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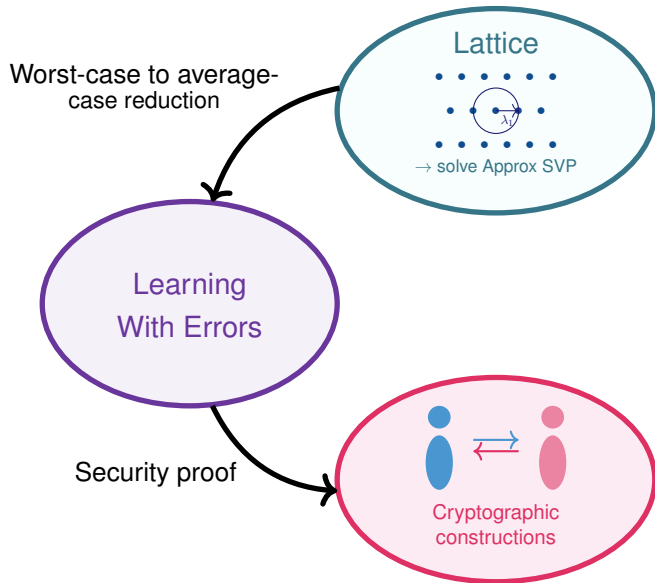
USING STRUCTURED VARIANTS IN LATTICE-BASED CRYPTOGRAPHY

Adeline Roux-Langlois

Normandie Univ, UNICAEN, ENSICAEN, CNRS, GREYC, Caen, FRANCE



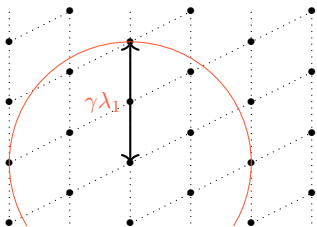
Using LWE to build provable constructions - theory



Approx Shortest Vector Problem (Approx SVP $_{\gamma}$)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n :

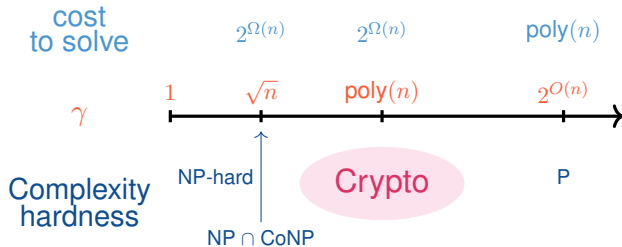
Output: find a non-zero vector $\mathbf{x} \in \mathcal{L}(\mathbf{B})$ such that $\|\mathbf{x}\| \leq \gamma \lambda_1(\mathcal{L}(\mathbf{B}))$



Lattice

$\mathcal{L}(\mathbf{B}) = \{\sum_{i=1}^n a_i \mathbf{b}_i, a_i \in \mathbb{Z}\}$, where the $(\mathbf{b}_i)_{1 \leq i \leq n}$'s, linearly independent vectors, are a **basis** of $\mathcal{L}(\mathbf{B})$.

Hardness of Approx SVP $_{\gamma}$

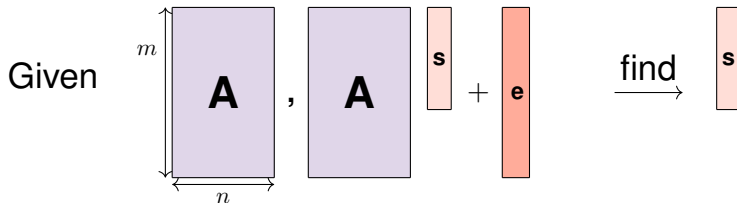


Conjecture

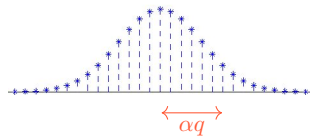
There is no polynomial time algorithm that approximates this lattice problem and its variants to within polynomial factors.

The Learning With Errors problem

$LWE_{\alpha, q}^n$



- ▶ $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m \times n})$,
- ▶ $\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$,
- ▶ $\mathbf{e} \leftarrow D_{\mathbb{Z}^m, \alpha q}$, small compared to q .



Discrete Gaussian error $D_{\mathbb{Z}, \alpha q}$

Search version: Given $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e})$, find \mathbf{s} .

Decision version: Distinguish from (\mathbf{A}, \mathbf{b}) with \mathbf{b} uniform.

Regev's encryption scheme

- ▶ **Parameters:** $n, m, q \in \mathbb{Z}, \alpha \in \mathbb{R}$,
- ▶ **Keys:** $\text{sk} = \mathbf{s}$ and $\text{pk} = (\mathbf{A}, \mathbf{b})$, with $\mathbf{b} = \mathbf{A} \mathbf{s} + \mathbf{e} \pmod q$
 where $\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$, $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m \times n})$, $\mathbf{e} \leftarrow D_{\mathbb{Z}^m, \alpha q}$.
- ▶ **Encryption** ($M \in \{0, 1\}$): Let $\mathbf{r} \leftarrow U(\{0, 1\}^m)$,

$$\mathbf{u}^T = \begin{matrix} \mathbf{r} \\ \mathbf{A} \end{matrix}, \quad v = \begin{matrix} \mathbf{r} \\ \mathbf{b} \end{matrix} + \lfloor q/2 \rfloor \cdot M$$

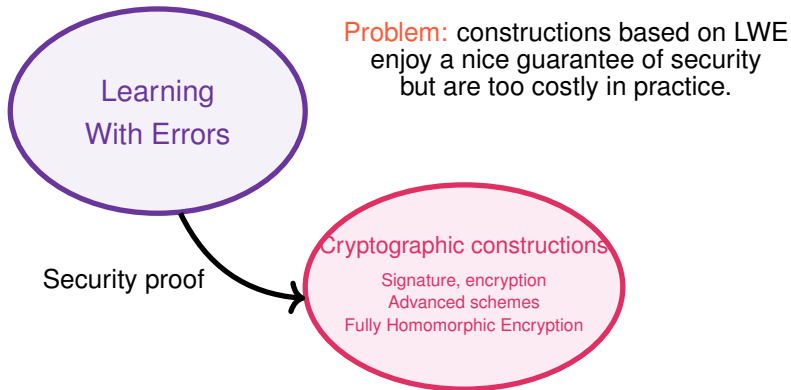
- ▶ **Decryption** of (\mathbf{u}, v) : compute $v - \mathbf{u}^T \mathbf{s}$,

$$\underbrace{\begin{matrix} \mathbf{r} \\ \mathbf{A} \mathbf{s} + \mathbf{e} \end{matrix}}_v + \lfloor q/2 \rfloor \cdot M - \underbrace{\begin{matrix} \mathbf{r} \\ \mathbf{A} \mathbf{s} \end{matrix}}_{\mathbf{u}^T \mathbf{s}} = \text{small} + \lfloor q/2 \rfloor \cdot M$$

If **close from 0**: return 0, if **close from** $\lfloor q/2 \rfloor$: return 1.

LWE hard \Rightarrow **Regev's scheme is IND-CPA secure.**

Hardness of LWE used as a foundation for many constructions.

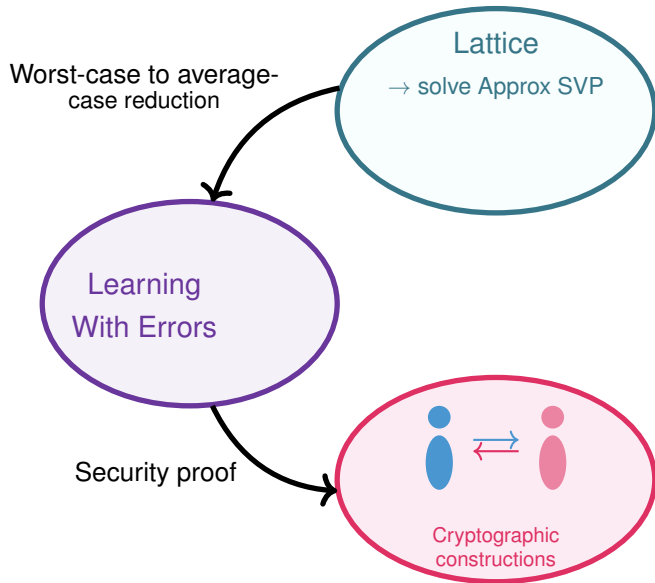


Solutions used today?

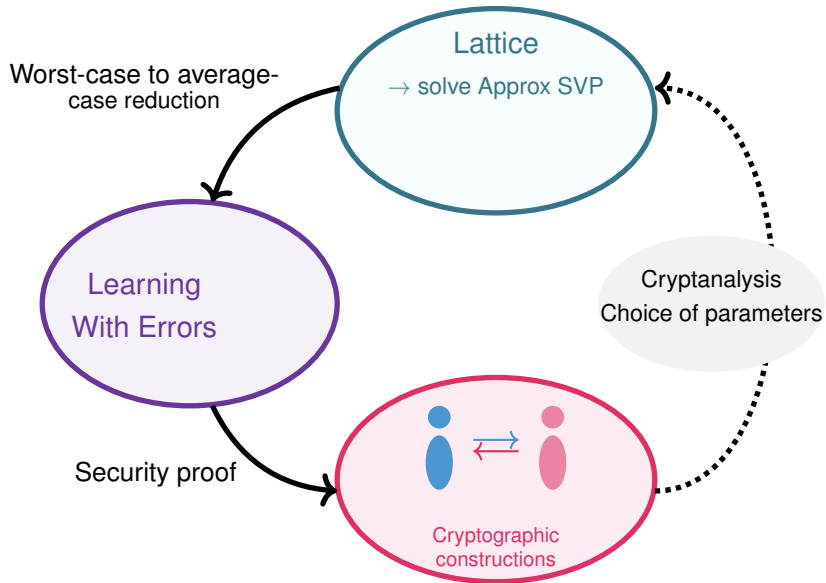
Among the 5 lattice-based finalists, 3 of them are based on (possibly structured) variants of LWE.

- ▶ Public Key Encryption
 - ▶ **Crystals - Kyber**: Module-LWE with both secret and noise chosen from a centered binomial distribution.
 - ▶ **Saber**: Module-LWR (deterministic variant).
 - ▶ **NTRU**
 - ▶ **FrodoKEM** (as alternate candidate): LWE but with smaller parameters.
- ▶ Signature
 - ▶ **Crystals - Dilithium**: Module-LWE with both secret and noise chosen in a small uniform interval, and Module-SIS.
 - ▶ **Falcon**: Ring-SIS on NTRU matrices.

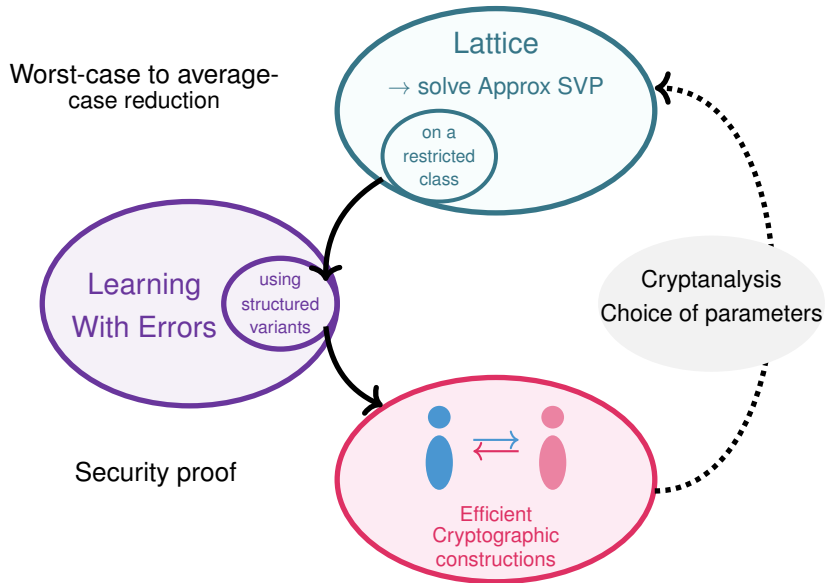
Using LWE to build constructions



Using LWE to build constructions in practice



Using LWE to build constructions in practice

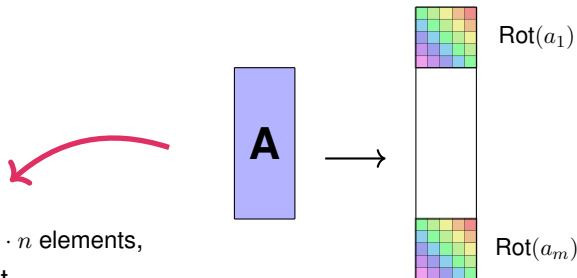


From SIS/LWE to structured variants

Problem: constructions based on LWE enjoy a nice guaranty of security but are too costly in practice.

→ replace \mathbb{Z}^n by a Ring, for example $R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$ ($n = 2^k$).

▶ Ring variants since 2006:



- ▶ Structured $\mathbf{A} \in \mathbb{Z}_q^{m \cdot n \times n}$ represented by $m \cdot n$ elements,
- ▶ Product with matrix/vector more efficient,
- ▶ Hardness of Ring-SIS, [Lyubashevsky and Micciancio 06] and [Peikert and Rosen 06]
- ▶ Hardness of Ring-LWE [Lyubashevsky, Peikert and Regev 10].

Idea: replace \mathbb{Z}^n by $R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$

where $n = 2^k$ then the polynomial $x^n + 1$ is irreducible.

Elements of this ring are polynomials of degree less than n .

R is a **cyclotomic ring**. R is also the ring of integer \mathcal{O}_K of an number field K :

- ▶ $K = \mathbb{Q}[x]/\langle x^n + 1 \rangle$: K is a cyclotomic field,
- ▶ $R = \mathbb{Z}[x]/\langle \phi_m(x) \rangle$ where ϕ_m is the m^{th} cyclotomic polynomial of degree $n = \varphi(m)$. Its roots are the m^{th} roots of unity $\zeta_m^j \in \mathbb{C}$, with $\zeta_m = e^{\frac{2i\pi}{m}}$.
(For $m = 2^{k+1}$, we have $\phi_m(x) = x^n + 1$.)
- ▶ Canonical embedding: $\sigma_K : \alpha \in K \mapsto ((\sigma(\alpha))_\sigma = (\alpha(\zeta_m^j))_j$.

Idea: replace \mathbb{Z}^n by $R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$

R is isomorph to \mathbb{Z}^n

Let $a \in R$, we have $a(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$,
the isomorphism $R \rightarrow \mathbb{Z}^n$ associate

the polynomial $a \in R$ to the vector $\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \in \mathbb{Z}^n$.

Idea: replace \mathbb{Z}^n by $R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$

Let's look at the product of two polynomials $x^n + 1$

▶ $a(x) = a_0 + a_1 \cdot x + \dots + a_{n-1} \cdot x^{n-1}$

▶ $s(x) = s_0 + a_1 \cdot x + \dots + a_{n-1} \cdot x^{n-1}$

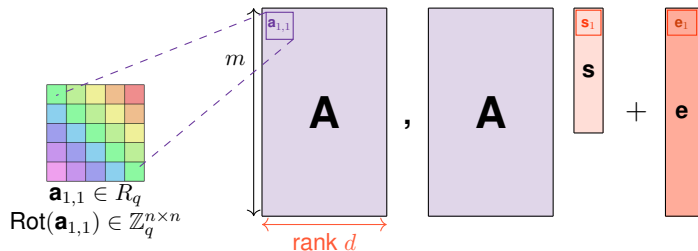
Using matrices, it gives the following block:

$$\begin{bmatrix} a_0 & -a_{n-1} & \cdots & -a_2 & -a_1 \\ a_1 & a_0 & \cdots & -a_3 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_0 & -a_{n-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{bmatrix}$$

Module LWE

Let K be a number field of degree n with R its ring of integers.
 Think of K as $\mathbb{Q}[x]/(x^n + 1)$ and of R as $\mathbb{Z}[x]/(x^n + 1)$ for $n = 2^k$.

Replace \mathbb{Z} by R , and \mathbb{Z}_q by $R_q = R/qR$.



- ▶ $\mathbf{A} \leftarrow U(R_q^{m \times d})$,
- ▶ $\mathbf{s} \leftarrow U(R_q^d)$,
- ▶ $e \in R^m$ small compared to q .

Special case $d = 1$
 is Ring-LWE

$R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$ and $R_q = R/qR$.

Module-SIS $_{q,m,\beta}$

Given $\mathbf{a}_1, \dots, \mathbf{a}_m \in R_q^d$ independent and uniform, find $z_1, \dots, z_m \in R$ such that $\sum_{i=1}^m \mathbf{a}_i \cdot z_i = 0 \pmod q$ and $0 < \|\mathbf{z}\| \leq \beta$.

Let $\alpha > 0$ and $\mathbf{s} \in (R_q)^d$, the distribution $A_{\mathbf{s}, D_{R, \alpha q}}^{(M)}$ is:

- ▶ $\mathbf{a} \in (R_q)^d$ uniform,
- ▶ e sampled from $D_{R, \alpha q}$,

Outputs: $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e)$.

Module-LWE $_{q, \nu_\alpha}$

Let $\mathbf{s} \in (R_q)^d$ uniform, distinguish between an arbitrary number of samples from $A_{\mathbf{s}, D_{R, \alpha q}}^{(M)}$ or the same number from $U((R_q)^d \times R_q)$.

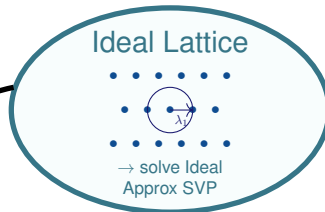
$$R = \mathbb{Z}[x]/\langle x^n + 1 \rangle \quad \text{and} \quad R_q = R/qR.$$

- ▶ An **ideal** I of R is an additive subgroup of R closed under multiplication by every elements of R .
- ▶ As R is isomorph to \mathbb{Z}^n , any ideal $I \in R$ defines an integer lattice $\Lambda(\mathbf{B})$ where $\mathbf{B} = \{g \bmod x^n + 1 : g \in I\}$.
- ▶ A subset $M \subseteq K^d$ is an **R -module** if it is closed under addition and multiplication by elements of R .
- ▶ A finite-type R -module: $M \subseteq R^d : \sum_{i=1}^D R \cdot \mathbf{b}_i, (\mathbf{b}_i) \in R^d,$
- ▶ $M = \sum_{i=1}^d I_i \cdot \mathbf{b}_i$ where I_i are ideals of R and (I_i, \mathbf{b}_i) is a pseudo-basis of M .
- ▶ As ideals, any module defines an integer module lattice.

Hardness of Ring Learning With Errors problem

Worst-case to average-
case reduction

- **Stehlé, Steinfeld, Tanaka and Xagawa 2009** - search
- **Lyubashevsky, Peikert, Regev 2010** - decisional
reduction both quantum, q poly



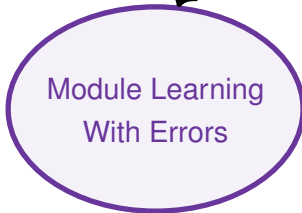
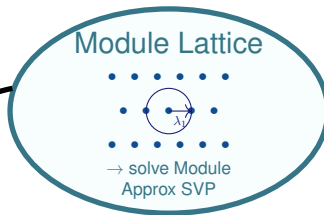
Ring Learning
With Errors

Self reductions

- **Applebaum, Cash, Peikert, Sahai 2009** - same error and secret

Worst-case to average-
case reduction

- **Langlois Stehlé 2015** - quantum, q poly
- Folklore: adapting **Peikert 2009** gives classical reduction **but** q exp and only search variant
- **Boudgoust, Jeudy, Roux-Langlois, Wen 2021** classical, q poly, decisional, **linear rank**

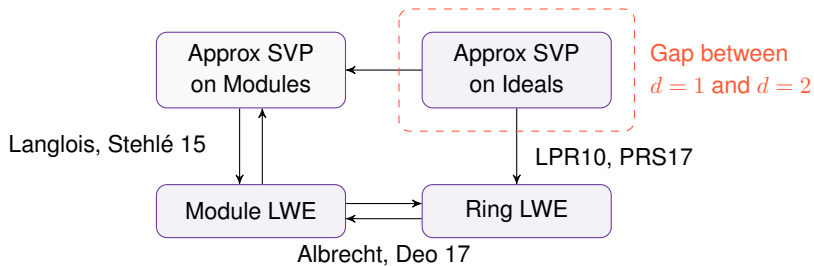


Self reductions

- **Applebaum, Cash, Peikert, Sahai 2009** - same error and secret
- **Boudgoust, Jeudy, Roux-Langlois, Wen 2022**: short error and secret distributions

Module or Rings?

► Hardness of the problem



- ▶ Choice of parameters
 - ▶ Example of Ring $R_q = \mathbb{Z}_q[x]/\langle x^n + 1 \rangle$
 - ▶ Constraints on parameters $n = 2^k, q = 1 \pmod{2n} \dots$

- ▶ An example of parameter set:
 - ▶ $n = 512 \Rightarrow 60$ bits of security,
 - ▶ $n = 1024 \Rightarrow 140$ bits of security,
 - ▶ $(n = 256, d = 3)$ gives $nd = 768$ which is "in between".

- ▶ Module LWE allows more flexibility.

From 2017 to 2024, NIST competition to develop new standards on post-quantum cryptography

2022 first results: **3 over 4 new standards** are lattice-based

- ▶ Kyber - encryption scheme based on Module-LWE,
- ▶ Dilithium - signature scheme based on Module SIS and LWE,
- ▶ Falcon - signature scheme based on NTRU and Ring-SIS.

Encryption scheme based on Ring-LWE

[Lyubashevsky, Peikert, Regev 2011]

KeyGen : The secret key is a small $s \in R$

The public key is $(a, b) = (a, b = a \cdot s + e) \in R_q^2$,
with $a \leftarrow U(R_q)$ and a small $e \in R$.

Enc : Given $m \in \{0, 1\}^n$, a message is a polynomial in R with coordinates in $\{0, 1\}$. Sample small r, e_1, e_2 in R and output

$$(a \cdot r + e_1, b \cdot r + e_2 + \lfloor q/2 \rfloor \cdot m) \in R_q \times R_q.$$

Dec : Given $(u, v) \in R_q \times R_q$, compute

$$v - u \cdot s = (r \cdot e - s \cdot e_1 + e_2) + b \lfloor q/2 \rfloor \cdot m$$

For each coordinate of m , the plaintext is 0 if the result is closer from 0 than $\lfloor q/2 \rfloor$, and 1 otherwise.

[Avanzi, Bos, Ducas, Kiltz, Lepoint, Lyubashevsky, Schanck, Schwabe, Seiler, Stehle]

- ▶ Kyber relies on Module-LWE,
- ▶ Use $R_q = \mathbb{Z}_q[x]/\langle x^{256} + 1 \rangle$ with $q = 7681$.
- ▶ The **small elements** follow a binomial distribution B_η :
For some positive integer η , sample $\{(a_i, b_i)\}_{i=1}^\eta \leftarrow (\{0, 1\}^2)^\eta$ and output $\sum_{i=1}^\eta (a_i - b_i)$.
- ▶ The uniform public key is generated given a *seed* and a function PARSE,
- ▶ Multiplication operations uses NTT - Number Theoretic Transform - which is a variant of the FFT in rings,
- ▶ Size of ciphertext is compressed by keeping only high order bits.

Current timings (ECDH)
Public key around 32 bytes
Efficiency comparable in
terms of cycles.

Kyber-512

Sizes (in bytes)	Haswell cycles (ref)	Haswell cycles (avx2)
sk: 1632	gen: 122684	gen: 33856
pk: 800	enc: 154524	enc: 45200
ct: 768	dec: 187960	dec: 34572

Kyber-768

Sizes (in bytes)	Haswell cycles (ref)	Haswell cycles (avx2)
sk: 2400	gen: 199408	gen: 52732
pk: 1184	enc: 235260	enc: 67624
ct: 1088	dec: 274900	dec: 53156

Kyber-1024

Sizes (in bytes)	Haswell cycles (ref)	Haswell cycles (avx2)
sk: 3168	gen: 307148	gen: 73544
pk: 1568	enc: 346648	enc: 97324
ct: 1568	dec: 396584	dec: 79128

Choice of parameters

- ▶ Parameters used by Kyber:
 - ▶ $n = 256$ and $d = 2, 3, 4$ giving three levels of security: 512, 768, 1024,
 - ▶ $q = 7681$

Choice of parameters

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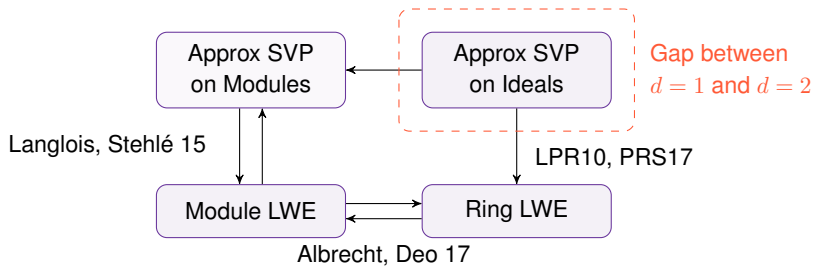
- ▶ How do they choose the parameters?
 - ▶ By considering the LWE instance with dimension nd ,
 - ▶ and the "lattice estimator" [Albrecht, Player, Scott 2015],

Choice of parameters

- ▶ Parameters used by Kyber:
 - ▶ $n = 256$ and $d = 2, 3, 4$ giving three levels of security: 512, 768, 1024,
 - ▶ $q = 7681$
- ▶ How do they choose the parameters?
 - ▶ By considering the LWE instance with dimension nd ,
 - ▶ and the "lattice estimator" [Albrecht, Player, Scott 2015],
- ▶ There is no consideration of the structure!
 - ▶ Why?
 - ▶ Because we don't know how...

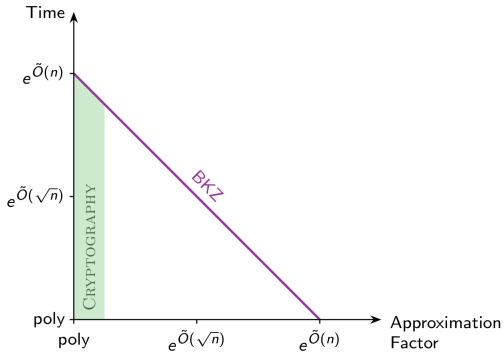
Approx Ideal SVP seems to be the easiest

► Hardness of the problem



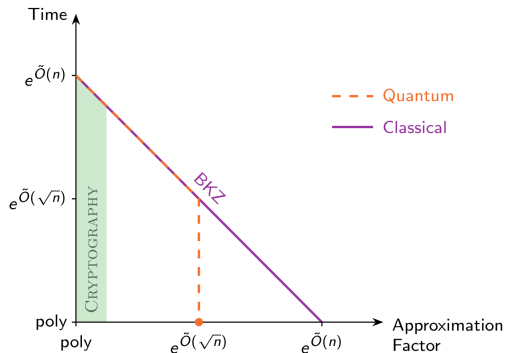
- ▶ For a long time, no algorithm manages to exploit the structure of Ideal SVP.
- ▶ 2014: Quantum algorithm computing (\mathcal{S} -)units, class groups in polynomial time!
[EHKS14,BS16]
- ▶ Followed by a long series of cryptanalysis works.
[CGS14,CDPR16,CDW17/21,PHS19,BR20,BLNR22,BL21,BEFHY22]

¹Thanks to Olivier Bernard and Andrea Lesavourey for part of the slides (particularly to Olivier for the `tikz` picture!)



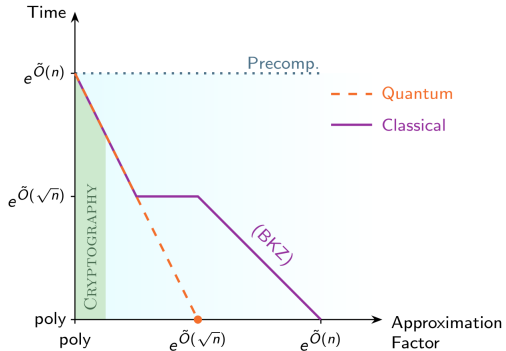
1. Schnorr's hierarchy (*unstructured*)

Algebraic cryptanalysis of Ideal-SVP



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2. CDW algorithm [Cramer, Ducas, Wesolowski 17/21]: uses short *Stickelberger* relations.

Algebraic cryptanalysis of Ideal-SVP



1. Schnorr's hierarchy (*unstructured*)
2. CDW algorithm [Cramer, Ducas, Wesolowski 17/21]: uses short *Stickelberger* relations.
3. PHS and Twisted-PHS [Pellet-Mary, Hanrot, Stehlé 19, Bernard, Roux-Langlois 20, Bernard, Lesavouvey, Nguyen, Roux-Langlois 22]: *S-unit attacks*.

Solving Approx Ideal SVP

Consider an intermediate problem.

Short Generator Principal ideal Problem (SG-PIP):

Given a principal ideal $I = (g)$ such that g is short, retrieve g .

$${}^2\text{Log}_K : x \mapsto (\ln |\sigma_1(x)|, \dots, \ln |\sigma_n(x)|)$$

Consider an intermediate problem.

Short Generator Principal ideal Problem (SG-PIP):

Given a principal ideal $I = (g)$ such that g is short, retrieve g .

1. Find a generator $h = gu$ of I ($u \in \mathcal{O}_K^\times$)

Can be done in polynomial time with a quantum computer

2. Find g given h .

Use the Log-embedding² and the Log-unit lattice $\text{Log}(\mathcal{O}_K^\times)$

² $\text{Log}_K : x \mapsto (\ln |\sigma_1(x)|, \dots, \ln |\sigma_n(x)|)$

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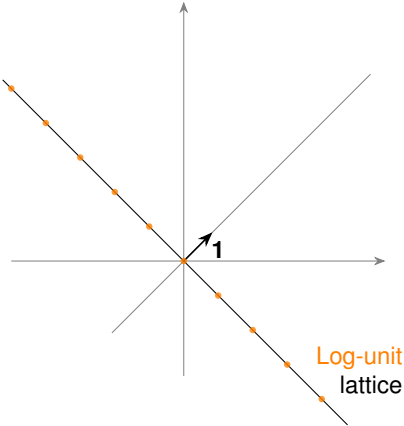
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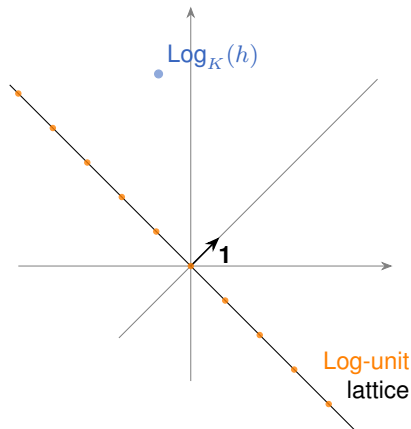
Use the Log-embedding² and the Log-unit lattice $\text{Log}(\mathcal{O}_K^\times)$

- ▶ [Cramer, Ducas, Peikert, Regev 2016] quantum polynomial-time or classical $2^{n^{2/3+\epsilon}}$ -time algorithm to solve SG-PIP over cyclotomic fields.

² $\text{Log}_K : x \mapsto (\ln |\sigma_1(x)|, \dots, \ln |\sigma_n(x)|)$

View of the algorithm





Let I be a challenge ideal.

1. Quantum decomposition

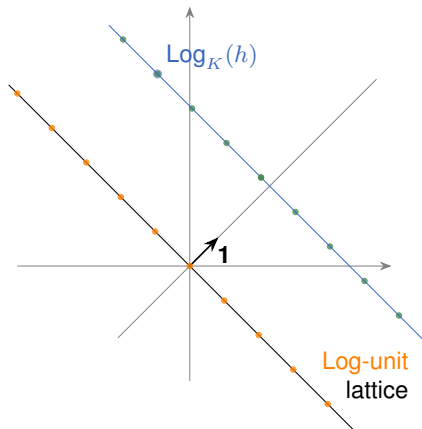
Apply Log_K

$$\text{Log}_K(h) = \text{Log}_K(g) + \text{Log}_K(u) \in$$

$$\text{Log}_K(g) + \text{Log}_K(\mathcal{O}_K^\times)$$

$$h = g \cdot u$$

View of the algorithm



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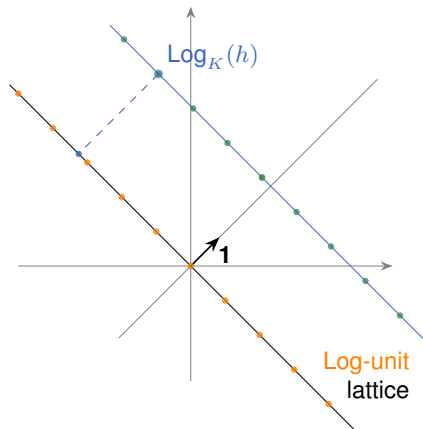
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2. *Short* coset representative ?

$$h = g \cdot u$$



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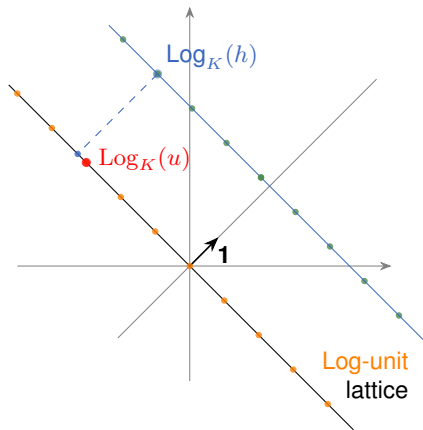
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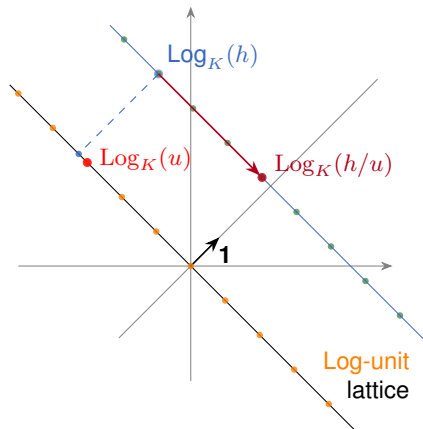
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1. **Quantum** decomposition

Apply Log_K

$$\text{Log}_K(h) = \text{Log}_K(g) + \text{Log}_K(u) \in$$

$$\text{Log}_K(g) + \text{Log}_K(\mathcal{O}_K^\times)$$

2. *Short* coset representative ?

3. Hope this is *short* in I .

$$h = g \cdot u$$

$$(h/u) = g$$

SVP of general ideals

Consider K a number field, I an ideal and S a set of prime ideals.

1. Compute a S -generator of I , i.e. h s.t. $(h) = I \prod_{\mathfrak{p} \in S} \mathfrak{p}^{v_{\mathfrak{p}}}$
2. Reduce h

Two variants for step 2.

1. First reduce $\prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}}$; then find a generator with the Log-embedding.
→ [Cramer, Ducas, Wesolowski 2017] cyclotomic fields, subexponential approximation factor
2. Use the Log- S -embedding³ to reduce everything.
→ [Pellet-Mary, Hanrot, Stehlé 2019] **all number fields**, **exponential preprocessing**, subexponential approximation factor
→ [Bernard, Roux-Langlois 2020] other def. of $\text{Log}_{K,S}$, same asymptotic results, **good results in practice for cyclotomics up to dimensions 70**.

³ $\text{Log}_{K,S} : x \mapsto (\ln|\sigma_1(x)|, \dots, \ln|\sigma_n(x)|, -v_{\mathfrak{p}_1}(x)\ln(N(\mathfrak{p}_1)), \dots, -v_{\mathfrak{p}_r}(x)\ln(N(\mathfrak{p}_r)))$

Can we extend these good results to higher dimensions ?

Two major obstructions for experiments:

- ▶ Decomposition $(h) = I \cdot \prod_{\mathfrak{p} \in S} \mathfrak{p}^{v_{\mathfrak{p}}}$
- ▶ Group of S -units $(s) = \prod_{\mathfrak{p} \in S} \mathfrak{p}^{e_{\mathfrak{p}}}$

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- ▶ Obtain explicit short basis of S_m
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- ▶ Free family of short S -units

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Allows us to approximate $\text{Log}(\mathcal{O}_{K,S}^{\times})$ with a full-rank sublattice

- ▶ Cyclotomic units
- ▶ Explicit Stickelberger generators
- ▶ Real $S \cap K_m^+$ -units \rightarrow only part sub-exponential; dimension $n/2$
- ▶ 2-saturation to reduce the index

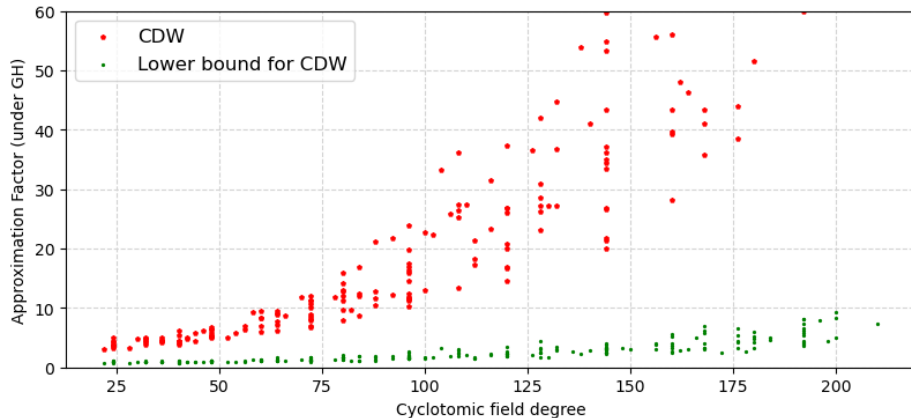
Experimental results⁴

Cyclotomic fields with almost all conductors, up to dimension 210.
Simulated targets in the Log-space

⁴Code available at <https://github.com/ob3rnard/Tw-Sti>.

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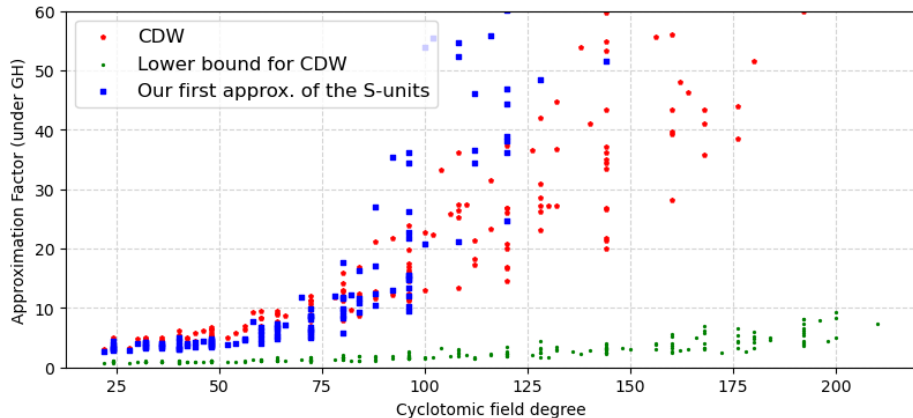
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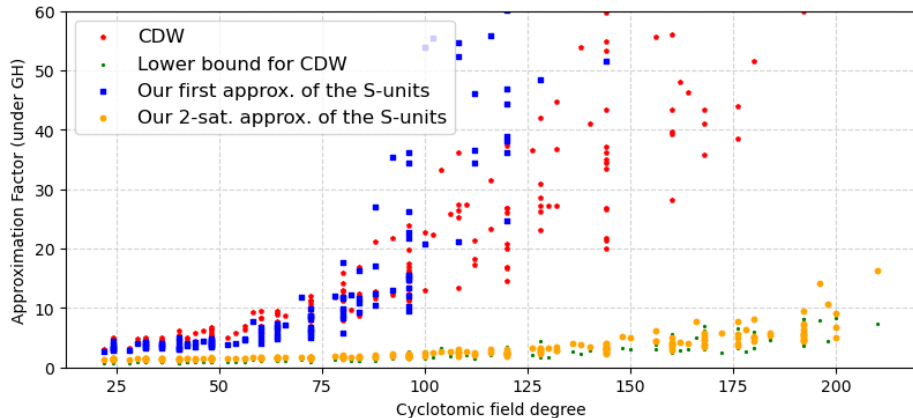
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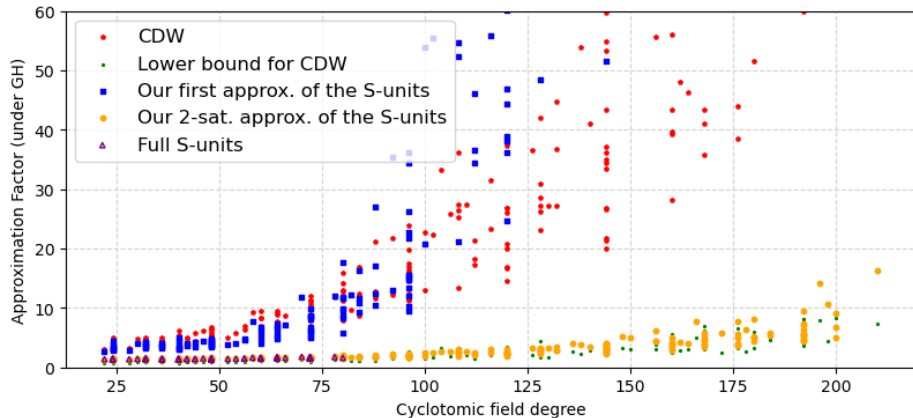
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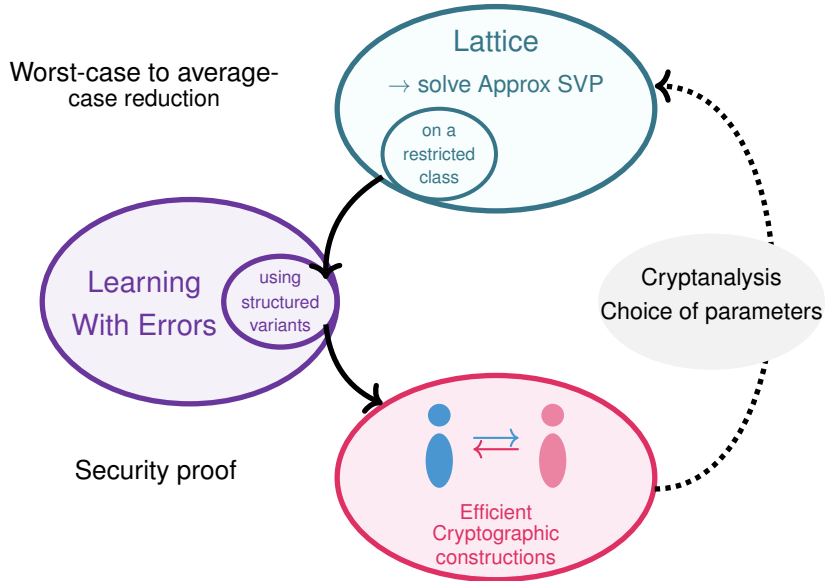
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Using LWE to build constructions in practice



- ▶ Lattice-based cryptography allows to build efficient constructions such as encryption or signature schemes with a security based on the hardness of difficult algorithmic problems on lattices.
- ▶ Three schemes (Kyber, Dilithium and Falcon) will be standardise by the NIST, together with a hash-based signature.
Two of them are based on Module-LWE.
- ▶ Approx Ideal SVP seems to be the easier problem to try to solve → the results of recent attacks does not impact the security of lattice-based constructions.