{Z},\alpha q}$

Search version: Given $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e})$, find $\mathbf{s}$.
Decision version: Distinguish from $(\mathbf{A}, \mathbf{b})$ with $\mathbf{b}$ uniform.

Equivalence between the two variants

LWE sample: $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \mod q)$ with short $\mathbf{e}$.

- Easy reduction : from **decision** to **search**
  - find **s** $\Rightarrow$ distinguish **b** uniform or **b** LWE sample,

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  - find $\mathbf{s} \Rightarrow$ distinguish $\mathbf{b}$ uniform or $\mathbf{b}$ LWE sample,
  - Given $(\mathbf{A}, \mathbf{b})$, find the oracle to find $\mathbf{s}$, compute $\mathbf{b} - \mathbf{A}\mathbf{s}$:

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  - Given $(\mathbf{A}, \mathbf{b})$, find the oracle to find **s**, compute $\mathbf{b} - \mathbf{A}\mathbf{s}$:
    - if it is small, then **b** is an LWE sample,
    - if it looks uniform, then **b** is uniform.

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- 2nd reduction: from **search** to **decision**
  - Distinguish **b** uniform from **b** LWE sample $\Rightarrow$ find **s**,
  - Given $(\mathbf{A}, \mathbf{b})$ use the oracle to find each coordinate of **s**: for all s_1^*, choose u uniform in $\mathbb{Z}_q$ and modify $(\mathbf{A}, \mathbf{b})$ as follow:

$$(\mathbf{a}, b) + (u, 0, \ldots, 0, us_1^*) = (\mathbf{a}', \langle \mathbf{a}', \mathbf{s} \rangle + e + u(s_1^* - s_1)),.$$

    - if $s_1^* = s_1$ it stays a LWE sample,
    - else **b** will be uniform.

Short Integer Solution problem [Ajtai 1996]

For $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$:

SIS$_\beta$

$$\mathbf{x}\,\mathbf{A} = \mathbf{0} \bmod q$$

Goal: Given $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$,
find $\mathbf{x}$ s.t. $0 < \|\mathbf{x}\| \leq \beta$.

LWE$_\alpha$

$$\left(\underbrace{\mathbf{A}}_{n}, \; \mathbf{A}\,\mathbf{s} + \mathbf{e}\right) \quad (\mathbf{A} \text{ with } m \text{ rows})$$

$\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$, $\mathbf{e} \leftarrow D_{\mathbb{Z}^m,\alpha q}$.

Goal: Given $(\mathbf{A}, \mathbf{A}\,\mathbf{s} + \mathbf{e})$,
find $\mathbf{s}$.

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For $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$:

SIS$_\beta$

$$\mathbf{x} \; \mathbf{A} = \mathbf{0} \bmod q$$

Solve SVP in

$$\Lambda_q^\perp(\mathbf{A}) = \{\mathbf{x} \in \mathbb{Z}^m | \mathbf{x}^T \mathbf{A} = 0 \bmod q\}$$

LWE$_\alpha$

$$\left(\underbrace{\mathbf{A}}_{n}, \; \mathbf{A}\,\mathbf{s} + \mathbf{e} \right), \quad \mathbf{A} \text{ of size } m \times n$$

$\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$, $\mathbf{e} \leftarrow D_{\mathbb{Z}^m,\alpha q}$.

Solve CVP in

$$\Lambda_q(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Z}^m : \mathbf{y} = \mathbf{A}\,\mathbf{s} \bmod q$$
for some $\mathbf{s} \in \mathbb{Z}^n\}$

Hardness of LWE

- Exhaustive search
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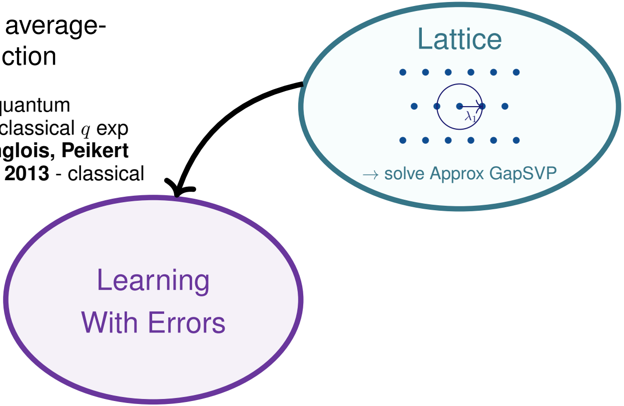
- How to do better?
  - LWE is a lattice problem: consider
    $$\Lambda_q(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Z}^m : \mathbf{y} = \mathbf{A}\mathbf{s} \bmod q \text{ for } \mathbf{s} \in \mathbb{Z}^n\}.$$
    Solving LWE $\Leftrightarrow$ solving CVP in this lattice.
  - Cost: $\left(\frac{n \log q}{\log^2 \alpha}\right)^{\frac{n \log q}{\log^2 \alpha}}$.

Hardness of the Learning With Errors problem

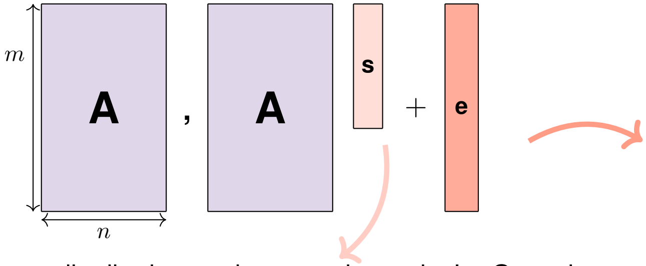
Worst-case to average-case reduction

- **Regev 2005** - quantum
- **Peikert 2009** - classical q exp
- **Brakerski, Langlois, Peikert Regev, Stehlé 2013** - classical



LWE variants

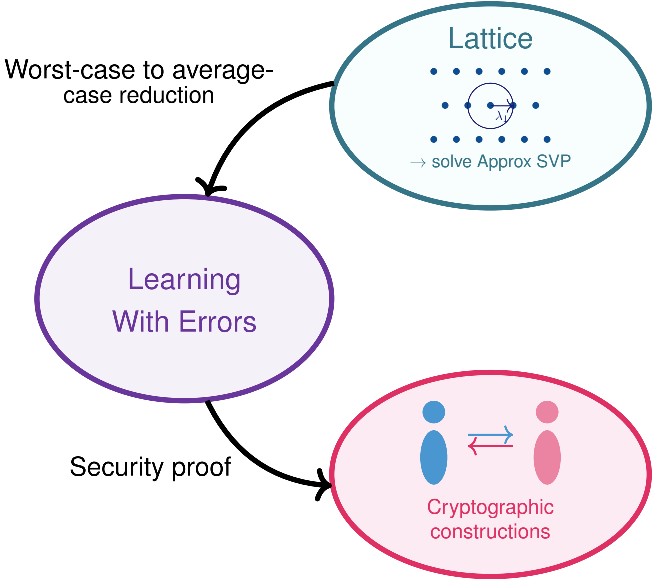
Choose another distribution for the secret or the error.
Regev 2009: uniform secret and gaussian error.



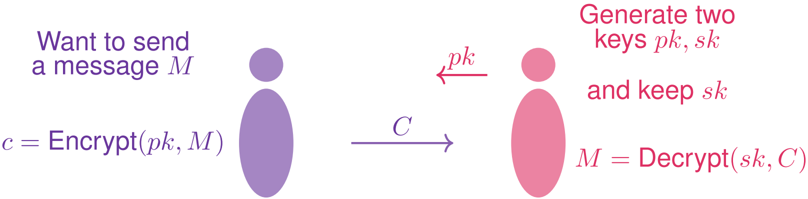
- Gaussian (continue, discretize, discrete ...),
- Uniform in small interval,
- Binary under conditions.

- Same distribution as the error: in particular Gaussian,
- Binary (Unif in $\{0,1\}^n$),
- Entropic.

Using LWE to build provable constructions - theory



Public key encryption - definition



An encryption scheme is defined by three algorithms **(KeyGen, Enc, Dec)**:

- The key generation algorithm **KeyGen** takes as input a security parameter λ and outputs the public and the secret keys (pk, sk).
- The encryption algorithm **Enc** takes as input the public key pk and a message m and outputs $c = Enc(pk, m)$,
- The decryption algorithm **Dec** takes as input the secret key sk and a ciphertext c and outputs $m = Dec(sk, c)$,

such that $Dec(sk, (Enc(pk, m)) = m$.

Regev's encryption scheme

- **Parameters**: $n, m, q \in \mathbb{Z}$, $\alpha \in \mathbb{R}$,
- **Keys**: sk = $\mathbf{s}$ and pk = $(\mathbf{A}, \mathbf{b})$, with $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
  where $\mathbf{s} \hookleftarrow U(\mathbb{Z}_q^n)$, $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m\times n})$, $\mathbf{e} \hookleftarrow D_{\mathbb{Z}^m,\alpha q}$.

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$$\mathbf{u}^T = \mathbf{r}\,\mathbf{A}, \quad v = \mathbf{r}\,\mathbf{b} + \lfloor q/2 \rceil \cdot M$$

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- **Decryption** of $(\mathbf{u}, v)$: compute $v - \mathbf{u}^T\mathbf{s}$,

$$\underbrace{\mathbf{r}\left[\mathbf{A}\,\mathbf{s} + \mathbf{e}\right] + \lfloor q/2 \rceil \cdot M}_{v} - \underbrace{\mathbf{r}\,\mathbf{A}\,\mathbf{s}}_{\mathbf{u}^T\mathbf{s}} = \text{small} + \lfloor q/2 \rceil \cdot M$$

If **close from** 0: return 0, if **close from** $\lfloor q/2 \rceil$: return 1.

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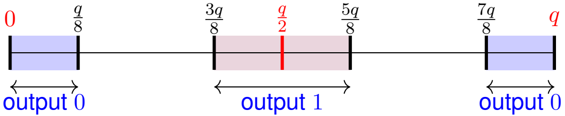
LWE hard $\Rightarrow$ Regev's scheme is IND-CPA secure.

Correction

The randomness **r** is uniformly chosen in $\{0,1\}^m$,
and **e** is sampled from a discrete gaussian of parameter $\alpha q \leq q/(8m)$,
then, with overwhealming probability,

$$\Big|\sum_{i\leq m} r_i e_i\Big| \leq \|\mathbf{r}\| \cdot \|\mathbf{e}\| \leq \sqrt{m} \cdot \frac{q}{8\sqrt{m}} = \frac{q}{8}.$$

$v - \mathbf{u}^T\mathbf{s}$ is either close from 0, either close from $\lfloor q/2 \rfloor$, which allows to find M.



IND-CPA security

To define the security, we use a game between a challenger and an adversary. We define two experiments Exp_b for $b \in \{0, 1\}$:

$\mathcal{C}$hallenger		$\mathcal{A}$dversary
Generate (pk, sk)	$\xrightarrow{pk}$	
	$\xleftarrow{M_0, M_1}$	Choose M_0, M_1
	$\xrightarrow{c \leftarrow \mathsf{Enc}(pk, M_b)}$	
		output b'

$$Adv^{CPA}(\mathcal{A}) = |\Pr[\mathcal{A} \rightarrow^{Exp_0} 1] - \Pr[\mathcal{A} \rightarrow^{Exp_1} 1]|.$$

IND-CPA security

Goal of the proof: show that if an adversary succeed in attacking the encryption scheme with a non-negligible advantage, then the challenger can use it to solve a difficult problem (here LWE).

Decision LWE can be seen as a game:

$\mathcal{C}$		$\mathcal{B}$
$\mathbf{A} \longleftarrow U(\mathbb{Z}_q^{m\times n})$		
RAND $(b=0)$: $\mathbf{b} \leftarrow U(\mathbb{Z}_q^{m})$		
LWE $(b=1)$: $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$	$\xrightarrow{(\mathbf{A},\mathbf{b})}$	
		output b'

$$Adv(\mathcal{B}) = \left|\Pr[\mathcal{B} \xrightarrow{RAND} 1] - \Pr[\mathcal{B} \xrightarrow{LWE} 1]\right|.$$

Leftover Hash Lemma

Let $m, n, q \geq 1$ be integers such that $m \geq 4n \log q$ and q prime, and let $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m \times n})$ and $\mathbf{r} \hookleftarrow U(\{0,1\}^m)$. Then $(\mathbf{A}, \mathbf{r}^T\mathbf{A})$ has statistical distance $\leq 2^{-n}$ from the uniform distribution on $\mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^n$.

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- Statistical distance : $\Delta(D_1, D_2) = \frac{1}{2} \sum_x |D_1(x) - D_2(x)|$.

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- Statistical distance : $\Delta(D_1, D_2) = \frac{1}{2}\sum_x |D_1(x) - D_2(x)|$.
- For any algorithm $\mathcal{A}$, we have $|\Pr[\mathcal{A}(D_1) = 1] - \Pr[\mathcal{A}(D_2) = 1]| \leq \Delta(D_1, D_2)$.
  $\Delta(D_1, D_2)$ small $\Rightarrow$ D_1 and D_2 are statistically indistinguishable.

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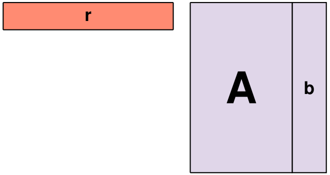
The LHL implies that (**(A b)** , **r** **(A b)**) is indistinguishable from uniform.



IND-CPA security proof

Idea: we start from an LWE instance, and build an instance of the IND-CPA experiment, then we use the answer of the adversary to solve LWE.
We use the following IND-CPA game:

$\mathcal{B}$		$\mathcal{A}$
$(sk = \mathbf{s}, pk = (\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}) \leftarrow KeyGen(.)$	$\xrightarrow{pk=(\mathbf{A},\mathbf{b})}$	
chooses b	$\xleftarrow{m_0,m_1}$	Chooses m_0, m_1,
computes $(c_1, c_2) \leftarrow Enc(pk, m_b)$	$\xrightarrow{c_1,c_2}$	Computes a bit b'
		if $b = b'$ then output Win

We want to show that if LWE is hard, then there exists a negligible function $negl$ such that:

$$\Pr[\mathcal{A} \text{ Win}] \leq 1/2 + negl(n).$$

IND-CPA security proof

$\mathcal{B}$ wants to solve decisional LWE using $\mathcal{A}$.

$\mathcal{C}$		$\mathcal{B}$		$\mathcal{A}$
RAND: $\mathbf{b}$ unif				
LWE: $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$	$\xrightarrow{(\mathbf{A},\mathbf{b})}$		$\xrightarrow{(\mathbf{A},\mathbf{b})}$	m_0, m_1,
		choose b	$\xleftarrow{m_0,m_1}$	
		$(\mathbf{r}^T\mathbf{A}, \mathbf{r}^T\mathbf{b} + q/2 \cdot m_b)$	$\xrightarrow{c_1,c_2}$	Computes b'
		if $b = b'$ output 1	$\xleftarrow{b'}$	
		else output 0		

For $\mathcal{B}$:

- RAND: $\mathbf{b}$ is uniform then c_2 is uniform. $\mathcal{A}$ cannot distinguish between the two cases, its advantage is equals to zero, the probability that $\mathcal{B}$ outputs 1 is $1/2$.

$$\Pr[\mathcal{B} \xrightarrow{RAND} 1] = 1/2,$$

IND-CPA security proof

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$\mathcal{C}$		$\mathcal{B}$		$\mathcal{A}$
RAND: **b** unif				
LWE: $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$	$\xrightarrow{(\mathbf{A},\mathbf{b})}$		$\xrightarrow{(\mathbf{A},\mathbf{b})}$	m_0, m_1,
		choose b	$\xleftarrow{m_0,m_1}$	
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		if $b = b'$ output 1	$\xleftarrow{b'}$	
		else output 0		

For $\mathcal{B}$:

- LWE: $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$ and then the ciphertext is exactly a ciphertext from the Regev encryption scheme. The probability that $\mathcal{B}$ outputs 1 is exactly the success probability of $\mathcal{A}$ in the encryption scheme security game (as it has the same view of the experiment).

$$\Pr[\mathcal{B} \xrightarrow{LWE} 1] = \Pr[\mathcal{A}\ \text{win}],$$

IND-CPA security proof

To conclude, we have:

$$\Pr[\mathcal{B} \xrightarrow{RAND} 1] = 1/2,$$

$$\Pr[\mathcal{B} \xrightarrow{LWE} 1] = \Pr[\mathcal{A} \text{ win}],$$

then:

$$\begin{aligned} Adv(\mathcal{B}) &= |\Pr[\mathcal{B} \xrightarrow{RAND} 1] - \Pr[\mathcal{B} \xrightarrow{LWE} 1]| \\ &= |\Pr[\mathcal{A} \text{ win}] - 1/2| \end{aligned}$$

If $\mathcal{A}$ succeeds with a non-negligible probability, then there exists ε such that $\Pr[\mathcal{A} \text{ win}] \geq 1/2 + \varepsilon$, then $Adv(\mathcal{B}) \geq \varepsilon$ which implies that there exists a distinguisher able to solve the decisional LWE problem.

Using LWE

Hardness of LWE used as a foundation for many constructions.

Learning
With Errors

Problem: constructions based on LWE
enjoy a nice guarantee of security
but are too costly in practice.

Security proof

Cryptographic constructions

Signature, encryption
Advanced schemes
Fully Homomorphic Encryption

Solutions used today?