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Schelling paradox in a system of cities

Vincent Boitier* Emmanuel Auvray†

Abstract

In this theoretical article, we develop a unified framework that encapsulates: i)- a system of heterogeneous cities, ii)- heterogeneous workers, iii)- the presence of agglomeration economies and congestion forces, and iv)- heterogeneous city composition preferences. We provide a full analytical characterization of the decentralized and centralized economies. Contrary to conventional wisdom, we show that social mixing can constitute a unique and stable equilibrium. We also disentangle the different economic factors that shape within- and across-city social/income dispersion. We also find that individuals' preferences be overturned at an aggregate scale, which is consistent with Schelling paradoxical findings. We then offer a rationale for these counterintuitive results. Last, we demonstrate that the decentralized economy is not optimal but can be restored by local subsidies.

JEL Classification: R12, J15, C72 ,D62

Keywords: Social mixing, systems of cities, heterogeneous workers, asymmetric preferences, optimality, restoration of optimality

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1 Introduction

How does the assignment of heterogeneous agents across heterogeneous cities operate? Can social mixing constitute a unique and stable spatial equilibrium? Can individuals' preferences be overturned at an aggregate scale? If so, how to rationalize these paradoxical results à la Schelling? What economic factors drive within- and across-city inequalities? Are large towns more unequal than small ones? How to explain the incomplete sorting between attractiveness of sites, firms, and workers? Is a spatial equilibrium optimal?¹ If not, how to restore efficiency?

We attempt to address these questions in a comprehensive and unified framework. Toward that goal, we build a system of cities with the following key ingredients. The number of cities is finite, and cities are heterogeneous according to a level of amenities. In each city there is a monocentric city with a continuum of locations. Absentee landlords own houses that are supplied by a competitive market. Firms are exogenously located in the city center, do not consume any space, and reward employees for their work. The system is also populated by an infinite number of workers distributed across two populations (rich vs. poor, black vs. white, etc.). Workers are heterogeneous in terms of income and transportation costs. They also have heterogeneous preferences regarding city composition, as in Miyao (1978, 1979). This means that workers are engaged with (possibly complex) intra- and inter-group externalities.

In such a setting land is not allocated by the bid rent theory, and in the labor market wages are exogenously determined. A spatial equilibrium is a vector of spatial distributions that verifies a variational inequality. This characterization allows us to derive reasonable conditions for uniqueness and (static) stability of a spatial equilibrium (see Boitier (2020)). The within- and across-city income that causes a social dispersion is summarized by a single statistics corresponding to a dissimilarity index.

From this new setting we derive valuable results. First, we derive closed-form solutions for the endogenous variables. The spatial distributions of workers and the dissimilarity index are analytical in the decentralized and centralized economies. Having closed-form solutions is crucial and advantageous. This enables conducting a detailed and robust comparative statistical analysis. This also allows a transparent identification of the driving forces of the workers' location. Due to its tractability, the model can also be a guide for empirical studies by helping to derive testable predictions that link the obtained dissimilarity index to measurable variables.

Second, we demonstrate that, contrary to conventional wisdom, social mixing (i.e. incomplete sorting) can be a unique and stable spatial equilibrium. Moreover, we advocate

¹This general question encompasses the following subsequent questions: Are cities too big or too small? Are cities too segregated? Is the system of cities too segregated?

that this configuration prevails under a large set of plausible parameter restrictions. This means that the situation frequently emerges as an outcome that reconciles the spatial theory with empirical evidence.

Third, we unveil the determinants of within- and across-city inequalities. We also underline that social mixing is a consequence of three distinguishable components. The first component is a demographic component. It posits that social mixing is improved when a population of workers grows faster than the other. The second factor captures the degree of differences between cities in terms of attractiveness. This component predicts that a system of cities composed with similar (different) amenable cities exhibits low (high) social mixing. The third factor is the combination of income inequalities, transportation costs, and city composition preferences. It clearly shows that there is a relationship between segregation by income and segregation by preferences. In particular, depending on the intra- and inter- group externalities, city composition preferences can mitigate (magnify) income segregation.

Fourth, we find that macroeconomic patterns can deviate from microeconomic preferences. Notably, a society in which there is a desire for a mixed environment generates less social mixing than a society in which there is a preference for a segregated environment. In so doing, we obtain results in line with Schelling (1971, 1978). We then offer a rationale for these counterintuitive findings. We identify city composition preference acts as agglomeration economies and congestion effects in the urban model. In summary, the paradoxical results à la Schelling are intuitively explained by the standard forces present in urban economics.

Fifth, we show that the decentralized economy never achieves an optimal allocation. This is because workers disregard some local externalities compared to what the social planner internalizes. In addition, we investigate two different aspects of inefficiency. On the one hand, we ask whether cities are underpopulated or overpopulated. We highlight that this depends on the nature of the city composition preferences. When these preferences are symmetric, large towns are oversized and small cities are undersized. If the suitable degree of asymmetry is introduced, the converse situation prevails. Large towns are too small and small cities are too big. This counterintuitive result concurs with Albouy et al. (2019). On the other hand, we determine whether social mixing is too high or too low in the system of cities. Again, the answer relies on the feature of city composition preferences.

Last, we stress that economists have a remedy to restore optimality. To eliminate inefficiency, local subsidies are sufficient. These are standard instruments in public economics. We show that local actors can introduce local subsidies. If local subsidies are implemented by competitive local land developers, inefficiency vanishes. This concurs with the results of Henderson and Becker (2000) and Albouy et al. (2019). Local subsidies can also be managed by a central government.

This article contributes to urban economics from different perspectives. It improves the design of spatial models. This is the first attempt to take into account heterogeneous sites, heterogeneous workers, and city composition preferences. Behrens and Robert-Nicoud (2015b) and Albouy et al. (2019) consider a system of cities with heterogeneous sites and a continuum of heterogeneous workers. But there are no neighborhood composition preferences. In addition, to derive results they need to assume that land rent is independent of the workers' composition of the city. These drawbacks do not exist in our framework. Land rent depends on the composition of the city. In Schelling models of segregation (see Zhang (2004a), Zhang (2004b), Pans and Vriend (2007), O'Sullivan (2009), Zhang (2011), Grauwil et al. (2012) and Boustan (2013)), social preferences are operative but economic factors are absent. Labor market is neutralized. Transportation costs are not integrated. Land is not allocated according to the bid rent theory.

The present article states that a mixed configuration can be unique and stable. This finding is at odds with the literature. In the Schelling model of segregation it has been well advocated that segregation is the only stable equilibrium. In the case of systems of cities (see Abdel-Rahman and Anas (2004)), Seegert (2011), Behrens and Robert-Nicoud (2015a) and Albouy et al. (2019)), the literature predicts a strict sorting of workers, and there is no social mixing. However, this finding also accords with empirical evidence. It is very uncommon to observe a fully segregated system of cities in the data.

The article also offers paradoxical results à la Schelling. Notably, attractive (adverse) social preferences can generate less (more) social mixing. In fact, the framework provides a reason for these findings. It highlights standard agglomeration and congestion forces as credible rationales. This contrasts with the literature. In the standard Schelling model, social relationships are too bulky, precluding the possibility of obtaining closed-form results. The model is like a black box, and no robust comparative statistical analysis may be available. Consequently, it is difficult to establish what drives paradoxical findings in these standard models. More globally, the fact that our setting can be analytically solved is important. This permits a transparent identification of the driving forces of the workers' location. This allows disentangling the respective roles of demographic, spatial (natural amenities/disamenities), and economic factors. This also enables guiding empirical studies.

The article joins the debate of policy regulation. Particularly, at which level must an optimal policy be implemented? In standard systems of cities, efficiency is restored at the city-scale with the concurrence of local land developers. Such a policy is possible in our framework. Optimality can also be achieved by local subsidies that are managed by a non-local government.

To conclude, the article complements some results found in the literature. As in Boitier and Auvray (2020), there is a relationship between segregation by income and segregation by preference, and social preferences can mitigate income segregation. In line with Albouy

et al. (2019), large cities can be undersized, whereas small cities can be overpopulated.

The article is organized as follows. Section 2 presents the new framework. Section 3 provides the conclusions.

2 System of cities with city composition preferences

In what follows we develop a system of cities in which heterogeneous workers have preferences over city composition. In Section 2.1, we describe the setup. In Section 2.2, we characterize the spatial equilibrium within cities. In Sections 2.3-2.7, we analyze the spatial equilibrium of the system of cities. In Sections 2.8 and 2.9, we tackle the question of optimality.

2.1 Environment

2.1.1 Geography

We follow Boitier (2020) for the geography. The economy consists in a system of C cities. Cities are distributed on the interval $\mathcal{Y} = [0, M(C + 1)]$ with $M > 0$ in such a way that the city c 's location from 0 is $Mc \in \mathcal{Y}$ for all $c \in \mathcal{C} = \{1, \dots, C\}$. Each city is endowed with a level of amenities expressed by a . $a : \mathcal{C} \rightarrow [\underline{a}, \bar{a}]$ is a bounded and ordered application with $\underline{a}, \bar{a} \in \mathbb{R}$. By ordered, we mean $a_1 \geq \dots \geq a_c \geq \dots \geq a_C$. Moreover, each city has an internal structure in the sense that it is composed of a continuum of locations denoted by $x \in \mathcal{X}$ with $\mathcal{X} = [0, \infty)$. Cities are monocentric: $x = 0$ is the Business District (BD) where all firms are exogenously located. Accordingly, x also represents both the distance to BD and access to jobs.

2.1.2 Populations

The economy is populated by a mass M of heterogeneous workers distributed in two populations (e.g. rich vs. poor, whites vs. blacks, young vs. old, males vs. females).² In a given population $p \in \{r, b\}$ there are m_p homogeneous workers so that $m_r + m_b = M = 1$. Considering r (b), we mean that workers are red (blue).

2.1.3 Spatial distributions of workers

There are two notions of spatial distributions: the spatial distribution of workers across cities and the spatial distribution of workers within cities.

²Note that our framework is very general such that segregation/integration grasps different forms: income, age, gender, etc. This means that segregation/integration by ethnicity is a particular case in the present setting.

Workers belonging to population $p \in \{r, b\}$ are spatially distributed across cities according to $m_p \in \Delta_p^C$ so that:

$$\Delta_p^C = \{m_p = (m_{p,1}, \dots, m_{p,C}) \in \mathbb{R}_+^C : \sum_{c=1}^C m_{p,c} = m_p\} \quad \forall p \in \{r, b\}$$

where $m_{p,c}$ is the mass of workers belonging to population $p \in \{r, b\}$ and located in city $c \in \mathcal{C}$.

Based on the presence of $m_{r,c}$ red workers in city c and $m_{b,c}$ blue workers in city c , the workers are spatially distributed in this city following two endogenous distributions denoted by μ_r, μ_b so that:

$$\begin{cases} \mu_r : \mathcal{X} \rightarrow \mathbb{R}_+ \\ \int_{\mathcal{X}} \mu_r(x) dx = m_{r,c} \end{cases} \quad \begin{cases} \mu_b : \mathcal{X} \rightarrow \mathbb{R}_+ \\ \int_{\mathcal{X}} \mu_b(x) dx = m_{b,c} \end{cases}$$

$\mu_r(x)$ is the density of red employees residing in x and $\mu_b(x)$ is the density of blue workers residing in x .

2.1.4 Segregation index

In this article we demonstrate that a spatial equilibrium is characterized by a situation where workers are unevenly dispersed across cities. To gauge the magnitude of this dispersion, we need an appropriate measure. The literature provides different dimensions of residential segregation: evenness, exposure, clustering, centralization, and concentration (see Duncan and Duncan (1955), Massey and Denton (1988) and Massey et al. (1996)). Here, we focus on evenness by computing the standard dissimilarity index (see James and Taeuber (1985)):

$$I_D = \frac{1}{2} \sum_{c=1}^C \left| \frac{m_{r,c}}{m_r} - \frac{m_{b,c}}{m_b} \right|$$

I_D measures the proportion of red workers that move in order to have an even city. If $I_D = 0$, then red and blue workers are evenly distributed across cities. This case is labeled as “perfectly mixed” (see Figure 2.b). Conversely, if $I_D = 1$, then the spatial segregation is maximized, and red and blue workers do not share common cities. In this case, the system of cities is said to be “segregated” (see Figure 2.a). There is a continuum of configurations between these two polar opposite cases (i.e. mixed systems of cities, see Figure 2.c). In particular, $0 < I_D^1 < I_D^2 < 1$ means that the system of city 1 shows higher social mixing than the system of cities 2.

2.1.5 Workers' characteristics

Workers are associated with the same preferences. They face the same two-tier utility function denoted by \mathcal{U} and expressed by:³

$$\mathcal{U}(z, a, m) = U(z) + a + \rho(m)$$

with U being the utility function associated to the consumption of z the amount of composite consumer goods used as numéraire. ρ is another utility function that describes workers' city composition preferences (see Section 2.1.6 for an explicit expression). U is the lower-tier utility function, whereas $a + \rho(m)$ is the upper-tier utility function. As usual, U is of the form:

$$U(z) = z$$

that is, U is simply linear.

Moreover, workers belonging to population $p \in \{r, b\}$ are endowed with one unit of labor, earn a wage $\omega_p > 0$, bear transportation costs $0 \leq \tau < 1$, and pay a rent $R(x)$ per unit of land to absentee landlords.⁴ Commuting costs can have two similar interpretations. τ can be viewed as the fraction of time that a worker spends in transports. Equivalently, τ can correspond to the frequency of trips to the BD made by workers. Therefore, the budget constraint of workers is the following:

$$z + R(x) = \omega_p(1 - \tau x) \quad \forall p \in \{r, b\}$$

$\omega_p(1 - \tau x)$ can be viewed as “iceberg” transportation costs (see Murata and Thisse (2005) and Behrens and Murata (2009)). In that case, $U_p(x)$ the (instantaneous) utility of the workers belonging to population $p \in \{r, b\}$ and residing in location x becomes the following:

$$U_p(x) = \omega_p(1 - \tau x) - R(x) \quad \forall p \in \{r, b\}$$

Note that consumption must be positive in equilibrium and so $z_p = U_p > 0$ must be satisfied in equilibrium. Also note that, as workers are endowed with a single unit of land, land consumption is exogenous. Consequently, there is no income effect on land consumption. However, this is inconsequential as the income effect remains operative in the model through transportation costs $-\omega_r\tau$ and $-\omega_b\tau$. In addition, embedding the Cobb–Douglas function in the framework could be cumbersome.

³In spatial economics, it is standard to use a two-tier utility function when the setting is complex and encapsulates intertwined decisions. Behrens and Murata (2009) adopt a two-tier utility function in a monopolistic competition urban model. Arkolakis et al. (2012) consider a two-tier utility function in a trade model with multiple sectors.

⁴In the core of the text, ω_p for all $p \in \{r, b\}$ is exogenous. See our Online Appendix for a case where wages are derived from a Cobb–Douglas production function. This allows us to study the case where the economy has several industries. See also our Online Appendix for a discussion about the role of absentee landlords in the framework.

2.1.6 Workers' city composition preferences

We now specify the form of the utility function ρ . Workers have preferences with respect to city composition:

$$\rho_{r,c}(m) = -\phi_{rr}m_{r,c} - \phi_{rb}m_{b,c}$$

and

$$\rho_{b,c}(m) = -\phi_{br}m_{r,c} - \phi_{bb}m_{b,c}$$

with $\phi_{rr}, \phi_{rb}, \phi_{br}, \phi_{bb} \in \mathbb{R}$ being some parameters.

To understand this better, let us consider some concrete examples. If $\phi_{kj} = 0$, then workers of type k are neutral concerning the presence of workers of type j in the city. If $\phi_{kj} < 0$ ($\phi_{kj} > 0$), then workers of type k prefer (dislike) cities where workers of type j live, as any city inhabited by workers of type j corresponds to an increase (a decrease) in the utility function. If $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = 0$, then there are no city composition externalities. In that case, we say that workers are color-neutral. If $\phi_{rr}, \phi_{bb} > 0$ ($\phi_{rr}, \phi_{bb} < 0$), then the intra-group externalities are negative (positive), and we say that workers have aversion/rejection (attraction). If $\phi_{rb}, \phi_{br} > 0$ ($\phi_{rb}, \phi_{br} < 0$), then the inter-group externalities are negative (positive). In that case, we say that workers have mutual aversion/rejection (mutual attraction).

Note that the preferences are fairly standards. We strictly follow the seminal articles of Sakoda (1971), Zhang (2004a) and Grauwin et al. (2012) in assuming that workers linearly consider the proportion of workers when choosing their location. Therefore, the number of neighbors has a monotonic effect on workers' utility function, which differs from the tipping point theory (see Card et al. (2008)). The fact that neighborhood externalities are operative at the city scale is also standard. We strictly follow the seminal articles of Miyao (1978, 1979).

We use linear preferences for two reasons. On the one hand, the preferences have nice properties. See Section 2.3.2 for more details. On the other hand, the preferences are flexible as workers can be engaged with asymmetric intra- and inter-group local interactions. The parameters may be positive, null, or negative, and may also differ according to the populations. This encapsulates a wider range of possible social patterns than with other preferences. For example, other preferences (including the original Schelling utility function (see Schelling (1971, 1978)), asymmetric peak utility functions (see Zhang (2011)), etc.) capture a symmetric desire for a mixed environment only. However, the use of linear preferences has a negative side effect: city composition preferences depend on the level of red and blue workers. One may expect that the city dependance could be captured by the

percentage of workers. Linear preferences also preclude the framework to model “tipping points”. Empirical studies report that a population of households may leave neighborhoods after that another population of households overcomes a certain point (see Card et al. (2008) and Boustan (2013)). Unfortunately, including such an ingredient makes the model very bulky. This rules out uniqueness. This also eliminates the possibility of deriving explicit and robust results. See our Online Appendix for more details.

In addition, the preferences are empirically supported. The relevant literature has established that individuals’ preferences regarding the composition of the city play a significant role in determining housing choice (see, among others, Schuman et al. (1997), Krysan and Farley (2002), Charles (2003)). For example, several studies document that the ethnic characteristics of individuals and neighborhoods influence decisions on location and relocation. Van Ham and Feijten (2008) and Van Ham and Clark (2009) also show that socio-economic factors (e.g. income and education) play a role. In addition, several studies (see Ihlanfeldt and Scafidi (2002) and Clark (2009)) point out that the effects are cities specific. They also document that the geographical distribution of minorities is mixed, even assuming that residents are free to move in and out of cities.

As outlined previously, we are relatively uncertain about the values of the parameters. We do not impose any restriction. The parameters may be positive, null, or negative, and may also differ from population to population. However, we reckon that the following cases should be highlighted:⁵

1. $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = 0$
2. $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = \psi$ with $\psi > 0$
3. $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = -\psi$ with $\psi > 0$
4. $\phi_{rr} = \phi_{bb} = -\psi$ and $\phi_{rb} = \phi_{br} = \psi$ and $\psi > 0$
5. $\phi_{rr} = \phi_{bb} = \phi_{br} = 0$ and $\phi_{rb} = \psi$ with $\psi > 0$

The first case corresponds to the situation without any group population preference. The second case appears when preferences are symmetric, and when workers have both own and mutual aversion. The third case is the inverse of the second one, that is, workers have both own and mutual attraction. The fourth case is an asymmetric one in which workers have own attraction but mutual rejection.⁶ This is the inverse of the third case. It can be

⁵Alternatives are present in the Online Appendix.

⁶It is possible to consider a case in which there is own rejection and mutual attraction. This sounds reasonable if the two groups are males and females. Sakoda (1971) considers a similar case called Boy-Girl.

interpreted as a case where workers have now a preference for a segregated environment. The last case is a sub-case of the fourth one. In fact, this is the seminal case of Zhang (2004a) where a population A is color-neutral and a population B rejects population A . As before, workers have a desire for a segregated city.

2.1.7 Workers' utility functions and workers' choices

To sum up, plugging the expressions of U_p and $\rho_{p,c}$ into the two-tier utility functions gives:

$$\mathcal{U}_{r,c}(x, m_r, m_b) = \omega_r(1 - \tau x) - R(x) + a_c - \phi_{rr}m_{r,c} - \phi_{rb}m_{b,c}$$

and

$$\mathcal{U}_{b,c}(x, m_r, m_b) = \omega_b(1 - \tau x) - R(x) + a_c - \phi_{br}m_{r,c} - \phi_{bb}m_{b,c}$$

where \mathcal{U}_r and \mathcal{U}_b are the new notations of the utility functions of workers.

Under this environment, workers play the following two-step location game:

1. They choose in which city to live.
2. They choose where to reside in their selected city.

Usually, the game is solved by backward induction. In Stage 2, workers face the following optimization problems:

$$\max_{x \in \mathcal{X}} \{ \omega_r(1 - \tau x) - R(x) \}$$

and

$$\max_{x \in \mathcal{X}} \{ \omega_b(1 - \tau x) - R(x) \}$$

The results in Stage 2 are determined in Section 2.2. In Stage 1, workers share the following maximization programs:

$$\max_{c \in \mathcal{C}} \{ \omega_r(1 - \tau x^*) - R(x^*) + a_c - \phi_{rr}m_{r,c} - \phi_{rb}m_{b,c} \}$$

and

$$\max_{c \in \mathcal{C}} \{ \omega_b(1 - \tau x^*) - R(x^*) + a_c - \phi_{br}m_{r,c} - \phi_{bb}m_{b,c} \}$$

where $*$ denotes the optimal solution in Stage 2. The results in Stage 1 are pinned down in Sections 2.3-2.7.

2.2 Equilibrium in Stage 2

In this section, we characterize the spatial equilibrium within cities. In Section 2.2.1, we clarify the notion of a spatial equilibrium. Our analysis is based on standard bid rent models (see Fujita (1989) and Fujita and Thisse (2003) for comprehensive frameworks). In Section 2.2.2, we explicitly determine such a spatial equilibrium. We derive the closed-form solutions for μ_r^*, μ_b^* the internal densities of workers, R^* land rents, \check{x} the equilibrium city size and v_r, v_b the indirect utility functions of workers.

2.2.1 Definition

To pin down the location of households in the city, it is common to use the bid rent theory (see Fujita (1989)). The latter consists in determining the maximum rent that a worker (blue or red) would pay for living in a given location. The bid rent function of the red workers residing in location x and in city c is given by:

$$\Psi_{r,c}^*(x, v_{r,c}) = \omega_r(1 - \tau x) - v_{r,c}$$

Similarly, the bid rent function of the blue workers residing in x and in city c is expressed by:

$$\Psi_{b,c}^*(x, v_{b,c}) = \omega_b(1 - \tau x) - v_{b,c}$$

As a consequence, a spatial equilibrium in city $c \in \mathcal{C}$ is defined in the following manner:

Definition 1 *For a fixed city $c \in \mathcal{C}$, an equilibrium in Stage 2 is a 5-tuple $(\mu_{r,c}^*, \mu_{b,c}^*)$ and $(v_{r,c}, v_{b,c}, R_c^*(x))$ so that:*

$$R_c^*(x) = \max \left\{ \Psi_{r,c}^*(x, v_{r,c}), \Psi_{b,c}^*(x, v_{b,c}), 0 \right\} \quad (1)$$

with

$$\Psi_{p,c}^*(x, v_{p,c}) = \omega_p(1 - \tau x) - v_{p,c}, \forall p \in \{r, b\} \quad (2)$$

$$\mathcal{U}_{r,c}(x, m_r, m_b) = v_{r,c} \quad (3)$$

$$\mathcal{U}_{b,c}(x, m_r, m_b) = v_{b,c} \quad (4)$$

$$\mu_{r,c}^*(x) + \mu_{b,c}^*(x) = 1 \quad (5)$$

$$\int_{\text{Supp}(\mu_{r,c}^*)} \mu_{r,c}^*(x) dx = m_{r,c} \quad (6)$$

$$\int_{\text{Supp}(\mu_{b,c}^*)} \mu_{b,c}^*(x) dx = m_{b,c} \quad (7)$$

These equations are simple. Equation (1) indicates that land is allocated to the highest bid rent. Equation (2) represents the bid rent function of the workers belonging to population $p \in \{r, b\}$ and residing in location x . Equations (3)–(4) are the non-arbitrage conditions which show that all agents reach the same utility level in each population of workers. Equation (5) satisfies the land-market equilibrium, and equations (6)–(7) meet the total population constraint in each population of workers. Last, we assume that land intensity equals 1 and the agricultural rent is 0. These elements do not influence the nature of the results.

2.2.2 Results

We now determine the spatial equilibrium for Stage 2. We also derive the conditions under which such a spatial equilibrium exists. Using Definition 1, we obtain the following:

Proposition 1

Assume that $\omega_r > \omega_b$. Assume that the following holds:

$$\omega_r(1 - \tau m_r) - \omega_b \tau m_b > 0 \quad (8)$$

$$\omega_b [1 - \tau(m_r + m_b)] > 0 \quad (9)$$

then red workers live near their workplace while blue workers reside on the fringes of the city.⁷ The associated city size is pinned down by:

$$\tilde{x}_c = m_{r,c} + m_{b,c}$$

Similarly, the associated land rent is given by (see Figure 1.a):

$$R_c^*(x) = \begin{cases} \omega_r \tau m_{r,c} + \omega_b \tau m_{b,c} - \omega_r \tau x & \forall x \in [0, m_{r,c}] \\ \omega_b \tau m_{r,c} + \omega_b \tau m_{b,c} - \omega_b \tau x & \forall x \in [m_{r,c}, m_{r,c} + m_{b,c}] \end{cases}$$

Last, the associated indirect utility functions are:

$$v_{r,c} = \omega_r + a_c - (\omega_r \tau + \phi_{rr})m_{r,c} - (\omega_b \tau + \phi_{rb})m_{b,c}$$

and

$$v_{b,c} = \omega_b + a_c - (\omega_b \tau + \phi_{br})m_{r,c} - (\omega_b \tau + \phi_{bb})m_{b,c}$$

Assume that $\omega_r = \omega_b = \omega$. Assume that the following is complied:

$$\omega(1 - \tau) > 0$$

⁷In the rest of the article, we assume that $\omega_r \geq \omega_b$.

then red and blue workers live in the same locations, and the spatial equilibrium boils down to (see Figure 1.b):

$$\begin{aligned}
R_c^*(x) &= \omega\tau(m_c - x) \quad \forall x \in [0, m_c] \\
v_{r,c} &= \omega + a_c - (\omega\tau + \phi_{rr})m_{r,c} - (\omega\tau + \phi_{rb})m_{b,c} \\
v_{b,c} &= \omega + a_c - (\omega\tau + \phi_{br})m_{r,c} - (\omega\tau + \phi_{bb})m_{b,c}
\end{aligned}$$

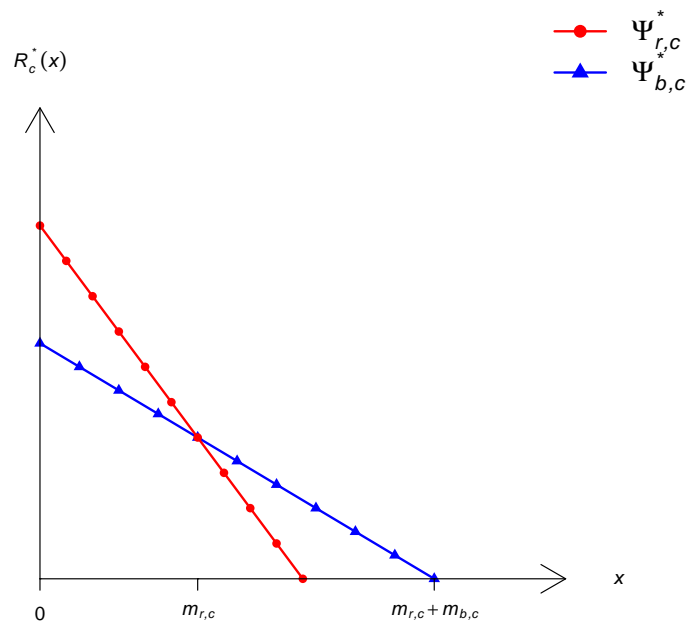
Sufficient conditions (8) and (9) ensure that consumption is positive in equilibrium, irrespective of the spatial distributions of workers. More precisely, equilibrium consumption in city $c \in \mathcal{C}$ for workers $p \in \{r, b\}$ is given by:

$$z_{p,c}^* = \omega_p - \omega_p\tau x^* - R_c^*(x^*)$$

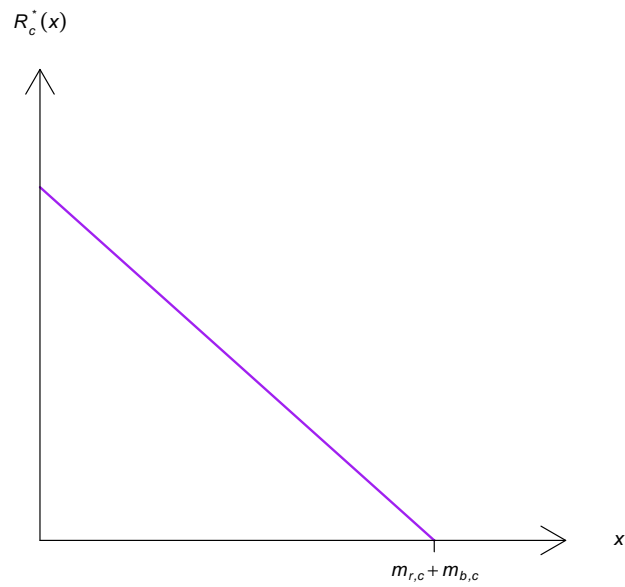
When $\omega_r > \omega_b$, Proposition 1 establishes that red workers live near their workplace, that is, on the interval $[0, m_{r,c}]$. Therefore, using R_c^* , equilibrium consumption for red workers becomes: $z_{r,c}^* = \omega_r(1 - \tau m_{r,c}) - \omega_b\tau m_{b,c}$. As $m_{b,c} \leq m_b$ and $m_{r,c} \leq m_r$, to ensure that consumption is positive in equilibrium, it is sufficient to set (8). A similar reasoning applies for $z_{b,c}^*$ which gives (9), and for the case $\omega_r = \omega_b$. Intuitively, conditions (8)-(9) simply stipulate that wages are high enough to overcome commuting costs and land rents whatever the city size.

As usual, R_c^* is a decreasing function with respect to x (see Figures 1.a and 1.b). This is because workers face a trade-off between accessibility and land prices, when choosing their location. They want to live in the BD to minimize their transportation costs. However, they also anticipate that more workers want to reside near the city center. This increases land prices. To avoid this, some workers have an incentive to live farther away. Specifically, R^* must decrease with respect to x to balance the two aforementioned forces.

The internal pattern depends on R_c^* , that is, on wages. When $\omega_r > \omega_b$, red workers live near their workplace, while blue workers reside on the fringes of the city (see Figure 1.a). The explanation for this is straightforward and standard. There are two conflicting forces in monocentric city models (see Alonso (1964), Mills (1967) and Muth (1969)). These two forces are summarized by the ratio: time transportation costs/housing consumption. If the ratio rises with incomes, rich workers live at central places as time commuting costs outweigh housing consumption benefits. By contrast, when the ratio falls with wages, rich workers live in the suburbs as housing consumption effects dominate time commuting costs. In our framework, land consumption is eliminated since workers consume a single unit of land. The location of workers only depends on time commuting costs. As red workers earn more than blue workers, they bear higher transportation costs. Consequently, they are ready to pay to be marginally close to the BD, and they outbid for locations that are close to the BD (see Figure 1.a). When incomes are identical (i.e., $\omega_r = \omega_b$), there is social mixing within cities. This means that red and blue workers inhabit in the same places. As



(a) Land rents when $\omega_r > \omega_b$



(b) Land rents when $\omega_r = \omega_b$

Figure 1

workers face the same transportation costs, they share the same incentives, and so locate in the same locations.

It is worth noting that the configuration obtained in Proposition 1 is consistent with real-world observations. Since Hohenberg and Lees (1986), scholars distinguish two different configurations. On the one hand, there is a pattern commonly encountered in US cities. Such a pattern is characterized by the fact that incomes rise with distance to BD. On the other hand, there is another pattern observed in European cities where incomes fall with distance to the city center. Our model predicts European configurations only.

Last, Proposition 1 derives the closed-form expressions for the indirect utility functions. This is key for the spatial equilibrium in Stage 1. To have a better understanding, we interpret how these utilities are affected by changes in wages and populations. For simplicity, we focus on the indirect utility function of red workers. A similar analysis can be done for blue workers. Eliminating the parameters ϕ , note the following:

$$v_{r,c} = \omega_r(1 - \tau x^*) + a_c - R^*(x^*) = \omega_r + a_c - \omega_r \tau m_{r,c} - \omega_b \tau m_{b,c}$$

We then operate a brief comparative statics analysis.

An increase in ω_r has two effects. There is a transportation costs effect and there is an income effect that transits through the land market. When red workers earn more, this mechanically increases commuting costs. In turn, this lowers consumption and so the utility function. In addition, as ω_r is improved, red workers are richer. As a consequence, they are ready to pay more for the unit of land. The bid rents of red workers in each location increase that rises land rents. The utility function decreases as a result.

When ω_r is improved, only the effect in the land market is operative. Land rents are positively affected as blue workers become wealthier.

An increase in $m_{r,c}$ has two effects. There is an expansion effect and there is a competition effect. First, when more red workers inhabit city $c \in \mathcal{C}$, the city expands and $[0, m_{r,c}]$ increases. This prompts some red worker to live farther away, and so experience higher commuting costs. Second, an increase in $m_{r,c}$ also means more competition in the land market. This puts an upward pressure on land rents, and $v_{r,c}$ diminishes as an outcome.

Likewise, an increase in $m_{b,c}$ generates higher land rents that decreases the utility function of red workers.

2.3 Equilibrium in Stage 1

Hereafter, we clarify the notion of a spatial equilibrium in the system of cities. We then provide a discussion about the forms of the indirect utility functions. Last, we offer a review of the results in Stage 1.

2.3.1 Definition

A spatial equilibrium in Stage 1 is defined as follows:

Definition 2 A spatial distribution $(m_r^*, m_b^*) \in \Delta_r^C \times \Delta_b^C$ is a spatial equilibrium in Stage 1 if:

$$\omega_r - (\omega_r\tau + \phi_{rr})m_{r,c}^* - (\omega_b\tau + \phi_{rb})m_{b,c}^* = v_r^* \quad (10)$$

and

$$\omega_b - (\omega_b\tau + \phi_{br})m_{r,c}^* - (\omega_b\tau + \phi_{bb})m_{b,c}^* = v_b^* \quad (11)$$

with $v_r^*, v_b^* \in \mathbb{R}$ being some constants.

An equilibrium is a situation where each worker receives the same population-specific utility level wherever its location. This means that red (blue) workers achieve in equilibrium the utility level v_r^* (v_b^*) such that $v_r^* \neq v_b^*$. In such a configuration, unilateral deviations of strategies are impossible.

2.3.2 The form of the indirect utility functions

Several important comments are in order.

After Stage 2, the indirect utility functions depend on $m_{r,c}$ and $m_{b,c}$ the local densities of workers living in c . This means that workers consider the city characteristics when choosing their place of residence. This also indicates the presence of agglomeration and congestion forces. Agglomeration forces are positive urban externalities and result from the preferences. Congestion forces are negative urban externalities and are twofold. As before, they can stem from preferences. But they are also the consequences of wages and transportation costs. If $m_{r,c}$ and $m_{b,c}$ are high in c , then the city is large, implying high aggregated transportation costs and land rent. As usual, agglomeration forces constitute a source of spatial concentration. Similarly, congestion forces constitute a source of spatial dispersion. Indeed, workers face a trade-off if $\phi_{rr}, \phi_{rb}, \phi_{br}, \phi_{bb} > 0$. The trade-off is captured by the decreasing relationships between $v_{r,c}$ and both c through a_c , the attractiveness of the city, and $m_{r,c}$ and $m_{b,c}$, the number of workers. This represents the aspiration to live in attractive cities and the desire to avoid congestion in large cities. Hence, we expect that in equilibrium the workers' distribution is never degenerated. This is intuitive. Workers want to live in attractive cities. However, as the same time, they anticipate that these cities will be larger and will generate higher transportation and land costs. This will prompt some workers to deviate by living in less attractive cities.

Last, the fact that the indirect utility functions are linear has four strong advantages. The first advantage is that linear dependence leads to a well-posed equilibrium. We show

that the equilibrium exists, that it is unique and stable under reasonable conditions. The second advantage is that the linearity of (10) and (11) has the appeal of simplicity and generates analytical results. Having closed-form solutions is key and very convenient. This enables conducting a detailed and robust comparative statistical analysis. This also allows a transparent identification of the driving forces of the workers' location, namely the relationship between individual preferences and global configurations is unequivocal. The third advantage is that the model can be a guide for empirical studies by helping to derive testable predictions. The last advantage is that the model can include economic factors and individual preferences. Segregation models (see, among others, Zhang (2004a), Zhang (2004b), Pans and Vriend (2007), O'Sullivan (2009), Zhang (2011) and Grauwil et al. (2012)) focus on a particular aspect: the individual preferences. There are no economic factors. The complex interactions coming from the labor market and the land market are absent from the analysis.

2.3.3 Results in Stage 1: A summary

In Sections 2.4-2.7, we provide a thorough characterization of the decentralized economy. In a nutshell,

i)- we determine some sufficient conditions for uniqueness and stability in Section 2.4.

ii)- we establish the existence of a mixed system of cities in Section 2.5.

iii)- we give some explicit solutions for the spatial distributions of workers in Section 2.6. Notably, we derive counterintuitive results where the preferences of workers at the individual level can be overturned at the city scale.

iv)- we derive a closed-form solution for I_D the dissimilarity index in Section 2.7. We identify the main drivers of social mixing. We show that the model can make testable predictions. We also derive additional counterintuitive results. We finally point out that preferences can mitigate income segregation.

2.4 Under which conditions is an equilibrium well posed?

An equilibrium of the form (10)–(11) is said to be well posed if it exists, is unique and stable. However, existence is a second-order problem. Hereafter, we focus on uniqueness and stability which are more challenging. The notion of uniqueness does not need to be specified. By contrast, the notion of stability is more complex and needs some clarifications. In the present article, we adopt a static definition of stability. In particular, we

define stability as follows:

Definition 3

A spatial equilibrium $(m_r^*, m_b^*) \in \Delta_r^C \times \Delta_b^C$ is stable if:

$$\sum_{c=1}^C v_r(c, m'_{r,c}, m'_{b,c}) (m_{r,c}^* - m'_{r,c}) + \sum_{c=1}^C v_b(c, m'_{r,c}, m'_{b,c}) (m_{b,c}^* - m'_{b,c}) > 0$$

$\forall (m'_r, m'_b) \in \Delta_r^C \times \Delta_b^C$ with $(m_r^*, m_b^*) \neq (m'_r, m'_b)$.

A spatial equilibrium is stable (more precisely globally evolutionarily stable) if the equilibrium is strictly robust to an ϵ -perturbation. If the inequality is not strict (i.e. \geq instead of $>$), then the equilibrium is said to be globally neutrally stable. This notion of stability dates back to Maynard Smith and Price (1973). Then, it has been popularized by Sandholm (2001) and Hofbauer and Sandholm (2009). Using Definition 3, we obtain the following:⁸

Proposition 2

If the following conditions are met:

$$-(\omega_r \tau + \omega_b \tau + \phi_{rr} + \phi_{bb}) < 0 \tag{12}$$

$$(\omega_r \tau + \phi_{rr})(\omega_b \tau + \phi_{bb}) > (\omega_b \tau + \phi_{rb})(\omega_r \tau + \phi_{br}) \tag{13}$$

then a spatial equilibrium is unique and stable.

The conditions we underline in Proposition 2 are sufficient but not necessary conditions. If they are not satisfied, it does not mean that the system of cities does not admit a unique and stable spatial equilibrium. Other conditions stating uniqueness and stability may be available.

The way we derive this condition is interesting in itself. To prove Proposition 2 we adopt a fairly indirect strategy. This strategy comprises the following steps. The first step is to rewrite the spatial equilibrium as a variational inequality (VI hereafter). The second step is to demonstrate that the variational inequality admits a unique solution under a monotony assumption about the vector of the utility functions $\mathbf{v} = (v_r, v_b)$. The third step is to remark that the assumption of monotony is equivalent to a negative definiteness condition on an interaction matrix denoted by Φ . The last step is to determine the suitable parameter restriction to comply with this negative definiteness condition, which is workable

⁸In Proposition 1, we make the distinction between the case $\omega_r > \omega_b$ and the case $\omega_r = \omega_b$. For Propositions 2–7, we do not need to make this distinction. To find the case in which workers have similar incomes, it is sufficient to set $\omega_r = \omega_b = \omega$ and $\tilde{\omega} = 0$ in the different propositions.

as the matrix is 2×2 . The strategy we follow is that of Boitier (2020). In this companion article, he demonstrate that systems of cities are in fact population games and sometimes potential games.

Uniqueness is desirable as it eliminates the “equilibrium selection problem”. In standard models of systems of cities (see Abdel-Rahman and Anas (2004), Seegert (2011), Behrens and Robert-Nicoud (2015a) and Albouy et al. (2019)), uniqueness is never an outcome. Rather, there is a continuum of equilibria that can be Pareto-ranked (see Henderson (1974)). Similarly, uniqueness is rarely a feature of Schelling models of segregation (see, within a large literature, Zhang (2004a), Zhang (2004b), Pancs and Vriend (2007), O’Sullivan (2009) and Zhang (2011)). As reviewed by Boustan (2013), the main characteristic of tipping models is multiple equilibria. For example, Grauwin et al. (2012) consider different utility functions and show that the spatial configuration is likely to be non-unique.

Here, uniqueness and stability emerge due to the combination of two ingredients. The first ingredient is the use of a finite number of types. In standard systems of cities, heterogeneity is modeled by a continuum, which creates multiplicity. As suggested by Proposition 2, uniqueness depends on the intra- and inter-group interactions. When there is a continuum of heterogeneity, these interactions are too numerous and deeply intertwined, and uniqueness fails as a result.⁹ With populations of workers, these drawbacks disappear. Intra- and inter-group interactions are less complex such that it is possible to determine the clear cut-offs for uniqueness and stability. The second ingredient is the linearity of city composition preferences. To see this, we need to have a look in Proof 2. As previously mentioned, uniqueness and stability stem from the property of an interaction matrix denoted by Φ . The latter is defined as follows:

$$\Phi(m_r, m_b) = \begin{pmatrix} \frac{\partial v_r}{\partial m_r} & \frac{\partial v_r}{\partial m_b} \\ \frac{\partial v_b}{\partial m_r} & \frac{\partial v_b}{\partial m_b} \end{pmatrix} = \begin{pmatrix} -(\omega_r \tau + \phi_{rr}) & -(\omega_b \tau + \phi_{rb}) \\ -(\omega_b \tau + \phi_{br}) & -(\omega_b \tau + \phi_{bb}) \end{pmatrix}$$

where $\frac{\partial v_p}{\partial m_p}$ captures the marginal effect of an increase in $m_{p,c}$ on v_p for all $p \in \mathcal{P}$. In sum, Φ is a sort of Jacobian matrix built with the indirect utility functions. When city composition preferences are linear, the marginal effects are independent of workers’ densities, and rely on exogenous parameters. Therefore, preferences have a monotonic effect in the model. This permits to eliminate multiplicity and instability by placing suitable assumptions on the set preferences. If we assume non-linear dependency in (10) and (11), monotonicity

⁹The use of a continuum has another shortcoming: wages and land rents must be independent of the workers’ composition of the city (see Behrens and Robert-Nicoud (2015a) for a discussion). In our framework, wages and land rents can depend on the composition of the city. Again this is because we manipulate finite populations of workers.

vanishes. The marginal effects now depend on the densities of workers that are endogenous variables. Therefore, the condition on Φ is rarely verified, and multiplicity emerges most of the time (see our Online Appendix for additional examples).

Our sufficient conditions for uniqueness and stability have the appeal of tractability. They are based on the transportation costs of workers and the degree of neighborhood externalities. This simplicity has many advantages. Notably, it permits a transparent analysis of the driving forces that generate uniqueness and stability. To show this, let us consider several examples.

When there are no city externalities $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = 0$. In that case, the economy encapsulates congestion forces captured by $-\omega_r\tau$ and $-\omega_b\tau$. The conditions for uniqueness collapse to:

$$-(\omega_r\tau + \omega_b\tau) < 0$$

and

$$(\omega_r - \omega_b)\tau\omega_b\tau > 0$$

The first condition is verified as $\omega_r, \omega_b > 0$. The second condition is also satisfied because $\omega_r > \omega_b$. As a result, the basic system of cities is well posed as it necessarily leads to a unique and stable equilibrium.

When city preferences are symmetric and negative $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = \psi$ with $\psi > 0$, the system of cities also encompasses congestion forces summarized by $-\omega_r\tau$, $-\omega_b\tau$ and ψ . The conditions are as follows:

$$-(\omega_r\tau + \omega_b\tau + 2\psi) < 0$$

and

$$(\omega_r - \omega_b)\tau(\omega_b\tau + \psi) > 0$$

As before, the first and second parts are fulfilled as $\omega_r > \omega_b > 0$ and $\psi > 0$. Uniqueness and stability hold when intra- and inter-group externalities are negative and similar. In other words, uniqueness and stability appear when congestion forces $-\omega_r\tau$ and $-\omega_b\tau$ dominate agglomeration economies $-\psi$.

The converse situation $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = -\psi$ with $\psi > 0$ gives:

$$-(\omega_r\tau + \omega_b\tau - 2\psi) < 0$$

and

$$(\omega_r - \omega_b)\tau(\omega_b\tau - \psi) > 0$$

The first equation is satisfied if $2\psi < \omega_r\tau + \omega_b\tau$. The second equation holds if $\psi < \omega_b\tau$. As $\omega_b \leq \omega_r$, if the second equation is satisfied then the first one is also complied. Thus, when interactions are positive and equivalent, the intra- and inter-group externalities must not be too large to ensure uniqueness and stability. That is, ψ must be bounded by $\omega_b\tau$.

When workers have a preference for a segregated environment so that $\phi_{rr} = \phi_{bb} = -\psi$ and $\phi_{rb} = \phi_{br} = \psi$ and $\psi > 0$, the conditions can be rewritten as follows:

$$-(\omega_r\tau + \omega_b\tau - 2\psi) < 0$$

and

$$(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi > 0$$

To obtain uniqueness and stability, the externalities must be bounded such that $2\psi < \omega_r\tau + \omega_b\tau$ and $\psi < \frac{\tilde{\omega}\tau\omega_b\tau}{\omega_r\tau + 3\omega_b}$.

In summary, this suggests that uniqueness and stability prevail when there are:

1. similar congestion forces
2. congestion forces dominate agglomeration economies

Conversely, multiplicity seems to appear when they are:

1. strong asymmetric negative intra- and inter-group externalities
2. agglomeration forces outweigh congestion effects

2.5 Mixed system of cities

With equilibria (10)–(11), different spatial configurations are possible. Notably, the system of cities can be said to be:

1. “segregated” in the sense that red and blue workers are (totally) separated into different cities (see Figure 2.a). In that case, the dissimilarity index is necessarily $I_D = 1$ and the segregation level in the system of cities is maximized.
2. “perfectly mixed” in the sense that red and blue workers share the same cities and in the same proportion (see Figure 2.b). In that case, the dissimilarity index is necessarily $I_D = 0$, and the segregation level in the system of cities is minimized.
3. “mixed” in the sense that red and blue workers share the same cities but in different proportions (see Figure 2.c). In that case, the dissimilarity index verifies $I_D \in]0, 1[$.

The model can support the existence of all three situations, namely segregated, mixed, and perfectly mixed situations. In this article we focus on mixed (and perfectly mixed) situations only. This is because it is uncommon to observe fully segregated systems of cities in the data. By contrast, empirical evidence documents non-extreme segregation, suggesting a mixed system of cities (see Ihlanfeldt and Scafidi (2002) and Clark (2009)). For instance, Combes et al. (2012) and Eeckhout et al. (2014) document an incomplete sorting of skilled workers across cities.

Using Proposition 1, we find the following.

Proposition 3

If the following conditions are verified:

$$\frac{(\phi_{bb} - \phi_{rb})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} > -\frac{m_r}{C} \quad (14)$$

$$\frac{(\tilde{\omega}\tau + \phi_{rr} - \phi_{br})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} > -\frac{m_b}{C} \quad (15)$$

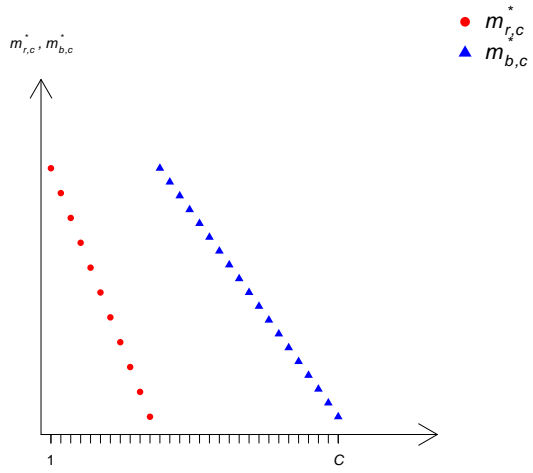
with $\tilde{\omega} = \omega_r - \omega_b$ and $\bar{a} = \frac{1}{C} \sum_{c=1}^C a_c$, then the system of cities is mixed. In addition, if the following holds:

$$(\phi_{bb} - \phi_{rb})m_b = (\tilde{\omega}\tau + \phi_{rr} - \phi_{br})m_r \quad (16)$$

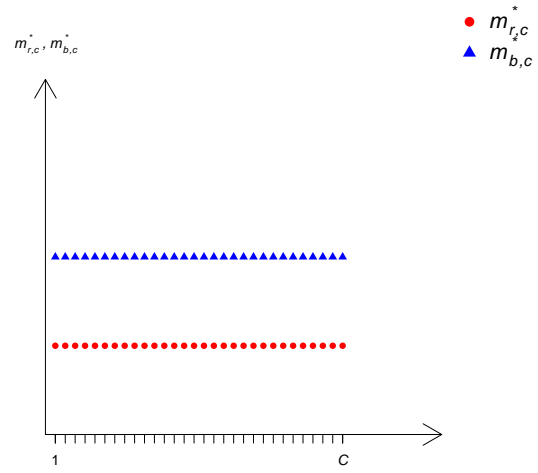
then the city is perfectly mixed.

Conditions (14)-(15) are sufficient to ensure that the system of cities is mixed. To attempt to have a rationale, we explain our strategy. Using equations (10)-(11), we determine the closed-form solutions for $m_{r,c}^*$, $m_{b,c}^*$ and m_c^* . Then, we look for the parameter restrictions such that $Supp(m_r^*) = Supp(m_b^*) = \mathcal{C}$. We finally find (14) and (15). Intuitively, these conditions indicate that agglomeration and dispersions forces must not be too large to ensure social mixing. When agglomerations forces are strong, workers inhabit in few cities. In the extreme case that agglomerations economies are very strong, a spatial equilibrium can be characterized by $m_{r,1} = m_r$ and $m_{b,1} = m_b$ and $m_{p,c'} = 0 \quad \forall p \in \{r, b\}$ and $c' \neq 1$. By contrast, when dispersions forces are large, workers have an incentive to live in two distinct spans located at the edges of the system of cities. Between these polar situations, there are many cases where agglomeration and dispersion forces are small enough to generate social mixing.

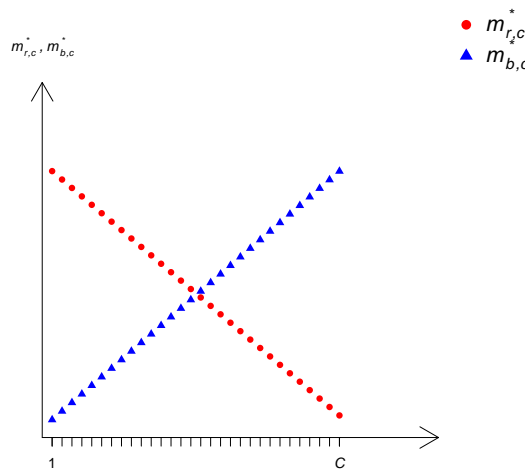
The spatial patterns prevail under different probabilities. A perfectly mixed system of cities constitutes an equilibrium if the knife-edge condition (16) is verified, meaning that this configuration rarely appears. Alternatively, a large set of plausible parameter restrictions can be easily found for the emergence of mixed systems of cities (see (14)-(15)). This



(a) Segregated system of cities



(b) Perfectly mixed system of cities



(c) Mixed system of cities

Figure 2

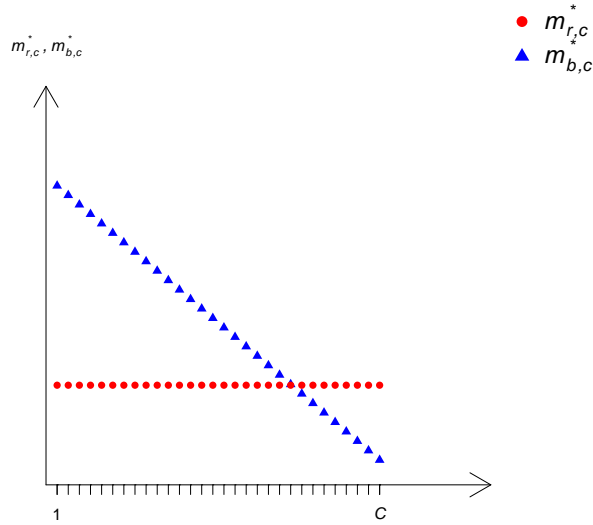


Figure 3: Perfectly mixed system of cities

means that mixed systems of cities frequently emerge as an outcome, which is realistic when confronting the model to the features of the data (see Combes et al. (2012) and Eeckhout et al. (2014)).

A counterintuitive result is that condition (16) is not fulfilled when workers have a preference for an integrated environment (i.e. $\phi_{rr} = \phi_{bb} = \phi_{rb} = \phi_{br} = \psi$ with $\psi < 0$). Conversely, the condition can hold when $\phi_{rb} > 0$ and $\phi_{br} > 0$. Put differently, a perfectly mixed system of cities can be achieved when workers have mutual rejections. This constitutes a paradoxical result à la Schelling.

Another important result prevails when workers have same incomes (e.g. $\omega_r = \omega_b$). In that case, a mixed system of cities can emerge. In particular, if $(\phi_{bb} - \phi_{rb})m_b = (\phi_{rr} - \phi_{br})m_r$ then the system of cities becomes perfectly mixed. From Proposition 1, it also appears that red and blue workers have the same locations within cities. Thus, when red and blue workers share similar wages, there is both social mixing across and within cities. To our best knowledge, this is the first framework that is able to make this prediction.

Coupled with Proposition 2, Proposition 3 posits the existence of a unique and stable mixed configuration. This constitutes *per se* a contribution as it is at odds with the urban literature. The Schelling model of segregation proposes that segregation is the only stable equilibrium. Mixed cities rarely occur in these frameworks and are rarely a stable

equilibrium. Haw and Hogan (2020) demonstrate that stable mixed configurations are only possible under restrictive assumptions. Namely, the minority must be small and the tolerance threshold must be large. Similarly, in standard systems of cities, there is no social mixing in the sense that there is a strict partition of type across cities (see among others Mori and Turrini (2005) and Okubo et al. (2010)). This is because these models exhibit positive assortative matching (see Sattinger (1993)) due to the supermodularity in type and city size. The present framework is able to generate a unique, stable and incomplete sorting. Such a result is the consequence of the inclusion of asymmetric agglomeration and dispersion forces. As previously stated, workers can be engaged with asymmetric intra- and inter-group local interactions. This asymmetry introduces the suitable degree of freedom such that workers inhabit the cities of the system.

2.6 What does a spatial equilibrium look like?

Using Proof 3, we provide a full exposition of the spatial equilibrium in the system of cities. We derive the closed-form solutions for the densities of workers. We then study peculiar cases to stress the links between the microeconomic motives of workers and the macroeconomic configurations. Notably, we highlight counterintuitive results in which the desire of workers at the individual level can be overturned at the city level.

Obtaining closed-form solutions for $m_{r,c}$, $m_{b,c}$ and m_c is key. In effect, standard Schelling models of segregation derive results from numerical simulations. The rare analytical results concern the parameter restrictions under which solutions are segregated and stable.¹⁰ As a consequence, these model seem to be black boxes where robust comparative statics analysis are not available. Here, our framework can overcome these issues. Technically, this is due to the linearity of the indirect utility preferences and the fact that the number of populations of workers is finite.

Proposition 4

A mixed system of cities is characterized by the following spatial distributions:

$$m_{r,c}^* = \frac{m_r}{C} + \frac{(\phi_{bb} - \phi_{rb})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}$$

$$m_{b,c}^* = \frac{m_b}{C} + \frac{(\tilde{\omega}\tau + \phi_{rr} - \phi_{br})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}$$

$$m_c^* = \frac{1}{C} + \frac{(\tilde{\omega}\tau + \phi_{rr} + \phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}$$

¹⁰“Despite the important insight revealed in Schelling’s simulation of the checkerboard model, for many years social scientists were unable to rigorously analyze the model, primarily because of the lack of suitable mathematical tools” in Zhang (2011).

where $m_c^* = m_{r,c}^* + m_{b,c}^*$ is the city size of city $c \in \mathcal{C}$.

Three comments are in order.

First, the spatial distributions are linear, implying that $m_{r,c}^*$, $m_{b,c}^*$ and m_c^* can either increase, decrease, or be constant with respect to attractiveness.

Second, Proposition 4 can predict situations that are consistent with real-world observations. Empirical studies report that the size of cities is positively correlated with inequality. Combes et al. (2012) and Eeckhout et al. (2014) document that large cities are more unequal locations than small cities. Similarly, Glaeser et al. (2009) underline that differences in terms of skill account for one-third of the variation in Gini coefficients. These features appear in the model if the following holds:

$$\frac{(\tilde{\omega}\tau + \phi_{rr} + \phi_{bb} - \phi_{rb} - \phi_{br})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} > 0$$

and

$$\frac{m_{r,c}}{m_{r,c} + m_{b,c}} = \frac{\frac{m_r}{C} + \frac{(\phi_{bb} - \phi_{rb})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}}{\frac{1}{C} + \frac{(\tilde{\omega}\tau + \phi_{rr} + \phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}} \quad \text{decreases with } c$$

The first condition states that the residential density gradient of workers is negative, which means that most (less) attractive cities are large (small) cities. The second condition implies that most (less) attractive cities are more (less) unequal places.

Third, even if the distributions are analytical, they remain fairly complex. The relationships between income inequalities captured by $\tilde{\omega}$, transportation costs, and social preferences are intertwined. To have a better understanding of the behaviors of $m_{r,c}^*$, $m_{b,c}^*$ and m_c^* , we consider particular examples.

2.6.1 No city composition preferences

When workers have no city composition preferences (i.e. $\phi_{rr} = \phi_{bb} = \phi_{rb} = \phi_{br} = 0$), the spatial distribution is reduced to:

$$m_{r,c}^* = \frac{m_r}{C}, \quad m_{b,c}^* = \frac{m_b}{C} + \frac{a_c - \bar{a}}{\omega_b\tau}, \quad m_c^* = \frac{1}{C} + \frac{a_c - \bar{a}}{\omega_b\tau}$$

Red workers are evenly distributed across cities (see Figure 3). By contrast, the density gradient of blue workers is negative, meaning that they reside in attractive cities in a larger proportion. To have a rationale for that, remind that the indirect utility function of red workers collapses to:

$$v_{r,c} = \omega_r + a_c - \omega_r\tau m_{r,c} - \omega_b\tau m_{b,c}$$

Red workers face a trade-off. On the one hand, they have desires to inhabit in most attractive cities. These incentives are summarized by $\omega_r + a_c$. On the other hand, they

want to live in cities where global population is low as any increase in $m_{r,c}$ (and $m_{b,c}$) corresponds to higher commuting costs and land prices (see Section 2.2.2). As red workers earn more than blue workers, this prompts red workers to disperse more across the system of cities than blue workers. The fact that red workers are evenly distributed across cities is a specificity of the model which has no particular interpretation. This may be considered as another paradoxical result. Similarly, blue workers face a standard trade-off between accessibility and aggregate transportation costs. They want to live in a_1 , the most attractive city. However, they also anticipate that more blue workers will reside this city. The city becomes larger, and this increases transportation costs. To avoid this, some blue workers have the incentive to live farther away. In particular, because an equilibrium is a state in which each blue worker shares the same equilibrium payoff regardless of the city, $m_{b,c}^*$ must decrease with respect to c to cancel out the two aforementioned forces. Last, note that city size m_c^* is positively correlated with attractiveness. We observe the largest cities on the most attractive sites and the smallest cities on the least amenable sites. This state of the world is straightforward and in line with empirical evidence.

2.6.2 Own and mutual rejections

Assume that workers have symmetric own and mutual rejections so that $\phi_{rr} = \phi_{bb} = \phi_{rb} = \phi_{br} = \psi$ with $\psi > 0$. This yields the following:

$$m_{r,c}^* = \frac{m_r}{C}, \quad m_{b,c}^* = \frac{m_b}{C} + \frac{a_c - \bar{a}}{\omega_b \tau + \psi}, \quad m_c^* = \frac{1}{C} + \frac{a_c - \bar{a}}{\omega_b \tau + \psi}$$

As before, the distribution of red workers is constant across cities. The spatial distribution of blue workers also decreases with respect to attractiveness. But the density gradient is now lower, and blue workers are more dispersed in the system of cities. In the extreme case that ψ is very high, the spatial distribution becomes even flatter. These results are intuitive as own and mutual rejections act like congestion/dispersion forces.

Another interesting configuration appears when own and mutual rejections are asymmetric: $\phi_{rr} = \phi_{rb} = \psi$ and $\phi_{bb} = \phi_{br} = \varphi$ with $\psi, \varphi > 0$. In that case, the endogenous spatial distribution collapses to:

$$m_{r,c}^* = \frac{m_r}{C} + \frac{(\varphi - \psi)(a_c - \bar{a})}{\tilde{\omega} \tau (\omega_b \tau - \varphi)}, \quad m_{b,c}^* = \frac{m_b}{C} + \frac{[\tilde{\omega} \tau - (\varphi - \psi)](a_c - \bar{a})}{\tilde{\omega} \tau (\omega_b \tau - \varphi)}, \quad m_c^* = \frac{1}{C} + \frac{a_c - \bar{a}}{\omega_b \tau - \varphi}$$

The link between city size and attractiveness is still positive. But now the density gradients of the spatial distributions of red and blue workers can be either positive or negative. The resulting configuration depends on the relative asymmetry in preferences captured by $\varphi - \psi$ and the magnitude of income inequalities and transportation costs expressed by $\tilde{\omega} \tau$.

2.6.3 Own and mutual attractions

Consider that workers have symmetric own and mutual attractions (i.e., $\phi_{rr} = \phi_{bb} = \phi_{rb} = \phi_{br} = -\psi$ with $\psi > 0$). This leads to:

$$m_{r,c}^* = \frac{m_r}{C}, \quad m_{b,c}^* = \frac{m_b}{C} + \frac{a_c - \bar{a}}{\omega_b \tau - \psi}, \quad m_c^* = \frac{1}{C} + \frac{a_c - \bar{a}}{\omega_b \tau - \psi}$$

This is the converse of the case where workers have own and mutual rejections. The spatial distribution of blue workers is steeper. This is because own and mutual attractions act like agglomeration/concentration forces.

2.6.4 Own attractions and mutual aversions

If workers prefer a segregated environment such that $\phi_{rr} = \phi_{bb} = -\psi$ and $\phi_{rb} = \phi_{br} = \psi$ and $\psi > 0$, then:

$$m_{r,c}^* = \frac{m_r}{C} - \frac{2\psi(a_c - \bar{a})}{(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi}, \quad m_{b,c}^* = \frac{m_b}{C} + \frac{(\tilde{\omega}\tau - 2\psi)(a_c - \bar{a})}{(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi}$$

$$m_c^* = \frac{1}{C} + \frac{(\tilde{\omega}\tau - 4\psi)(a_c - \bar{a})}{(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi}$$

Under uniqueness $(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi > 0$, the density gradient of blue workers is negative, but the density gradient of red workers is positive. This is intuitive as workers have a desire for a segregated city composition.

2.6.5 No income inequality

When workers have identical incomes, the distributions strongly depend on city composition preferences. For symmetric preferences $\phi_{rr} = \phi_{bb} = \phi$ and $\phi_{rb} = \phi_{br} = \psi$, we obtain:

$$m_{r,c}^* = \frac{m_r}{C} - \frac{(\phi - \psi)(a_c - \bar{a})}{(\tilde{\omega}\tau + \phi^2) + \phi^2 - \psi^2}, \quad m_{b,c}^* = \frac{m_b}{C} - \frac{(\phi - \psi)(a_c - \bar{a})}{(\tilde{\omega}\tau + \phi^2) + \phi^2 - \psi^2}$$

$$m_c^* = \frac{1}{C} - \frac{2(\phi - \psi)(a_c - \bar{a})}{(\tilde{\omega}\tau + \phi^2) + \phi^2 - \psi^2}$$

Red and blue workers share the same spatial density gradient, and the system of cities becomes perfectly mixed if $m_r = m_b$. The density gradient of workers should be negative when $\phi - \psi > 0$. This is intuitive. $\phi > 0$ captures own dispersion and $\psi > 0$ measures cross dispersion. When the ‘‘own’’ effect outweighs the ‘‘cross’’ one, this naturally pushes toward more dispersion.

2.7 Looking into the black box: What drives social mixing in a system of cities?

In what follows, we derive the expression for I_D the dissimilarity index. We then attempt to identify the main drivers of social mixing.

Proposition 5

The (endogenous) dissimilarity index is described by:

$$I_D = \Omega_M \times \Gamma_{\omega, \phi} \times \Psi_C$$

where

$$\Omega_M = \frac{1}{2m_r m_b}$$

$$\Gamma_{\omega, \phi} = \left| \frac{(\phi_{bb} - \phi_{rb}) m_b - (\tilde{\omega}\tau + \phi_{rr} - \phi_{br}) m_r}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} \right|$$

$$\Psi_C = \sum_{c=1}^C |a_c - \bar{a}|$$

with $\bar{a} = \frac{1}{C} \sum_{c=1}^C a_c$.

The linearity of (10) and (11) is crucial for Proposition 5. This allows obtaining closed-form solutions for the spatial distributions $m_{r,c}$ and $m_{b,c}$ and so for the dissimilarity index I_D . This also enables conducting a transparent and robust comparative statics analysis. Note that the model is able to generate a continuum of dissimilarity indexes. Therefore, the model is flexible enough to replicate empirical evidence. Having a closed-form solution is also important for empirical studies. Our framework can be a guide for empirical studies. In particular, the model offers the following testable prediction:

$$\ln I_D = \alpha \ln \Omega_M + \beta \ln \Gamma_{\omega, \phi} + \gamma \ln \Psi_C + \epsilon \quad (17)$$

that links the dissimilarity index to measurable variables. In particular, when workers have no city composition preferences, equation (17) collapses to:

$$\ln I_D = \zeta + \alpha \ln m_b + \beta \ln \omega_b + \gamma \ln \tau + \theta \ln \Psi_C + \epsilon \quad (18)$$

where the model predicts the following: $\zeta = -\ln 2$, $\beta = \gamma = -1$ and $\theta = 1$. ω_b , m_b and τ can be built using data on wages, populations and commuting costs. ω_b can correspond to the average wages of workers over a certain period. The same applies for m_b . τ can be approximated by the average fraction of time that a worker spends in transports. Even if measuring the quality of sites is a hard task, Ψ_C can be approximated with observable data. In particular, $a_c = \prod_{i \in \mathcal{I}} a_{i,c}$ can be constructed by all the desirable and observable

characteristics $a_{i,c}$ of the city c . $a_{i,c}$ can encapsulate mean temperature, the percentage of water surface, etc (see the US Department of Agricultural county-level amenity data). It is also possible to include in $a_{i,c}$ a proxy suggested by Ahlfeldt (2013) and Saiz et al. (2018). The latter takes the number of outside geocode pictures taken by residents at a given location.¹¹ In that case, equation (18) can be estimated using standard methods.¹²

The endogenous dissimilarity index is unambiguous. It is built up with three components. The first component Ω_M captures the demography of the system of cities. Ω_M summarizes both the size of the system and the inequality in terms of population size. The second term $\Gamma_{\omega,\phi}$ is the by-product of the intra- and inter-group externalities. This is keeping in mind that these interactions encapsulate income inequalities, transportation costs, and neighborhood preferences. The last component Ψ_C measures the degree of city differences in terms of amenities. The fact that the dissimilarity index directly depends on Ψ_C is consistent with empirical evidence. For instance, Ellison and Glaeser (1999) indicate that differences in locational fundamentals account for one-fifth of the observed spatial distribution of households.

The interpretation of the first and third terms is clear. An increase in the size of the population $p \in \{r, b\}$ decreases the value of the dissimilarity index, that is, an increase in demography causes more social mixing in the model. Similarly, social mixing is directly tied to the attractiveness of cities. In particular, a system of cities that shows low heterogeneity (i.e. Ψ_C low) generates high social mixing, other things being equal. In other words, the combination of income inequalities, transportation costs, and neighborhood preference will increase when cities exhibit large differences in terms of amenity.

The second part of I_D is much more complex and so needs more careful attention.¹³ We begin by underlying a general comment. We then offer more specific insights considering peculiar sub-cases. Other sub-cases are possible and leads to the same qualitative results (see our Online Appendix).

¹¹Obviously, there does not exist a set of $a_{i,c}$ that takes into account the full spectrum of amenities of cities.

¹²The measurement $\Gamma_{\omega,\phi}$ is more challenging. Technically, ϕ captures the effect on the utility function of an increase in the level of red/blue workers. An approximation of ϕ could be obtained by declarative data (see Davis and Smith (1993) and Charles (2003)).

¹³A corollary of Proposition 3 is present in Proposition 5. A perfectly mixed system of cities implies $I_D = 0$. This prevails when condition (16) is complied.

2.7.1 Segregation by income vs. segregation by preferences

The interaction between income inequalities and city composition shapes social mixing. In particular, there is a relationship between segregation by income and segregation by color preferences. It is also worth noting that Boitier and Auvray (2020) also find such a relationship. The main difference with Boitier and Auvray (2020) is that they obtain this mechanism within the city, not in a system of cities. When city composition preferences are neutralized, the level of social mixing is driven by income inequalities such that:

$$\Gamma_{\omega,\phi} = \frac{\tilde{\omega}m_r}{\omega_r\omega_b - \omega_b\tau\omega_b}$$

Therefore, city composition preferences magnify segregation by income when:

$$\frac{\tilde{\omega}m_r}{\omega_r\omega_b - \omega_b\tau\omega_b} < \left| \frac{(\phi_{bb} - \phi_{rb})m_b - (\tilde{\omega}\tau + \phi_{rr} - \phi_{br})m_r}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} \right|$$

Conversely, the preferences mitigate segregation by income if and only if:

$$\frac{\tilde{\omega}m_r}{\omega_r\omega_b - \omega_b\tau\omega_b} > \left| \frac{(\phi_{bb} - \phi_{rb})m_b - (\tilde{\omega}\tau + \phi_{rr} - \phi_{br})m_r}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} \right|$$

The above condition is satisfied when there are own and mutual rejections. In consequence, color preferences can mitigate income segregation, which has empirical support (see Rosenthal and Ross (2015)).

2.7.2 Some specific cases

As a starting point, assume that workers are color-neutral $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = 0$. If so, the dissimilarity index becomes the following:

$$I_D = \frac{1}{2m_b} \times \frac{1}{\omega_b\tau} \times \Psi_C \quad (19)$$

This version of the dissimilarity index constitutes our reference point.

Then assume that workers have symmetric and negative city preferences $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = \psi$ with $\psi > 0$. The dissimilarity index boils down to:

$$I_D = \frac{1}{2m_b} \times \frac{1}{\omega_b\tau + \psi} \times \Psi_C$$

The above index is lower than (19). This result is paradoxical as it states that a situation in which there is no city composition preferences generates less social mixing (i.e. more segregation) than a situation where workers experience mutual rejection. Another counter-intuitive result lies in the effect of ψ . An increase in ψ unambiguously leads to more social

mixing, and, at the limit, $I_D \approx 0$ when ψ is large. An increase in mutual rejection improves social mixing, and the system can converge toward a perfectly mixed configuration. Again, this finding is paradoxical as individuals' preferences seem to be overturned at an aggregate scale.

Equivalent results are present when workers have symmetric own and mutual attractions $\phi_{rr} = \phi_{rb} = \phi_{br} = \phi_{bb} = \psi$ with $\psi < 0$:

$$I_D = \frac{1}{2m_b} \times \frac{1}{|\omega_b\tau - \psi|} \times \Psi_C$$

The dissimilarity index is larger than (19). In the same manner, an increase in ψ improves the value of I_D . An increase in the desire for a mixed social environment leads to more segregation. This is equivalent to the Schelling paradox, since macroeconomic patterns strongly deviate from microeconomic preferences.

These counterintuitive effects can be rationalized as follows. When $\psi > 0$, then the city composition preferences act as congestion forces. This prompts workers to disperse across cities. The distribution of blue workers becomes flatter. In turn, the dissimilarity index decreases. By contrast, if $\psi < 0$, the converse situation exists. Social preferences act as agglomeration economies. This prompts (blue) workers to flock to the most attractive city. The spatial distribution of blue workers becomes steeper. As a result, the dissimilarity index increases.

Finally, when workers have a symmetric preference for a segregated environment such that $\phi_{rr} = \phi_{bb} = -\psi$ and $\phi_{rb} = \phi_{br} = \psi$ and $\psi > 0$, then:

$$I_D = \frac{1}{2m_r m_b} \times \left| \frac{2\psi m_b + (\tilde{\omega}\tau - 2\psi)m_r}{(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi} \right| \times \Psi_C$$

Under multiplicity, the previous comments remain valid. However, under uniqueness and stability (i.e. $(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi > 0$ and so $\tilde{\omega}\tau - 2\psi > 0$), the paradoxical effects disappear. It can be readily verified that I_D is larger than the one in (19), and an increase in color preferences at the microeconomic level now generates less social mixing in the system of cities. Boitier and Auvray (2020) find a similar result. They underline that an increase in racial preferences at the microeconomic level generates more social mixing in the city. Note that the strategy adopted in Boitier and Auvray (2020) is different from ours. They study the spatial equilibrium within the city only (there is no system of cities) and in the particular case of Zhang (2004a). Moreover, they obtain a spatial equilibrium that is not unique and that is not stable. Here, we go a step forward. We suggest that the result obtained in Boitier and Auvray (2020) is tied to the regime of uniqueness. When there is

multiplicity, an increase in racial preferences at the microeconomic level can generate more social mixing. By contrast, when uniqueness holds, the counterintuitive effect is canceled out.

2.8 Optimality

Hereafter, we build the problem for the social planner. We then show that the decentralized economy cannot sustain the social allocation. However, in Section 2.9, we offer two remedies to restore optimality. We also show that cities can be oversized or undersized. We last underline that social mixing can be too high or too low.

2.8.1 The social planner criterion

The social planner is utilitarian. He determines a spatial distribution for red workers and a spatial distribution for blue workers that maximize the total welfare:

$$\max_{m_r \in \Delta_r^C, m_b \in \Delta_b^C} \mathcal{S}(m_r, m_b)$$

with

$$\mathcal{S}(m_r, m_b) = \sum_{c=1}^C v_{r,c} m_{r,c} + \sum_{c=1}^C v_{b,c} m_{b,c}$$

that is

$$\begin{aligned} \mathcal{S}(m_r, m_b) = & \sum_{c=1}^C (\omega_r + a_c) m_{r,c} + (\omega_b + a_c) m_{b,c} - (\omega_r \tau + \phi_{rr}) m_{r,c}^2 - (\omega_b \tau + \phi_{bb}) m_{b,c}^2 \dots \\ & \dots - (2\omega_b \tau + \phi_{rb} + \phi_{br}) m_{r,c} m_{b,c} \end{aligned} \quad (20)$$

using (10) and (11). The criteria for the social planner are simply the indirect utility functions deduced from Stage 2.

Proposition 6

If the following conditions are verified:

$$\begin{aligned} \frac{(2\phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r \tau + \phi_{rr})(\omega_b \tau + \phi_{bb}) - (2\omega_b \tau + \phi_{rb} + \phi_{br})^2} &> -\frac{m_r}{C} \\ \frac{(2\tilde{\omega} \tau + 2\phi_{rr} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r \tau + \phi_{rr})(\omega_b \tau + \phi_{bb}) - (2\omega_b \tau + \phi_{rb} + \phi_{br})^2} &> -\frac{m_b}{C} \end{aligned}$$

then the social planner predicts a mixed system of cities pinned down by:

$$\tilde{m}_{r,c} = \frac{m_r}{C} + \frac{(2\phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r \tau + \phi_{rr})(\omega_b \tau + \phi_{bb}) - (2\omega_b \tau + \phi_{rb} + \phi_{br})^2}$$

$$\begin{aligned}\tilde{m}_{b,c} &= \frac{m_b}{C} + \frac{(2\tilde{\omega}\tau + 2\phi_{rr} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} \\ \tilde{m}_c &= \frac{1}{C} + \frac{2(\tilde{\omega}\tau + \phi_{rr} + \phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} \\ \tilde{I}_D &= \frac{1}{2m_r m_b} \times \tilde{\Gamma}_{\omega,\phi} \times \sum_{c=1}^C |a_c - \bar{a}|\end{aligned}$$

with

$$\tilde{\Gamma}_{\omega,\phi} = \left| \frac{(2\phi_{bb} - \phi_{rb} - \phi_{br})m_b - (2\tilde{\omega}\tau + 2\phi_{rr} - \phi_{rb} - \phi_{br})m_r}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} \right|$$

The investigation of the results of Proposition 6 is done in the following subsections.

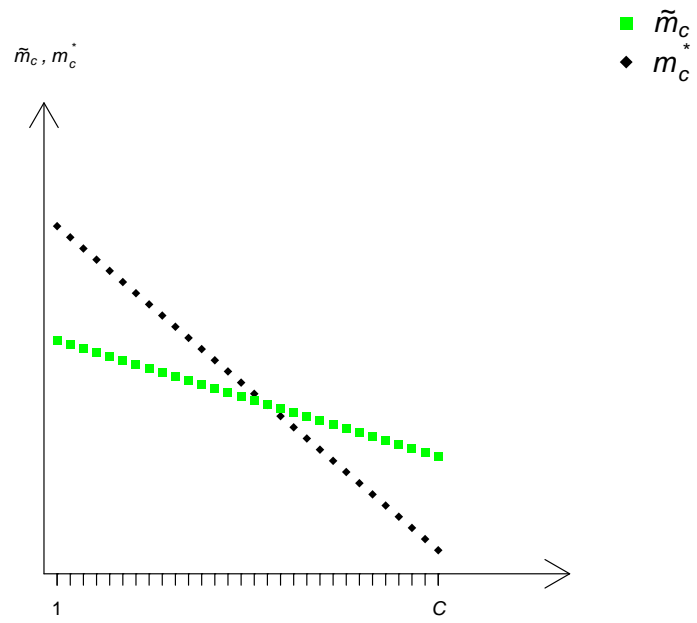
2.8.2 What are the roots of inefficiency?

A decentralized economy never achieves an optimal location. To disentangle the different causes leading to non-optimality, we note that the first-order conditions of the social planner program are expressed by:

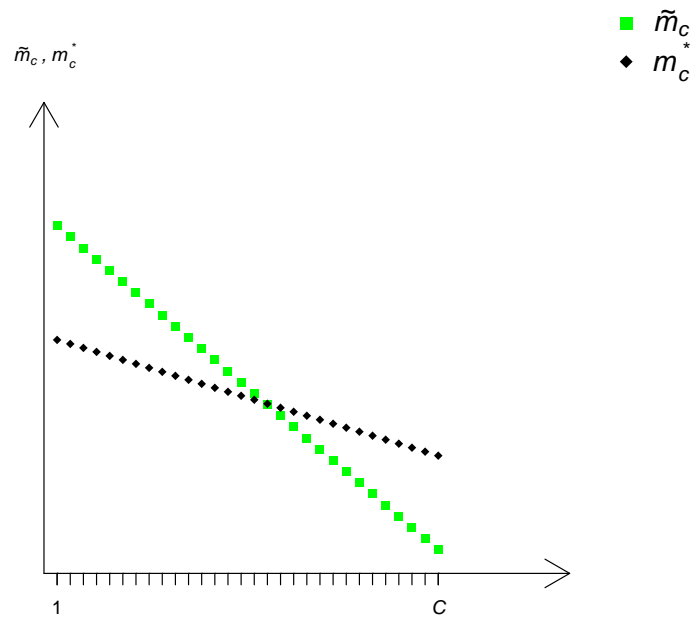
$$\omega_r + a_c - 2(\omega_r\tau + \phi_{rr})\tilde{m}_{r,c} - (2\omega_b\tau + \phi_{rb} + \phi_{br})\tilde{m}_{b,c} = \tilde{v}_r \quad (21)$$

$$\omega_b + a_c - (2\omega_b\tau + \phi_{rb} + \phi_{br})\tilde{m}_{r,c} - 2(\omega_b\tau + \phi_{bb})\tilde{m}_{b,c} = \tilde{v}_b \quad (22)$$

with $\tilde{v}_r, \tilde{v}_b \in \mathbb{R}$. The conditions have standard interpretations. The conditions state that the optimal/social distributions of workers is that where the marginal benefits arising from wages and preferences offset the increased commuting costs and the increased land rents due to an added individual. In line with Arnott (1979), the conditions also imply that the marginal benefits and the marginal costs are equal across cities. In addition, a glance at (10)-(11) and (21)-(22) suggests that the origins of inefficiency are twofold. The first (market) failure arises through the intra-group externalities. Workers do not properly internalize their effects on the utilities of others. The “private” behaviors $-(\omega_r\tau + \phi_{rr})$ and $-(\omega_b\tau + \phi_{bb})$ do not coincide with the “social” behaviors $-2(\omega_r\tau + \phi_{rr})$ and $-2(\omega_b\tau + \phi_{bb})$. The second (market) failure arises from the inter-group externalities. Workers take into account $-(\omega_b\tau + \phi_{rb})$ and $-(\omega_b\tau + \phi_{br})$, while the social planner pays attention to $-(2\omega_b\tau + \phi_{rb} + \phi_{br})$. In a nutshell, the decentralized economy cannot sustain the social allocation because of the preference of externalities, which constitutes a standard origin for inefficiency (see Fujita (1989)).



(a) Oversized big cities and undersized small cities



(b) Undersized big cities and oversized small cities

Figure 4

2.8.3 Are cities oversized or undersized?

In urban economics, a consensus is that cities are perceived as overpopulated for two reasons.¹⁴ First, cities are considered as too large by scholars. Many models conclude that city size is excessive to what is socially desirable. “Despite their economic importance, cities are often perceived as being too large [...] Urban economics textbooks often present the view that cities are oversized as fact.”, in Albouy et al. (2019). Second, cities are also perceived as too sprawl by people. Empirical studies suggest that urban expansion implies high commuting costs, more pollution and more destruction of open spaces (see Glaeser and Kahn (2004)). That is why, many policies have been implemented in order to restrict spatial expansion, such as taxes, urban growth boundaries or zoning programs (see Brueckner (2000) for a review about the stakes in the urban sprawl debate).

Recently, this common view has been challenged by Albouy et al. (2019). These authors develop a system of cities with heterogeneous sites, homogeneous agents, fiscal externality, and where there are both agglomeration and congestion forces. In their model the indirect utility function is peaked, so much so that congestion economies dominate agglomeration economies beyond some city size. They show that, if the fiscal externality is larger than the urban externality, larger cities are undersized, whereas smaller cities are oversized. Seegert (2011) also points out the effect of land regulations on city size according to different amenities (production and quality of life amenities).

Similar results hold in our framework (see Figures 4.1 and 4.2). What is appealing is that we do not need fiscal externalities to generate these features. This naturally occurs in our framework. When workers are color-neutral, we obtain:¹⁵

$$\tilde{m}_c = \frac{1}{C} + \frac{a_c - \bar{a}}{2\omega_b\tau}, \quad \tilde{m}_c > m_c^* \Leftrightarrow a_c - \bar{a} < 0$$

The city is underpopulated (overpopulated) when $a_c - \bar{a} < 0$: the level of amenities of the city is lower (larger) than the mean of attractiveness in the system of cities. As $a_1 \geq \dots \geq a_c \geq \dots \geq a_C$, in equilibrium, cities with a high level of attractiveness are oversized, whereas cities with a low level of attractiveness are undersized. Such a result remains valid when workers have symmetric own and mutual rejections:

$$\tilde{m}_c = \frac{1}{C} + \frac{a_c - \bar{a}}{2(\omega_b\tau + \psi)}, \quad \tilde{m}_c > m_c^* \Leftrightarrow a_c - \bar{a} < 0$$

when workers have symmetric own and mutual attractions:¹⁶

$$\tilde{m}_c = \frac{1}{C} + \frac{a_c - \bar{a}}{2(\omega_b\tau - \psi)}, \quad \tilde{m}_c > m_c^* \Leftrightarrow a_c - \bar{a} < 0$$

¹⁴Note that $m_c = m_{r,c} + m_{b,c}$ indicates the number of inhabitants in city $c \in \mathcal{C}$, and also measures city expansion as city size is $\tilde{x} = m_c$.

¹⁵ $\tilde{m}_c > m_c^*$ means that city c is undersized. $\tilde{m}_c < m_c^*$ means that city c is oversized.

¹⁶Here we suppose the uniqueness of a spatial equilibrium, i.e. $\omega_b\tau > \psi$ is verified.

when workers have symmetric own attractions and mutual rejections:¹⁷

$$\tilde{m}_c = \frac{1}{C} + \frac{(\tilde{\omega}\tau - 4\psi)(a_c - \bar{a})}{(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi}, \quad \tilde{m}_c > m_c^* \Leftrightarrow a_c - \bar{a} < 0$$

By contrast, if $\phi_{rr} = \phi_{rb} = \psi$ and $\phi_{bb} = \phi_{br} = \varphi$ with $\psi, \varphi > 0$ and if the following is satisfied:

$$\frac{1}{4\tilde{\omega}\tau(\omega_b\tau - \varphi) + \psi\varphi - (\psi + \varphi)^2} > \frac{1}{2\tilde{\omega}\tau(\omega_b\tau - \varphi)}$$

then

$$\tilde{m}_c > m_c^* \Leftrightarrow a_c - \bar{a} > 0$$

Large cities are undersized, and small cities are oversized. Thus, to replicate the result of Albouy et al. (2019) in our framework, some degree of asymmetry between preferences is required.

2.8.4 Is social mixing too high or too low?

It is well acknowledged that segregation/integration can create a public economic policy issue. The famous Tiebout (1956)'s model concludes that households locate according to their public goods preferences. Many studies report that segregation and mixing shape school choice and school spending (see Nechyba (2003)). More globally, many countries, including the Netherlands and Germany, have implemented policies to promote multiculturalism. Here, we add another result in the literature. We show that the social planner sets a system of cities that exhibits more (less) social mixing compared to what prevails in the decentralized economy. As before, the form of the dissimilarity index is rather complex. To obtain some insights, we consider various peculiar examples. When workers have no city composition preferences, we find:

$$\tilde{\Gamma}_{\omega,\phi} = \frac{m_r}{2\omega_b\tau} < \Gamma_{\omega,\phi}^* = \frac{m_r}{\omega_b\tau}$$

The centralized economy achieves more social mixing than the decentralized economy. Again, the reason lies with the strength of congestion costs. In fact, congestion forces are stronger in the centralized economy. This prompts the social planner to implement more dispersed spatial distributions, causing more social mixing. This finding is robust to the inclusion of symmetric own and mutual rejections:

$$\tilde{\Gamma}_{\omega,\phi} = \frac{m_r}{2(\omega_b\tau + \psi)} < \Gamma_{\omega,\phi}^* = \frac{m_r}{\omega_b\tau + \psi}$$

to the inclusion of symmetric own and mutual attractions:

$$\tilde{\Gamma}_{\omega,\phi} = \frac{m_r}{2|\omega_b\tau - \psi|} < \Gamma_{\omega,\phi}^* = \frac{m_r}{|\omega_b\tau - \psi|}$$

¹⁷Again, we suppose uniqueness of a spatial equilibrium, i.e. $(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi > 0$ is verified.

or to the inclusion of symmetric own attractions and mutual rejections:

$$\tilde{\Gamma}_{\omega,\phi} = \frac{1}{2} \times \left| \frac{2\psi m_b + (\tilde{\omega}\tau - 2\psi)m_r}{(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi} \right| < \Gamma_{\omega,\phi}^* = \left| \frac{2\psi m_b + (\tilde{\omega}\tau - 2\psi)m_r}{(\tilde{\omega}\tau - 3\psi)\omega_b\tau - \omega_r\tau\psi} \right|$$

Conversely, if $\phi_{rr} = \phi_{rb} = \psi$ and $\phi_{bb} = \phi_{br} = \varphi$ with $\psi, \varphi > 0$, then the following can be verified:

$$\Gamma_{\omega,\phi}^* = \left| \frac{(\varphi - \psi)m_b - [\tilde{\omega}\tau - (\varphi - \psi)]m_r}{\tilde{\omega}\tau(\omega_b\tau - \varphi)} \right| < \tilde{\Gamma}_{\omega,\phi} = \left| \frac{(\varphi - \psi)m_b - [2\tilde{\omega}\tau - (\varphi - \psi)]m_r}{4\tilde{\omega}\tau(\omega_b\tau - \varphi) + \psi\varphi - (\psi + \varphi)^2} \right|$$

As before, to have an optimal allocation generating less social mixing, one needs to introduce the suitable degree of asymmetry between preferences.

2.9 Restoration of optimality

The natural question to ask is whether the economist has a ready policy to restore the efficiency of the decentralized economy. In a system of cities, Henderson (1974), Henderson and Becker (2000), and Albouy et al. (2019) examine the effect of subsidies managed by competitive local land developers (See Section 2.9.1). In urban economics, it is common to explore the influence of subsidies introduced by a (central) government (See Section 2.9.2).

2.9.1 A possible instrument

A standard remedy in a system of cities is to introduce subsidies implemented by local land developers. Henderson (1974) and Henderson and Becker (2000) assess the virtue of such an instrument with homogeneous workers and homogeneous cities. Albouy et al. (2019) shows that this result remains valid when cities are heterogeneous. Therefore, we test if such subsidies can restore the optimality in our framework. To do that, we assume that the system of cities is now populated by C local land developers that act as competitive agents (i.e. there is no collusion/cooperation). These local land developers receive land rent and can attract workers by setting subsidies. We assume that local land developers behave in a perfectly information framework such that they can discriminate workers based on their type. Consequently, they offer two different subsidies: $s_{r,c} \in \mathbb{R}$ for red workers and $s_{b,c} \in \mathbb{R}$ for blue workers. Under his new setup, the program of the local land developer that owns city $c \in \mathcal{C}$ is the following:

$$\max_{m_{r,c}, m_{b,c}} \{ALR_c - s_{r,c}m_{r,c} - s_{b,c}m_{b,c}\}$$

under the following constraints:

$$\omega_r - (\omega_r\tau + \phi_{rr})m_{r,c} - (\omega_b\tau + \phi_{rb})m_{b,c} + s_{r,c} = v_r$$

and

$$\omega_b - (\omega_b\tau + \phi_{br})m_{r,c} - (\omega_b\tau + \phi_{bb})m_{b,c} + s_{b,c} = v_b$$

ALR_c denotes aggregate land rents in city $c \in \mathcal{C}$ such that $ALR_c = ALR_{r,c} + ALR_{b,c}$. $ALR_{r,c}$ is the land rent paid by red workers in city c . It is given by:

$$ALR_{r,c} = \int_0^{m_{r,c}} R_c^*(x)dx = \frac{\omega_r\tau m_{r,c}^2}{2} + \omega_b\tau m_{r,c}m_{b,c}$$

Similarly, $ALR_{b,c}$ is the land rent paid by blue workers in city c such that:

$$ALR_{b,c} = \int_{m_{r,c}}^{m_{r,c}+m_{b,c}} R_c^*(x)dx = \frac{\omega_b\tau m_{b,c}^2}{2}$$

By substitution, the criterion collapses to:

$$\omega_r m_{r,c} + \omega_b m_{b,c} + a_c m_c - \frac{\omega_r\tau m_{r,c}^2 + \omega_b\tau m_{b,c}^2}{2} - \phi_{rr} m_{r,c}^2 - \phi_{bb} m_{b,c}^2 - (\omega_b\tau + \phi_{rb} + \phi_{br}) m_{r,c} m_{b,c} \quad (23)$$

In addition, as rents are now directly distributed to local land developers, land rents can be viewed as transfers between economic actors. This marginally modifies the problem of the social planner. Land rents are now excluded, and only commuting costs remain operative. To see that, ATC_c denotes the value for aggregate transportation costs in city $c \in \mathcal{C}$. ATC_c is the sum of $ATC_{r,c}$ and $ATC_{b,r}$ the aggregate transportation costs for red and blue workers. ATC_c are determined as follows:

$$ATC_{r,c} = \int_0^{m_{r,c}} \omega_r\tau x dx = \frac{\omega_r\tau m_{r,c}^2}{2}$$

and $ATC_{b,c}$ are given by:

$$ATC_{b,c} = \int_{m_{r,c}}^{m_{r,c}+m_{b,c}} \omega_b\tau x dx = \frac{\omega_b\tau (m_{b,c}^2 + 2m_{r,c}m_{b,c})}{2}$$

Under this new environment, the criterion of the social planner becomes:

$$\sum_{c=1}^C \left\{ \omega_r m_{r,c} + \omega_b m_{b,c} + a_c m_c - ATC_c - \phi_{rr} m_{r,c}^2 - \phi_{bb} m_{b,c}^2 - (\phi_{rb} + \phi_{br}) m_{r,c} m_{b,c} \right\}$$

that is

$$\sum_{c=1}^C \left\{ (\omega_r m_{r,c} + \omega_b m_{b,c} + a_c m_c - \frac{\omega_r\tau m_{r,c}^2 + \omega_b\tau m_{b,c}^2}{2} - \phi_{rr} m_{r,c}^2 - \phi_{bb} m_{b,c}^2 - (\omega_b\tau + \phi_{rb} + \phi_{br}) m_{r,c} m_{b,c}) \right\}$$

The criterion (23) is equivalent from that of the social planner. This implies that subsidies introduced by perfectly competitive land developers is sufficient to correct the inefficiency in the system of cities.¹⁸

¹⁸If land rents are not considered as transfers, this results does not remain valid. The criterion (23) is different from that of the social planner (see equation (20)). In that case, subsidies introduced by perfectly competitive land developers is not sufficient to correct the inefficiency in the system of cities.

2.9.2 Another possible instrument

In urban economics, another standard remedy is to consider local subsidies implemented by a government. As for local land developers, the government has perfect information and can discriminate against workers according to their type. Consequently, the government sets two subsidies $\tilde{s}_{r,c} \in \mathbb{R}$ and $\tilde{s}_{b,c} \in \mathbb{R}$ in each city $c \in \mathcal{C}$. These subsidies are financed by a lump-sum tax that is applicable to all workers. The tax is denoted by $\tilde{s} \in \mathbb{R}$ and defined as follows:

$$\bar{s} = \frac{1}{M} \sum_{c=1}^C \{\tilde{s}_{r,c} + \tilde{s}_{b,c}\}$$

Under this new environment, the indirect utility functions boils down to the following

$$v_{r,c} = \omega_r - (\omega_r\tau + \phi_{rr})m_{r,c}^* - (\omega_b\tau + \phi_{rb})m_{b,c}^* + \tilde{s}_{r,c} - \tilde{s}$$

and

$$v_{b,c} = \omega_b - (\omega_b\tau + \phi_{br})m_{r,c}^* - (\omega_b\tau + \phi_{bb})m_{b,c}^* + \tilde{s}_{r,c} - \tilde{s}$$

Using Proposition 6, we obtain the following.

Proposition 7

If the following is fulfilled:

$$\tilde{s}_{r,c} = -(\omega_r\tau + \phi_{rr})m_{r,c} - (\omega_b\tau + \phi_{br})m_{b,c} \quad \tilde{s}_{b,c} = -(\omega_b\tau + \phi_{rb})m_{r,c} - (\omega_b\tau + \phi_{bb})m_{b,c}$$

then the decentralized economy achieves efficient allocation.

The use of local subsidies implemented by a government is sufficient to eliminate inefficiencies. The values of the subsidies are intuitive. They correspond to what workers disregard (see (10)–(11) and (21)–(22)).

3 Conclusions

We build a unified framework for a system of heterogeneous cities with heterogeneous workers. We offer a full analytical characterization of the decentralized and centralized economies. Closed-form solutions help to derive testable predictions that link the standard dissimilarity index to measurable variables. Contrary to conventional wisdom, we show that social mixing can constitute a unique and stable equilibrium. We also emphasize that such a configuration appears under a wide set of parameters. Consequently, social mixing frequently emerges as an outcome in our new framework, which is in accordance with the data. We disentangle the different economic factors that shape within- and across-city social/income dispersion. We underline the different effects of demographic factors,

amenities, income inequalities, transportation costs, and social preference. We also find that individuals' preferences be overturned at an aggregate scale, which is consistent with Schelling paradoxical findings. We then offer a rationale for these counterintuitive results. Last, we demonstrate that the decentralized economy is not optimal. Depending on the nature of social preferences, large (small) cities are oversized (undersized), and the level of social mixing can be too high or too low. However, it is possible to restore efficiency through local subsidies managed by a central government or by local land developers.

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A Proofs

Proof 1

Consider a city $c \in \mathcal{C}$. Assume that $\omega_r > \omega_b$. If so, red workers bear higher transportation costs than blue workers. This yields:

$$\left| \frac{\partial \Psi_{r,c}(x, v_{r,c})}{\partial x} \right| > \left| \frac{\partial \Psi_{b,c}(x, v_{b,c})}{\partial x} \right|$$

which implies that red workers live near their workplace while blue workers reside on the fringes of the city such that:

$$\mu_{r,c}^*(x) = \begin{cases} 1 & \forall x \in [0, \tilde{x}_c] \\ 0 & \forall x \in [\tilde{x}_c, \check{x}_c] \end{cases}$$

and

$$\mu_{b,c}^*(x) = \begin{cases} 0 & \forall x \in [0, \tilde{x}_c] \\ 1 & \forall x \in [\tilde{x}_c, \check{x}_c] \end{cases}$$

On the fringes of the city, the following must hold:

$$\Psi_{b,c}(\check{x}_c, v_{b,c}) = 0 \Leftrightarrow v_{b,c} = \omega_b - \omega_b \tau \check{x}_c$$

Then, computing $\int_0^{\check{x}_c} \mu_{b,c}^*(x) dx = m_{b,c}$ gives $\check{x}_c = m_{r,c} + m_{b,c}$, and implies the following:

$$v_{b,c} = \omega_b - \omega_b \tau m_{r,c} - \omega_b \tau m_{b,c}$$

that is

$$v_{b,c} = \omega_b + a_c - (\omega_b \tau + \phi_{br}) m_{r,c} - (\omega_b \tau + \phi_{bb}) m_{b,c}$$

by abuse of notation. In addition, \tilde{x}_c verifies the following:

$$\Psi_{r,c}(\tilde{x}_c, v_{r,c}) = \Psi_{b,c}(\tilde{x}_c, v_{b,c}) \Leftrightarrow \tilde{v}_c = \tilde{\omega} - \tilde{\omega} \tau \tilde{x}_c$$

with $\tilde{v} = v_r - v_b$ and $\tilde{\omega} = \omega_r - \omega_b$. Then, $\int_0^{\tilde{x}_c} \mu_{r,c}^*(x) dx = m_{r,c}$ gives $\tilde{x}_c = m_{r,c}$, and yields the following:

$$\tilde{v}_c = \tilde{\omega} - \tilde{\omega} \tau m_{r,c}$$

As a consequence, we obtain:

$$v_{r,c} = \omega_r - \omega_r \tau m_{r,c} - \omega_b \tau m_{b,c}$$

that is

$$v_{r,c} = \omega_r + a_c - (\omega_r \tau + \phi_{rr}) m_{r,c} - (\omega_b \tau + \phi_{rb}) m_{b,c}$$

by abuse of notation. Using the previous results, the following is verified:

$$R_c^*(x) = \begin{cases} \omega_r \tau m_{r,c} + \omega_b \tau m_{b,c} - \omega_r \tau x & \forall x \in [0, m_{r,c}] \\ \omega_b \tau m_{r,c} + \omega_b \tau m_{b,c} - \omega_b \tau x & \forall x \in [m_{r,c}, m_{r,c} + m_{b,c}] \end{cases}$$

and consumption functions are given by:

$$z_{r,c} = \omega_r - \omega_r \tau m_{r,c} - \omega_b \tau m_{b,c}$$

$$z_{b,c} = \omega_b - \omega_b \tau m_{r,c} - \omega_b \tau m_{b,c}$$

Last, we assume that the following holds:

$$\omega_r(1 - \tau m_r) - \omega_b \tau m_b > 0$$

$$\omega_b(1 - \tau m_b) - \omega_b \tau m_r > 0$$

to ensure positive consumption.

Assume that $\omega_r = \omega_b = \omega$. Adapting the previous part gives:

$$v_{r,c} = \omega + a_c - (\omega \tau + \phi_{rr})m_{r,c} - (\omega \tau + \phi_{rb})m_{b,c}$$

$$v_{b,c} = \omega + a_c - (\omega \tau + \phi_{br})m_{r,c} - (\omega \tau + \phi_{bb})m_{b,c}$$

$$R_c^*(x) = \omega \tau (m_c - x) \quad \forall x \in [0, m_c]$$

and

$$z_c = \omega(1 - \tau m_c) > 0 \Rightarrow \omega(1 - \tau) > 0$$

as $m_c \leq M = 1$. \diamond

Proof 2: Uniqueness

The proof for uniqueness is operated according to different steps.

• Step 0: Some reformulations

Define the following:

$$v_r(c, m_{r,c}, m_{b,c}) = \omega_r + a_c - (\omega_r \tau + \phi_{rr})m_{r,c} - (\omega_b \tau + \phi_{rb})m_{b,c}$$

and

$$v_b(c, m_{r,c}, m_{b,c}) = \omega_b + a_c - (\omega_b \tau + \phi_{br})m_{r,c} - (\omega_b \tau + \phi_{bb})m_{b,c}$$

• **Step 1: Rewriting the spatial equilibrium as a VI**

Note that if $m_r^* \in \Delta_r^C$ is a spatial equilibrium then it satisfies:

$$\text{Supp}(m_r^*) \subset \arg \max_{c \in \mathcal{C}} v_r(c, m_{r,c}^*, m_{b,c}^*)$$

where $\text{Supp}(m_r^*) = \{c \in \mathcal{C} : m_{r,c}^* > 0\}$ is the support of m_r^* . Equivalently, m_r^* satisfies the following VI:

$$\sum_{c=1}^C v_r(c, m_{r,c}^*, m_{b,c}^*) (m_{r,c}^* - m'_{r,c}) \geq 0 \quad \forall m'_r \in \Delta_r^C$$

Similarly, if $m_b^* \in \Delta_b^C$ is a spatial equilibrium then it satisfies

$$\text{Supp}(m_b^*) \subset \arg \max_{c \in \mathcal{C}} v_b(c, m_{r,c}^*, m_{b,c}^*)$$

where $\text{Supp}(m_b^*) = \{c \in \mathcal{C} : m_{b,c}^* > 0\}$ is the support of m_b^* . That is, m_b^* verifies the following VI:

$$\sum_{c=1}^C v_b(c, m_{r,c}^*, m_{b,c}^*) [m_{b,c}^* - m'_{b,c}] \geq 0 \quad \forall m'_b \in \Delta_b^C$$

Summing the VI yields:

$$\sum_{c=1}^C v_r(c, m_{r,c}^*, m_{b,c}^*) (m_{r,c}^* - m'_{r,c}) + \sum_{c=1}^C v_b(c, m_{r,c}^*, m_{b,c}^*) (m_{b,c}^* - m'_{b,c}) \geq 0$$

$\forall m'_r \in \Delta_r^C$ and $m'_b \in \Delta_b^C$.

Step 2: Definition of the monotonicity

$v = (v_r, v_b)$ is said to be strictly monotone if:

$$\begin{aligned} & \sum_{c=1}^C [v_r(c, m_{r,c}^*, m_{b,c}^*) - v_r(c, m'_{r,c}, m'_{b,c})] (m_{r,c}^* - m'_{r,c}) \dots \\ & \dots + \sum_{c=1}^C [v_b(c, m_{r,c}^*, m_{b,c}^*) - v_b(c, m'_{r,c}, m'_{b,c})] (m_{b,c}^* - m'_{b,c}) < 0 \end{aligned}$$

$\forall (m_r^*, m_b^*), (m'_r, m'_b) \in \Delta_r^C \times \Delta_b^C$ with $(m_r^*, m_b^*) \neq (m'_r, m'_b)$.

Step 3: Monotonicity as a sufficient condition for uniqueness

Let $(m_r^*, m_b^*), (m'_r, m'_b) \in \Delta_r^C \times \Delta_b^C$ two spatial equilibria with $(m_r^*, m_b^*) \neq (m'_r, m'_b)$. By definition, we have:

$$\sum_{c=1}^C v_r(c, m_{r,c}^*, m_{b,c}^*) (m_{r,c}^* - m'_{r,c}) + \sum_{c=1}^C v_b(c, m_{r,c}^*, m_{b,c}^*) (m_{b,c}^* - m'_{b,c}) \geq 0$$

and

$$\sum_{c=1}^C v_r(c, m'_{r,c}, m'_{b,c}) (m'_{r,c} - m^*_{r,c}) + \sum_{c=1}^C v_b(c, m'_{r,c}, m'_{b,c}) (m'_{b,c} - m^*_{b,c}) \geq 0$$

Using the two VI gives:

$$\begin{aligned} & \sum_{c=1}^C \left[v_r(c, m^*_{r,c}, m^*_{b,c}) - v_r(c, m'_{r,c}, m'_{b,c}) \right] (m^*_{r,c} - m'_{r,c}) \dots \\ & \dots + \sum_{c=1}^C \left[v_b(c, m^*_{r,c}, m^*_{b,c}) - v_b(c, m'_{r,c}, m'_{b,c}) \right] (m^*_{b,c} - m'_{b,c}) \geq 0 \end{aligned}$$

Assume that v is strictly monotone such that:

$$\begin{aligned} & \sum_{c=1}^C \left[v_r(c, m^*_{r,c}, m^*_{b,c}) - v_r(c, m'_{r,c}, m'_{b,c}) \right] (m^*_{r,c} - m'_{r,c}) \dots \\ & \dots + \sum_{c=1}^C \left[v_b(c, m^*_{r,c}, m^*_{b,c}) - v_b(c, m'_{r,c}, m'_{b,c}) \right] (m^*_{b,c} - m'_{b,c}) < 0 \end{aligned}$$

This implies $(m_r^*, m_b^*) = (m'_r, m'_b)$, that is, a spatial equilibrium $(m_r^*, m_b^*) \in \Delta_r^C \times \Delta_b^C$ is unique.

Step 4: Some sufficient conditions for that v to be strictly monotone

Define the following matrices:

$$\Phi(m_r, m_b) = \begin{pmatrix} \frac{\partial v_r}{\partial m_r} & \frac{\partial v_r}{\partial m_b} \\ \frac{\partial v_b}{\partial m_r} & \frac{\partial v_b}{\partial m_b} \end{pmatrix} = \begin{pmatrix} -(\omega_r \tau + \phi_{rr}) & -(\omega_b \tau + \phi_{rb}) \\ -(\omega_b \tau + \phi_{br}) & -(\omega_b \tau + \phi_{bb}) \end{pmatrix}$$

and

$$\Phi(m_r, m_b) + \Phi(m_r, m_b)^T = \begin{pmatrix} -2(\omega_r \tau + \phi_{rr}) & -(2\omega_b \tau + \phi_{rb} + \phi_{br}) \\ -(2\omega_b \tau + \phi_{rb} + \phi_{br}) & -2(\omega_b \tau + \phi_{bb}) \end{pmatrix}$$

As v_r and v_b are differentiable and using the Euler-Lagrange theorem, if the following holds:

$$\Phi(m_r, m_b) + \Phi(m_r, m_b)^T \quad \text{is definite negative}$$

then v is strictly monotone. Therefore, if the following conditions are verified:

$$-2(\omega_r \tau + \omega_b \tau + \phi_{rr} + \phi_{bb}) < 0, \quad 4(\omega_r \tau + \phi_{rr})(\omega_b \tau + \phi_{bb}) > (2\omega_b \tau + \phi_{rb} + \phi_{br})^2$$

then $\Phi(m_r, m_b) + \Phi(m_r, m_b)^T$ is negative definite, and a spatial equilibrium is unique. Equivalently, if the following conditions are verified:

$$-(\omega_r \tau + \phi_{rr}) < 0, \quad (\omega_r \tau + \phi_{rr})(\omega_b \tau + \phi_{bb}) > (\omega_b \tau + \phi_{rb})(\omega_b \tau + \phi_{br})$$

then $\Phi(m_r, m_b) + \Phi(m_r, m_b)^T$ is negative definite, and a spatial equilibrium is unique. \diamond

Proof 2: Stability

A spatial equilibrium $(m_r^*, m_b^*) \in \Delta_r^C \times \Delta_b^C$ is stable if:

$$\sum_{c=1}^C v_r(c, m'_{r,c}, m'_{b,c}) (m_{r,c}^* - m'_{r,c}) + \sum_{c=1}^C v_b(c, m'_{r,c}, m'_{b,c}) (m_{b,c}^* - m'_{b,c}) > 0$$

$\forall (m'_r, m'_b) \in \Delta_r^C \times \Delta_b^C$ with $(m_r^*, m_b^*) \neq (m'_r, m'_b)$. Since (m_r^*, m_b^*) is a spatial equilibrium, it satisfies:

$$\sum_{c=1}^C v_r(c, m_{r,c}^*, m_{b,c}^*) (m_{r,c}^* - m'_{r,c}) + \sum_{c=1}^C v_b(c, m_{r,c}^*, m_{b,c}^*) (m_{b,c}^* - m'_{b,c}) \geq 0$$

$\forall m'_r \in \Delta_r^C$ and $m'_b \in \Delta_b^C$. In addition, assume that v is strictly monotone such that:

$$\begin{aligned} & \sum_{c=1}^C \left[v_r(c, m_{r,c}^*, m_{b,c}^*) - v_r(c, m'_{r,c}, m'_{b,c}) \right] (m_{r,c}^* - m'_{r,c}) \dots \\ & \dots + \sum_{c=1}^C \left[v_b(c, m_{r,c}^*, m_{b,c}^*) - v_b(c, m'_{r,c}, m'_{b,c}) \right] (m_{b,c}^* - m'_{b,c}) < 0 \end{aligned}$$

Adding these relationships leads to:

$$\sum_{c=1}^C v_r(c, m'_{r,c}, m'_{b,c}) (m_{r,c}^* - m'_{r,c}) + \sum_{c=1}^C v_b(c, m'_{r,c}, m'_{b,c}) (m_{b,c}^* - m'_{b,c}) > 0$$

which implies that (m_r^*, m_b^*) is stable. Then use “Proof 2: Uniqueness” to complete the proof. \diamond

Proof 3

Consider the following system:

$$v_{r,c} = \omega_r + a_c - \varphi_{rr} m_{r,c} - \varphi_{rb} m_{b,c}$$

$$v_{b,c} = \omega_b + a_c - \varphi_{br} m_{w,c} - \varphi_{bb} m_{b,c}$$

with $\varphi_{rr} = \omega_r \tau + \phi_{rr}$, $\varphi_{bb} = \omega_b \tau + \phi_{bb}$, $\varphi_{rb} = \omega_b \tau + \phi_{rb}$ and $\varphi_{br} = \omega_b \tau + \phi_{br}$. Assume that $m_{r,c} > 0$ and $m_{b,c} > 0 \quad \forall c \in \mathcal{C}$. Summing $v_{b,c}$ over $c \in \mathcal{C}$ and noting that $v_{b,c} = v_b^*$ in equilibrium gives:

$$v_b^* = \frac{\omega_b C + \sum_{c=1}^C a_c - \varphi_{br} m_r - \varphi_{bb} m_b}{C}$$

Doing the same for $v_{r,c}$ yields:

$$v_r^* = \frac{\omega_r C + \sum_{c=1}^C a_c - \varphi_{rr} m_r - \varphi_{rb} m_b}{C}$$

After simple algebra, the following is verified in equilibrium:

$$m_{b,c}^* = \frac{\omega_b + a_c - v_b^*}{\varphi_{bb}} - \frac{\varphi_{br}}{\varphi_{bb}} m_{r,c}^*$$

Similarly for $m_{r,c}^*$:

$$m_{r,c}^* = \frac{\omega_r + a_c - v_r^*}{\varphi_{rr}} - \frac{\varphi_{br}}{\varphi_{rr}} m_{b,c}^*$$

In addition, plugging $m_{r,c}^*$ into the expression of $m_{b,c}^*$ leads to:

$$m_{b,c}^* = \frac{(\omega_b + a_c - v_b^*)\varphi_{rr} - (\omega_r + a_c - v_r^*)\varphi_{br}}{\varphi_{rr}\varphi_{bb} - \varphi_{rb}\varphi_{br}}$$

Likewise, plugging $m_{b,c}^*$ into the expression of $m_{r,c}^*$ leads to:

$$m_{r,c}^* = \frac{(\omega_r + a_c - v_r^*)\varphi_{bb} - (\omega_b + a_c - v_b^*)\varphi_{rb}}{\varphi_{rr}\varphi_{bb} - \varphi_{rb}\varphi_{br}}$$

Using the values of v_b^* and v_r^* gives:

$$\omega_b + a_c - v_b^* = \frac{Ca_c - \sum_{c=1}^C a_c + \varphi_{br}m_r + \varphi_{bb}m_b}{C}$$

and

$$\omega_r + a_c - v_r^* = \frac{Ca_c - \sum_{c=1}^C a_c + \varphi_{rr}m_r + \varphi_{rb}m_b}{C}$$

Introducing these relationships into $m_{b,c}^*$ leads to:

$$m_{b,c}^* = \frac{m_b}{C} + \frac{(\varphi_{rr} - \varphi_{br}) \left(Ca_c - \sum_{c=1}^C a_c \right)}{(\varphi_{rr}\varphi_{bb} - \varphi_{rb}\varphi_{br}) C}$$

Likewise for $m_{r,c}^*$:

$$m_{r,c}^* = \frac{m_r}{C} + \frac{(\varphi_{bb} - \varphi_{rb}) \left(Ca_c - \sum_{c=1}^C a_c \right)}{(\varphi_{rr}\varphi_{bb} - \varphi_{rb}\varphi_{br}) C}$$

These two equations collapse to the following:

$$m_{b,c}^* = \frac{m_b}{C} + \frac{(\tilde{\omega}\tau + \phi_{rr} - \phi_{br})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}$$

$$m_{r,c}^* = \frac{m_r}{C} + \frac{(\phi_{bb} - \phi_{rb})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}$$

and

$$m_c^* = \frac{1}{C} + \frac{(\tilde{\omega}\tau + \phi_{rr} + \phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})}$$

as $\varphi_{rr} = \omega_r\tau + \phi_{rr}$, $\varphi_{bb} = \omega_b\tau + \phi_{bb}$, $\varphi_{rb} = \omega_b\tau + \phi_{rb}$ and $\varphi_{br} = \omega_r\tau + \phi_{br}$ and $\bar{a} = \frac{1}{C} \sum_{c=1}^C a_c$.

To conclude, a mixed system of cities exists if that is $m_{r,c}^* > 0$ and $m_{b,c}^* > 0 \quad \forall c \in \mathcal{C}$, if and only if:

$$\frac{(\tilde{\omega}\tau + \phi_{rr} - \phi_{br})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} > -\frac{m_b}{C}$$

$$\frac{(\phi_{bb} - \phi_{rb})(a_c - \bar{a})}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} > -\frac{m_r}{C}$$

$\forall c \in \mathcal{C}$. \diamond

Proof 4

Trivial using Proof 3. \diamond

Proof 5

The dissimilarity index is the following:

$$I_D = \frac{1}{2} \sum_{c=1}^C \left| \frac{m_{r,c}^*}{m_r} - \frac{m_{b,c}^*}{m_b} \right|$$

Plugging the expressions $m_{r,c}^*$ and $m_{b,c}^*$ (obtained from Proof 3) into I_D gives:

$$I_D = \frac{1}{2m_r m_b C} \times \Gamma_{\omega, \phi} \times \sum_{c=1}^C |a_c - \bar{a}|$$

with Γ being defined as follows:

$$\Gamma_{\omega, \phi} = \left| \frac{(\varphi_{bb} - \varphi_{rb}) m_b - (\varphi_{rr} - \varphi_{br}) m_r}{\varphi_{rr}\varphi_{bb} - \varphi_{rb}\varphi_{br}} \right|$$

This element collapses to:

$$\Gamma_{\omega, \phi} = \left| \frac{(\phi_{bb} - \phi_{rb}) m_b - (\tilde{\omega}\tau + \phi_{rr} - \phi_{br}) m_r}{(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})} \right|$$

as $\varphi_{rr} = \omega_r\tau + \phi_{rr}$, $\varphi_{bb} = \omega_b\tau + \phi_{bb}$, $\varphi_{rb} = \omega_b\tau + \phi_{rb}$ and $\varphi_{br} = \omega_b\tau + \phi_{br}$. \diamond

Proof 6

The Lagrangian for the social planner problem is:

$$\mathcal{L}(m, \xi) = \mathcal{S}(m_r, m_b) + \zeta_r \left(m_r - \sum_{c=1}^C m_{r,c} \right) + \sum_{c=1}^C \mu_{r,c} m_{r,c} + \zeta_b \left(m_b - \sum_{c=1}^C m_{b,c} \right) + \sum_{c=1}^C \mu_{b,c} m_{b,c}$$

$\forall m = (m_r, m_b) \in \Delta_r^C \times \Delta_b^C$ and $\xi = (\zeta_r, \zeta_b, \mu_r, \mu_b) \in \mathbb{R}^{2+2C}$. The Kuhn and Tucker first order conditions (KT-FOC hereafter) are:

$$\omega_r + a_c - 2(\omega_r\tau + \phi_{rr})\tilde{m}_{r,c} - (2\omega_b\tau + \phi_{rb} + \phi_{br})\tilde{m}_{b,c} = \zeta_r - \mu_{r,c}$$

$$\omega_b + a_c - (2\omega_b\tau + \phi_{rb} + \phi_{br})\tilde{m}_{r,c} - 2(\omega_b\tau + \phi_{bb})\tilde{m}_{b,c} = \zeta_b - \mu_{b,c}$$

$$\mu_{r,c}\tilde{m}_{r,c} = 0, \quad \mu_{b,c}\tilde{m}_{b,c} = 0$$

$$\begin{aligned} \mu_{r,c} &\geq 0, & \mu_{b,c} &\geq 0 \\ \sum_{c=1}^C \tilde{m}_{r,c} &= m_r, & \sum_{c=1}^C \tilde{m}_{b,c} &= m_b \end{aligned}$$

$\forall c \in \mathcal{C}$. Then, compute the Hessian matrix denoted by $\tilde{\Phi}$ such that:

$$\tilde{\Phi}(m_r, m_b) = \begin{pmatrix} -2(\omega_r\tau + \phi_{rr}) & -(2\omega_b\tau + \phi_{rb} + \phi_{br}) \\ -(2\omega_b\tau + \phi_{rb} + \phi_{br}) & -2(\omega_b\tau + \phi_{bb}) \end{pmatrix}$$

Note that $\tilde{\Phi}(m_r, m_b) = \Phi(m_r, m_b) + \Phi(m_r, m_b)^T$. Therefore, following Proof 2, $\tilde{\Phi}$ is negative definite if:

$$-(\omega_r\tau + \omega_b\tau + \phi_{rr} + \phi_{bb}) < 0 \quad \text{and} \quad (\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) > (\omega_b\tau + \phi_{rb})(\omega_b\tau + \phi_{br})$$

or

$$-2(\omega_r\tau + \omega_b\tau + \phi_{rr} + \phi_{bb}) < 0 \quad \text{and} \quad 4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) > (2\omega_b\tau + \phi_{rb} + \phi_{br})^2$$

Last, assume that $\tilde{m}_{r,c} > 0$ and $\tilde{m}_{b,c} > 0 \forall c \in \mathcal{C}$. The KT-FOC collapse to:

$$\begin{aligned} \omega_r + a_c - 2(\omega_r\tau + \phi_{rr})\tilde{m}_{r,c} - (2\omega_b\tau + \phi_{rb} + \phi_{br})\tilde{m}_{b,c} &= \zeta_r \\ \omega_b + a_c - (2\omega_b\tau + \phi_{rb} + \phi_{br})\tilde{m}_{r,c} - 2(\omega_b\tau + \phi_{bb})\tilde{m}_{b,c} &= \zeta_b \\ \sum_{c=1}^C \tilde{m}_{r,c} &= m_r, & \sum_{c=1}^C \tilde{m}_{b,c} &= m_b \end{aligned}$$

Using Proof 3 and by identification, we find:

$$\begin{aligned} \tilde{m}_{r,c} &= \frac{m_r}{C} + \frac{(2\phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} \\ \tilde{m}_{b,c} &= \frac{m_b}{C} + \frac{(2\tilde{\omega}\tau + 2\phi_{rr} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} \\ \tilde{m}_c &= \frac{1}{C} + \frac{2(\tilde{\omega}\tau + \phi_{rr} + \phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} \\ \tilde{I}_D &= \frac{1}{2m_r m_b C} \times \tilde{\Gamma}_{\omega, \phi} \times \sum_{c=1}^C |a_c - \bar{a}| \end{aligned}$$

with

$$\tilde{\Gamma}_{\omega, \phi} = \left| \frac{(2\phi_{bb} - \phi_{rb} - \phi_{br})m_b - (2\tilde{\omega}\tau + 2\phi_{rr} - \phi_{rb} - \phi_{br})m_r}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} \right|$$

and a mixed system of cities exists if and only if:

$$\frac{(2\phi_{bb} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} > -\frac{m_r}{C}$$

$$\frac{(2\tilde{\omega}\tau + 2\phi_{rr} - \phi_{rb} - \phi_{br})(a_c - \bar{a})}{4(\omega_r\tau + \phi_{rr})(\omega_b\tau + \phi_{bb}) - (2\omega_b\tau + \phi_{rb} + \phi_{br})^2} > -\frac{m_b}{C}$$

$\forall c \in \mathcal{C}$. \diamond

Proof 7

The KT-FOC for the social planner problem are given by:

$$\omega_r + a_c - 2(\omega_r\tau + \phi_{rr})m_{r,c} - (2\omega_b\tau + \phi_{rb} + \phi_{br})m_{b,c} = \tilde{v}_r$$

and

$$\omega_b + a_c - (2\omega_b\tau + \phi_{rb} + \phi_{br})m_{r,c} - 2(\omega_b\tau + \phi_{bb})m_{b,c} = \tilde{v}_b$$

The indirect utility functions are described by:

$$\omega_r - (\omega_r\tau + \phi_{rr})m_{r,c} - (\omega_b\tau + \phi_{rb})m_{b,c} + \tilde{s}_{r,c} = v_r^* + \tilde{s}$$

and

$$\omega_b - (\omega_b\tau + \phi_{br})m_{r,c} - (\omega_b\tau + \phi_{bb})m_{b,c} + \tilde{s}_{b,c} = v_b^* + \tilde{s}$$

Therefore, if the following holds:

$$\tilde{s}_{r,c} = -(\omega_r\tau + \phi_{rr})m_{r,c} - (\omega_b\tau + \phi_{br})m_{b,c} \quad \tilde{s}_{b,c} = -(\omega_b\tau + \phi_{rb})m_{r,c} - (\omega_b\tau + \phi_{bb})m_{b,c}$$

then

$$\omega_r + a_c - 2(\omega_r\tau + \phi_{rr})m_{r,c} - (2\omega_b\tau + \phi_{rb} + \phi_{br})m_{b,c} = v_r^* + \tilde{s}$$

$$\omega_b + a_c - (2\omega_b\tau + \phi_{rb} + \phi_{br})m_{r,c} - 2(\omega_b\tau + \phi_{bb})m_{b,c} = v_b^* + \tilde{s}$$

and optimality is restored. Last, we need to check that consumption functions are positive in equilibrium. To do that, note that:

$$\tilde{s} = -\frac{(\omega_r\tau + \omega_b\tau + \phi_{rr} + \phi_{rb})m_r + (2\omega_b\tau + \phi_{bb} + \phi_{br})m_b}{M}$$

$$z_{r,c} = \omega_r - (2\omega_r\tau + \phi_{rr})m_{r,c} - (2\omega_b\tau + \phi_{br})m_{b,c} + \frac{(\omega_r\tau + \omega_b\tau + \phi_{rr} + \phi_{rb})m_r + (2\omega_b\tau + \phi_{bb} + \phi_{br})m_b}{M}$$

and

$$z_{b,c} = \omega_b - (2\omega_b\tau + \phi_{rb})m_{r,c} - (2\omega_b\tau + \phi_{bb})m_{b,c} + \frac{(\omega_r\tau + \omega_b\tau + \phi_{rr} + \phi_{rb})m_r + (2\omega_b\tau + \phi_{bb} + \phi_{br})m_b}{M}$$

Sufficient conditions for that $z_{r,c} > 0$ and $z_{b,c} > 0 \quad \forall c \in \mathcal{C}$ are the following:

$$\omega_r + \frac{(\omega_r\tau + \omega_b\tau + \phi_{rr} + \phi_{rb})m_r + (2\omega_b\tau + \phi_{bb} + \phi_{br})m_b}{M} > (2\omega_r\tau + \phi_{rr})m_r + (2\omega_b\tau + \phi_{br})m_b$$

and

$$\omega_b + \frac{(\omega_r\tau + \omega_b\tau + \phi_{rr} + \phi_{rb})m_r + (2\omega_b\tau + \phi_{bb} + \phi_{br})m_b}{M} > (2\omega_b\tau + \phi_{rb})m_r + (2\omega_b\tau + \phi_{bb})m_b$$

\diamond