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Vincent Gauthier[a], Renaud Masson[a], Mihail Gărăjeu[b,*], Thomas Helfer[a]

[a] *CEA, DES, IRESNE, DEC, Cadarache F-13108 Saint-Paul-lez-Durance, France*
[b] *Aix Marseille Univ, CNRS, Centrale Marseille, LMA, F-13453 Marseille, France*

Abstract

In this work, a mean-field micromechanical approach is proposed to determine the effective behaviour of a microstructure where damage occurs in one phase. Our approach fits into the framework of generalised standard materials proposed by Germain et al. (1983) by defining an incremental potential in the spirit of Lahellec and Suquet (2007), made up of two potentials: the free energy and the dissipation, given in this work by Lorentz and Andrieux (2003). It results in the definition of an elastic nonlinear problem at each time increment but, due to the damage evolution, this incremental potential is non convex. The proposed incremental approach also deviates from Lahellec and Suquet (2007) by considering in the damaged phase two sub-phases characterised by two damage states at the beginning of each time step. Next, this elastic nonlinear problem is regularised by introducing a linear comparison composite whose features are optimised by taking inspiration from the Full Optimised Second Order approach of Ponte-Castañeda (2016). However, the reduction of the number of degrees of freedom of the problem is realised by considering an isotropic linear comparison medium.

The abilities of the proposed approach are assessed through a particular case. The composite is made up of two elastic phases : the matrix and the inclusions. The inclusions are subjected to a monotonous increasing swelling while the boundary of the RVE is stress free. For this problem, the resulting solution is given by closed-form expressions. These analytical developments are then used to obtain the effective response of the composite whose dependency on physical parameters (volume fraction of inclusions, toughness of the matrix,

*Corresponding author
Email address: mihai.garajeu@univ-amu.fr (Mihail Gărăjeu)

elastic contrast ...) is then analysed. The analysis is essentially based on an energy balance between the relaxed stored elastic energy and the dissipated energy. Explicit expressions of the critical values of the unique loading parameter (swelling of the inclusions) related to the onset of damage and the failure of the matrix phase are also given. The theoretical effective response compares well with the results of full fields calculations.
Keywords: Brittle damage, Homogenisation, Incremental potential, Second order methods, Particulate composites, Swelling, Standard generalised materials

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1. Introduction

Any material is predisposed to undergo damage, providing that the loading prescribed be high enough. The damage corresponds to the alteration of material properties and is related to the presence of micro-defects. The spatial distribution of these micro-defects is often related to the sources of material heterogeneity such as the dispersion of rigid inclusions in a plastically flowing matrix that leads to the germination of voids (ductile failure) or, as we will see later, the germination of micro-defects related to strain incompatibilities induced by an increase in temperature (brittle failure). These micro-defects will propagate until the final failure of the considered material.

Damage models are able to represent these two successive steps, namely the onset of micro-defects and their propagation. The former is described via the definition of a threshold function (in a similar way to plasticity models). The latter is modelled via evolution laws and, depending on the considered framework, may be derived from different functions that ensure that the dissipated energy is positive. For instance, within the framework of generalised standard materials, they are derived from a convex dissipation potential (see for instance Halphen and Nguyen (1975); Lorentz and Benallal (2005)). More generally, the threshold function can be directly introduced (see Chaboche et al. (2009)). In addition, damage models may be cast into two groups, depending on the form of the potentials they are made up. The first group corresponds to local damage models where damage at a given point is related to the fields at this point (see Francfort and Marigo (1993) for instance). The second group corresponds to non local damage models, studied by Francfort and Marigo (1998); Lorentz and Benallal (2005); Miehe et al. (2010), models where the damage at a given point depends

also on the fields in nearby points (see also De Lorenzis and Gerasimov (2020)). They are of great interest for numerical simulations and encompass phase field models initially proposed by Bourdin et al. (2000) which are currently the topic of many research works.

However, damage models were mainly developed for structures. What about damage models for microstructures? Some works were carried out in order to link the damage to material flaws: micro cracks for fragile damage in Zhu et al. (2008); Kondo and Dormieux (2009); Chen et al. (2022) or porosity for ductile damage in Rice and Tracey (1969); Gurson (1977); Gărăjeu et al. (2000). Those flaws define a phase of its own and the effective properties of the composite are assessed via mean field homogenisation. Most recently, many works have used the phase field modelling of damage to simulate by full-field approaches the evolution of damage inside microstructures. Let us quote for instance Chen et al. (2019); Ma and Sun (2020); Li et al. (2012); Sharma et al. (2018) which put in place fast-fourier transform computations for a phase field modelling of damage. One can also refers to Schneider (2019); Hossain et al. (2014); Michel and Suquet (2022) that tackle the problem of the estimation of the effective toughness of heterogeneous materials. With the exception of Chen et al. (2022) which will be discussed later, no attempt has been made to determine by mean field homogenisation the effective response of a composite where the onset of damage and its propagation throughout one of the phases is assessed by a damage model, making it nonlinear and, above all, time dependent as the evolution versus time of the damage is sought.

Fortunately, the resolution of time evolution problems in mean field homogenisation have been marked by a major inflection with the work of Lahellec and Suquet (2007). Based upon an implicit time integration scheme, an incremental potential was first proposed by Lahellec and Suquet (2007). This approach, that fits into the framework of generalised standard material of Germain et al. (1983), has then been applied to elastoplastic composites so that the resulting problem for each time increment is a nonlinear elastic problem. Taking advantage of the variational formulation of the nonlinear elastic problem, a linear comparison composite, referred to as LCC thereafter, is introduced and the features of this LCC are optimised to deliver bounds or estimates of the effective behaviour. In that specific scope of nonlinear elasticity (or equivalently viscoplasticity), one can quote the variational upper bound which was first put forward in Ponte-Castañeda (1991). Afterwards, new approaches

were proposed in Ponte-Castañeda (1996, 2002, 2016), that introduce a polarisation in the LCC, and that are exact to the second order in the heterogeneity contrast. They now take into account the fields fluctuations in each phase. On the contrary to previous approaches such as Ponte-Castañeda (1996, 2002), the one developed in Ponte-Castañeda (2016), called FO-SO (Full Optimised Second Order), leads to estimates that are fully stationary exact to second-order in the contrast and exhibits no duality gaps. Finally, the combination of the incremental formulation and the use of nonlinear homogenisation theories have led to many recent developments since the first proposal of Lahellec and Suquet (2007) to derive the effective properties of elastoplastic composites.

The present work seeks to combine the incremental variational potential and the second-order procedure to supply a new model describing the time evolution of damage in two-phase composites. Its abilities and limitations are assessed through a specific case: a particulate composite with inclusions subjected to a monotonous increasing swelling. This question was recently addressed by Chen et al. (2022). However, the analytical model reported in Chen et al. (2022) is based on the solution of the composite sphere with the assumption of the spherical symmetry in order to describe concentric damaged zones around the inclusions, whereas our work does not formulate any assumption regarding the form of the damage field. Moreover, in their work Chen et al. (2022) supposed *a priori* that only the shear modulus is affected by damage, whereas in the general model proposed here, both bulk and shear modulus are prone to damage (even if in the application presented in the Section 4 only the bulk modulus is affected by damage). Another difference is that the present work fits into the framework of variational methods of homogenisation. This is why a dissipation potential is defined beforehand in the framework of standard generalised materials using a damage isotropic model from Lorentz and Andrieux (2003).

The structure of this article is the following. In Section 2, the incremental nonlinear homogenisation problem is defined. To that end, following the work of Lahellec and Suquet (2007) the incremental potential of the problem is written down by considering the separation of the matrix in two subdomains, each one being characterised by a different damage evolution. A suitable problem for homogenisation is obtained supposing that the damage fields are uniform in each subdomain. The effective nonlinear response is then obtained.

As the matrix softens with damage, the resulting elastic nonlinear problem is non convex and is consequently regularised with a linearisation approach that is exposed in Section 3 and which is inspired by the Full Optimised Second Order approach of Ponte-Castañeda (2016). The stationary conditions that stem from the linearisation technique involve the expression of the first and second order moments of the strain field in the matrix.

Eventually, the previous stationary equations coming from the linearisation of the effective response are applied to the test case of the matrix/inclusion composite (as presented above) in Section 4. The dependency of the macroscopic response on physical parameters such as the toughness of the matrix as well as the volume fraction of inclusions is analysed.

2. Incremental nonlinear homogenisation problem

We consider a composite material made up of two homogeneous isotropic phases: an inclusion phase (indexed by (2)) which is assumed to be randomly distributed in a matrix phase (indexed by (1)). The representative volume element (RVE) of the composite is denoted V and each of the two phases occupy domains $V^{(r)}$ associated to the characteristic function $\chi^{(r)}$ and the volume fraction $c^{(r)} = \langle \chi^{(r)}(\boldsymbol{x}) \rangle$, $r = 1, 2$, where $\langle \cdot \rangle$ denotes the volume average operator over V, $\langle \cdot \rangle = \dfrac{1}{|V|} \displaystyle\int_V \cdot \, \mathrm{d}\boldsymbol{x}$. Likewise, $\langle \cdot \rangle^{(r)}$ denotes the volume average over a phase (r).

The behaviour of the inclusion phase is thermoelastic and derives from the strain potential

$$w^{(2)}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_0) = \frac{1}{2} \boldsymbol{\varepsilon} : \boldsymbol{C}^{(2)} : \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0 : \boldsymbol{C}^{(2)} : \boldsymbol{\varepsilon}, \tag{1}$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain (symmetric gradient of the displacement field), $\boldsymbol{\varepsilon}_0$ a stress-free strain which may for instance corresponds to a thermal strain mismatch between the different phases and $\boldsymbol{C}^{(2)}$ is the elastic stiffness isotropic tensor with elastic moduli $\kappa^{(2)}$ and $\mu^{(2)}$:

$$\boldsymbol{C}^{(2)} = 3\kappa^{(2)} \boldsymbol{J} + 2\mu^{(2)} \boldsymbol{K}. \tag{2}$$

The fourth order tensors $\boldsymbol{J}$ and $\boldsymbol{K}$ are the projectors onto the space of spherical and deviatoric tensors, respectively:

$$\boldsymbol{J} = \frac{1}{3} \boldsymbol{\delta} \otimes \boldsymbol{\delta}, \qquad \boldsymbol{K} = \boldsymbol{I} - \boldsymbol{J},$$

$\boldsymbol{\delta}$ being the second order identity tensor and $\boldsymbol{I}$ the identity tensor over the space of symetric second order tensors, $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

The matrix phase exhibits a damageable behaviour which can be derived within the framework of generalised standard materials of Halphen and Nguyen (1975); Germain et al. (1983). Its constitutive relations are obtained from two thermodynamic potentials, the free-energy $\psi^{(1)}(\boldsymbol{\varepsilon}, d)$ and the dissipation potential $\varphi^{(1)}(\dot{d})$, which are convex functions of the state variables $(\boldsymbol{\varepsilon}, d)$ and their time-derivative. Moreover $\varphi^{(1)}$ has a minimum in zero and $\varphi^{(1)}(0) = 0$. In this work the internal variable d is the damage field and $\varphi^{(1)}$ is a positively homogeneous function of degree one. The Cauchy stress denoted as $\boldsymbol{\sigma}$ and the driving force Y associated to the the state variable d are given by:

$$\boldsymbol{\sigma} = \frac{\partial \psi^{(1)}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, d), \qquad Y = -\frac{\partial \psi^{(1)}}{\partial d}(\boldsymbol{\varepsilon}, d).$$

According to the standard generalised approach, Halphen and Nguyen (1975); Germain et al. (1983), the thermodynamic force Y belongs to the sub-gradient of the dissipation potential (i.e. assumption of normal evolution), namely $Y \in \partial\varphi^{(1)}(\dot{d})$. The explicit expressions for the free energy $\psi^{(1)}(\boldsymbol{\varepsilon}, d)$ and the dissipation potential $\varphi^{(1)}(\dot{d})$ will be given in Section 2.2.

In this study, the question of interest is the evolution of the damage when the RVE is subject to a monotonous increasing loading. Denoting by t the time, $t = 0$ is chosen for the origin of time for a RVE that is undamaged and free of any loading. Classically, a uniform strain condition is prescribed to the boundary of the RVE, $\boldsymbol{u}(\boldsymbol{x}, t) = \bar{\boldsymbol{\varepsilon}}(t).\boldsymbol{x}$, $\boldsymbol{x} \in \partial V$, where $\boldsymbol{u}$ is the displacement field and $\bar{\boldsymbol{\varepsilon}}(t)$ is a second order symmetric tensor. In addition, a uniform spherical swelling is prescribed to the inclusions, $\boldsymbol{\varepsilon}_0(\boldsymbol{x}, t) = \varepsilon_0^{(2)}(t)\boldsymbol{\delta}$, $\boldsymbol{x} \in V^{(2)}$. The eigenvalues of the overall strain $\bar{\boldsymbol{\varepsilon}}(t)$ and the inclusions swelling $\varepsilon_0^{(2)}(t)$ are nil at $t = 0$ and are monotonously increasing functions of t.

2.1. Effective response and incremental potential

The free-energy ψ and the dissipation potential φ of the composite are given by:

$$\psi(\boldsymbol{x}, \boldsymbol{\varepsilon}, d, \boldsymbol{\varepsilon}_0) = \chi^{(1)}(\boldsymbol{x})\psi^{(1)}(\boldsymbol{\varepsilon}, d) + \chi^{(2)}(\boldsymbol{x})w^{(2)}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_0), \tag{3}$$

$$\varphi(\boldsymbol{x}, \dot{d}) = \chi^{(1)}(\boldsymbol{x})\varphi^{(1)}(\dot{d}). \tag{4}$$

The localisation problem consists in the determination of the local fields $\boldsymbol{\sigma}(\boldsymbol{x},t)$, $\boldsymbol{\varepsilon}(\boldsymbol{x},t)$ and $d(\boldsymbol{x},t)$, nil at $t=0$, and solutions at any instant t in the range $t\in[0,\,T]$ of the following system (in order to simplify the writing, we drop the dependency of the fields with respect to spatial and temporal variables $\boldsymbol{x}$ and t):

$$\left\{\begin{array}{l} \boldsymbol{\sigma}=\dfrac{\partial\psi}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},d,\boldsymbol{\varepsilon}_0),\\ -\dfrac{\partial\psi}{\partial d}(\boldsymbol{\varepsilon},d,\boldsymbol{\varepsilon}_0)\in\partial\varphi(\dot{d}),\\ \operatorname{div}\boldsymbol{\sigma}=0,\\ \boldsymbol{u}(\boldsymbol{x},t)=\bar{\boldsymbol{\varepsilon}}(t).\boldsymbol{x},\quad \boldsymbol{x}\in\partial V\quad\text{and}\quad t\in[0,T] \end{array}\right. \tag{5}$$

Given the loading functions $(\bar{\boldsymbol{\varepsilon}}(t),\,\varepsilon_0^{(2)}(t))$, the effective response of the RVE is the time history of the overall stress $\bar{\boldsymbol{\sigma}}(t)$, $t\in[0,\,T]$, defined as the average over V of the stress field: $\bar{\boldsymbol{\sigma}}(t)=\langle\boldsymbol{\sigma}(\boldsymbol{x},t)\rangle$. Following Mialon (1986), the local problem (5) can be integrated in time by using an implicit Euler method. For doing this, a discretisation of the time interval is considered : $t_0=0<t_1<t_2<\ldots<t_n=T$ and on each time step $[t_n,\,t_{n+1}]$ of length $\Delta t=t_{n+1}-t_n$ the incremental potential $w_{inc}(\boldsymbol{x},\,\boldsymbol{\varepsilon},\,d,\,\boldsymbol{\varepsilon}_0)$ associated to the free-energy (3) and to the dissipation potential (4) is defined by:

$$w_{inc}(\boldsymbol{x},\,\boldsymbol{\varepsilon},\,d,\,\boldsymbol{\varepsilon}_0)=\psi(\boldsymbol{x},\,\boldsymbol{\varepsilon},\,d,\,\boldsymbol{\varepsilon}_0)+\varphi\left(\boldsymbol{x},\,d-d_n(\boldsymbol{x})\right), \tag{6}$$

where $d_n(\boldsymbol{x})=d(\boldsymbol{x},\,t_n)$.

In the framework of generalised standard materials, Mialon (1986) has demonstrated that at a given point $\boldsymbol{x}$, the value $d_{n+1}(\boldsymbol{x})$ of the internal variable d at time t_{n+1} is given by the minimisation of this incremental potential, namely:

$$d_{n+1}(\boldsymbol{x})-d_n(\boldsymbol{x})=\operatorname{Arg}\min_{\Delta d}\{\ w_{inc}(\boldsymbol{x},\,\boldsymbol{\varepsilon}_{n+1}(\boldsymbol{x}),\,d_n(\boldsymbol{x})+\Delta d(\boldsymbol{x}),\,\boldsymbol{\varepsilon}_0(t_{n+1}))\}. \tag{7}$$

In classical damage theories, the dissipation potential φ is unbounded when the damage variable d decreases. As a consequence, when searching to minimise the incremental potential, the solution is necessarily positive, i.e. $d-d_n(\boldsymbol{x})\geq 0$. This is the reason why for the damage theories, the incremental potential can be considered as the restriction of the function (6) to the domain $d>d_n(\boldsymbol{x})$. In that case, the usual stationary condition:

$$\left(\frac{\partial w_{inc}}{\partial d}\right)_{\boldsymbol{\varepsilon},\,d}=0\Longleftrightarrow\frac{\partial\psi}{\partial d}(\boldsymbol{x},\,\boldsymbol{\varepsilon}_{n+1}(\boldsymbol{x}),d_{n+1}(\boldsymbol{x}),\,\varepsilon_0(t_{n+1}))+\frac{\partial\varphi}{\partial d}(\boldsymbol{x},\,d_{n+1}(\boldsymbol{x})-d_n(\boldsymbol{x}))=0 \tag{8}$$

is extended by continuity for $d = d_n(\boldsymbol{x})$ (corresponding for instance to the elastic branch).

The problem (5) is written at the instant t_{n+1} :

$$\left\{\begin{array}{l} \boldsymbol{\sigma}_{n+1} = \dfrac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}_{n+1}, d_{n+1}, \boldsymbol{\varepsilon}_0(t_{n+1})), \\ d_{n+1}(\boldsymbol{x}) - d_n(\boldsymbol{x}) = \mathrm{Arg}\min\limits_{\Delta d}\{\ w_{inc}(\boldsymbol{x},\, \boldsymbol{\varepsilon}_{n+1},\, d_n + \Delta d,\, \boldsymbol{\varepsilon}_0(t_{n+1}))\}, \\ \mathrm{div}\, \boldsymbol{\sigma}_{n+1} = 0, \\ \boldsymbol{u}(\boldsymbol{x}, t_{n+1}) = \bar{\boldsymbol{\varepsilon}}(t_{n+1}).\boldsymbol{x}, \quad \boldsymbol{x} \in \partial V \end{array}\right. \tag{9}$$

where $\boldsymbol{\sigma}_{n+1}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}, t_{n+1})$ and $\boldsymbol{\varepsilon}_{n+1}(\boldsymbol{x}) = \boldsymbol{\varepsilon}(\boldsymbol{x}, t_{n+1})$ are unknown fields while the corresponding fields $\boldsymbol{\sigma}_n(\boldsymbol{x})$, $\boldsymbol{\varepsilon}_n(\boldsymbol{x})$ are supposed to be known at time t_n.

The effective incremental potential (Lahellec and Suquet (2007)) is defined as:

$$\widetilde{w}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}_0) = \inf_{\boldsymbol{\varepsilon} \in \mathcal{K}}\ \langle \inf_{d}\, w_{inc}(\boldsymbol{x},\, \boldsymbol{\varepsilon},\, d,\, \boldsymbol{\varepsilon}_0)\rangle. \tag{10}$$

where $\mathcal{K}$ is the set of kinematically admissible strain fields:

$$\mathcal{K} = \left\{ \boldsymbol{\varepsilon} \,|\, \exists \boldsymbol{v} \text{ such that } \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}\right) \text{ in } V \text{ and } \boldsymbol{v}(\boldsymbol{x}) = \bar{\boldsymbol{\varepsilon}}.\boldsymbol{x} \text{ on } \partial V \right\}$$

The result established in Lahellec and Suquet (2007), with the incremental potential w_{inc} that is a differentiable function with respect to d, shows that the average of the stress field at time t_{n+1} is related to the macroscopic strain $\bar{\boldsymbol{\varepsilon}}(t_{n+1})$ by a stationary relation:

$$\bar{\boldsymbol{\sigma}}(t_{n+1}) = \langle \boldsymbol{\sigma}_{n+1} \rangle = \frac{\partial \widetilde{w}}{\partial \bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\varepsilon}}(t_{n+1}),\, \varepsilon_0(t_{n+1})), \tag{11}$$

In the present case, their result can still be obtained since $\frac{\partial w_{inc}}{\partial d} = 0$ (the stationary condition discussed above) can be written for any admissible damage state, $d \geq d_n(\boldsymbol{x})$.

Another difference with respect to the study of Lahellec and Suquet (2007) is that the problem addressed in this former work pertains to a composite material with viscoelastic phases where the infimum in (10) over the field $d(\boldsymbol{x})$ of the potential defined in (6) is still a convex function with respect to $\boldsymbol{\varepsilon}$. On the contrary, for the case studied here, the softening behaviour of the matrix implies the loss of this convexity property. Therefore the infimum in (10) over the strain field should be replaced by a stationary condition which ultimately leads to the same macroscopic stationary result (11). The use of a linear comparison composite is a way to address this issue, as it will be discussed in Section 3.

The problem studied here is simpler than the one considered in Lahellec and Suquet (2007), since the free-energy $\psi^{(2)}$ of the inclusion phase does not depend on the internal variable d, the field $d_n(\boldsymbol{x})$ being defined only in the matrix phase, $\boldsymbol{x} \in V^{(1)}$. Therefore, denoting by

$$w_{inc}^{(1)}(\boldsymbol{x}, \boldsymbol{\varepsilon}, d) = \psi^{(1)}(\boldsymbol{\varepsilon}, d) + \varphi^{(1)}\left(d - d_n(\boldsymbol{x})\right) \tag{12}$$

the incremental potential associated to the matrix phase, the effective incremental potential (10) can be written as:

$$\widetilde{w}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}_0) = \underset{\boldsymbol{\varepsilon} \in \mathcal{K}}{\text{stat}} \left\{ c^{(1)} \langle \inf_{d} w_{inc}^{(1)}(\boldsymbol{x}, \boldsymbol{\varepsilon},\, d) \rangle^{(1)} + c^{(2)} \langle w^{(2)}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_0) \rangle^{(2)} \right\}, \tag{13}$$

where, as explained above, "inf" has been substituted by "stat", which stands for the research of stationary points of the corresponding function. It is worth noting that $w_{inc}^{(1)}$ explicitly depends on $\boldsymbol{x}$ through the field d_n.

Equation (13) shows that the effective incremental potential can be seen as a "classical" effective potential for a two-phases nonlinear composite. Indeed, for a given strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$, and since the free energy and the dissipation potential of the matrix phase are two convex positive functions, the stationary condition for d in (13) gives a local equation for the field d, which turns out to be the consistency condition in the matrix:

$$d_{n+1}(\boldsymbol{x}) - d_n(\boldsymbol{x}) = \text{Arg}\min_{\Delta d} \{\ \psi^{(1)}(\boldsymbol{\varepsilon}(\boldsymbol{x}),\, d_n(\boldsymbol{x}) + \Delta d) + \varphi^{(1)}\left(\Delta d\right)\}. \tag{14}$$

In the next section, this equation is solved in order to obtain the expression of the field $d_{n+1}(\boldsymbol{x}) = d_n(\boldsymbol{x}) + \Delta d(\boldsymbol{x})$.

2.2. Damage evolution in the matrix phase

The local behaviour model used in this work was proposed in Lorentz and Benallal (2005). However, it is emphasised that this model is the "local" version of a more general formulation proposed in Lorentz and Benallal (2005). The two corresponding thermodynamic potentials are defined as

$$\left\{ \psi^{(1)}(\boldsymbol{\varepsilon}, d) = \frac{1}{2}(1-d)\, \boldsymbol{\varepsilon} : \boldsymbol{C}^{(1)} : \boldsymbol{\varepsilon} + \frac{k_Y\, d\,(d-1)}{1+\gamma-d} [0.3cm] \varphi^{(1)}(\dot{d}) = k_Y \dot{d} + I_{\mathbb{R}^+}(\dot{d}) \right. \tag{15}$$

with k_Y and γ material parameters (k_Y represents the dissipated energy by damage required for a complete failure of a unit volume, γ is a coefficient corresponding to the slope of the

stress/strain curve in softening regime) and $\boldsymbol{C}^{(1)}$ the elastic stiffness isotropic tensor of the matrix (with elastic moduli $\kappa^{(1)}$ and $\mu^{(1)}$). The values of the scalar field $d(\boldsymbol{x}, t)$, which represents the damage at position $\boldsymbol{x} \in V^{(1)}$ and instant t, belong to the interval [0, 1].

In the definition of the dissipation potential $\varphi^{(1)}$, the characteristic function $I_{\mathbb{R}^+}$ ($I_{\mathbb{R}^+}(x) = 0$ if $x \in \mathbb{R}^+$ and ∞ if $x \notin \mathbb{R}^+$) enforces the irreversibility of damage evolution and prevents the damage from decreasing. At any point $\boldsymbol{x}$ of the matrix phase ($\boldsymbol{x} \in V^{(1)}$) the damage state $d(\boldsymbol{x})$ is admissible if $d(\boldsymbol{x}) - d_n(\boldsymbol{x}) \geq 0$ while the incremental potential defined by Eq. (12) above reads:

$$w^{(1)}_{inc}(\boldsymbol{x},\, \boldsymbol{\varepsilon}(\boldsymbol{x}),\, d(\boldsymbol{x})) = \psi^{(1)}(\boldsymbol{\varepsilon}(\boldsymbol{x}),\, d(\boldsymbol{x})) + k_Y\,(d(\boldsymbol{x}) - d_n(\boldsymbol{x})). \tag{16}$$

For a given admissible strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$, the admissible damage state is obtained by derivation with respect to $d(\boldsymbol{x})$ of expression (16) of the incremental potential. It yields:

$$d(\boldsymbol{x}) \in [d_n(\boldsymbol{x});\, 1\,] : \quad \frac{\partial}{\partial d}\Big(w^{(1)}_{inc}(\boldsymbol{x},\, \boldsymbol{\epsilon}(\boldsymbol{x}),\, d)\Big)_{d=d(\boldsymbol{x})} = w_e(\boldsymbol{\varepsilon}(\boldsymbol{x})) - k_Y \frac{\gamma(1+\gamma)}{(1+\gamma-d(\boldsymbol{x}))^2} = 0, \tag{17}$$

where $w_e(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C}^{(1)} : \boldsymbol{\varepsilon}$ is the elastic energy density in the undamaged matrix corresponding to the strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$.

One denotes by $w_e^-(d_n(\boldsymbol{x}))$ the minimum value of this energy required for the damage to increase at point $\boldsymbol{x}$ and adds up to: $w_e^-(d_n(\boldsymbol{x})) = k_Y \frac{\gamma(1+\gamma)}{(1+\gamma-d_n(\boldsymbol{x}))^2}$. Likewise, $w_e^+ = k_Y \frac{(1+\gamma)}{\gamma}$ stands for the maximum energy value beyond which the matrix is fully damaged. With the two extremes value of w_e, the Equation (17) can be inverted and, taking into account the purely elastic and the fully damaged branches, it gives:

$$d(\boldsymbol{\varepsilon}(\boldsymbol{x}), d_n(\boldsymbol{x})) = \begin{cases} 1, & \text{if } w_e(\boldsymbol{\varepsilon}(\boldsymbol{x})) \geqslant w_e^+ \\ 1+\gamma - \sqrt{\dfrac{k_Y\gamma(1+\gamma)}{w_e(\boldsymbol{\varepsilon}(\boldsymbol{x})}}), & \text{if} \quad w_e^-(d_n(\boldsymbol{x})) \leqslant w_e(\boldsymbol{\varepsilon}(\boldsymbol{x})) \leqslant w_e^+ \\ d_n(\boldsymbol{x}), & \text{if } w_e(\boldsymbol{\varepsilon}(\boldsymbol{x})) \leq w_e^-(d_n(\boldsymbol{x})) \end{cases} \tag{18}$$

Finally, at a given point $\boldsymbol{x}$, it is remarked that substituting the variable $d(\boldsymbol{x})$ in Eq. (16) by the function $d(\boldsymbol{\varepsilon}(\boldsymbol{x}), d_n(\boldsymbol{x}))$ introduced above (Eq. (18)) defines the incremental potential as a sole function of the strain $\boldsymbol{\varepsilon}(\boldsymbol{x})$ and the damage at the beginning of the time step, $d_n(\boldsymbol{x})$, at this point. As discussed above, this function depends continuously on the strain field (see table 1 where the variable $\boldsymbol{x}$ has been omitted to shorten expressions).

	$w_e(\boldsymbol{\varepsilon}) \leq w_e^-(d_n)$	$w_e^-(d_n) \leq w_e(\boldsymbol{\varepsilon}) \leq w_e^+$	$w_e(\boldsymbol{\varepsilon}) \geq w_e^+$
$d(\boldsymbol{\varepsilon},\, d_n))$	d_n	$1+\gamma-\sqrt{\frac{k_Y\gamma(1+\gamma)}{w_e(\boldsymbol{\varepsilon}(\boldsymbol{x})}})$	1
$w_{inc}^{(1)}\Big(\boldsymbol{\varepsilon},\, d(\boldsymbol{\varepsilon},\, d_n)\Big)$	$\psi^{(1)}(\boldsymbol{\varepsilon},\, d_n)$	$\psi^{(1)}(\boldsymbol{\varepsilon},\, d(\boldsymbol{\varepsilon},\, d_n)) + k_Y\,(d(\boldsymbol{\varepsilon},\, d_n) - d_n)$	$\psi^{(1)}(\boldsymbol{\varepsilon},\, 1) = 0$

Table 1: The incremental potential as a function of the strain state $\boldsymbol{\varepsilon}$ and the damage at the beginning of the time step, d_n.

2.3. Piecewise uniform damage field

Equation (18) shows that the damage, at a given point of the matrix, for each time step, cannot decrease as it is driven by the elastic energy w_e. Moreover, the damage field explicitly depends on the spatial position $\boldsymbol{x}$ through the elastic energy threshold $w_e^-(d_n(\boldsymbol{x}))$. Therefore, the matrix could be viewed as a composite material containing an infinite number of phases, each of them being related to different damage values. To overcome this issue, a first attempt would be to consider a uniform damage field in the phase (1), but in the present study, as the damage is localised in specific sub-volumes of the matrix phase, the complementary part remaining elastic (or weakly damaged), the assumption that the field $d_n(\boldsymbol{x})$ can be approximated by two distinct values is more appropriate. Hence, we consider a partition of the matrix domain in two sub-domains $\hat{V}^{(1)}$ and $\check{V}^{(1)}$ within the damage evolution is different. In $\hat{V}^{(1)}$, damage is likely to reach high values (locally close to 1), while in $\check{V}^{(1)}$ damage remains low. The corresponding reference values of the damage are denoted by $\hat{d}_n$, in the damaged sub-volume $\hat{V}^{(1)}$ of the matrix phase and $\check{d}_n$ in its weakly damaged part $\check{V}^{(1)}$. The characteristic functions of these two sub-volumes are denoted by $\hat{\chi}^{(1)}(\boldsymbol{x})$ and $\check{\chi}^{(1)}(\boldsymbol{x})$ ($\boldsymbol{x} \in V^{(1)}$) respectively, so that:

$$\boldsymbol{x} \in V^{(1)} \quad d_n(\boldsymbol{x}) \approx \hat{\chi}^{(1)}(\boldsymbol{x})\,\hat{d}_n + \check{\chi}^{(1)}(\boldsymbol{x})\,\check{d}_n \tag{19}$$

For the homogenisation approach intended in this work, the exact definition of the subdomains $\hat{V}^{(1)}$ and $\check{V}^{(1)}$ and their characteristic function is not necessary, but it is important to be able to determine the volume fractions $\hat{\alpha}$ and $\check{\alpha}$ of these subdomains. We will see how to determine these volume fractions in Section 4. It is also pointed out that the two domains do not evolve with the damage and remain constant versus time. On the other hand, damage intensities in these two zones can potentially evolve differently with loading. We will see in

Section 4 that these distinct evolutions can be used to approximate the fact that some parts of the matrix are damaged by the prescribed loading, while others remain elastic. In the time integration process, $(\hat{d}_n, \check{d}_n)$ are the value associated to the damage in the sub-volumes $\hat{V}^{(1)}$ and $\check{V}^{(1)}$ at the end of the previous time step (for the first time step $\hat{d}_1 = \check{d}_1 = 0$) and their values at the end of the current time step $(n+1)$ remain to be determined.

If $\boldsymbol{x}$ belong to one of the two sub-volumes, for instance $\hat{V}^{(1)}$, the estimated value of the damage field $d(\boldsymbol{x})$ corresponding to the minimum in (13), as given by (18), does not depend explicitly on $\boldsymbol{x}$ and is denoted by $d(\boldsymbol{\varepsilon}(\boldsymbol{x}), \hat{d}_n)$. Likewise, the incremental potential associated to each sub-volume of the matrix phase (12) does not depend explicitly on $\boldsymbol{x}$ either. In the sub-volumes $\hat{V}^{(1)}$ and $\check{V}^{(1)}$, the potentials can be written as:

$$\text{in } \hat{V}^{(1)} : \quad \hat{w}^{(1)}(\boldsymbol{\varepsilon}) = \psi^{(1)}(\boldsymbol{\varepsilon}, d(\boldsymbol{\varepsilon}, \hat{d}_n)) + k_Y(d(\boldsymbol{\varepsilon}, \hat{d}_n) - \hat{d}_n) \tag{20}$$

and

$$\text{in } \check{V}^{(1)} : \quad \check{w}^{(1)}(\boldsymbol{\varepsilon}) = \psi^{(1)}(\boldsymbol{\varepsilon}, d(\boldsymbol{\varepsilon}, \check{d}_n)) + k_Y(d(\boldsymbol{\varepsilon}, \check{d}_n) - \check{d}_n). \tag{21}$$

These two potentials are formally the same, but they could have very different expressions depending on which branch of (18) is considered to express $d(\boldsymbol{\varepsilon}, \hat{d}_n)$ and $d(\boldsymbol{\varepsilon}, \check{d}_n)$.

For further developments, note at this stage that the derivative of the potentials $\hat{w}^{(1)}$ and $\check{w}^{(1)}$ with respect to the strain has to be calculated using the chain derivation rule and gives :

$$\frac{\partial \hat{w}^{(1)}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) = \Big(1 - d(\boldsymbol{\varepsilon}, \hat{d}_n)\Big)\, \boldsymbol{C}^{(1)} : \boldsymbol{\varepsilon} \quad \text{and} \quad \frac{\partial \check{w}^{(1)}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) = \big(1 - d(\boldsymbol{\varepsilon}, \check{d}_n)\big)\, \boldsymbol{C}^{(1)} : \boldsymbol{\varepsilon} \tag{22}$$

since the consistency condition (8) is satisfied.

The current approach differs from the incremental variational approach proposed initially in Lahellec and Suquet (2007) as it makes use of two different reference values, one for each subdomain, for the damage field. The damage value is chosen among the branches in the Equation (18). The last equations (22) above embody this difference. Another difference with respect to the approach of Lahellec and Suquet (2007) is the manner to manage the reference values of the internal variable. In the present approach, $\hat{d}_n$ and $\check{d}_n$ are the reference values for the damage field at the beginning of a time step. They are obtained at the end of the previous time step. At the end of the current time step they are updated accordingly to Relation (55). Rigorously, they depend on the second order moments of the strain field in

each sub-volume. These moments are not accessible by the homogenisation method chosen further since sub-volumes are not phases, but they can be estimated, as explained in Section 3.4.4.

2.4. Conclusions on the resulting homogenisation problem at $t = t_{n+1}$

With the assumption adopted in the previous subsection, the initial problem boils down to an elastic nonlinear homogenisation problem. Indeed, the nonlinear potential in $V^{(1)}$ is approximated by:

$$w_{inc}^{(1)}\big(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{x}), d(\boldsymbol{\varepsilon}(\boldsymbol{x}), d_n(\boldsymbol{x}))\big) \approx w^{(1)}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{x})) = \hat{\chi}(\boldsymbol{x})\, \hat{w}^{(1)}(\boldsymbol{\varepsilon}(\boldsymbol{x})) + \check{\chi}(\boldsymbol{x})\, \check{w}^{(1)}(\boldsymbol{\varepsilon}(\boldsymbol{x})) \tag{23}$$

Denoting by $\hat{\alpha} = |\hat{V}^{(1)}/V^{(1)}|$ and $\check{\alpha} = |\check{V}^{(1)}/V^{(1)}|$ the volume fractions of the two sub-volumes defined above, one remarks that:

$$\langle w^{(1)}(\boldsymbol{\varepsilon})\rangle^{(1)} = \hat{\alpha}\, \langle w^{(1)}(\boldsymbol{\varepsilon})\rangle^{\hat{V}^{(1)}} + \check{\alpha}\, \langle w^{(1)}(\boldsymbol{\varepsilon})\rangle^{\check{V}^{(1)}}. \tag{24}$$

Next, the effective incremental potential can be written:

$$\widetilde{w}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}_0) = \underset{\boldsymbol{\varepsilon}\in\mathcal{K}}{\operatorname{stat}} \left\{ c^{(1)} \langle w^{(1)}(\boldsymbol{\varepsilon})\rangle^{(1)} + c^{(2)} \langle w^{(2)}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_0)\rangle^{(2)} \right\} \tag{25}$$

This is a classical relation in nonlinear homogenisation for a two-phases composite made up of a thermoelastic inclusion phase and a nonlinear elastic matrix phase. Likewise Lahellec and Suquet (2007), the expression (23) of the nonlinear incremental potential of the matrix changes at each time step. However, some specific aspects require a highlight:

- According to what has been evidenced in Section 2.1, the major difference between the classical homogenisation theories and this work lies in the nonlinear incremental potential of the matrix that is not here a convex function with respect to the strain field. This issue is addressed by using a linearisation approach coupled with an error function, for which only stationary conditions can be written due to the loss of convexity of the nonlinear incremental potential of the matrix. This process allows for the definition of a linear comparison composite and makes the minimisation over the admissible strain fields in Equation (25) meaningful.

- Here, the two sub-volumes resulting of the division of the matrix display two different damage states at the beginning of each time step. Therefore, these sub-volumes are described by two different expressions of the nonlinear potential $w^{(1)}(\boldsymbol{\varepsilon})$. Once again, the linearisation approach and the choice of a uniform linear comparison composite in the matrix overcome this difficulty. However, the volume fraction of these two sub-volumes, denoted by $(\hat{\alpha}, \check{\alpha})$, would be chosen with particular care in Section 4.

We note for the following that volume fractions $(\hat{\alpha}, \check{\alpha})$ are both constrained by the relations: $\hat{\alpha} + \check{\alpha} = 1$, $0 \leq \hat{\alpha} \leq 1$ and $0 \leq \check{\alpha} \leq 1$.

3. Stationary variational estimate

In the previous section, we have shown that the mean field solution of the time discretized problem (9) on a given time step $[t_n; t_{n+1}]$ can be estimated by solving the elastic nonlinear homogenisation problem (25). Classically this is done in two steps: the first one demands a "linearisation" of the problem (25) for which it is chosen in this work to adapt the method originally proposed in Ponte-Castañeda (2016) while the second one makes use of well known linear homogenisation techniques such as the Mori and Tanaka (1973) scheme.

In a first part, the focus is made on the presentation of the linearisation approach in the case of a two phases composites. The value of the "error" function, defined as the difference between the nonlinear potential and the potential of the Linear Comparison Composite, is computed using two stationary points in the damageable phase, i.e. the matrix. The general theory Ponte-Castañeda (2016) defines a Linear Comparison Composite (later referred to as LCC) that is "optimal". Its definition requires the expression of the second order moments of the strain field in the LCC. This is why a second subsection is dedicated to the reminder of general results of linear homogenisation which lead to the expression of second order moments. The results are specified for the case of a two phases composite.

A third subsection is dedicated to the expansion of the stationary equations associated to the linearisation theory introduced in the previous subsection, for the case of a two phases composite with two stationary points. In the fourth subsection, this general framework is simplified by considering an isotropic LCC and a nil polarisation tensor, which leads to a secant modified approach with two stationary points.

3.1. Linearisation approach

As explained in the introduction, the linearisation approach adopted in this study was originally proposed by Ponte-Castañeda (1996) and then discussed and improved in a series of articles such as Ponte-Castañeda and Willis (1999); Ponte-Castañeda (2002) and Ponte-Castañeda (2016). In the present case, the strain potential of the inclusion phase, defined by Relation (1), is linear thermoelastic. That's why only the nonlinear potential $w^{(1)}(\boldsymbol{\varepsilon})$ should be "linearised". To that end, a linear thermoelastic comparison composite is considered in which the behaviour of the matrix stems from a linear thermoelastic potential $w^{(1)}_{LCC}$:

$$w^{(1)}_{LCC}(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}\colon\boldsymbol{C}\colon\boldsymbol{\varepsilon} + \boldsymbol{\tau}\colon\boldsymbol{\varepsilon} \tag{26}$$

where the fourth order tensor $\boldsymbol{C}$ and the polarisation $\boldsymbol{\tau}$ are uniform on the contrary to Ponte-Castañeda (2016) where the polarisation $\boldsymbol{\tau}$ is a convex combination of several polarisations. This amounts to considering two different polarisations in the two sub-volumes defined in Section 2 and causes the loss of the full optimised formulation developed in Ponte-Castañeda (2016). Nevertheless this loss is of no consequence for our work since the simplification of the stiffness tensor, as presented in Section 4, allows for the determination of each component of the stiffness tensor with only one polarisation. Plus, it would be possible to choose distinct values of stiffness and polarisation in the two sub-volumes which would be equivalent to the definition of a third phase in the linear comparison composite. This choice proves to be convenient whenever the spatial localisation of the damaged zone is known, which is the case for the nucleation of separated voids or the apparition of damage at the interface between phases. But these alternatives are left for future works and the choice of two identical tensors in the two subdomains of the matrix, $\hat{V}^{(1)}$ and $\check{V}^{(1)}$, is kept in the following.

The effective potential of the thermoelastic comparison composite is then:

$$\widetilde{w}_{LCC}(\boldsymbol{C},\boldsymbol{\tau},\bar{\boldsymbol{\varepsilon}},\boldsymbol{\varepsilon}_0) = \inf_{\boldsymbol{\varepsilon}\in\mathcal{K}}\left(c^{(1)}\left\langle\, w^{(1)}_{LCC}\left(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}\right)\right\rangle^{(1)} + c^{(2)}\left\langle w^{(2)}\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}_0\right)\right\rangle^{(2)}\right) \tag{27}$$

The field Δw, which stands for the discrepancy between the nonlinear and the thermoelastic potential in the matrix, is put forward:

$$\Delta w(\boldsymbol{x},\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) = \hat{\chi}(\boldsymbol{x})\,\Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) + \check{\chi}(\boldsymbol{x})\,\Delta\check{w}(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) \tag{28}$$

The fact that the error functions per sub-volume, $\Delta\hat{w}$ and $\Delta\check{w}$, do not depend explicitly on $\boldsymbol{x}$ is highlighted and:

$$\Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) = \hat{w}^{(1)}(\boldsymbol{\varepsilon}) - w^{(1)}_{LCC}(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) \tag{29}$$

The expression is similar for $\Delta\check{w}$.

A variational estimate of the effective potential, based on stationarity conditions, is proposed in Ponte-Castañeda (2002):

$$\widetilde{w}(\bar{\boldsymbol{\varepsilon}},\boldsymbol{\varepsilon}_0) = \underset{\boldsymbol{C},\boldsymbol{\tau}}{\text{stat}} \left[\widetilde{w}_{LCC}\left(\boldsymbol{C},\boldsymbol{\tau},\bar{\boldsymbol{\varepsilon}},\boldsymbol{\varepsilon}_0\right) + c^{(1)} \left(\hat{\alpha} \underset{\boldsymbol{\varepsilon}}{\text{stat}}\, \Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) + \check{\alpha} \underset{\boldsymbol{\varepsilon}}{\text{stat}}\, \Delta\check{w}(\boldsymbol{C},\boldsymbol{\tau},\boldsymbol{\varepsilon}) \right) \right] \tag{30}$$

for the case where two subdomains of the matrix phase, are considered, as is presented in this article. As explained in the previous Section 2, the two subdomains correspond to two distinct states of damage at the beginning of the time step (typically elastic or damaged) while the weighting coefficients ($\hat{\alpha}$, $\check{\alpha}$) may be regarded as the volume fraction of each sub-volume.

In view of the subsequent choices, it is chosen to attach only one stationary point to each subdomain. Denoting by $\hat{\boldsymbol{\varepsilon}}$ and $\check{\boldsymbol{\varepsilon}}$ these two stationary points, one writes:

$$V^{(1)}_{err}(\boldsymbol{C},\boldsymbol{\tau}) = \hat{\alpha}\,\Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}) + \check{\alpha}\,\Delta\check{w}(\boldsymbol{C},\boldsymbol{\tau},\check{\boldsymbol{\varepsilon}}), \tag{31}$$

and the variational estimate of the effective potential can be written as:

$$\widetilde{w}(\bar{\boldsymbol{\varepsilon}},\boldsymbol{\varepsilon}_0) = \underset{\boldsymbol{C},\boldsymbol{\tau}}{\text{stat}} \left[\widetilde{w}_{LCC}\left(\boldsymbol{C},\boldsymbol{\tau},\bar{\boldsymbol{\varepsilon}},\boldsymbol{\varepsilon}_0\right) + c^{(1)} V^{(1)}_{err}(\boldsymbol{C},\boldsymbol{\tau}) \right] \tag{32}$$

3.2. Effective properties of the thermoelastic comparison composite

The thermoelastic comparison composite is a two phases composite with the same microstructure as the considered heterogeneous material. Both phases are thermoelastic with the potentials defined by (26) in the matrix phase and (1) in the inclusions while uniform per phase polarisations are prescribed to the phases: $\boldsymbol{\tau}^{(1)} = \boldsymbol{\tau}$ in the matrix phase and $\boldsymbol{\tau}^{(2)} = -\boldsymbol{C}^{(2)} : \boldsymbol{\varepsilon}_0$ in the inclusion phase.

Following the works of Laws (1973) and Willis (1981), because of the linearity of the problem, the average of the strain over phase (r) in the thermoelastic composite can be written as:

$$\bar{\boldsymbol{\varepsilon}}^{(r)} = \boldsymbol{A}^{(r)} : \bar{\boldsymbol{\varepsilon}} + \boldsymbol{a}^{(r)} \tag{33}$$

where $\boldsymbol{A}^{(r)}$ and $\boldsymbol{a}^{(r)}$ are strain concentration tensors depending on the utilised homogenisation procedure. The effective behaviour is also thermoelastic and the constitutive relation can be written as:

$$\bar{\boldsymbol{\sigma}} = \tilde{\boldsymbol{C}} : \bar{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\tau}}. \tag{34}$$

where $\tilde{\boldsymbol{C}}$ and $\tilde{\boldsymbol{\tau}}$ are the effective stiffness tensor and the effective polarisation, which are given in terms of localisation tensors:

$$\begin{aligned} \tilde{C} &= \sum_r c^{(r)}\, \boldsymbol{C}^{(r)} \,:\, \boldsymbol{A}^{(r)} = c^{(1)}\, \boldsymbol{C} \,:\, \boldsymbol{A}^{(1)} + c^{(2)}\, \boldsymbol{C}^{(2)} \,:\, \boldsymbol{A}^{(2)}, \\ \tilde{\boldsymbol{\tau}} &= \sum_r c^{(r)} \left(\boldsymbol{A}^{(r)}\right)^T \,:\, \boldsymbol{\tau}^{(r)} = c^{(1)} \left(\boldsymbol{A}^{(1)}\right)^T \,:\, \boldsymbol{\tau} - c^{(2)} \left(\boldsymbol{A}^{(2)}\right)^T \,:\, \boldsymbol{C}^{(2)} : \boldsymbol{\varepsilon}_0 \end{aligned} \tag{35}$$

(for any fourth-order tensor $\boldsymbol{A}$, $\boldsymbol{A}^T$ denotes the transposed of $\boldsymbol{A}$). The expression of localisation tensors based on the work of Levin (1967) are given in Appendix A.

An equivalent formulation of the behaviour law given in (34) involving the effective stress-free strain may be put forward:

$$\begin{cases} \bar{\boldsymbol{\sigma}} = \tilde{\boldsymbol{C}} : \left(\bar{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{\varepsilon}}_0\right) \\ -\tilde{\boldsymbol{C}} : \tilde{\boldsymbol{\varepsilon}}_0 = \tilde{\boldsymbol{\tau}} \end{cases} \tag{36}$$

This alternate form of behaviour law will be used in Section 4 of this paper. Furthermore, the effective properties of a linear media can be estimated with specific homogenisation schemes, such as, in this paper, the Mori and Tanaka (1973) scheme.

The effective stress-strain relation (34) stems from the effective potential of the thermoelastic comparison composite which has the form (see for instance Ponte-Castañeda (2016) for further details):

$$\widetilde{w}_{LCC}\left(\boldsymbol{C},\, \boldsymbol{\tau},\, \bar{\boldsymbol{\varepsilon}},\, \boldsymbol{\varepsilon}_0\right) = \frac{1}{2}\bar{\boldsymbol{\varepsilon}} \,:\, \tilde{\boldsymbol{C}} \,:\, \bar{\boldsymbol{\varepsilon}} + \bar{\boldsymbol{\varepsilon}} \,:\, \tilde{\boldsymbol{\tau}} + \frac{1}{2}\sum_r c^{(r)} \boldsymbol{a}^{(r)} \,:\, \boldsymbol{\tau}^{(r)} \tag{37}$$

Using the approach proposed in Suquet (1995) and Ponte Castañeda and Suquet (1997), the first and second moments of the strain in the LCC can be obtained from $\widetilde{w}_{LCC}$:

$$\left\langle \boldsymbol{\varepsilon} \right\rangle_{LCC}^{(r)} = \frac{1}{c^{(r)}} \frac{\partial \widetilde{w}_{LCC}}{\partial \boldsymbol{\tau}^{(r)}} \tag{38}$$

and

$$\langle \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} \rangle_{LCC}^{(r)} = \frac{2}{c^{(r)}} \frac{\partial \widetilde{w}_{LCC}}{\partial \boldsymbol{C}^{(r)}} \tag{39}$$

Isotropic invariants. In the case of an isotropic two phases composite, such as in Section 4, the expressions (38) and (39) yielding the second order moments can be simplified by the introduction of two isotropic invariants: the hydrostatic strain and the von Mises equivalent strain. These two invariants of a given strain $\boldsymbol{\varepsilon}$ are defined by

$$\varepsilon_m = \frac{\mathrm{tr}(\boldsymbol{\varepsilon})}{3} \qquad \text{and} \qquad \varepsilon_{eq} = \sqrt{\frac{2}{3}\, \boldsymbol{e} \colon \boldsymbol{e}}, \tag{40}$$

$\boldsymbol{e}$ being the strain deviator, $\boldsymbol{e} = \boldsymbol{\varepsilon} - \varepsilon_m\, \boldsymbol{\delta}$. The second order moments of the hydrostatic and equivalent strain are obtained from (39) using the following projection rules on $\boldsymbol{J}$ and $\boldsymbol{K}$:

$$3\, \varepsilon_m^2 = \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} :: \boldsymbol{J} \qquad \text{and} \qquad \frac{3}{2}\, \varepsilon_{eq}^2 = \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} :: \boldsymbol{K},$$

where the operator :: means a sum over four indexes, $\boldsymbol{A} :: \boldsymbol{B} = A_{ijkl}\, B_{ijkl}$ (the scalar product of fourth order tensors). From these relations and noting that

$$\frac{\partial}{\partial \kappa^{(r)}} = 3\, \frac{\partial}{\partial \boldsymbol{C}^{(r)}} :: \boldsymbol{J} \qquad \text{and} \qquad \frac{\partial}{\partial \mu^{(r)}} = 2\, \frac{\partial}{\partial \boldsymbol{C}^{(r)}} :: \boldsymbol{K}$$

(38) and (39) can be rewritten:

$$\begin{cases} < \varepsilon_m^2 >_{LCC}^{(r)} = \dfrac{2}{9\, c^{(r)}} \dfrac{\partial \widetilde{w}_{LCC}}{\partial \kappa^{(r)}} \\ < \varepsilon_{eq}^2 >_{LCC}^{(r)} = \dfrac{2}{3\, c^{(r)}} \dfrac{\partial \widetilde{w}_{LCC}}{\partial \mu^{(r)}} \end{cases} \tag{41}$$

3.3. Stationary conditions

As announced in introduction of Section 3, this last subsection aims at getting the stationary conditions associated to the problem (32), which is a particular case of the theory laid out by Ponte-Castañeda (2016). Furthermore and as explained above, the matrix is split in two subdomains associated with one stationary point for each of them.

Likewise Ponte-Castañeda (2016), the stationary conditions are split in two sets. The first set is referred to as inner stationary conditions and stems from the optimisation of error functions in the matrix (31). The second set of conditions is related to the stationarity with respect to $\boldsymbol{C}$ and $\boldsymbol{\tau}$, as given in (32).

Another particularity of this approach is the way to update the damage value at the end of the time step. This point is of paramount importance in the following section as it determines how the damage evolves inside the matrix.

3.3.1. Inner stationary condition

A stationary point $\hat{\boldsymbol{\varepsilon}}$ of the function $\Delta\hat{w}$ is the solution of the equation

$$\frac{\partial \Delta\hat{w}}{\partial \boldsymbol{\varepsilon}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right)=0. \tag{42}$$

Each stationary point $\hat{\boldsymbol{\varepsilon}}$ depends on the modulus $\boldsymbol{C}$ and the polarisation $\boldsymbol{\tau}$ in the matrix phase but also on the value of the damage parameter at the beginning of the time step, $\hat{d}_n$. Injecting the expression of the derivative of the potential $\hat{w}^{(1)}$ with respect to the strain (Relation (22)) in the last equation yields:

$$\left[\left(1-d(\hat{\boldsymbol{\varepsilon}},\hat{d}_n)\right)\boldsymbol{C}^{(1)}-\boldsymbol{C}\right]:\hat{\boldsymbol{\varepsilon}}-\boldsymbol{\tau}=0. \tag{43}$$

Using only one solution of (43) denoted by $\hat{\boldsymbol{\varepsilon}}\left(\boldsymbol{C},\boldsymbol{\tau}\right)$ in the sub-volume $\hat{V}^{(1)}$ and similarly only one solution $\check{\boldsymbol{\varepsilon}}\left(\boldsymbol{C},\boldsymbol{\tau}\right)$ in the sub-volume $\check{V}^{(1)}$, we get for the function $V_{err}^{(1)}$ the expression (31).

3.3.2. Outer stationary condition

The stationary conditions with respect to $\boldsymbol{C}$ and $\boldsymbol{\tau}$ for the estimate (32) of the effective potential are now derived. Let's start with the stationary condition with respect to tensor $\boldsymbol{C}$ in the matrix phase:

$$\frac{d\widetilde{w}_{LCC}}{d\boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau},\bar{\boldsymbol{\varepsilon}},\boldsymbol{\varepsilon}_0\right)+c^{(1)}\left(\hat{\alpha}\,\frac{d}{d\boldsymbol{C}}\left(\Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}})\right)+\check{\alpha}\,\frac{d}{d\boldsymbol{C}}\left(\Delta\check{w}(\boldsymbol{C},\boldsymbol{\tau},\check{\boldsymbol{\varepsilon}})\right)\right)=0. \tag{44}$$

We remark that:

$$\frac{d}{d\boldsymbol{C}}\left(\Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}})\right)=\left[\frac{\partial\Delta\hat{w}}{\partial\boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right)+\frac{\partial\Delta\hat{w}}{\partial\boldsymbol{\varepsilon}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right):\frac{\partial\hat{\boldsymbol{\varepsilon}}}{\partial\boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau}\right)\right], \tag{45}$$

that simplifies to:

$$\frac{d}{d\boldsymbol{C}}\left(\Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}})\right)=\frac{\partial\Delta\hat{w}}{\partial\boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right)$$

using the fact that $\hat{\boldsymbol{\varepsilon}}\,(\boldsymbol{C},\boldsymbol{\tau})$ is a stationary point for $\Delta\hat{w}$. Given that only the linear potential $w^{(1)}_{LCC}$ depends explicitly on the elastic stiffness in the matrix $\boldsymbol{C}$ in the error function $\Delta\hat{w}$, we obtain:

$$\frac{d}{d\boldsymbol{C}}\left(\Delta\hat{w}(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}})\right) = -\frac{\partial w^{(1)}_{LCC}}{\partial \boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right).$$

A similar relation is obtained for the derivative of $\Delta\check{w}(\boldsymbol{C},\boldsymbol{\tau},\check{\boldsymbol{\varepsilon}})$ with respect to the tensor $\boldsymbol{C}$ in the matrix. From (38) and (39), recalling that $\boldsymbol{C}$ and $\boldsymbol{\tau}$ are the stiffness tensor and the polarisation of the matrix phase in the LCC, the stationary condition (44) becomes:

$$c^{(1)}\left(\frac{1}{2}\langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(1)}_{LCC} - \hat{\alpha}\frac{\partial w^{(1)}_{LCC}}{\partial \boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right) - \check{\alpha}\frac{\partial w^{(1)}_{LCC}}{\partial \boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau},\check{\boldsymbol{\varepsilon}}\right)\right) = 0. \tag{46}$$

Similar developments make possible the simplification of the stationary condition with respect to the polarisation, namely:

$$c^{(1)}\left(\langle\boldsymbol{\varepsilon}\rangle^{(1)}_{LCC} - \hat{\alpha}\,\frac{\partial w^{(1)}_{LCC}}{\partial \boldsymbol{\tau}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right) - \check{\alpha}\,\frac{\partial w^{(1)}_{LCC}}{\partial \boldsymbol{\tau}}\left(\boldsymbol{C},\boldsymbol{\tau},\check{\boldsymbol{\varepsilon}}\right)\right) = 0. \tag{47}$$

From the definition (26) of the potential $w^{(1)}_{LCC}$, calculated at the stationary point $\hat{\boldsymbol{\varepsilon}}$, it follows:

$$\frac{\partial w^{(1)}_{LCC}}{\partial \boldsymbol{C}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right) = \frac{1}{2}\hat{\boldsymbol{\varepsilon}}\otimes\hat{\boldsymbol{\varepsilon}} \tag{48}$$

and

$$\frac{\partial w^{(1)}_{LCC}}{\partial \boldsymbol{\tau}}\left(\boldsymbol{C},\boldsymbol{\tau},\hat{\boldsymbol{\varepsilon}}\right) = \hat{\boldsymbol{\varepsilon}} \tag{49}$$

Similar relations are derived at the stationary point $\check{\boldsymbol{\varepsilon}}$.

Finally, from the last four equations and since $\hat{\alpha}+\check{\alpha}=1$, the outer stationary conditions form a system of equations for $\boldsymbol{C}$ and $\boldsymbol{\tau}$:

$$\langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(1)}_{LCC} = \hat{\alpha}\,\hat{\boldsymbol{\varepsilon}}\otimes\hat{\boldsymbol{\varepsilon}} + \check{\alpha}\,\check{\boldsymbol{\varepsilon}}\otimes\check{\boldsymbol{\varepsilon}} \qquad \text{and} \qquad \langle\boldsymbol{\varepsilon}\rangle^{(1)}_{LCC} = \hat{\alpha}\,\hat{\boldsymbol{\varepsilon}} + \check{\alpha}\,\check{\boldsymbol{\varepsilon}}. \tag{50}$$

These two tensorial equations form a system of equations for $\boldsymbol{C}$ and $\boldsymbol{\tau}$, but as remarked in Ponte-Castañeda (2016), nothing ensures that the system is compatible. Indeed, direct calculations of the fluctuation covariance tensor in the matrix, in the linear comparison composite, give

$$\mathbb{C}^{(1)}_{\boldsymbol{\varepsilon}} = \left\langle\left(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}\rangle^{(1)}_{LCC}\right)\otimes\left(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}\rangle^{(1)}_{LCC}\right)\right\rangle = \hat{\alpha}(1-\hat{\alpha})\,(\hat{\boldsymbol{\varepsilon}}-\check{\boldsymbol{\varepsilon}})\otimes(\hat{\boldsymbol{\varepsilon}}-\check{\boldsymbol{\varepsilon}}).$$

This relation shows that the average of a rank 1 tensor is a rank 1 tensor, which is not true in general. For this reason, in Ponte-Castañeda (2016) a solution to this issue is proposed by considering a LCC polarisation tensor as a convex combination of several second order tensors. In the present work, in Section 3.4, it is chosen to reduce the number of unknowns and equations of the system (50), in such a way that it becomes compatible, by considering particular anisotropy for the tensors $\boldsymbol{C}$ and $\boldsymbol{\tau}$.

3.3.3. Effective response

In summary, the equations which allow to obtain the effective response are:

- the solution of the Equation (43) gives the stationary points for each sub-volume, $\hat{\boldsymbol{\varepsilon}}$ and $\check{\boldsymbol{\varepsilon}}$ respectively,
- the relations (38) and (39) allow to determine the first and second moments of the strain field in the LCC, when a homogenisation procedure is used for the thermoelastic comparison composite. These moments are function of $\boldsymbol{C}$ and $\boldsymbol{\tau}$.
- the nonlinear equations (50) are solved in order to obtain the stiffness tensor $\boldsymbol{C}$ and the polarisation $\boldsymbol{\tau}$.
- the effective potential $\widetilde{w}_{LCC}$ (37) and the localisation tensors whose expressions are given in Appendix A (Relation (A.3)) can be computed using a homogenisation procedure for given values of $\boldsymbol{C}$ and $\boldsymbol{\tau}$. Thus the effective behaviour is completely determined.

3.4. A modified secant approach with two stationary points

In what follows, the general framework presented in the previous section is addressed in the particular case where the stiffness tensor of the thermoelastic comparison composite is chosen isotropic, *i.e.* of the form:

$$\boldsymbol{C} = 3\,\kappa^{(1)}\,(1 - D_\kappa)\boldsymbol{J} + 2\,\mu^{(1)}\,(1 - D_\mu)\boldsymbol{K}. \tag{51}$$

As a consequence, only two scalars, D_κ and D_μ, has to be determined, instead of the whole stiffness tensor $\boldsymbol{C}$. This particular choice allows for different damage evolutions for the bulk

and shear elastic moduli and leads to simpler analytical expressions for both inner and outer stationary conditions. It is noteworthy that this choice concerns only the LCC, not the real behaviour of the matrix, which is still described by the Relation (15).

3.4.1. Outer stationary conditions

The first of the two outer stationary conditions (50) can be simplified since the stiffness tensor of the thermoelastic comparison composite (51) depends only on the two unknown scalar parameters D_κ and D_μ. Therefore, the stationary condition with respect to $\boldsymbol{C}$ has to be replaced by two stationary conditions with respect to D_κ and D_μ. Since

$$\frac{\partial}{\partial D_\kappa} = -3\kappa^{(1)} \frac{\partial}{\partial \boldsymbol{C}} ::\boldsymbol{J} \qquad \text{and} \qquad \frac{\partial}{\partial D_\mu} = -2\mu^{(1)} \frac{\partial}{\partial \boldsymbol{C}} ::\boldsymbol{K},$$

the new equations for D_κ and D_μ are obtained from (50_1) by projection on $\boldsymbol{J}$ and $\boldsymbol{K}$ and gives:

$$\langle \boldsymbol{\varepsilon}_m^2 \rangle_{LCC}^{(1)} = \hat{\alpha}(\hat{\boldsymbol{\varepsilon}}_m)^2 + \check{\alpha}(\check{\boldsymbol{\varepsilon}}_m)^2 \qquad \text{and} \qquad \langle \boldsymbol{\varepsilon}_{eq}^2 \rangle_{LCC}^{(1)} = \hat{\alpha}(\hat{\boldsymbol{\varepsilon}}_{eq})^2 + \check{\alpha}(\check{\boldsymbol{\varepsilon}}_{eq})^2 \tag{52}$$

Note that the stationary condition for $\boldsymbol{\tau}$ (50_2) is unchanged with this particular choice. Note also that (52) are two scalar equations and hence the possible incompatibility issue of the system (50_1) is removed. In addition, analytic expressions of outer stationary conditions (52) can be set up thanks to the estimation provided by linear schemes of homogenisation, such as Mori and Tanaka (1973) for this paper, or the more general bounds of Hashin and Shtrikman (1963).

3.4.2. Inner stationary conditions

As the linearised modulus of the matrix phase is isotropic, the tensorial inner stationary conditions (43), written for both sub-volumes $\hat{V}^{(1)}$ and $\check{V}^{(1)}$, can be projected onto the spaces of spherical and deviatoric tensors and the inner stationary conditions read:

$$\begin{cases} 3\kappa^{(1)} \left(D_\kappa - d(\hat{\boldsymbol{\varepsilon}}, \hat{d}_n) \right) \hat{\varepsilon}_m = \tau_m \\ 2\mu^{(1)} \left(D_\mu - d(\hat{\boldsymbol{\varepsilon}}, \hat{d}_n) \right) \hat{\boldsymbol{e}} = \boldsymbol{\tau}_d \end{cases} \quad \text{and} \quad \begin{cases} 3\kappa^{(1)} \left(D_\kappa - d(\check{\boldsymbol{\varepsilon}}, \check{d}_n) \right) \check{\varepsilon}_m = \tau_m \\ 2\mu^{(1)} \left(D_\mu - d(\check{\boldsymbol{\varepsilon}}, \check{d}_n) \right) \check{\boldsymbol{e}} = \boldsymbol{\tau}_d \end{cases} \tag{53}$$

where $\hat{\varepsilon}_m \boldsymbol{\delta}$ and $\tau_m \boldsymbol{\delta}$ are the spherical parts of the tensors $\hat{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\tau}$ while $\hat{\boldsymbol{e}}$ and $\boldsymbol{\tau}_d$ denote their deviatoric parts (same notations for $\check{\boldsymbol{\varepsilon}}$). In each system, the equations are coupled since the functions $d(\hat{\boldsymbol{\varepsilon}}, \hat{d}_n)$ and $d(\check{\boldsymbol{\varepsilon}}, \check{d}_n)$ appearing in (53) feature both the spherical and

deviatoric parts of $\hat{\boldsymbol{\varepsilon}}$ and $\check{\boldsymbol{\varepsilon}}$, respectively. The two systems of (53) differ by the value of initial damage state $(\hat{d}_n, \check{d}_n)$ and also by the expressions of the functions $d(\hat{\boldsymbol{\varepsilon}}, \hat{d}_n)$ and $d(\check{\boldsymbol{\varepsilon}}, \check{d}_n)$ which could be any of the branches detailed in Table 1. Each system in (53) could have several solutions depending on the expressions of the functions $d(\hat{\boldsymbol{\varepsilon}}, \hat{d}_n)$ and $d(\check{\boldsymbol{\varepsilon}}, \check{d}_n)$, which have to be consistent with the chosen branches.

3.4.3. Choice of a nil polarisation

A further simplification is obtained by choosing a nil polarisation tensor, $\boldsymbol{\tau} = 0$. Such a choice of a linear comparison composite is equivalent to a modified secant approach with the particularity that two stationary points for the strain field are defined in the matrix phase. In this case the stationary condition (50_2) related to $\boldsymbol{\tau}$ can no longer be obtained. The system of equations to be solve in order to obtain the stationary points $\hat{\boldsymbol{\varepsilon}}$ and $\check{\boldsymbol{\varepsilon}}$ and the LCC coefficients D_κ and D_μ is made up of the equations (52) and (53) in which $\boldsymbol{\tau} = 0$:

$$\begin{cases} \Big(D_\kappa - d(\hat{\varepsilon}_m, \hat{\varepsilon}_{eq}, \hat{d}_n)\Big)\, \hat{\varepsilon}_m = 0 \\ \Big(D_\mu - d(\hat{\varepsilon}_m, \hat{\varepsilon}_{eq}, \hat{d}_n)\Big)\, \hat{\boldsymbol{e}} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \big(D_\kappa - d(\check{\varepsilon}_m, \check{\varepsilon}_{eq}, \check{d}_n)\big)\, \check{\varepsilon}_m = 0 \\ \big(D_\mu - d(\check{\varepsilon}_m, \check{\varepsilon}_{eq}, \check{d}_n)\big)\, \check{\boldsymbol{e}} = 0 \end{cases} \tag{54}$$

Again, we emphasise that the functions $d(\hat{\varepsilon}_m, \hat{\varepsilon}_{eq}, \hat{d}_n)$ and $d(\check{\varepsilon}_m, \check{\varepsilon}_{eq}, \check{d}_n)$ are multi-branches functions and different expressions could be used, independently, in each one of these systems. Therefore these systems have several solutions and some choices have to be done.

Although several stationary points are considered for the establishment of stationary equations, such as for the second order approach, the fact that there is no stationary condition related to $\boldsymbol{\tau}$ makes our approach incompatible with one of the main features of second order approaches: the approach presented here is not exact to second-order in the phase contrasts. Notwithstanding, the choice consisting of a nil polarisation in the matrix and two components only for the stiffness tensor of the matrix in the LCC makes possible to determine altogether the unknowns of the LCC which precisely are these two components. In that extent, the proposed estimate is fully optimised, providing a strong enough simplification. We will see in Section 4.2 that these simplifications are relevant to the problem under consideration.

3.4.4. Update of the damage parameters

The last question regards the update of the damage parameters in each sub-volumes at the end of the time step. It should be obtained from (18) by replacing the energy $w_e(\boldsymbol{\varepsilon})$ with its average over this sub-volume. This average is expressed in terms of the second order isotropic moments of the strain field in this sub-volume. By comparison with the average relation and according to the relations (52) which give the second order isotropic moments in the matrix phase of the LCC, $(\hat{\boldsymbol{\varepsilon}}_m)^2$ and $(\hat{\boldsymbol{\varepsilon}}_{eq})^2$ will be identified with the second order isotropic moments over the sub-volume $\hat{V}^{(1)}$ (recall that $\hat{\alpha}$ is the volume fraction of this sub-volume in the matrix phase). That's why the estimate of the damage parameter in this sub-volume at the end of the time step reads:

$$\hat{d}_{n+1} = \begin{cases} 1, \quad \text{if} \quad w_e(\hat{\boldsymbol{\varepsilon}}) \geqslant w_e^+ \\ 1+\gamma-\sqrt{\frac{k_Y\gamma(1+\gamma)}{w_e(\hat{\boldsymbol{\varepsilon}})}}), \quad \text{if} \quad w_e^-(\hat{d}_n) \leqslant w_e(\hat{\boldsymbol{\varepsilon}}) \leqslant w_e^+ \\ \hat{d}_n, \quad \text{if } w_e(\hat{\boldsymbol{\varepsilon}}) \leq w_e^-(\hat{d}_n) \end{cases} \tag{55}$$

Similar considerations allow to define $\check{d}_{n+1}$ using the stationary point $\check{\boldsymbol{\varepsilon}}$ in subdomain $\check{V}^{(1)}$.

3.5. Final remarks

About the non-convexity of the incremental potential : As explained in Ponte-Castañeda (2016), the error functions, depending on the strain field, are not convex, although the linear and nonlinear potentials are. Here, the fact that, as explained in Section 2, the incremental potential is non convex does not therefore raise additional difficulties. As a result, the error function defined in Equation (31) remains non-convex so that, as in Ponte-Castañeda (2016), the stationary points are not necessarily minima for the error function, but only its local extrema. However, the non-convex character of the incremental potential probably needs to increase the number of possible stationary points.

About the linearisation procedure : The proposed linearisation approach in the previous sub-section corresponds to the modified secant approach as the polarisation is $\boldsymbol{\tau} = 0$. A more simple choice of the LCC can be done by considering $D_\mu = D_\kappa$. However, as later discussed in Section 4.2, only the choice $D_\mu \neq D_\kappa$ yields the expected predictions.

4. Application: an isotropic solid damaged by a dispersion of swelling inclusions

In this Section, the general framework presented in Section 3 is applied to a particular problem: the inclusions are subjected to an isotropic and monotonous increasing swelling $\varepsilon_0^{(2)}(t)\boldsymbol{\delta}$. In addition, the RVE is stress-free at its boundary. The adaptation of the general framework presented above to this problem sets up a particular "stationary estimate" meaning that it is suitable for the particular problem considered in this paper.

It is remarked here that the random distribution of the inclusions inside the matrix, together with the fact that the loading consists in the inclusions swelling, induce multiple seeds of damage onset. Davidge and Green (1968) carried out an experimental study of this question by considering a dispersion of spheres (ceramic) in a matrix made of glass. During the cooling of the two materials, linking radial cracks have been formed between the spheres. They are due to differences in the expansion coefficients of the two constituents. In parallel, Miehe et al. (2015) performed two-dimensional simulations of this problem with a phase-field approach and obtained this star-like structure of cracks (see Figure 20-b in this reference). Therefore, the damage will appear first in the vicinity of the inclusions and propagates to create a network that eventually leads to a matrix that cannot resist anymore to the swelling prescribed. This network spreads over the entire volume of the RVE, making this particular problem more suitable for homogenisation approach.

In parallel, inclusion swelling triggers a tensile in the matrix which leads ultimately to its total failure. After the failure of the matrix, the RVE swelling is the same as the swelling prescribed. This remark will prove itself important to check whether the proposed solution in this section features the correct ultimate behaviour.

4.1. A particular stationary estimate

In general, the systems (54) coupled with the conditions (52) have to be solved numerically which would make the procedure burdensome if all the branches of the functions $d(\hat{\varepsilon}_m, \hat{\varepsilon}_{eq}, \hat{d}_n)$ and $d(\check{\varepsilon}_m, \check{\varepsilon}_{eq}, \check{d}_n)$ are tested. In this section a particular approach, adapted to the problem under consideration, is made up of two subsequent steps and leads to simple analytical model.

The first step consists in choosing a stationary point purely spherical tensor and the

other purely deviatoric. More precisely, the purely spherical stationary point is associated with the sub-volume $\hat{V}^{(1)}$ and the equation for the stationary spherical tensor $\hat{\boldsymbol{\varepsilon}} = \hat{\varepsilon}_m \boldsymbol{\delta}$ in the sub-volume $\hat{V}^{(1)}$ becomes

$$D_\kappa - d(\hat{\varepsilon}_m, \hat{d}_n) = 0, \tag{56}$$

with

$$d(\hat{\varepsilon}_m, \hat{d}_n) = \begin{cases} 1, \quad \text{if} \quad |\hat{\varepsilon}_m| \geqslant \dfrac{\xi_m}{\gamma} \\ 1 + \gamma - \frac{\xi_m}{|\hat{\varepsilon}_m|}, \quad \text{if} \quad \frac{\xi_m}{1+\gamma-\hat{d}_n} \leqslant |\hat{\varepsilon}_m| \leqslant \frac{\xi_m}{\gamma} \quad \text{and} \quad \xi_m = \sqrt{\dfrac{2k_Y\gamma(1+\gamma)}{9\kappa^{(1)}}} \\ \hat{d}_n, \quad \text{if } |\hat{\varepsilon}_m| < \dfrac{\xi_m}{1+\gamma-\hat{d}_n} \end{cases} \tag{57}$$

The purely deviatoric stationary point $\check{\boldsymbol{\varepsilon}}$ is associated with the sub-volume $\check{V}^{(1)}$. Then, $\check{\boldsymbol{\varepsilon}} = \check{\boldsymbol{e}}$ and the equation for the stationary point becomes :

$$D_\mu - d(\check{\varepsilon}_{eq}, \check{d}_n) = 0 \tag{58}$$

with:

$$d(\check{\varepsilon}_{eq}, \check{d}_n) = \begin{cases} 1, \quad \text{if} \quad \check{\varepsilon}_{eq} \geqslant \dfrac{\xi_d}{\gamma} \\ 1 + \gamma - \frac{\xi_d}{\check{\varepsilon}_{eq}}, \quad \text{if} \quad \frac{\xi_d}{1+\gamma-\check{d}_n} \leqslant \check{\varepsilon}_{eq} \leqslant \frac{\xi_d}{\gamma} \quad \text{and} \quad \xi_d = \sqrt{\dfrac{2k_Y\,\gamma\,(1+\gamma)}{3\,\mu^{(1)}}} \\ \check{d}_n, \quad \text{if } \check{\varepsilon}_{eq} < \dfrac{\xi_d}{1+\gamma-\check{d}_n} \end{cases} \tag{59}$$

The equations (56) and (58) can be solved for each branch of (57) and (59). **The second step** consists in choosing the damage evolution branch in each sub-volume. In particular, making use of the second branch of (57) and (59) (i.e. the damage increases in the two sub-volumes) yields:

$$\frac{9}{2}\kappa^{(1)}\hat{\varepsilon}_m^2 = \frac{k_Y\,\gamma\,(1+\gamma)}{\Big(1+\gamma-D_\kappa\Big)^2} \quad \text{and} \quad \frac{3}{2}\mu^{(1)}\check{\varepsilon}_{eq}^2 = \frac{k_Y\,\gamma\,(1+\gamma)}{\Big(1+\gamma-D_\mu\Big)^2}. \tag{60}$$

These solutions hold provided that $\hat{d}_n < d(\hat{\varepsilon}_m, \hat{d}_n) \leq 1$ and $\check{d}_n < d(\check{\varepsilon}_{eq}, \check{d}_n) \leq 1$.

Conversely, if the third branch of (59) is used (i.e. the damage increases in the sub-volume $\check{V}^{(1)}$ but not in the sub-volume $\check{V}^{(1)}$), then $D_\mu = \check{d}_n$ and the second former equation must be replaced by an inequality:

$$0 \leq \frac{3}{2}\mu^{(1)}\check{\varepsilon}_{eq}^2 < \frac{k_Y\,\gamma\,(1+\gamma)}{\Big(1+\gamma-\check{d}_n\Big)^2}. \tag{61}$$

This inequation will prove itself useful when the propagation solution shall be developed further down on the section.

In parallel, the **"outer" stationary** conditions (52) reduce to:

$$< \varepsilon_m^2 >_{LCC}^{(1)} = \hat{\alpha}\, \hat{\varepsilon}_m^2 \quad \text{and} \quad < \varepsilon_{eq}^2 >_{LCC}^{(1)} = \check{\alpha}\, \check{\varepsilon}_{eq}^2 \tag{62}$$

since the stationary conditions involving the first order moments are no longer obtained. As explained above, this is due to the fact that the polarisation $\boldsymbol{\tau}$ was defined as zero in the matrix. We recall that the weighting coefficients $\hat{\alpha}$ and $\check{\alpha}$ follow the relations: $\hat{\alpha} + \check{\alpha} = 1$, $0 \leq \hat{\alpha} \leq 1$ and $0 \leq \check{\alpha} \leq 1$.

Injecting the inner conditions (60) in the outer ones (62) gives the final relations:

$$\frac{9}{2}\kappa^{(1)} < \varepsilon_m^2 >_{LCC}^{(1)} = \hat{\alpha}\, \frac{k_Y\, \gamma\, (1+\gamma)}{\Big(1+\gamma-D_\kappa\Big)^2} \quad \text{and} \quad \frac{3}{2}\mu^{(1)} < \varepsilon_{eq}^2 >_{LCC}^{(1)} = \check{\alpha}\, \frac{k_Y\, \gamma\, (1+\gamma)}{\Big(1+\gamma-D_\mu\Big)^2}. \tag{63}$$

Alternatively, if the inner conditions (61) are met, (62) yields:

$$\frac{9}{2}\kappa^{(1)} < \varepsilon_m^2 >_{LCC}^{(1)} = \hat{\alpha}\, \frac{k_Y\, \gamma\, (1+\gamma)}{\Big(1+\gamma-D_\kappa\Big)^2} \quad \text{and} \quad 0 \leq \frac{3}{2}\mu^{(1)} < \varepsilon_{eq}^2 >_{LCC}^{(1)} < \check{\alpha}\, \frac{k_Y\, \gamma\, (1+\gamma)}{\Big(1+\gamma-\check{d}_n\Big)^2} \tag{64}$$

The second order strain moments in the matrix phase of the LCC have been derived in the Appendix A (relations (A.10) and (A.11)). These moments read:

$$\begin{aligned} < \varepsilon_m^2 >_{LCC}^{(1)} &= \left(\frac{4\, c^{(2)}\, (1-D_\mu)\, \mu^{(1)}}{\beta(D_\mu, D_\kappa)}\, \varepsilon_0^{(2)} \right)^2 \\ < \varepsilon_{eq}^2 >_{LCC}^{(1)} &= c^{(2)} \left(\frac{6\, (1-D_\kappa)\, \kappa^{(1)}}{\beta(D_\mu, D_\kappa)}\, \varepsilon_0^{(2)} \right)^2 \end{aligned} \tag{65}$$

with:

$$\beta(D_\mu, D_\kappa) = 3\, \kappa^{(1)}\, (1-D_\kappa) + 4\, \mu^{(1)}\, (1-D_\mu)\, ((1-D_\kappa)\, c^{(1)}\, \eta + c^{(2)}). \tag{66}$$

($\eta = \kappa^{(1)}/\kappa^{(2)}$ denotes the contrast on the bulk moduli of the two phases). In a similar way, the macroscopic strain is given by the following relation, based on (A.7) in Appendix A:

$$\overline{\varepsilon}_m = \frac{c^{(2)} \Big(3\, \kappa^{(1)}\, (1-D_\kappa) + 4\, \mu^{(1)}\, (1-D_\mu) \Big)\, \varepsilon_0^{(2)}}{\beta(D_\mu, D_\kappa)} \tag{67}$$

We evidence in the following that the specific form for the stationary points we have considered so far leads to an evolution for the unknowns $(\hat{\varepsilon}_m, \check{\varepsilon}_{eq}, D_k, D_\mu)$, with respect to the prescribed inclusion swelling, consistent with all the previous conditions.

4.1.1. Initiation

At the onset of damage, namely: $D_\mu = D_\kappa = 0$, the values of the stationary points can be estimated with Equation (60) and turn out to be identical to the second order moments of the strain field (63). The index c refers to these critical values of stationary points:

$$\frac{9}{2}\,\kappa^{(1)}\,\hat{\varepsilon}^2_{m(c)} = \frac{k_Y\,\gamma}{1+\gamma} \quad \text{and} \quad \frac{3}{2}\,\mu^{(1)}\check{\varepsilon}^2_{eq(c)} = \frac{k_Y\,\gamma}{1+\gamma} \tag{68}$$

Injecting the outer stationary conditions (62) in (68) above yields:

$$\frac{9}{2}\,\kappa^{(1)}\,<\varepsilon^2_m>^{(1)}_{LCC} = \frac{\hat{\alpha} k_Y\,\gamma}{1+\gamma} \quad \text{and} \quad \frac{3}{2}\,\mu^{(1)}<\varepsilon^2_{eq}>^{(1)}_{LCC} = \frac{\check{\alpha}\,k_Y\,\gamma}{1+\gamma} \tag{69}$$

The combination of the expressions of the strain moments as a function of the loading (65) and this last relation (69) yields two possible values for the critical prescribed stress-free strain. The one related to the first stationary point reads:

$$\hat{\varepsilon}^{(2)}_{0(c)} = \frac{1}{c^{(2)}}\frac{\beta(0,0)}{4\,\mu^{(1)}}\sqrt{\frac{2\hat{\alpha}\,k_Y\,\gamma}{9\,(1+\gamma)\,\kappa^{(1)}}}. \tag{70}$$

while, the critical prescribed stress-free strain associated to the second stationary point is:

$$\check{\varepsilon}^{(2)}_{0(c)} = \frac{\beta(0,0)}{6\,\kappa^{(1)}}\sqrt{\frac{2\check{\alpha}\,k_Y\gamma}{3\,c^{(2)}\,(1+\gamma)\,\mu^{(1)}}} \tag{71}$$

In the last two equations $\beta(0,0)$ is the corresponding value obtained from (66) for $D_\mu = D_\kappa = 0$, namely: $\beta(0,0) = 3\,\kappa^{(1)} + 4\,\mu^{(1)}\,(c^{(1)}\,\eta + c^{(2)})$. Next, it is possible to determine a value for the two weighting coefficients $(\hat{\alpha}, \check{\alpha})$ that could be set in such a way that the two critical prescribed stress-free strains are equal (i.e. simultaneous onset of the damage in the two sub-volumes), namely:

$$\check{\varepsilon}^{(2)}_{0(c)} = \hat{\varepsilon}^{(2)}_{0(c)} = \varepsilon^{(2)}_{0(c)} \quad \Rightarrow \quad \frac{\hat{\alpha}}{\check{\alpha}} = \frac{4\,c^{(2)}\,\mu^{(1)}}{3\,\kappa^{(1)}}, \tag{72}$$

and since $\hat{\alpha} + \check{\alpha} = 1$, it gives:

$$\hat{\alpha} = \frac{4\,c^{(2)}}{3\frac{\kappa^{(1)}}{\mu^{(1)}} + 4\,c^{(2)}} \tag{73}$$

which is obviously consistent with the constraint $0 \leq \hat{\alpha} \leq 1$. The corresponding critical prescribed stress-free strain is:

$$\varepsilon^{(2)}_{0(c)} = \frac{\sqrt{2}}{3\sqrt{c^{(2)}}}\frac{\beta(0,0)/(4\,\mu^{(1)})}{\sqrt{\kappa^{(1)}\,(c^{(2)} + \frac{3\,\kappa^{(1)}}{4\,\mu^{(1)}})}}\sqrt{\frac{k_Y\gamma}{1+\gamma}} \tag{74}$$

while the macroscopic strain is the one of a purely elastic solid, namely:

$$\overline{\varepsilon}_{m(c)} = c^{(2)} \left(\frac{3\,\kappa^{(1)} + 4\,\mu^{(1)}}{\beta(0,0)} \right) \varepsilon_{0(c)}^{(2)} \tag{75}$$

4.1.2. Propagation

The two systems (57) and (59) are both divided in three branches. It is possible to choose first, for those two systems, the branch corresponding to the increase of the two damage variables (D_κ, D_μ). However, this solution described in detail in Appendix B does not yield satisfactory predictions. As explained in this appendix, this particular branch is equivalent to the choice of one unique stationary point in the matrix (i.e. one unique damage parameter $D = D_\kappa = D_\mu$), which shows a posteriori the interest of choosing two sub-volumes (i.e. two stationary points) in the matrix. Given the symmetry of the loading studied in this work, it is more relevant to allow only the evolution of the damage variable D_κ which is related to the bulk modulus of the matrix phase in the sub-volume $\hat{V}^{(1)}$; the sub-volume $\check{V}^{(1)}$ remaining purely elastic. So, the pair $(D_\mu = 0, 0 < D_\kappa \leq 1)$ is considered.

In that situation and according to the combination of the outer stationary conditions (62) with the expression of the second order moments (A.10), the second stationary point is simply given by the normalised form:

$$\frac{\check{\epsilon}_{eq}}{\check{\epsilon}_{eq(c)}} = \frac{\beta(0,0)\,(1 - D_\kappa)}{\beta(0, D_\kappa)} \frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}} \tag{76}$$

In order to be valid, the expression (76) must satisfy the consistency condition (61), which can be written as, for $D_\mu = 0$:

$$0 \leq \frac{3}{2}\,\mu^{(1)}\,\check{\epsilon}_{eq}^2 < \frac{k_Y\,\gamma}{1+\gamma} = \frac{3}{2}\,\mu^{(1)}\,\check{\epsilon}_{eq(c)}^2 \quad \Leftrightarrow \quad 0 \leq \check{\epsilon}_{eq}^2 < \check{\epsilon}_{eq(c)}^2. \tag{77}$$

This consistency condition is satisfied whenever D_κ increases as the stationary point $\check{\epsilon}_{eq}$, as defined by (76), can be shown to be a decreasing function of D_κ namely:

$$\frac{d}{d\,D_\kappa}\left(\frac{\check{\epsilon}_{eq}}{\check{\epsilon}_{eq(c)}}\right) = -\frac{4\,\mu^{(1)}\,c^{(2)}\,\beta(0,0)}{\beta^2(0,D_\kappa)} \frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}} < 0$$

while $\check{\epsilon}_{eq}(D_\kappa = 0) = \check{\epsilon}_{eq(c)}$ and $\check{\epsilon}_{eq}(D_\kappa = 1) = 0$

For a given prescribed stress-free strain $\varepsilon_0^{(2)}$, the only unknown is the scalar $D_\kappa \in [\hat{d}_n; 1]$. Combining the first relation of (A.10) with the first relation of (63), the following equation

is obtained:

$$\frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}} = \Big(\frac{1+\gamma}{1+\gamma-D_\kappa}\Big)\frac{\beta(0,D_\kappa)}{\beta(0,0)}, \tag{78}$$

whose solution is:

$$D_\kappa = \frac{(1+\gamma)\,\beta(0,0)\left(\frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}}-1\right)}{\beta(0,0)\,\frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}} - (1+\gamma)\,(3\,\kappa^{(1)}+4\,\mu^{(1)}\,c^{(1)}\,\eta)}. \tag{79}$$

The full solution includes at once the Equation (79) for the evolution of D_κ and the expression of the macroscopic response, which can be recast from (A.7) (see Appendix A) into the following form:

$$\frac{\overline{\varepsilon}_m}{\overline{\varepsilon}_{m(c)}} = \frac{\beta(0,0)}{\beta(0,D_\kappa)}\Big(\frac{3\,\kappa^{(1)}\,(1-D_\kappa)+4\,\mu^{(1)}}{3\,\kappa^{(1)}+4\,\mu^{(1)}}\Big)\,\frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}} \tag{80}$$

with $\overline{\varepsilon}_{m(c)}$ the macroscopic strain at the initiation of damage as given by the Relation (75).

Consistency of solution (79). The solution described in Relation (79) is valid if the damage parameter D_κ increases from 0 (initiation) to 1 (fully damaged matrix) with the prescribed stress-free strain.

First, we remark that the initial zero damage is recovered by Relation (79) at the initiation point, $\frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}} = 1$. In addition, by taking its derivative with respect to the normalised stress-free strain $\frac{\varepsilon_0^{(2)}}{\varepsilon_{0(c)}^{(2)}}$, the expression (79) of the damage variable D_κ as a function of the prescribed stress-free strain can be shown to increase if and only if:

$$\frac{1+\gamma}{\gamma}\Big(\frac{4\,\mu^{(1)}\,c^{(2)}}{\beta(0,0)}\Big) > 1 \tag{81}$$

In that situation, the fully damaged state $D_\kappa = 1$ is reached when the prescribed stress-free strain attains a limit value, denoted by $\varepsilon_{0(l)}^{(2)}$ and defined by:

$$\varepsilon_{0(l)}^{(2)} = \varepsilon_{0(c)}^{(2)}\left(\frac{1+\gamma}{\gamma}\Big(\frac{4\,\mu^{(1)}\,c^{(2)}}{\beta(0,0)}\Big)\right) \tag{82}$$

As explained in Appendix C, this inequality (81) may be viewed as a limit for the energy that can be dissipated through damage by the material. Its left term corresponds to the ratio between the dissipated energy through damage and the relaxed stored elastic energy within

the material at the onset of damage, between the sound and damaged state, referred to as $\mathcal{D}/\mathcal{W}_e$. As a matter of fact, it is not possible to store more elastic energy than the RVE can dissipate through damage. As illustrated in the next Section 4.2, this limit is dependent on the values of the materials parameters. The condition (81) ensures that the energy balance inequality holds true.

If the condition (81) is not satisfied ($\mathcal{D} \leq \mathcal{W}_e$), the propagation branch (79) is not valid. As demonstrated below, the only possible solution at $\varepsilon_0^{(2)} = \varepsilon_{0(c)}^{(2)+}$ is then $D_\kappa = 1$ which corresponds to a discontinuous evolution of the damage variable D_κ. In that situation, the limit stress-free strain $\varepsilon_{0(l)}^{(2)}$ equals $\varepsilon_{0(c)}^{(2)}$.

4.1.3. Fully damaged state

When the prescribed stress-free strain is larger than the limit $\varepsilon_{0(l)}^{(2)}$ ($D_\kappa = 1$), the inner condition (60) related to the first stationary point now becomes:

$$\varepsilon_0^{(2)} > \varepsilon_{0(l)}^{(2)} : \quad \frac{9}{2}\kappa^{(1)}\hat{\varepsilon}_m^2(\varepsilon_0^{(2)}) \geq \frac{k_Y\,(1+\gamma)}{\gamma} \tag{83}$$

while the inner condition related to the second stationary point remains the inequality (77).

The second order moments given by (A.10) and (A.11) reported in Appendix A are now written as:

$$\varepsilon_0^{(2)} > \varepsilon_{0(l)}^{(2)} : \quad <\varepsilon_m^2>_{LCC}^{(1)} = \left(\varepsilon_0^{(2)}\right)^2 \quad \text{and} \quad <\varepsilon_{eq}^2>_{LCC}^{(1)} = 0 \tag{84}$$

so that the outer stationary conditions (62) now reduce to:

$$\varepsilon_0^{(2)} > \varepsilon_{0(l)}^{(2)} : \quad \sqrt{\hat{\alpha}}\,\hat{\varepsilon}_m(\varepsilon_0^{(2)}) = \varepsilon_0^{(2)} \quad \text{and} \quad \check{\varepsilon}_{eq}(\varepsilon_0^{(2)}) = 0 \tag{85}$$

(these evolutions of the two stationary points ($\hat{\varepsilon}_m$, $\check{\varepsilon}_{eq}$) are shown to be consistent with the inner stationary conditions (83) and (77) in Appendix D). Finally, the macroscopic strain is simply given by Relation (67) with $D_\mu = 0$ and $D_\kappa = 1$, namely: $\overline{\varepsilon}_m = \varepsilon_0^{(2)}$ which is consistent with the expected result.

4.1.4. Final remarks

We conclude this section with the following three remarks:

1. Heterogeneity contrast: for the particular loading under consideration, Mori and Tanaka (1973) estimates of the first and second moments of the strain field in the matrix (see

relations (A.10) and (A.11)) do not depend on the inclusions shear modulus $\mu^{(2)}$. As a result, the composite answer depends only on the bulk modulus of the inclusions through the contrast $\eta = \kappa^{(1)}/\kappa^{(2)}$.

2. More general loadings: the loading prescribed to the inclusions was monotonous increasing in this work. As a consequence, it is possible to obtain an analytical and explicit expression for the evolution of the damage versus the loading prescribed to the inclusions. This evolution is available for all of the three subsequent stages: the elastic stage I ($D_\kappa = 0$), the propagation of damage II (Equation ((79)) and the stage of the fully damaged matrix III ($D_\kappa = 1$). As discussed above, the behaviour jumps from I to III whenever the elastic energy stored by the matrix cannot be dissipated through damage. For a more complex loading, it is required to compute the values of the two damage variables (D_μ, D_κ) for each time step $[t_n;\, t_{n+1}]$.
3. Volume fraction of the matrix sub-volumes: according to the Relation (73) and illustrated by the curve plotted in Figure 1, $\hat{\alpha}$ is an increasing function of the volume fraction $c^{(2)}$. It accounts for the fact that the more the inclusions, the wider the zone of the matrix sharing a contact interface with the inclusions. Thus, the damage zone is larger.

   However, the values of $\hat{\alpha}$ and $\check{\alpha}$ can be chosen arbitrarily, provided that $\hat{\alpha} + \check{\alpha} = 1$. It is noteworthy that the choice of $\hat{\alpha}$ and $\check{\alpha}$ affects both of the values of the critical loadings (70) and (71). Therefore, the ratio between the critical prescribed stress-free strain corresponding to the onset of damage in each matrix sub-volumes $\hat{V}$ and $\check{V}$:

$$\frac{\hat{\varepsilon}^{(2)}_{0(c)}}{\check{\varepsilon}^{(2)}_{0(c)}} = \sqrt{\frac{\hat{\alpha}}{\check{\alpha}}}\sqrt{\frac{3\,\kappa^{(1)}}{4\,c^{(2)}\,\mu^{(1)}}}$$

   is an increasing function of $\hat{\alpha}$ and the value (73) of $\hat{\alpha}$ corresponds to a ratio $\frac{\hat{\varepsilon}^{(2)}_{0(c)}}{\check{\varepsilon}^{(2)}_{0(c)}}$ equal to 1. If $\hat{\alpha}$ is taken less than this value, the critical prescribed stress-free strain $\hat{\varepsilon}^{(2)}_{0(c)}$ associated to the stationary point in the damaged matrix is less than the critical prescribed stress-free strain $\check{\varepsilon}^{(2)}_{0(c)}$ associated to the stationary point in the non-damaged matrix. Therefore the onset of damage will take place firstly in the domain $\hat{V}^{(1)}$ and the approach presented in this paper can still be done. On the contrary, if $\hat{\alpha}$ is greater than the value (73), the onset of damage will take place firstly in the domain $\check{V}^{(1)}$ and

it will affect the shear modulus. This is a different evolution of the damage which is not investigated in this paper.

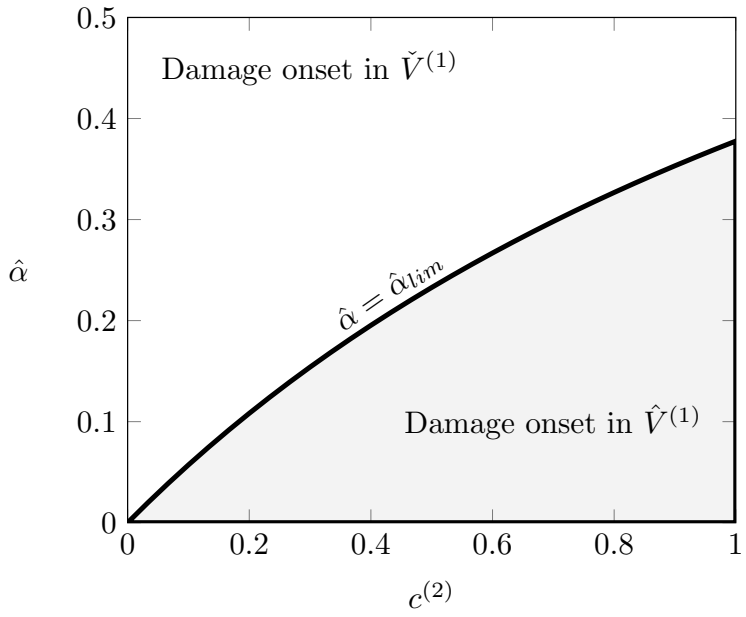

Figure 1: Plot of $\hat{\alpha}$ versus the volume fraction of inclusion ($\kappa^{(1)}/\mu^{(1)} \approx 2.2$). $\hat{\alpha} = \hat{\alpha}_{lim}$ corresponds to a ratio $\hat{\varepsilon}^{(2)}_{0(c)}/\check{\varepsilon}^{(2)}_{0(c)}$ equal to 1.

4.2. Numerical results

In this last section, the particular stationary estimate defined in 4.1 is applied to the considered problem, an isotropic solid with damage induced by a dispersion of swelling inclusions. With this model, the onset of the damage is given by (70) with the constraint (72) on the volume fraction of the damaged volume $\hat{\alpha}$. The subsequent evolution of the damage parameter and the macroscopic strain response have been established previously.

The dependency of the damage evolution in the matrix and the macroscopic response on the parameters of the model, such as the volume fraction of inclusions or the softening coefficient γ, is assessed.

4.2.1. Outlook of the macroscopic response

The model prediction of the evolution of the purely spherical macroscopic strain response $\bar{\varepsilon}_m$ as a function of the stress-free strain $\varepsilon_0^{(2)}$ prescribed to the inclusions can be seen on

	$\kappa^{(i)}$	$\mu^{(i)}$	$c^{(i)}$	γ	k_Y
$i = 1$ (matrix)	$\kappa^{(1)} = 167$ GPa	76 GPa	0,8	0.03	$3,935 \times 10^{-4}$J/m^3
$i = 2$ (inclusions)	$\kappa^{(2)} = \kappa^{(1)}/\eta$		0,2		

Table 2: General material parameters. The composite answer does not depend on the inclusions shear modulus. As a result its value is not reported. The heterogeneity contrast η will be fixed hereafter.

Figure 2. The three subsequent stages described in Figure 2 are correctly predicted by this first model. The first one stands for the elastic stage where the damage in the matrix is zero. Then, the second stage is reached as soon as the prescribed swelling reaches the critical value $\varepsilon^{(2)}_{0(c)}$. Afterwards, the RVE features a linear behaviour with a slope value depending on the material parameters whose influence will be assessed in the following subsection. Ultimately, the matrix fails and the swelling of the RVE is then the same as the swelling prescribed. From that moment on, there is no more a tensile in the matrix which can't accommodate anymore the swelling of the inclusions. As explained above, it is absolutely necessary to define two

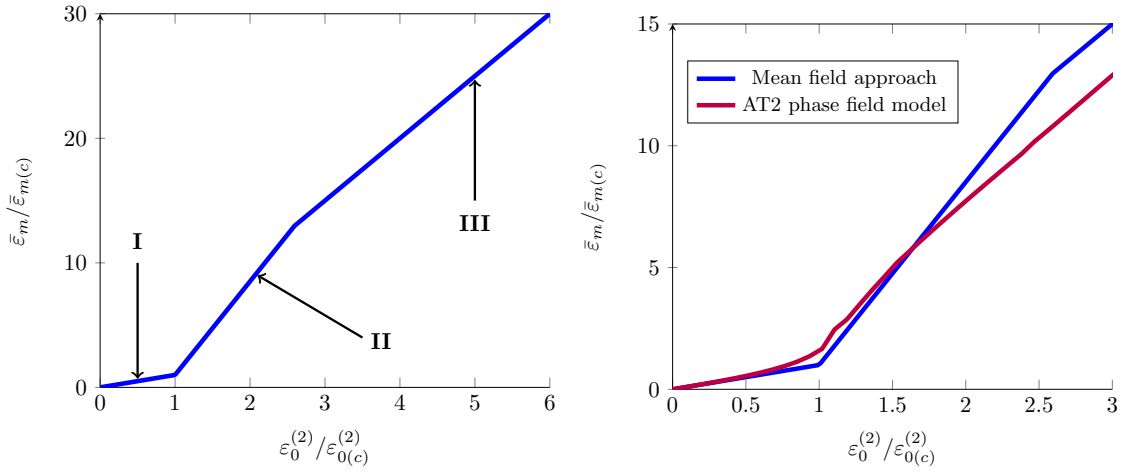

Figure 2: **Left**: Plot of the purely spherical macroscopic strain (from (79)) versus the swelling prescribed to the inclusions. **I**: Purely elastic; **II**: Damage propagation; **III**: Fully damaged. The prescribed swelling and the macroscopic strain are normalized by their values at the onset of damage. **Right**: Comparison of the macroscopic response for the mean field approach and a AT2 phase field approach adapted in Gauthier et al. (2022)

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stationary points (i.e. two sub-volumes) and to choose different evolutions for the damage

parameters of these two sub-volumes to obtain such satisfactory results (i.e. choosing a single stationary point which is equivalent to a modified secant approach does not reproduce stage III). Interestingly, explicit expressions of the critical values of the unique loading parameter (swelling of the inclusions) related to the onset of damage (Relation (74)) and the failure of the matrix phase (Relation (82) when the stored elastic energy is lower than the energy the RVE can dissipate through damage) are also given as a function of the data. The effect of the material parameters on the macroscopic response are studied in the next subsections. If these material parameters are chosen such that the stored elastic energy exceeds the energy that the RVE can dissipate, we will see that the curve displayed in Figure 2 is discontinuous (no more stage II, discontinuous transition from stage I to stage III).

Finally, the proposed mean field model is compared with full field computations performed with a model that differs from the local damage model presented in this work. Indeed, the latter still requires a regularisation, as explained in details in Lorentz and Benallal (2005). Nevertheless, other regularised damage models exist in literature, such as the phase field model that was adapted in Gauthier et al. (2022) for the problem of a heterogeneous microstructure. In order to compare the predictions of the present mean field model with the results of full field computations using the phase field model, the two models must be "correlated". This is done by choosing a set of material parameters defined in table 2, which makes that the local damage model used in this work displays the same critical swelling for the onset of damage as that in Gauthier et al. (2022). A complete description of the methodology that resulted in the plot of the macroscopic response in Figure 2 with the phase field model may be found in Gauthier et al. (2022). Then, the comparison between the two approaches, regarding the macroscopic response, points out that the presence of the three subsequent stages, with their corresponding slopes, seems to be the distinguishing feature of the macroscopic response of a microstructure subjected to a differential swelling.

4.2.2. Influence of the softening coefficient γ (dissipated energy)

The material parameters for this subsection are given in table 2, save the value of γ that is modified while the heterogeneity contrast is fixed to the unit ($\eta = 1$). The parameter γ is related to the dissipated energy in the sense that an increasing value of γ reduces the

dissipated energy (see Lorentz and Benallal (2005)). As a consequence, the ratio between the dissipated energy and the stored elastic energy at the onset of damage ($\mathcal{D}/\mathcal{W}_e$, see Relation (81)) is a decreasing function of γ. Therefore, beyond a critical value, $\gamma = 0.081$ for the considered data, the dissipated energy $\mathcal{D}$ will be lower than the stored elastic one $\mathcal{W}_e$ so that, as explained in Section 4.1, the time evolution of the damage parameter D_κ will present a discontinuity.

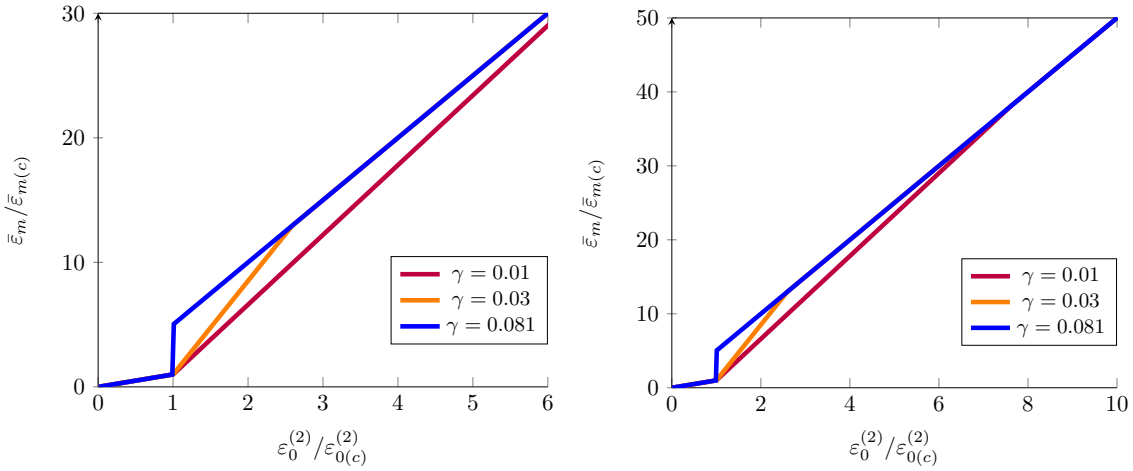

Figure 3: Plot of the macroscopic strain (80) versus the swelling prescribed, for three distinct values of γ related to the dissipated energy. **Left**: Low swellings. **Right**: High swellings. The prescribed swelling and the macroscopic strain are normalized by their values at the onset of damage. The material data are given in table 2 save the value of γ.

The existence of this critical value for γ can be clearly observed on the curves representing the normalised effective strain versus the normalised prescribed swelling to the inclusions (see Figure 3). First of all, the closer γ gets to its critical value (making the left term in (81) equal to 1), the higher the slope value becomes. In addition and as expected, when γ reaches its critical value, a discontinuity may be observed at the onset of damage. This entails D_κ being instantaneously set to 1 after the onset of damage, see Figure 4.

4.2.3. Influence of the inclusions volume fraction $c^{(2)}$

The material parameters are given in table 2, save the value of $c^{(2)}$, which is now modified while the heterogeneity contrast is still fixed to the unit ($\eta = 1$). In Figure 5-left, the diminution of the volume fraction of inclusions implies that the onset of damage occurs later.

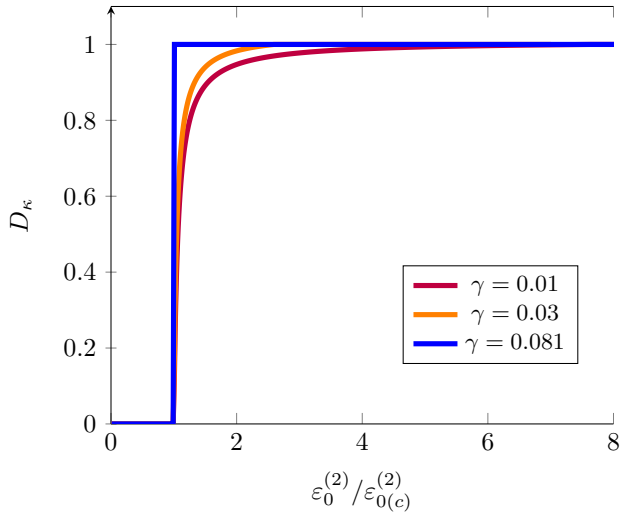

Figure 4: Plot of the damage evolution (from (79)) versus the swelling prescribed, for three distinct values of γ. The prescribed swelling is normalized by its value at the onset of damage. The material data are given in table 2 save the value of γ.

Indeed, for less inclusions but the same prescribed swelling, the strain elastic energy is lower and then it is required to reach a higher value of swelling to trigger the damage in the matrix. That statement can be sustained by the observation of Figure 5-right where, for a given swelling prescribed, the damage is lower for a lower fraction of inclusions. Concerning the propagation, Relation (81) shows that the ratio $\mathcal{D}/\mathcal{W}_e$ between the dissipated energy and the stored elastic energy at the onset of damage is an increasing function of the volume fraction of inclusions, $c^{(2)}$. Consequently, the slopes related to the propagation phase on Figure 5 decrease with the volume fraction of inclusions while there exists a limit value of the volume fraction of inclusions, equal to 0.077 for the considered data, below which the effective response and the damage parameter feature a discontinuity.

4.2.4. Influence of the heterogeneity contrast $\eta = \kappa^{(1)}/\kappa^{(2)}$

The material parameters are given in table 2 and we now study the effect of the heterogeneity contrast $\eta = \kappa^{(1)}/\kappa^{(2)}$ on the composite answer. As plotted in Figure 6, the contrast has a strong influence on the onset and propagation of damage. Indeed, for an infinite contrast that corresponds to rigid inclusions ($\kappa^{(2)} \to 0$), the critical swelling value is minimum and equals $3,9 \times 10^{-4}$ while a vanishing contrast (porous inclusions) leads to an infinite value

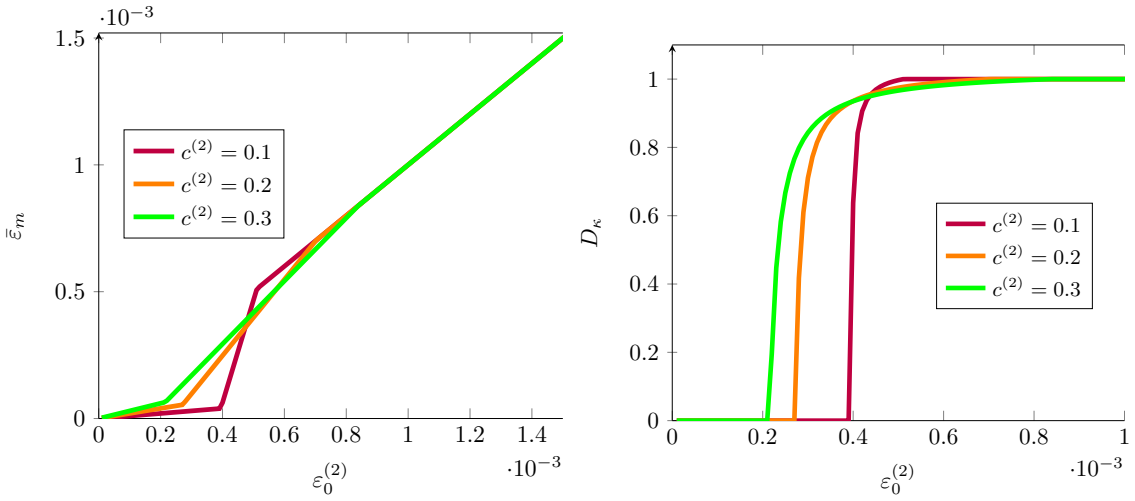

Figure 5: Evolutions of the macroscopic strain (80) and of damage (79) as a function of the swelling prescribed, for three distinct values of inclusions volume fraction $c^{(2)}$. The material data are given in table 2 save the value of $c^{(2)}$.

for the critical swelling, which is equivalent to stating that the inclusions exert no stresses to the matrix.

Regarding the propagation of damage, it is highlighted on the right plot in Figure 6 that a higher contrast may cause an unstable propagation of damage that is depicted by a discontinuity of the macroscopic response. The limit value of η for the unstable damage propagation is yielded by (81) and equals 5,3.

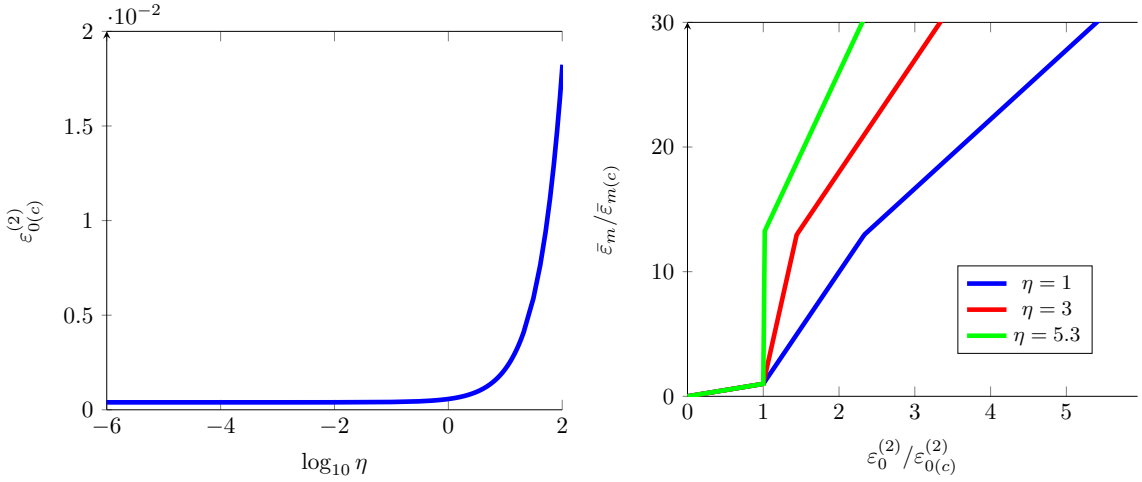

Figure 6: **Left**: Evolution of the critical swelling versus the elastic contrast macroscopic strain (74). **Right**: Evolution of macroscopic strain (80) as a function of the swelling prescribed, for three values of contrast.

5. Conclusions, future works

The work presented in this paper represents a first contribution (as far as the authors know) of the establishment of a micromechanical damage model at the intersection between the damage mechanics and the nonlinear homogenisation.

The damage model stems from the seminal work of Lorentz and Benallal (2005). The proposed approach in this paper first defines an elastic nonlinear homogenisation problem based on an incremental potential like in Lahellec and Suquet (2007), which is here non convex because of the softening behaviour in the matrix. The matrix is split in two zones corresponding to two damage states which may evolve differently. On the other hand, our approach makes use of a variational estimate inspired by the FOSO approach reported in (Ponte-Castañeda (2016)). However, we choose to narrow down the number of degrees of freedom by setting the polarisation in the matrix to zero and by considering an isotropic stiffness tensor in the LCC. Those simplifications are consistent with the two zones approach exposed in this work. *In fine*, the retained linearisation method is like a secant modified approach but with two stationary points. This approach regularises the incremental potential and defines a well-posed problem as the resulting LCC is convex.

The abilities of the general theory are assessed through a case test: the study of the effective response of a composite, stress free at the boundary, with an elastic fragile matrix reinforced with inclusions subjected to a monotonous increasing isotropic swelling. The corresponding macroscopic response displays the expected three subsequent stages, in accordance with the damage model adapted in this work. Those stages stand for the elastic answer, the propagation of damage within the matrix and, eventually, the matrix fully broken. It is imperative to define (at least) two stationary points (i.e. at least two sub-volumes) and to choose different evolutions for the damage parameters of these two sub-volumes to obtain such satisfactory results. Furthermore, the dependency of the macroscopic response on physical parameters, such as toughness or volume fraction of inclusions is provided. This particular loading allows us to derive the solution analytically. These analytical developments demonstrate that the model provides an answer which may be discontinuous to ensure the consistency with the balance between the initial stored elastic energy and the final dissipated one. Notwithstanding, future work should be carried out for more complex loadings, such as

non isotropic loadings (traction, shear, ...), in order to back up the choices expressed in this work such as the volume fraction of the matrix involved in the damage evolution (the $\hat{\alpha}$ parameter). Another challenge would be to consider non-monotonic loadings, which will require the use of an expression of the stored elastic energy differentiating traction and compression.

Acknowledgements

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Appendix A. Analytical expression of the effective behaviour and second order moments for a two phases isotropic linear composite

The relations laid out in this Appendix stem from the relations in section 3.2 and are useful to set up the results in section 4. Thus, the isotropic stiffness tensors are given by:

$$\boldsymbol{C} = 3\kappa\boldsymbol{J} + 2\mu\boldsymbol{K} \quad \text{and} \quad \boldsymbol{C}^{(2)} = 3\kappa^{(2)}\boldsymbol{J} + 2\mu^{(2)}\boldsymbol{K} \tag{A.1}$$

while the prescribed loading consists in a stress-free strain $\boldsymbol{\varepsilon}_0^{(2)} = \varepsilon_{0m}^{(2)}\boldsymbol{\delta}$ in the inclusions, and a stress free on the boundary of the RVE: $\boldsymbol{\sigma} = 0$. Furthermore, there is a nil polarisation in the matrix: $\tau_m = 0$ and the polarisation and the stress-free strain in the inclusion are related through:

$$\tau_m^{(2)} = -3\kappa^{(2)}\varepsilon_{0m}^{(2)} \tag{A.2}$$

First, let us remind an important and general result: the expression of the localisation tensors $\boldsymbol{A}^{(r)}$ and $\boldsymbol{a}^{(r)}$ in the case of a two phases composites Levin (1967). Both of these relations involve the effective stiffness tensor $\tilde{\boldsymbol{C}}$:

$$\begin{cases} \boldsymbol{A}^{(r)} = \boldsymbol{I} + \frac{1}{c^{(r)}}\left(\boldsymbol{C}^{(r)} - \boldsymbol{C}^{(s)}\right)^{-1} : \left(\tilde{\boldsymbol{C}} - \left\langle \boldsymbol{C} \right\rangle\right) \\ \boldsymbol{a}^{(r)} = \frac{1}{c^{(r)}}\left(\boldsymbol{C}^{(r)} - \boldsymbol{C}^{(s)}\right)^{-1} : \left(\tilde{\boldsymbol{C}} - \left\langle \boldsymbol{C} \right\rangle\right) : \left(\boldsymbol{C}^{(r)} - \boldsymbol{C}^{(s)}\right)^{-1} : \left(\boldsymbol{\tau}^{(r)} - \boldsymbol{\tau}^{(s)}\right) \end{cases} \tag{A.3}$$

for $r \neq s,\ r,s \in \{1,2\}$ and where $\langle \boldsymbol{C} \rangle = c^{(1)}\boldsymbol{C} + c^{(2)}\boldsymbol{C}^{(2)}$.

In this article, our choice inclines towards the Mori and Tanaka (1973) scheme to estimate the effective stiffness tensor. Furthermore, considering the symmetry of our problem (isotropic loading and isotropic behavior of the phases), only the component along $\boldsymbol{J}$ of $\tilde{\boldsymbol{C}}$ is useful to us. Plus, only the inclusion phase is thermoelastic. The retained expression of effective bulk modulus is:

$$\tilde{\kappa}_{MT} = \frac{4\mu < \kappa > +3\,\kappa\,\kappa^{(2)}}{4\mu + 3\,\kappa\,\kappa^{(2)} < \frac{1}{\kappa} >} \tag{A.4}$$

with $< \kappa >= c^{(1)}\kappa + c^{(2)}\kappa^{(2)}$ and $< \frac{1}{\kappa} >= c^{(1)}\frac{1}{\kappa} + c^{(2)}\frac{1}{\kappa^{(2)}}$.

Likewise, the symmetry of the loading entails a simplification of the form of the localization tensors which are now isotropic and "spherical". That is, given the general form $\boldsymbol{A}^{(r)} = A_m^{(r)}\boldsymbol{J} + A_d^{(r)}\boldsymbol{K}$ and $\boldsymbol{a}^{(r)} = a_m^{(r)}\boldsymbol{\delta} + \boldsymbol{a}_d^{(r)}$, $r \in [1,2]$, only remains $A_m^{(r)}$ and $a_m^{(r)}$.

The combination of (A.4) and the first relation in (A.3) yields:

$$A_m^{(2)} = \frac{3\kappa + 4\mu}{4\mu + 3\,\kappa\,\kappa^{(2)} < \frac{1}{\kappa} >} \tag{A.5}$$

while the combination of (A.4), the second relation in (A.3) and of (A.2) yields:

$$a_m^{(2)} = \frac{3\,c^{(1)}\,\kappa^{(2)}\varepsilon_{0m}^{(2)}}{4\mu + 3\,\kappa\,\kappa^{(2)} < \frac{1}{\kappa} >} \tag{A.6}$$

Once the effective polarisation estimated from the second equation in (35), the effective stress-free strain (spherical given the prescribed loading), is derived from the second equation in (36):

$$\tilde{\varepsilon}_{0m} = \kappa^{(2)}\frac{c^{(2)}\left(3\,\kappa + 4\,\mu\right)\varepsilon_{0m}^{(2)}}{4\mu < \kappa > +3\,\kappa\,\kappa^{(2)}} \tag{A.7}$$

Then, the combination of the localisation tensors, (A.5) and (A.6) and the expression of the energy (37) gives the following form:

$$\begin{aligned}\widetilde{w}(\overline{\varepsilon}_m, \varepsilon_{0m}^{(2)}) = \quad & \frac{9}{2}\frac{1}{4\mu + 3\,\kappa\,\kappa^{(2)} < \frac{1}{\kappa} >}\Big[\big(4\mu < \kappa > +3\,\kappa\,\kappa^{(2)}\big)\,\overline{\varepsilon}_m^2 \\ & -2c^{(2)}\kappa^{(2)}\big(3\kappa + 4\mu\big)\varepsilon_{0m}^{(2)}\overline{\varepsilon}_m - 3c^{(1)}c^{(2)}(\kappa^{(2)}\,\varepsilon_{0m}^{(2)})^2\Big]\end{aligned} \tag{A.8}$$

Given that the stress vanishes on the RVE boundary, the first equation of (36) yields that:

$$\overline{\varepsilon}_m = \tilde{\varepsilon}_{0m} \tag{A.9}$$

In this case, the second order moments based on relations (41) expand the following way:

$$< \varepsilon_m^2 >_{LCC}^{(1)} = \frac{2}{9c^{(1)}} \frac{\partial \widetilde{w}}{\partial \kappa}(\tilde{\varepsilon}_{0m}, \varepsilon_{0m}^{(2)}) = \left(\frac{4\, c^{(2)}\, \kappa^{(2)}\, \mu}{3\kappa\, \kappa^{(2)} + 4\mu < \kappa >} \varepsilon_{0m}^{(2)} \right)^2 \tag{A.10}$$

$$< \varepsilon_{eq}^2 >_{LCC}^{(1)} = \frac{2}{3c^{(1)}} \frac{\partial \widetilde{w}}{\partial \mu}(\tilde{\varepsilon}_{0m}, \varepsilon_{0m}^{(2)}) = c^{(2)} \left(\frac{6\ \kappa\kappa^{(2)}}{3\kappa\kappa^{(2)} + 4\mu < \kappa >} \varepsilon_{0m}^{(2)} \right)^2 \tag{A.11}$$

In parallel, the stored energy (A.8) becomes:

$$\widetilde{w}(\tilde{\varepsilon}_{0m}, \varepsilon_{0m}^{(2)}) = \frac{9}{2} c^{(2)} \kappa^{(2)} (\varepsilon_{0m}^{(2)})^2 \left[\frac{4c^{(1)} \mu\kappa}{3\kappa\kappa^{(2)} + 4\mu < \kappa >} - 1 \right] \tag{A.12}$$

which can be turned equivalently into:

$$\widetilde{w}(\tilde{\varepsilon}_{0m}, \varepsilon_{0m}^{(2)}) = -\frac{9}{2} \frac{3\kappa + 4c^{(2)}\mu}{3\kappa\kappa^{(2)} + 4\mu < \kappa >} c^{(2)} \left(\kappa^{(2)} \varepsilon_{0m}^{(2)} \right)^2 \tag{A.13}$$

Appendix B. Propagation of damage for two increasing damage variables

When both of the damage variables increase within the matrix during the propagation phase, we explain here why the corresponding estimates are not relevant.

Injecting relations (A.10) in (63), we obtain after some algebraic simplifications the following system of two nonlinear equations:

$$\begin{aligned} 3\sqrt{8}\, c^{(2)} (1 - D_\mu) \left(1 + \gamma - D_\kappa \right) |\varepsilon_0^{(2)}| &= \frac{\beta(D_\mu, D_\kappa)}{\mu^{(1)} \sqrt{\kappa^{(1)}}} \sqrt{\hat{\alpha}\, k\, \gamma\, (1+\gamma)} \\ 3\sqrt{6\, c^{(2)}} (1 - D_\kappa) \left(1 + \gamma - D_\mu \right) |\varepsilon_0^{(2)}| &= \frac{\beta(D_\mu, D_\kappa)}{\kappa^{(1)} \sqrt{\mu^{(1)}}} \sqrt{\check{\alpha}\, k\, \gamma\, (1+\gamma)} \end{aligned} \tag{B.1}$$

where the pair of unknowns (D_μ, D_κ) have to be searched in the interval $[\check{d}_n; 1] \times [\hat{d}_n; 1]$, for a given prescribed stress-free strain $\varepsilon_0^{(2)}$. As the two weighting coefficients ($\hat{\alpha}$, $\check{\alpha}$) satisfy the relation (72), this system of equations is in fact equivalent to:

$$(1 - D_\mu) \left(1 + \gamma - D_\kappa \right) |\varepsilon_0^{(2)}| = (1 - D_\kappa) \left(1 + \gamma - D_\mu \right) |\varepsilon_0^{(2)}| = \mathfrak{C}\, \beta(D_\mu, D_\kappa) \tag{B.2}$$

with the right member $\mathfrak{C}$ defined by:

$$\mathfrak{C} = \frac{\sqrt{\hat{\alpha}\, k\, \gamma\, (1+\gamma)}}{3\, c^{(2)}\, \mu^{(1)} \sqrt{8\, \kappa^{(1)}}} = \frac{\sqrt{\check{\alpha}\, k\, \gamma\, (1+\gamma)}}{3\, \kappa^{(1)}\, \sqrt{6\, c^{(2)}\, \mu^{(1)}}} \tag{B.3}$$

From these last relations we can deduce that, for a non zero swelling of the inclusions, the two solutions (D_μ, D_κ) must satisfy:

$$(1-D_\mu)\left(1+\gamma-D_\kappa\right)=(1-D_\kappa)\left(1+\gamma-D_\mu\right),$$

relation that is satisfied if and only if $D_\mu = D_\kappa = D$, the unique unknown D being solution of the following polynomial equation of second degree:

$$(1-D)\left(1+\gamma-D\right)|\varepsilon_0^{(2)}|=\frac{\sqrt{\hat{\alpha}\,k\,\gamma\,(1+\gamma)}}{3\,c^{(2)}\,\mu^{(1)}\sqrt{8\,\kappa^{(1)}}}\,\beta(D,\,D). \tag{B.4}$$

If from the beginning, the matrix of the LCC would have been defined with only one damage variable, namely $\boldsymbol{C}=(1-D):\boldsymbol{C}^{(1)}$ (i.e. one single stationary point or equivalently no sub-volumes in the matrix), the outer stationary condition (63) would have been unique and would have been written:

$$\frac{9}{2}\kappa^{(1)}<\varepsilon_m^2>_{LCC}^{(1)}+\frac{3}{2}\mu^{(1)}<\varepsilon_{eq}^2>_{LCC}^{(1)}=\frac{k_Y\,\gamma\,(1+\gamma)}{\left(1+\gamma-D\right)^2} \tag{B.5}$$

while the inner stationary conditions (56) and (58) would be also replaced by a unique relation:

$$D-d(\boldsymbol{\varepsilon},d_n)=0 \tag{B.6}$$

This solution is similar to a modified secant approach and can be easily shown to be equivalent to the particular branch considered in this appendix.

Unfortunately, this solution does not yield the correct slope in the limit of the fully damaged state. Therefore, starting from the general relation (67) in the particular situation where ($D_\mu = D_\kappa = D$), the overall strain reads:

$$\overline{\varepsilon}_m=\frac{c^{(2)}\left(3\,\kappa^{(1)}+4\,\mu^{(1)}\right)\varepsilon_0^{(2)}}{3\,\kappa^{(1)}+4\,\mu^{(1)}\left(c^{(1)}\,(1-D)\,\eta+c^{(2)}\right)}\xrightarrow{D\to 1}\frac{c^{(2)}\left(3\,\kappa^{(1)}+4\,\mu^{(1)}\right)\varepsilon_0^{(2)}}{3\,\kappa^{(1)}+4\,c^{(2)}\,\mu^{(1)}}\neq 1 \tag{B.7}$$

This branch does not yield satisfactory solutions.

Appendix C. Interpretation of the inequality (81) as an energy balance

It is noteworthy to account for the inequality in (81) with an energy balance. More precisely, one aims at comparing the variation of the stored elastic energy at $\varepsilon_{0(c)}^{(2)}$ between $D_\kappa = 0$ and $D_\kappa = 1$ with the energy dissipated by damage propagation.

According to (A.12), the elastic energy stored for $\varepsilon^{(2)}_{0(c)}$ prescribed and a nil damage value ($\kappa = \kappa^{(1)}$) can be written as:

$$\widetilde{w}_I(\tilde{\varepsilon}_{0(c)}, \varepsilon^{(2)}_{0(c)})V = \frac{9}{2}c^{(2)}\kappa^{(1)}(\varepsilon^{(2)}_{0(c)})^2\left[\frac{4c^{(1)}\mu^{(1)}}{\beta(0,0)} - \frac{\kappa^{(2)}}{\kappa^{(1)}}\right]V \tag{C.1}$$

while for $D_\kappa = 1$ ($\kappa = 0$) and the same loading prescribed:

$$\widetilde{w}_F(\tilde{\varepsilon}_{0(c)}, \varepsilon^{(2)}_{0(c)})V = -\frac{9}{2}c^{(2)}\kappa^{(2)}(\varepsilon^{(2)}_{0(c)})^2 V \tag{C.2}$$

The energy balance, for a dissipated energy by damage $\mathcal{D}$, and a relaxed stored elastic energy $\mathcal{W}_e$ can be written as:

$$\mathcal{W}_e < \mathcal{D} \quad \Leftrightarrow \quad \mathcal{D}/\mathcal{W}_e > 1 \tag{C.3}$$

with:

$$\mathcal{W}_e = -\left(\widetilde{w}_F(\tilde{\varepsilon}_{0(c)}, \varepsilon^{(2)}_{0(c)}) - \widetilde{w}_I(\tilde{\varepsilon}_{0(c)}, \varepsilon^{(2)}_{0(c)})\right)V = \frac{9}{2}c^{(2)}\kappa^{(1)}(\varepsilon^{(2)}_{0(c)})^2\frac{4c^{(1)}\mu^{(1)}}{\beta(0,0)}\,V \tag{C.4}$$

and:

$$\mathcal{D} = \hat{\alpha}Vc^{(1)}k_Y. \tag{C.5}$$

Injecting the expression of $\varepsilon^{(2)}_{0(c)}$ given in (70) in the expression of the relaxed stored elastic energy, one gets:

$$\mathcal{D}/\mathcal{W}_e = \frac{4c^{(2)}\mu^{(1)}}{\beta(0,0)}\frac{1+\gamma}{\gamma} > 1 \tag{C.6}$$

which is indeed equivalent to the condition (81).

Appendix D. Fully damaged state, consistency of the stationary points with the inner stationary conditions

When the prescribed stress-free strain exceeds the fully damaged state limit, $\varepsilon^{(2)}_{0(l)}$, the expressions of the stationary points with respect to the prescribed stress-free strain are given by relations (85). Are these expressions of the two stationary points consistent with their corresponding inner stationary conditions ?

Obviously it is for the second stationary point as given by relation (85)-right: as this stationary point $\check{\varepsilon}_{eq}$ is zero, the inequality (77) is satisfied. For the first stationary point $\hat{\varepsilon}_m$,

it is first remarked that it increases with the prescribed stress-free strain (see relation (85)). Therefore, the inner stationary conditions (83) are satisfied if:

$$\frac{9}{2}\kappa^{(1)}\hat{\varepsilon}_m^2(\varepsilon_{0(l)}^{(2)}) \geq \frac{k_Y\,(1+\gamma)}{\gamma}. \tag{D.1}$$

If $\mathcal{D} > \mathcal{W}_e$ (condition (81) is satisfied and the propagation branch does exist), the stationary point $\hat{\varepsilon}_m$ satisfies the following relation when the stress-free prescribed strain reaches the limit value $\varepsilon_{0(l)}^{(2)}$:

$$\frac{9}{2}\,\kappa^{(1)}\,\hat{\varepsilon}_m^2(\varepsilon_{0(l)}^{(2)}) = k_Y\,\frac{(1+\gamma)}{\gamma} \tag{D.2}$$

and then the condition (D.1) above is satisfied. Conversely, if $\mathcal{D} \leq \mathcal{W}_e$ (condition (81) is not satisfied), the limit stress-free strain $\varepsilon_{0(l)}^{(2)}$ equals $\varepsilon_{0(c)}^{(2)}$ so that:

$$\frac{9}{2}\,\kappa^{(1)}\,\hat{\varepsilon}_m^2(\varepsilon_{0(l)}^{(2)}) = \frac{9}{2}\,\kappa^{(1)}\,\hat{\varepsilon}_m^2(\varepsilon_{0(c)}^{(2)}) = k_Y\,\frac{\gamma}{1+\gamma}\,\underbrace{\left(\frac{\beta(0,0)}{4\,\mu^{(1)}\,c^{(2)}}\right)^2}_{>(\frac{1+\gamma}{\gamma})^2} > k_Y\,\frac{1+\gamma}{\gamma}. \tag{D.3}$$

and then the condition (D.1) above is also satisfied.

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