



HAL
open science

ON THE STRONG β -HYBRID SOLUTION OF AN N-PERSON GAME

Bertrand Crettez, Rabia Nessah, Tarik Tazdaït

► **To cite this version:**

Bertrand Crettez, Rabia Nessah, Tarik Tazdaït. ON THE STRONG β -HYBRID SOLUTION OF AN N-PERSON GAME. *Theory and Decision*, 2023, 94 (3), pp.363-377. 10.1007/s11238-022-09900-0 . hal-04204632

HAL Id: hal-04204632

<https://hal.science/hal-04204632>

Submitted on 12 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE STRONG β -HYBRID SOLUTION OF AN N-PERSON GAME

Bertrand Crettez* Rabia Nessah[†] Tarik Tazdait^{‡§}

November 21, 2022

Abstract

We propose a new notion of coalitional equilibria, the strong β -hybrid solution, which is a refinement of the hybrid solution introduced by Zhao. Zhao's solution is well suited to study situations where people cooperate within coalitions but where coalitions compete with one another. This paper's solution, as opposed to the hybrid solution, assigns to each coalition a strategy profile that is strongly-Pareto optimal. Moreover, like the β -core, deviations by subcoalitions of any existing coalition are deterred by the threat of a unique counter-strategy available to the non-deviating players. Zhao proved the existence of existence of strong β -hybrid solution for transferable utility games with compact and convex strategy spaces and concave continuous payoff functions. Here, we extend his result to non-transferable utility games.

Keywords: α -core, β -core, N -person game, coalition structure, hybrid solution, strong hybrid solution

1 INTRODUCTION

To study cooperation in strategic non-transferable utility games, one can rely on the concepts of coalitional equilibria. This paper refines one such concept, namely Zhao [1992]'s hybrid solution. A hybrid solution is a strategy profile that is stable against deviations by every subcoalition of any coalition included in a given partition of the players set. More precisely, any subcoalition of a

*Université Panthéon-Assas, Paris II, CRED, EA 7321, 21 Rue Valette - 75005 Paris, France. contact: bertrand.crettez@u-paris2.fr.

[†]IESEG School of Management, LEM, CNRS, Lille, France. contact: r.nessah@ieseg.fr.

[‡]CIREA, CNRS, EHESS, Ecole des Ponts, Nogent sur Marne, France. contact: tazdait@centre-cired.fr.

[§]We thank an anonymous referee as well as editor for stimulating and constructive comments on a previous version of this work.

given coalition can be prevented from not choosing the solution because the players remaining in the coalition can use suitable blocking strategies. Zhao (*ibid*) showed the existence of the hybrid solution (see also Yang and Yuan [2019] for some refinements).

Zhao's solution concept is especially well suited to study the cases where there is cooperation inside coalitions, but competition between coalitions, as well as in the cases where only some coalitions are worth considering. Indeed, in many "real-world" situations some coalitions are unlikely to form so different are the views of their members on the issue at hand.¹ Interestingly, in the *Journal of Economics Perspective*' Winter 2000 Forecasts for the Future of Economics, Allen [2000] has picked hybrid games as one of the three important areas in economic theory in the 21st century (the other two being network games and information economics).

This paper refines the hybrid solution for strategic non-transferable utility games by introducing a new coalitional equilibrium concept, the *strong β -hybrid solution*. Like the hybrid solution, the strong β -hybrid solution is a strategy profile that is stable against deviations by any subcoalition of each coalition belonging to a coalition structure. In contrast to the hybrid solution, however, the deviations are prevented by a *unique* blocking counter-strategy available to the remaining agents of each coalition (all the members of the other coalitions being assumed to stick to the solution strategies). The requirement of the uniqueness of the deterring strategy is connected to the notion of β -core introduced by Aumann [1961, 1959]). An action profile is in the β -core if no coalition can guarantee a higher payoff for each of its members by deviating from this profile, because coalition outsiders can choose a dominant blocking strategy. But the strong β -hybrid solution differs from Aumann's notion in that it pertains to a given coalition structure, and is robust to the deviations undertaken by subcoalitions where only *some* members may strictly benefit from deviating (the other members being unaffected by breaking away from the coalition to which they belong).²

The strong β -hybrid solution is also connected to what Crettez et al. [2021] call the strong hybrid solution, which can be thought-off as a strong α -hybrid solution. The term α refers to Aumann's notion of α -core. An action profile is in the α -core if no coalition can guarantee a higher payoff for each of its members by choosing another strategy profile, independently of the actions taken by the other players. In the same vein, an action profile is a strong hybrid solution if no subcoalition of a coalition of a given coalition structure can choose an alternative profile that strictly benefits at least one of its members. Any such deviation is indeed deterred by suitable counter-

¹By contrast, the coalitional equilibria concepts initially defined by Aumann, namely the α and β cores require that *each* conceivable coalition should not break away from a preably agreed strategy profile.

²That is, not all the members of a deviating sub-coalition need to be strictly better off when opting out of a coalition. Some members can be willing to leave the coalition in order to help others, as long as they do not suffer any loss.

strategies available to the players remaining in the coalition. So the main difference between the strong β -hybrid solution and the strong hybrid one is that in the former blocking strategies are unique, whereas they may depend on the deviations in the latter.

We prove the existence of the strong β -hybrid solution for games with compact and convex strategy spaces and concave payoff functions. Our results imply the existence of the β -core for non-transferable utility games. This is specially interesting since we know from Scarf [1971]’s counter-example that a game can have a non-empty α -core, but an empty β -core. To the best of our knowledge, there are not many proofs of the existence of the β -core for non-transferable utility games.³ The closest paper to the present one is Zhao [1999b] who shows inter alia the existence of a *strong β -hybrid equilibrium* for transferable utility games, where players have continuous concave payoffs defined on compact and convex strategy sets and where a strong separability assumption holds (see, *ibid*, Theorem IV pp. 162).⁴ Here, we extend Zhao’s results to non-transferable utility games, without relying on his strong separability assumption.

The remainder of the paper unfolds as follows. Section 2 sets out formal definitions and establishes some properties of certain coalitional equilibrium concepts. Section 3 establishes the existence of the strong β -hybrid solution for games with compact and convex strategy spaces and concave continuous payoff functions. Section 4 presents some examples of this solution. Section 5 briefly concludes the paper.

2 DEFINITIONS

In this section, we first present our setting. Then, we state some definitions and properties for coalitional equilibria.

2.1 PRELIMINARIES

Consider an n -person game $G = \langle I, X_i, u_i \rangle$ where $I = \{1, \dots, n\}$ is the set of players, X_i (a nonempty set in $\mathbb{R}^{l(i)}$, $l(i) \geq 1$) is player i ’s strategy set and $u_i : X \rightarrow \mathbb{R}$ is player i ’s payoff function, with $X = \prod_{i \in I} X_i$.

³Relying on Scarf [1971], Yano [1990] provided a proof of the existence of the local β -core. Zhao [1999a, 1999b] present general existence results of β -cores for transferable utility games.

⁴The strong separability assumption is satisfied if “the outsider’s action that best punishes (as a second mover) coalition S is also the action that best punishes (as the first mover) each member of the coalition.” (see Zhao, *ibid*, p. 157). Zhao’s approach and the strong separability assumption are used by Meinhardt [2002] to establish the existence of β -equilibrium for common pool games.

For each nonempty coalition S of players, and for any subset K of S , let $-S$ and $S - K$ be the sets defined by $-S = \{i \in I : i \notin S\}$ and $S - K = \{j \in S : j \notin K\}$, respectively. If S is the singleton $\{i\}$, we denote by $-i$ the set of all players but player i . We also denote by $X_S = \prod_{i \in S} X_i$ the set of strategy profiles of coalition S 's members. A coalition structure $\mathfrak{C} = \{P_1, \dots, P_m\}$ is a partition of I . That is, $\bigcup_{i=1, \dots, m} P_i = I$ and $P_i \cap P_j = \emptyset$, for each $i \neq j$. For all coalition structures $\mathfrak{C} = \{P_1, \dots, P_m\}$, we can express any strategy profile $x = (x_1, \dots, x_n) \in X$ as $x = (x_{P_1}, x_{P_2}, \dots, x_{P_m})$, where $x_{P_i} \in X_{P_i}$.

A coalition structure \mathfrak{C} being given, we say that an n -person game G is compact, continuous and concave if for all $i \in I$, X_i is compact, convex, u_i is continuous on X , and the function $z_S \mapsto u_i(x_S, x_{-S})$ is concave for all coalitions S in \mathfrak{C} .

For any two vectors $a, b \in \mathbb{R}^n$, we write $a \succeq b$ if $a_i \geq b_i$ for all $i = 1, \dots, n$, $a \gg b$ if $a_i > b_i$ for all $i = 1, \dots, n$ and $a \succ b$ if $a \succeq b$ and $a \neq b$.

2.2 COALITIONAL EQUILIBRIA

We first recall Zhao's Hybrid solution.

DEFINITION 2.1 (Zhao [1992]) *Given a coalition structure \mathfrak{C} , a feasible strategy $\bar{x} \in X$ is a **hybrid-solution** if, for every coalition S in \mathfrak{C} and for any subcoalition K of S there is no strategy profile x_K in X_K such that $\underline{u}_K(x_K, \bar{x}_{-S}) \gg u_K(\bar{x})$, where $\underline{u}_K(x_K, \bar{x}_{-S}) = \{\inf_{x_{S-K}} u_i(x_K, x_{S-K}, \bar{x}_{-S}), i \in K\}$.*

A hybrid solution is thus a strategy profile that is stable against deviations by every subcoalition of any coalition included in a *given* partition of the players set. More precisely, any subcoalition of a given coalition S can be prevented from deviating from the solution by suitable counter-strategies available to the players remaining in the coalition, assuming that all the other players (in the other coalitions) stick to the solution. The next definition was introduced in Crettez et al. [2021].

DEFINITION 2.2 *Given a coalition structure \mathfrak{C} , a strategy profile \bar{x} in X is a **strong hybrid solution** of game G if for every coalition S in \mathfrak{C} and for any subcoalition K of S , there does not exist x_K in X_K such that for each x_{S-K} in X_{S-K} $u_K(x_K, x_{S-K}, \bar{x}_{-S}) \succ u_K(\bar{x})$, with $u_K(\cdot) = (u_i(\cdot), i \in K)$.*

The main difference between the hybrid and the strong solutions is that in the latter there are no possible deviations for which only some members of the deviating subcoalition benefit from breaking away from the coalition (the other deviant members' payoff remaining unchanged).

Before introducing our notion of strong β -hybrid solution we now recall the formal definition of the β -core of an n -person game introduced by Aumann [1961]. A strategy \bar{x} is in the β -core if for every coalition K , players in $-K$ can choose a strategy \tilde{x}_{-K} such that for each deviation x_K of K , the payoff of any member of K at (x_K, \tilde{x}_{-K}) cannot be higher than the value obtained when all the players stick to \bar{x}_K .

Hence what matters for the β -core is that whoever may be the remaining agents in a given coalition, they can always use the *same* strategy to deter the other coalition members from breaking away (and this property holds for every possible coalition).

We next introduce our notion of strong β -hybrid solution.

DEFINITION 2.3 Given a coalition structure \mathfrak{C} , a strategy profile $\bar{x} \in X$ is a **strong β -hybrid solution** of game G if, for every $S \in \mathfrak{C}$, for any sub-coalition K in S , there exists a strategy $\tilde{x}_{S-K} \in X_{S-K}$ such that there is no deviation $x_K \in X_K$ satisfying $u_K(x_K, \tilde{x}_{S-K}, \bar{x}_{-S}) \succ u_K(\bar{x})$, where $u_K(\cdot) = (u_i(\cdot), i \in K)$.

Thus \bar{x} is a strong β -hybrid solution if, for every coalition $S \in \mathfrak{C}$ and for any sub-coalition $K \subseteq S$, the players in $S - K$ have a strategy \tilde{x}_{S-K} such that for each deviation x_K of K , the payoffs of players in K at $(x_K, \tilde{x}_{S-K}, \bar{x}_{-S})$ cannot be higher than what they would obtain by playing \bar{x}_K when the players of $-K$ play \bar{x}_{-K} . This equilibrium is stable against deviations from any coalition $K \subseteq S$ and for any $S \in \mathfrak{C}$.

The strong β -hybrid solution encapsulates three ideas. Firstly, only a given collection of coalitions matters (that is, not all coalitions need to be considered). Secondly, a subcoalition of an admissible coalition is prevented from deviating from a given strategy by a uniform counter-strategy available to the individuals remaining in the coalition. This is the β -core flavor of the notion (a unique counter-strategy is used by the individuals remaining in the coalition).⁵ Thirdly, deviating strategies needs not bring about a strict increase in the payoffs of *all* its members. Some members' payoffs are indeed allowed to be unaffected by the deviating strategy. To put it differently, we believe that some form of support or altruism should be included in the definition of cooperation at work within a coalition. Thus, a player may agree to deviate in order to help others, even if she does not strictly benefit from the deviation (altruism, or concern for others, however, is limited in that a player shall not self-sacrifice to benefit others). To put it another way, the strong-hybrid solution is strongly Pareto-optimal.

⁵By contrast, the hybrid solution and the strong hybrid solution have an α -core flavor.

It directly follows from the definitions above that any strong β -hybrid solution is both a strong and a hybrid solution. The two next two results also follow immediately from the definition of strong β -hybrid solution.

Property 2.1 *Strong β -hybrid solutions are individually rational, i.e., for each deviation x_i of every player i in any coalition S , there exists a punishment strategy x_{S-i} of players in $S - i$ such that player i cannot be better off when players of $-S$ play \bar{x}_{-S} .*

Property 2.2 *Strong β -hybrid solutions are such that for every coalition $S \in \mathfrak{C}$, the strategy profile \bar{x}_S is a solution to the following multicriteria optimization problem $\langle X_S, \{u_i(\cdot, \bar{x}_{-S}), \}_{i \in S} \rangle$.*

We next consider the relationship between the strong β -hybrid solution and strong Nash equilibrium.

DEFINITION 2.4 (Aumann [1959]) A strategy profile $\bar{x} \in X$ is a **strong Nash equilibrium** of game G if, for every coalition S , there is no $y_S \in X_S$ such that $u_S(y_S, \bar{x}_{-S}) \succ u_S(\bar{x})$.

That is, a strategy profile is a strong equilibrium if no coalition can profitably deviate from this profile. A straightforward implication is that any strong equilibrium is both Pareto efficient and stable with regard to the deviation of any coalition. We then have:

Property 2.3 *Any strong equilibrium is a strong β -hybrid solution.*

We now consider the C -absolute optimal solution (Nessah and Tazdaït [2013]), which is an extension of the absolute optimal solution in multiobjective programming (Zhao [1983]);

DEFINITION 2.5 (Nessah and Tazdaït [2013]) Given a coalition structure \mathfrak{C} , a feasible strategy $\bar{x} \in X$ is an **absolute optimal solution relatively to \mathfrak{C}** (or \mathfrak{C} -absolute optimal solution) of game G if, for every coalition $S \in \mathfrak{C}$, we have $u_j(y_S, \bar{x}_{-S}) \leq u_j(\bar{x})$, for each $j \in S$ and $y_S \in X_S$.

Hence, a strategy profile \bar{x} is an absolute optimal solution relatively to \mathfrak{C} if no player in any coalition S in \mathfrak{C} , can be better off when players in S deviate from their \mathfrak{C} -absolute optimal strategy profile \bar{x}_S . This equilibrium is stable against deviation from any player in to S . The following result follows directly from the definition.

Property 2.4 *Any absolute optimal solution relatively to \mathfrak{C} is a strong β -hybrid solution.*

3 EXISTENCE RESULTS

We now establish a general existence result for the strong β -hybrid solution. To do this we introduce the correspondence $B_{\mathfrak{C}} : X \rightrightarrows X$

$$B_{\mathfrak{C}}(x) = \prod_{S \in \mathfrak{C}} B_S(x)$$

where for each $x \in X$, $S \in \mathfrak{C}$ (a coalition structure \mathfrak{C} being given):

$$B_S(x) = \left\{ \bar{x}_S \in A_S(x) : \sum_{i \in S} u_i(z_S, x_{-S}) \leq \sum_{i \in S} u_i(\bar{x}_S, x_{-S}), \forall z_S \in A_S(x) \right\}$$

and

$$A_S(x) = \left\{ \bar{x}_S \in X_S : \forall K \subsetneq S, \exists z_{S-K} \in X_{S-K}, \text{ such that } \forall y_K \in X_K \right. \\ \left. u_i(y_K, z_{S-K}, x_{-S}) \leq u_i(\bar{x}_S, x_{-S}), \forall i \in K \right\}.$$

The sets $A_S(x)$ include all the strategy profiles \bar{x}_S for which any subcoalition K of S is deterred from not choosing \bar{x}_K by a unique strategy z_{S-K} available to the players in $S - K$ (all the remaining players playing x_{-S}). The sets $B_S(x)$ include all the elements of $A_S(x)$ that maximizes the sum of the payoffs of coalition S 's players (the maximization being performed on the set $A_S(x)$). The correspondence $B_{\mathfrak{C}}$ plays key a role, because as we shall see below its fixed-points are strong β -hybrid-solutions. To show the existence of these solutions, it is then enough to show that correspondence $B_{\mathfrak{C}}$ has a fixed point. The next Theorem provides sufficient conditions for that property to hold.

THEOREM 3.1 *Let a coalition structure \mathfrak{C} be given. Assume that game G is compact, continuous and concave and that the correspondence $A_{\mathfrak{C}}$ (where $A_{\mathfrak{C}}(x) = \prod_{S \in \mathfrak{C}} A_S(x)$) is nonempty valued and lower hemicontinuous. Then game G has at least one strong β -hybrid solution.*

Interestingly, this result implies the existence of the β -core for non-transferable utility games. This is specially interesting since we know from Scarf [1971]'s counter-example that a game can have a non-empty α -core, but an empty β -core.

To prove theorem 3.1, we need two lemmata. The first one relates to the existence of a strong β -hybrid solution to the fixed-points of correspondence $B_{\mathfrak{C}}$.

LEMMA 3.1 *If $\bar{x} \in X$ is a fixed point of $B_{\mathfrak{C}}$, then \bar{x} is a strong β -hybrid solution.*

PROOF. Let $\bar{x} \in X$ be a fixed point of $B_{\mathfrak{C}}$. Since $\bar{x} \in B_{\mathfrak{C}}(\bar{x})$, then $\bar{x} \in A_{\mathfrak{C}}(\bar{x})$ and for each $z \in A_{\mathfrak{C}}(\bar{x})$, we have

$$\sum_{i \in S} u_i(z_S, \bar{x}_{-S}) \leq \sum_{i \in S} u_i(\bar{x}), \text{ for each } S \in \mathfrak{C}. \quad (3.1)$$

Furthermore, as $\bar{x} \in A_{\mathfrak{C}}(\bar{x})$ then for each $S \in \mathfrak{C}$, $K \subsetneq S$, there exists $z_{S-K} \in X_{S-K}$ such that for any $y_K \in X_K$,

$$u_i(y_K, z_{S-K}, \bar{x}_{-S}) \leq u_i(\bar{x}), \forall i \in K. \quad (3.2)$$

If \bar{x} is not a strong β -hybrid solution for game G , then there exist a coalition $S_0 \in \mathfrak{C}$, a sub-coalition $K_0 \subseteq S_0$, such that for each $z_{S_0-K_0} \in X_{S_0-K_0}$ there is a strategy $y_{K_0} \in X_{K_0}$ that satisfies the following inequalities

$$\begin{cases} 1) \text{ for each } i \in K_0, & u_i(\bar{x}) \leq u_i(y_{K_0}, z_{S_0-K_0}, \bar{x}_{-S_0}), \\ 2) \text{ there exists } j \in K_0, & u_j(\bar{x}) < u_j(y_{K_0}, z_{S_0-K_0}, \bar{x}_{-S_0}). \end{cases} \quad (3.3)$$

Then, we deduce that for each $z_{S_0-K_0} \in X_{S_0-K_0}$ there is a strategy $y_{K_0} \in X_{K_0}$ such that

$$\sum_{i \in K_0} u_i(\bar{x}) < \sum_{i \in K_0} u_i(y_{K_0}, z_{S_0-K_0}, \bar{x}_{-S_0}). \quad (3.4)$$

We now distinguish two cases, depending on whether $K_0 = S_0$ or $K_0 \subsetneq S_0$

- 1) Suppose $K_0 = S_0$. Since by (3.3), for each $i \in S_0$ we have $u_i(\bar{x}) \leq u_i(y_{S_0}, \bar{x}_{-S_0})$, for some $y_{S_0} \in X_{S_0}$ and $\bar{x}_{S_0} \in A_{S_0}(\bar{x})$, then by (3.2), $y_{S_0} \in A_{S_0}(\bar{x})$ and consequently by (3.1) and (3.4), we obtain

$$\sum_{i \in S_0} u_i(z_{S_0}, \bar{x}_{-S_0}) \leq \sum_{i \in S_0} u_i(\bar{x}) < \sum_{i \in S_0} u_i(y_{S_0}, \bar{x}_{-S_0}), \text{ for each } z_{S_0} \in A_{S_0}(\bar{x}),$$

which is impossible for $z_{S_0} = y_{S_0}$.

- 2) Suppose $K_0 \subsetneq S_0$. Then by (3.3), for each $z_{S_0-K_0} \in X_{S_0-K_0}$ there exists y_{K_0} in X_{K_0} and $j \in K_0$ such that $u_j(\bar{x}) < u_j(y_{K_0}, z_{S_0-K_0}, \bar{x}_{-S_0})$, which contradicts inequality (3.2).

This completes the proof. ■

Before presenting three other Lemmata needed in the proof of Theorem 3.1 we recall two necessary and sufficient conditions for a correspondence $\Gamma : X \rightrightarrows X$ to be upper hemicontinuous and lower hemicontinuous, respectively.

REMARK 3.1 As $X \subset \prod_{i \in I} \mathbb{R}^{l(i)}$ is first countable,⁶ by Theorem 17.20 2 of Aliprantis and Border [2006] a correspondence $\Gamma : X \rightrightarrows X$ is upper hemicontinuous on X if, and only if, whenever a sequence $\{x^p\}_p$ converges to x and there is a sequence $\{\bar{x}^p\}_p$ such that $\bar{x}^p \in \Gamma(x^p)$ for each $p \geq 1$, then the sequence $\{\bar{x}^p\}_p$ converges to $\bar{x} \in \Gamma(x)$.

Moreover, also because $X \subset \prod_{i \in I} \mathbb{R}^{l(i)}$ is first countable, by Theorem 17.21 2 of Aliprantis and Border [2006] a correspondence $\Gamma : X \rightrightarrows X$ is lower hemicontinuous on X if, and only if, for every $x \in X$, $\bar{x} \in \Gamma(x)$ and any sequence $\{x^p\}_{p \geq 0} \subset X$ converging to x , there exist a subsequence $\{\bar{x}^{\varphi(p)}\}_{p \geq 1} \subset X$ converging to \bar{x} such that $\bar{x}^{\varphi(p)} \in \Gamma(x^{\varphi(p)})$.⁷

LEMMA 3.2 *Suppose that game G is compact, continuous and concave. Also suppose that the correspondence $A_{\mathcal{C}}$ is nonempty valued. Then it is compact, convex valued, and upper hemicontinuous on X .*

PROOF. Since the functions u_i are continuous over the compact set X , the correspondence $A_{\mathcal{C}}$ is compact valued. To show that $A_{\mathcal{C}}$ is upper hemicontinuous it is sufficient to show that whenever a sequence $\{x^p\}_p$ converges to x and there is a sequence $\{\bar{x}^p\}_p$ such that $\bar{x}^p \in A_{\mathcal{C}}(x^p)$ for each $p \geq 1$, then the sequence $\{\bar{x}^p\}_p$ converges to $\bar{x} \in A_{\mathcal{C}}(x)$. Moreover, since $A_{\mathcal{C}}$ is a product of correspondences, by Theorem 17.28 1 Aliprantis and Border [2006]), to show that it is continuous it suffices to show that $A_S : X \rightrightarrows X_S$ is upper hemicontinuous for all S . Let then S in \mathcal{C} be given. For each K in C and for each p there exists z_{S-K}^p such that for all $y_K \in X_K$, $u_i(y_K, z_{S-K}^p, x_{-S}^p) \leq u_i(\bar{x}_S^p, x_S^p)$ for all $i \in K$. Since $z_{S-K}^p \in X_{S-K}$ for all p and X_{S-K} is compact, we can assume without loss of generality that z_{S-K}^p converges to z_{S-K} in X_{S-K} . As the payoff function u_i are continuous it holds that for all $y_K \in X_K$, $u_i(y_K, z_{S-K}, x_{-S}) \leq u_i(\bar{x}_S, x_S)$ for all $i \in K$. But this proves that \bar{x}_S is in $A_S(x)$. Therefore A_S is upper hemicontinuous. Finally since $z_S \mapsto u_i(z_S, x_{-S})$ is concave for each $i \in S$ and for each S , we deduce that $A_{\mathcal{C}}$ is convex valued. ■

LEMMA 3.3 *Suppose that the game G is compact, continuous and concave and the correspondence $A_{\mathcal{C}}$ is nonempty valued. Then, the correspondence $B_{\mathcal{C}}$ is compact and convex valued.*

PROOF. Let $x \in X$. By assumption the set $A_{\mathcal{C}}(x)$ is nonempty and from Lemma 3.2 it is compact. Then, as the function $z_S \mapsto \sum_{i \in S} u_i(z_S, x_{-S})$ is continuous over the compact set $A_{\mathcal{C}}(x)$, there exists $\bar{z} \in A_{\mathcal{C}}(x)$ such that $\sum_{i \in S} u_i(z_S, x_{-S}) \leq \sum_{i \in S} u_i(\bar{z}_S, x_{-S})$, for each $S \in \mathcal{C}$ and $z \in A_{\mathcal{C}}(x)$. Then, $\bar{z} \in B_{\mathcal{C}}(x)$. By continuity of u_i and compactness of $A_{\mathcal{C}}(x)$, $B_{\mathcal{C}}(x)$ is compact

⁶A topological space is called first countable if every point has a countable neighborhood basis.

⁷The definitions apply mutatis mutandis to correspondences $\Gamma X \rightrightarrows Y$ where Y is a subset of \mathbb{R}^l .

(see the arguments used in the previous Lemma). Since $z \mapsto \sum_{i \in S} u_i(z_S, x_{-S})$ is concave and $A_{\mathfrak{C}}(x)$ is convex, it follows that $B_{\mathfrak{C}}(x)$ is convex as well. ■

LEMMA 3.4 *Suppose that the game G is compact, continuous and that the correspondence $A_{\mathfrak{C}}$ is nonempty valued and lower hemicontinuous on X . Then, the correspondence $B_{\mathfrak{C}}$ is upper hemicontinuous on X .*

PROOF. To establish that $B_{\mathfrak{C}}$ is upper hemicontinuous on X it suffices to show that whenever a sequence $\{x^p\}_p$ converges to x and there is a sequence $\{\bar{x}^p\}_p$ such that $\bar{x}^p \in B_{\mathfrak{C}}(x^p)$ for each $p \geq 1$, then the sequence $\{\bar{x}^p\}_p$ converges to $\bar{x} \in B_{\mathfrak{C}}(x)$. Let then $\{x^p\}_p$ and $\{\bar{x}^p\}_p$ be two sequences such that $\{x^p\}_p$ converges to x and such that $\bar{x}^p \in B_{\mathfrak{C}}(x^p)$ for each $p \geq 1$. Without loss of generality, we can suppose that $\{\bar{x}^p\}_p$ converges to \bar{x} .

Since $\bar{x}^p \in B_{\mathfrak{C}}(x^p)$ for all p then $\bar{x}^p \in A_{\mathfrak{C}}(x^p)$ for all p as well. As from Lemma 3.2 $A_{\mathfrak{C}}$ is upper hemicontinuous on X , then \bar{x} is in $A_{\mathfrak{C}}(x)$.

Now suppose that $\bar{x} \notin B_{\mathfrak{C}}(x) = \prod_{S \in \mathfrak{C}} B_S(x)$. Then for some $S_0 \in \mathfrak{C}$, $\bar{x}_{S_0} \notin B_{S_0}(x)$. As $\bar{x}_{S_0} \in A_{S_0}(x)$, then there exists $\tilde{z}_{S_0} \in A_{S_0}(x)$ such that $\sum_{i \in S_0} u_i(\bar{x}_{S_0}, x_{-S_0}) < \sum_{i \in S_0} u_i(\tilde{z}_{S_0}, x_{-S_0})$.

Since $\tilde{z}_{S_0} \in A_{S_0}(x)$ and the sequence $\{x^p\}_p$ converges to x , then as $A_{\mathfrak{C}}$ is lower hemicontinuous, there exists a subsequence $\{\tilde{z}_{S_0}^{\varphi(p)}\}_{p \geq 1} \subset X_{S_0}$ converging to \tilde{z}_{S_0} such that

$$\tilde{z}_{S_0}^{\varphi(p)} \in A_{S_0}(x^{\varphi(p)}), \text{ for each } p. \quad (3.5)$$

That is, for each p , for any sub-coalition $K \subsetneq S_0$, there exists $z_{S_0-K}^{\varphi(p)} \in X_{S_0-K}$ such that for each $y_K \in X_K$ we have

$$u_i(y_K, z_{S_0-K}^{\varphi(p)}, x_{-S_0}^{\varphi(p)}) \leq u_i(\tilde{z}_{S_0}^{\varphi(p)}, x_{-S_0}^{\varphi(p)}), \text{ for each } i \in K.$$

Now, as $\bar{x}^p \in B_{\mathfrak{C}}(x^p)$ for all p , this implies that $\bar{x}_{S_0}^p \in B_{S_0}(x^p)$ for all p . Then $\bar{x}_{S_0}^p \in A_{S_0}(x^p)$ and for each $z_{S_0} \in A_{S_0}(x^p)$, we have $\sum_{i \in S_0} u_i(z_{S_0}, x_{-S_0}^p) \leq \sum_{i \in S_0} u_i(\bar{x}_{S_0}^p, x_{-S_0}^p)$. In particular, $\bar{x}_{S_0}^{\varphi(p)} \in A_{S_0}(x^{\varphi(p)})$ and for each $z_{S_0} \in A_{S_0}(x^{\varphi(p)})$ we have

$$\sum_{i \in S_0} u_i(z_{S_0}, x_{-S_0}^{\varphi(p)}) \leq \sum_{i \in S_0} u_i(\bar{x}_{S_0}^{\varphi(p)}, x_{-S_0}^{\varphi(p)}).$$

By (3.5), we deduce that

$$\sum_{i \in S_0} u_i(\tilde{z}_{S_0}^{\varphi(p)}, x_{-S_0}^{\varphi(p)}) \leq \sum_{i \in S_0} u_i(\bar{x}_{S_0}^{\varphi(p)}, x_{-S_0}^{\varphi(p)}). \quad (3.6)$$

By continuity of the payoff functions u_i and since the sequences $\{\tilde{z}_{S_0}^{\varphi(p)}\}_p$, $\{x^{\varphi(p)}\}_p$, and $\{\bar{x}^{\varphi(p)}\}_p$ converges to \tilde{z}_{S_0} , x and \bar{x} , respectively, we obtain $\sum_{i \in S_0} u_i(\tilde{z}_{S_0}, x_{-S_0}) \leq \sum_{i \in S_0} u_i(\bar{x}_{S_0}, x_{-S_0})$ which contradicts the inequality $\sum_{i \in S_0} u_i(\bar{x}_{S_0}, x_{-S_0}) < \sum_{i \in S_0} u_i(\tilde{z}_{S_0}, x_{-S_0})$. ■

PROOF OF THEOREM 3.1. By Lemmas 3.2-3.4, we deduce that the correspondence $B_{\mathfrak{C}} : X \rightrightarrows X$ is nonempty, compact, convex valued and upper hemicontinuous on X . Then, by Kakutani-Fan-Glicksberg Theorem (see Aliprantis and Border [2006], Corollary 17.55), the correspondence $B_{\mathfrak{C}}$ has a fixed point and by Lemma 3.1, this fixed point is a strong β -hybrid solution. ■

In order to apply Theorem 3.1 one needs to check that the correspondence $A_{\mathfrak{C}}$ is both nonempty and lower hemicontinuous. The ensuing two remarks address these conditions in turn.

REMARK 3.2 Let the coalition structure \mathfrak{C} be such that each coalition S_j contains no more than two players, i.e., $S_j = \{j_1, j_2\}$. Then if the game G is compact and continuous, then $A_{\mathfrak{C}}$ is nonempty valued. To see this let x be in X . Consider any coalition S_j . Under our assumptions one can show by standard arguments that the game where S_j are the sole players (all the other players sticking to the strategy x_{-S_j}) has a Nash equilibrium \bar{x}_{S_j} . Now let $K = \{j_2\}$. Set $z_{S_j - \{j_2\}} = \bar{x}_{j_1}$. Then by definition of Nash equilibrium, for all y_{j_2} in X_{j_2} we have

$$u_{j_2}(y_{j_2}, z_{S_j - \{j_2\}}, x_{-S}) \leq u_{j_2}(\bar{x}_{j_2}, \bar{x}_{j_1}, x_{-S}). \quad (3.7)$$

The same reasoning applies to $K = \{j_1\}$ and thus one sees that \bar{x}_{S_j} is in A_{S_j} .

REMARK 3.3 Recall that $A_{\mathfrak{C}}$ is nonempty valued if and only if, for every $S \in \mathfrak{C}$ and every $x \in X$, there exists $\bar{x}_S \in X_S$ satisfying the following property: for any sub-coalition $K \subsetneq S$, there exists $z_{S-K} \in X_{S-K}$ such that for each $y_K \in X_K$, we have

$$u_i(y_K, z_{S-K}, x_{-S}) \leq u_i(\bar{x}_S, x_{-S}), \text{ for each } i \in K. \quad (3.8)$$

We shall say that $A_{\mathfrak{C}}$ is *strictly-nonempty valued* if inequality (3.8) is strict. The strict nonemptiness of $A_{\mathfrak{C}}$ means that for every coalition $S \in \mathfrak{C}$ and every $x \in X$, there exists a strategy \bar{x}_S such that for any proper sub-coalition K of S ($K \subsetneq S$), coalition $S - K$ can punish any deviation z_K of coalition K by choosing a strategy profile ensuring that the payoff of each subcoalition K member is strictly lower than what it would be with (\bar{x}_S, x_{-S}) . The following result connects strict nonemptiness and lower hemicontinuity of $A_{\mathfrak{C}}$.

PROPOSITION 3.1 *Let a coalition structure \mathfrak{C} be given. Assume that game G is continuous. If $A_{\mathfrak{C}}$ is strictly-nonempty valued, then it is lower hemicontinuous on X .*

PROOF. Let $S \in \mathfrak{C}$, $x \in X$ and $\bar{x}_S \in X_S$ be as in Remark 3.3 (that is, for any sub-coalition $K \subsetneq S$, there exists $z_{S-K} \in X_{S-K}$, for each $y_K \in X_K$ such that: $u_i(y_K, z_{S-K}, x_{-S}) < u_i(\bar{x}_S, x_{-S})$, for each $i \in K$). Choose $\epsilon > 0$ such that for any sub-coalition $K \subsetneq S$, there exists

$z_{S-K} \in X_{S-K}$ satisfying: $u_i(y_K, z_{S-K}, x_{-S}) + \frac{\epsilon}{2} < u_i(\bar{x}_S, x_{-S}) - \frac{\epsilon}{2}$, for each $y_K \in X_K$ and for each $i \in K$. By continuity of u_i , there is a neighborhood $\mathcal{N}(x_{-S})$ of x_{-S} that satisfies the following property: for each $x'_{-S} \in \mathcal{N}(x_{-S})$, for any sub-coalition $K \subsetneq S$, there exists $z_{S-K} \in X_{S-K}$ such that for each $y_K \in X_K$

$$u_i(y_K, z_{S-K}, x'_{-S}) + \frac{\epsilon}{2} < u_i(\bar{x}_S, x'_{-S}). \quad (3.9)$$

Let $\{x^p\}_p$ be any sequence converging to x . Define by $\{\bar{x}_S^p\}_p$ the constant sequence $\bar{x}_S^p = \bar{x}_S$, for each $p \geq 1$. Since $\{x^p\}_p$ converges to x , then for any neighborhood $\mathcal{N}(x_{-S})$ of x_{-S} there is $l_0 \in \mathbb{N}$ such that: $\forall p > l_0, x^p_{-S} \in \mathcal{N}(x_{-S})$. Therefore by (3.9), we obtain that for each $p > l_0$ $u_i(y_K, z_{S-K}, x^p_{-S}) \leq u_i(\bar{x}_S^p, x^p_{-S})$. Since the argument above applies to all coalitions S in \mathfrak{C} and since by Theorem 17.28 2 in Aliprantis and Border [2006] the product of a finite family of lower hemicontinuous correspondences is lower hemicontinuous the Proposition follows. ■

COROLLARY 3.1 *Game G has at least one strong β -hybrid solution if it is compact, continuous and concave and if the correspondence $A_{\mathfrak{C}}$ is strictly-nonempty valued.*

The non-emptiness of correspondence $A_{\mathfrak{C}}$, whether strict or not is a major property for the existence of a strong β -hybrid solution. When does correspondence $A_{\mathfrak{C}}$ has empty value? Emptiness of $A_{\mathfrak{C}}$ occurs when there is a certain coalition S , and a certain subcoalition K of S in which there is at least one individual who can improve her payoff, whatever the counter-strategies used by the remaining players in $S - K$. In that connection we see that strong β -equilibrium are unlikely to exist whenever players' decisions have limited negative effects on the payoffs of their teammates. That is because, in that case there is no effective deterrence of deviating behaviors.

4 EXAMPLES

This section gives three examples of games that are compact, continuous, concave and for which there exists a strong β -hybrid equilibrium because in each case the correspondence $A_{\mathfrak{C}}$ is strictly-nonempty valued (that is, Corollary 3.1 applies in each example).

4.1 A garbage game

Consider a version of Shapley and Shubik [1969]'s garbage game, in which four neighbors decide on where to dump their garbage. Assume that each neighbor has a quantity of garbage equal to one. Let $x_{ij} \geq 0$ be the quantity of garbage that neighbor i locates in the garden of neighbor j . For each $i, j, 0 \leq \sum_{j=1}^4 x_{ij} \leq 1, 0 \leq x_{ij}$.

Suppose that all agents i have the same payoff function

$$u_i(x) = -\alpha \left(\sum_{j=1}^4 x_{ji} \right), \quad 0 < \alpha \quad (4.1)$$

and assume that there are only two coalitions. The first coalition includes neighbors 1, 2, 3 while the second only includes neighbor 4.

Let x be in \mathbb{R}^{16} and such that for each i, j , $0 \leq \sum_{j=1}^4 x_{ij} \leq 1$, $0 \leq x_{ij}$.

Let us concentrate on the coalition $\{1, 2, 3\}$. Set $\bar{x}_{ij} = 0$, $i, j = 1, 2, 3$. Consider the subcoalition $K = \{1, 2\}$. Also define $z_{31} = z_{32} = 1/2$. Then, players 1 and 2 do not break away from coalition S if

$$-\alpha(1 + x_{41}) < -\alpha x_{41}. \quad (4.2)$$

That conditions always holds since $\alpha > 0$. The same argument applies for any subcoalition K of $S = \{1, 2, 3\}$. Then $A_{\mathcal{C}}$ is strictly-nonempty valued. One can check that any vector \bar{x} in \mathbb{R}_+^{16} such that $\bar{x}_{ij} = 0$, $i, j = 1, 2, 3$, $\bar{x}_{ij} = 1$, $i = 1, 2, 3$, $j = 4$, and $\sum_{i=4, j=1, 2, 3} \bar{x}_{ij} = 1$ is a strong β -hybrid equilibrium. This equilibrium is sustained by the deterring strategy $\bar{z}_{ij} = 1/2$, $i, j = 1, 2, 3$.

4.2 A contribution game

Let us consider the following six players contribution game in which each player can either contribute to a collective action or retract his contribution. Let $I = \{1, 2, 3, 4, 5, 6\}$ be the set of players, $X_i = [0, 1]$ be the strategy space of player's i , and player i 's payoff function be given by

$$u_i(x) = -ix_i + \alpha \left(\sum_{j \neq i} x_j \right), \quad \alpha > 0, \quad i = 1, \dots, 6.$$

Let the coalition structure \mathcal{C} be defined by $\mathcal{C} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$.

Let us check that the set $A_{\mathcal{C}}$ is strictly-nonempty valued. To see this, let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in [0, 1]^6$ be given. Consider player i and the coalition S to which she belongs. Set $\bar{x}_S = (1, 1, 1)$. Let K be a subcoalition of S to which i may belong. The best gain that she can get by deviating from \bar{x}_S is equal to $\alpha(\sum_{k \in K} x_k + \sum_{k \in S-K} z_k + \sum_{k \in -S} x_k)$. Now suppose that player k in $S - K$ chooses $z_k = 0$. Then deviating from \bar{x}_S is a bad choice for player i whenever $\alpha(\sum_{k \in K} x_k + \sum_{k \in -S} x_k) < -i + \alpha(3 + \sum_{k \in -S} x_k)$. This inequality is satisfied for all i when $6 < \alpha$. Therefore, whenever $\alpha < 6$ the correspondence $A_{\mathcal{C}}$ is strictly-nonempty valued. One can check that $\bar{x}_i = 1$ for all i is a strong β -hybrid equilibrium. This equilibrium is sustained

by the deterring strategy $\bar{z}_k = 0$. Interestingly, if α is less than 6 it is no longer true that $A_{\mathcal{C}}$ is non-empty. In that case, it is impossible to deter player 6 from deviating in a profitable way.

4.3 A game with environmental externalities

Consider a game with n identical agents (*e.g.*, countries or firms). Assume that these agents produce and consume the same good and that production is a polluting activity. Let y_i denote the production of agent i and let x_i be the emission of pollutants resulting from this production. Also assume that $y_i = g_i(x_i)$ and that $u_i(x) = g_i(x_i) - v_i(\sum_{j \in I} x_j)$, where $v_i(\cdot)$ is increasing on \mathbb{R}_+ . Thus agent i 's payoff increases with the consumption of the produced good ($g_i(x_i)$) and decreases with global pollution ($\sum_{j \in I} x_j$). Further, assume that $X_i = \{x_i : 0 \leq x_i \leq x_i^0\}$ (x_i^0 is agent i 's maximum emission level).

More specifically, suppose as in Chander [2007] that $g(x_i) = \sqrt{x_i}$, $v_i(\sum_{j \in I} x_j) = \sum_{j \in I} x_j$ so that $u_i(x) = \sqrt{x_i} - \sum_{j \in I} x_j$. Also set $X_i = \{x_i : 0 \leq x_i \leq 1\}$. It is straightforward to show that the game defined above is compact, continuous and concave. Let us show that the correspondence $A_{\mathcal{C}}$ is strictly-nonempty valued.

To do this, consider any coalition S in \mathcal{C} , where \mathcal{C} is a given coalition structure. Denote by s the cardinality of S and let χ_{-S} be defined by $\chi_{-S} = \sum_{i \in -S} x_i$, where $x_i \in X_i$ for all i in $-S$. Then one can show that the maximum of

$$\sum_{i \in S} u_i(x_S, x_{-S}) = \sum_{i \in S} (\sqrt{x_i} - s x_i - \chi_{-S})$$

with respect to $x_S \in X_S$ is achieved at $\bar{x}_i = \frac{1}{4s^2}$, for all $i \in S$.

Define \bar{x} by $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X$ with $\bar{x}_i = \frac{1}{4s^2}$, $i \in S$, $s = |S|$. If $s = 1$, for every $S \in \mathcal{C}$, the correspondence $A_{\mathcal{C}}$ is strictly-non empty valued. If $s \geq 2$ for some $S \in \mathcal{C}$, consider any proper sub-coalition $K \subsetneq S$ with cardinality k , where $k = 1, \dots, s-1$. Then, for each $K \subsetneq S$ and $i \in K$, we have for each $y_K \in X_K$

$$\begin{aligned} \min_{z_{S-K} \in X_{S-K}} u_i(y_K, z_{S-K}, x_{-S}) &= \left[\sqrt{y_i} - \sum_{j \in K} y_j - (s-k) - \chi_{-S} \right] \\ &\leq \frac{1}{4} - s + k - \chi_{-S} \end{aligned}$$

where the $\bar{z}_j = 1$ for all j in $S - K$. Since $\frac{1}{4} + k < s + \frac{1}{4s}$, the correspondence $A_{\mathcal{C}}$ is strictly-nonempty valued. Now for each S , and for all i in S set $\bar{x}_i = \frac{1}{4s^2}$, where we recall that s is the cardinality of S . Then one can check that the strategy profile \bar{x} is a strong β -hybrid equilibrium (because we have actually proved that \bar{x} is fixed-point of the correspondance $B_{\mathcal{C}}$).⁸

⁸To extend this existence result to more general specifications of the game, we simply need to make sure that for each

5 CONCLUSION

We have proposed a notion of coalitional equilibrium which is a refinement of the hybrid solution. Like this solution, the strong β -hybrid solution is well suited to study situations where people cooperate within coalitions but where coalitions compete with one another. In our solution, as opposed to the hybrid solution, the strategy profile assigned to each coalition is strongly-Pareto optimal. To put it another way, all possible gains of cooperation at the coalition level have been exhausted. We have also proposed an existence result for the strong β -hybrid solution in general games.

There are at least three issues that can be studied in further research. The first one is to study whether ideas used in the analysis of transferable utility games can be transposed to non-transferable utility game. In particular, it remains to be seen if Zhao's strong separability condition for a non-empty TU β -core can be extended to NTU normal form games. The second one is the existence of a strong β -hybrid solution in games with discontinuous payoffs. Another interesting issue is the existence of this solution in games with general strategy spaces, and with an arbitrarily large number of players.

Declarations

The authors have no relevant financial or non-financial interests to disclose.

References

- ALLEN N. (2000). The future of microeconomic theory, *Journal of Economic Perspectives*, 14, 143-150.
- ALIPRANTIS, C.B. AND BORDER, K.C. (2006). *Infinite dimensional analysis: A Hitch-hiker's Guide*, Third Edition, Springer.
- AUMANN, R.J. (1959). Acceptable points in general cooperative n -person games. In: Tucker A, Luce D (Eds.) *Contributions to the Theory of Games* 4, Annals of Mathematics Studies Vol. 40, Princeton University Press, Princeton.
- AUMANN, R.J. (1961). The core of a cooperative game without side payments, *Transactions of the American Mathematical Society*, **98**, 539–552.
- CHANDER, P. (2007). The gamma-core and coalition formation, *International Journal of Game Theory*, **35**, 539–556.

$x \in X$ and for each $i \in I$, $v_i''(\sum_{j \in I} x_j) \geq 0$ and $g_i''(x_i) \leq 0$. Under these conditions the functions $u_i(x)$ are concave on X and we can easily prove as with Chander's specification above that the correspondence is strictly-nonempty valued.

- CRETTEZ, B., NESSAH, R. AND TAZDAÏT, T. (2021). On the strong hybrid solution of an n -person game, *Mathematical Social Sciences*, 0(0), pp. 0-19
- KAJII A. (1992). A generalization of Scarf's theorem: An α -core existence theorem without transitivity or completeness, *Journal of Economic Theory*, **56**, 194-205.
- MEINHARDT H. (2002). *Cooperative decision making in common pool situations*. Lecture Notes in Economics and Mathematical Systems 517. Berlin, Germany: Springer.
- NESSAH, R., TAZDAÏT, T. (2013). Absolute optimal solution for a compact and convex game, *European Journal of Operational Research*, **224**, 353–361.
- SCARF, H.E. (1971). On the existence of a cooperative solution for a general class of N -person games. *Journal of Economic Theory*, **3**, 169–187.
- SHAPLEY, L. AND M. SHUBIK (1971). On the core of an economic system with externalities. *The American Economic Review*, 41:678–684.
- YANG, Z., YUAN, G. X. (2019). Some generalizations of Zhao's theorem: Hybrid solutions and weak hybrid solutions for games with nonordered preferences, *Journal of Mathematical Economics*, **84**, 94–100.
- YANO, M. (1990). A local theory of cooperative games, *International Journal of Game Theory*, **19**, 301–324.
- ZHAO, J. (1983). Results on the absolute solutions of multiobjective mathematical programming, *Journal of Beijing Institute of Aeronautics and Astronautics*, **3**, 107–114.
- ZHAO, J. (1992). The hybrid solutions of an n -person game, *Games and Economic Behavior*, **4**, 145–160.
- ZHAO, J. (1996). The hybrid equilibria and core selection in exchange economies with externalities, *Journal of Mathematical Economics*, **26**, 387–407.
- ZHAO, J. (1999a). The existence of TU α -core in normal form games. *International Journal of Game Theory*, **28**, 25–34.
- ZHAO, J. (1999b). A β -core existence result and its application to oligopoly markets. *Games and Economic Behavior*, **27**, 153–168.
- ZHAO, J. (2018 a). TU oligopoly games and industrial cooperation, in *Handbook of Game Theory and Industrial Organization*, Corchón Luis C. and Marini Marco A. (ed.), Cheltenham, UK: Edward Elgar.