Integral representation formula for linear non-autonomous difference-delay equations

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Abstract

This note states and proves an integral representation formula for the continuous solutions of a linear non-autonomous difference delay system in terms of the Lebesgue-Stieltjes integral depending on the initial data of the system. It gives a precise and correct statement of various formulations in the literature, and extends it to the time-varying case.

Keywords: linear systems, difference delay systems, integral representation, Volterra equations.

1. Statement of the result and comments

Let us consider a linear non-autonomous difference-delay system of the form:

$$y(t) = \sum_{j=1}^{N} D_j(t) y(t-\tau_j), \qquad t > s, \qquad y(s+\theta) = \phi(\theta) \text{ for } -\tau_N \leq \theta \leq 0, \tag{1.1}$$

where d, N are positive integers and the delays $\tau_1 < \cdots < \tau_N$ are strictly positive real numbers, while $D_1(t), \ldots, D_N(t)$ are complex[2] $d \times d$ matrices depending continuously on time t, the real number $s \in \mathbb{R}$ is the initial time and the function $\phi : [-\tau_N, 0] \to \mathbb{C}^d$ the *initial data*.

A *solution* is a map $y : [s-\tau_N, +\infty) \to \mathbb{C}^d$ that satisfies (1.1). Clearly, in order to get a continuous solution, the initial data which is but the restriction of y to the interval $[-\tau_N, 0]$ must satisfy a compatibility condition, namely ϕ should belong to the set C_s defined by

$$C_s := \{\phi \in C^0([-\tau_N, 0], \mathbb{C}^d) : \phi(0) = \sum_{j=1}^{N} D_j(s)\phi(-\tau_j)\}; \tag{1.2}$$

here and below, $C^0(E, F)$ indicates continuous maps from E into F.

Conversely, given ϕ in C_s, an easy recursion shows that System (1.1) has a unique solution y with initial data ϕ, and that this solution is continuous. It is customary (see *e.e* [1]) to express, for various linear "functional" dynamical systems, such a general solution through a representation integral formula involving the initial condition $\phi \in C_s$ and the so-called *fundamental solution* of

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[2]We treat the case where coefficients (in the matrices D_j as well as in the solutions) are taken in $\mathbb{C}$; the one where they are taken in $\mathbb{R}$ works exactly in the same way.

the system. Here, the fundamental solution can be viewed as a particular non-continuous matrix-valued solution of System (1.1), it is the unique $X : \mathbb{R}^2 \to \mathbb{C}^{d\times d}$ satisfying

$$X(t,s) = \begin{cases} 0 \text{ for } t < s, \\ I_d + \sum_{j=1}^{N} D_j(t) X(t-\tau_j, s) \text{ for } t \geq s. \end{cases} \tag{1.3}$$

Arguing inductively, it is easy to check that X uniquely exists and is continuous at each (t,s) such that $t - s \notin \mathcal{F}$, where $\mathcal{F}$ is the positive lattice in $[0, +\infty)$ generated by the numbers τ_ℓ:

$$\mathcal{F} := \left\{ \sum_{\ell=1}^{N} n_\ell \, \tau_\ell \,, \ (n_1, \ldots, n_N) \in \mathbb{N}^N \right\}. \tag{1.4}$$

Clearly, X has a bounded jump across each line $t - s = \mathfrak{f}$ for $\mathfrak{f} \in \mathcal{F}$; in fact, a moment's thinking will convince the reader that

$$X(t,s) \ = \ - \sum_{\mathfrak{f} \in [0, t-s] \cap \mathcal{F}} \mathfrak{C}_\mathfrak{f}(t) \,, \quad s \leq t \,, \tag{1.5}$$

where each $\mathfrak{C}_\mathfrak{f}(.)$ is continuous $\mathbb{R} \to \mathbb{C}^{d\times d}$ (additional smoothness assumptions for the maps $D_j(.)$ would transfer to each $\mathfrak{C}_\mathfrak{f}$). One can see that $\mathfrak{C}_\mathfrak{f}(t)$ is a finite sum of products of $D_j(t - \mathfrak{f}')$, where $\mathfrak{f}'$ ranges over the elements of $\mathcal{F}$ whose defining integers n_ℓ in (1.4) do not exceed those defining $\mathfrak{f}$, the empty product being the identity matrix. A precise expression for $\mathfrak{C}_\mathfrak{f}$ can be obtained by reasoning as in [2, Sec. 3.2] or [3, Sec. 4.5], but we will not need it.

The announced representation formula is now given by the following theorem.

Theorem 1 (representation formula). *For $s \in \mathbb{R}$ and ϕ in C_s the solution $y \in C^0([s - \tau_N, +\infty), \mathbb{C}^d)$ to (1.1) is*

$$y(t) = - \sum_{j=1}^{N} \int_{s^-}^{(s+\tau_j)^-} d_\alpha X(t,\alpha) D_j(\alpha) \phi(\alpha - \tau_j - s), \qquad t \geq s \,, \tag{1.6}$$

where X was defined in (1.3).

The integrals $\int_{s^-}^{(s+\tau_j)^-}$ in Equation (1.6) have to be understood as Lebesgue-Stieltjes integrals on the intervals $[s, s + \tau_j)$. These integrals are well defined because, for fixed t, the function $X(t, \cdot)$ is locally of bounded variation. The notions of Lebesgue-Stieltjes integral and functions with bounded variations are recalled in Section 2.

Representation formulae like (1.6) are instrumental in many instances to deal with linear functional dynamical systems. It is, for instance, often the starting point to provide sufficient and necessary criteria for exponential stability, working in the frequency domain with the use of Laplace transform: in [4], or [1] or the forthcoming [5] by the authors, exponential stability is explored this way for difference-delay system, or various functional differential equations, either time invariant or time varying. These results, in turn, are relevant in studying the stability of 1-D hyperbolic PDE's, see *e.g.* [6, Theorem 3.5 and Theorem 3.8] and [7].

The rest of this note is organised as follows. After the remark below, a stand-alone proof of Theorem 1 is given in Section 3, while Section 2 contains the prerequisites and preliminaries results.

In particular, Section 2.1 gathers some needed notions on functions with bounded variations and the Lebesgue-Stieltjes integral, that the reader might be unfamiliar with. Section 2.2 develops the theory of Volterra integral equations of second kind with B^∞ and it states existence and uniqueness of solutions for this type of equation *via* the resolvent of a Stieltjes-Volterra kernel. The proofs of the fundamental results on Volterra integral equations are given in Appendix A.

Motivation for the present note

The representation formula stated in Theorem 1 above is valid for continuous solutions of the time-varying difference-delay System (1.1) with continuous coefficients.

We are giving a careful statement of this formula, together with a detailed proof, although similar formulae appear in the literature for various classes of linear dynamical systems that are not radically different from (1.1). We believe that this useful for two reasons. On the one hand the time-varying case is not treated in the literature, and stating the formula for the time-varying case requires some adjustments. On the other hand, and more importantly, even in the case of linear autonomous difference-delay equations, the representation formulas that can be found in the literature (see *e.g.* [4] or [8], or also [1] for more general Volterra equations) seem to have issues in their very statement, probably due to some misprints in the indices and integral bounds[3], and it was also impossible to find a correct detailed proof of them: for instance, the proof given in [1] in the case of general Volterra integral equations has some important gaps that require to be filled, see the first paragraph of Section 2.2 and Footnote 4. The present note is at the same time an attempt to overcome some inaccuracies in the literature and an extension to the non-autonomous case.

2. Summary on functions with bounded variations, the Lebesgue-Stieltjes integral and Volterra integral equations with B^∞ kernel

Recall that the real and complex fields are denoted by $\mathbb{R}$ and $\mathbb{C}$. For d a strictly positive integer, we write $\|\cdot\|$ for Euclidean norm on $\mathbb{C}^d$ and $|||\cdot|||$ for the norm of a matrix $M \in \mathbb{C}^{d\times d}$:

$$|||M||| = \sup_{\|x\|=1} \|Mx\|.$$

We put I_d for the $d \times d$ identity matrix.

2.1. Functions with bounded variations and the Lebesgue-Stieltjes integral

For I a real interval and $f : I \to \mathbb{R}$ a function, the *total variation* of f on I is defined as

$$W_I(f) := \sup_{\substack{x_0<x_1<\cdots<x_N \\ x_i\in I, N\in\mathbb{N}}} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| < \infty. \tag{2.1}$$

The space $BV(I)$ of functions with *bounded variation* on I consists of those f such that $W_I(f) < \infty$, endowed with the norm $\|f\|_{BV(I)} = W_I(f) + |f(d)|$ where $d \in I$ is arbitrary but fixed. Different

[3]For instance, the formula in [4, Lemma 3.4], stated for time-invariant systems of the same type as (1.1) (with possibly an infinite number of delays) does not agree with (1.6), and a test on systems with two delays will convince the reader that it is faulty because the integration is not done in the right interval; however we can mention that for one delay, our formula agrees with the one in [4].

d give rise to equivalent norms for which $BV(I)$ is a Banach space, and $\|.\|_{BV(I)}$ is stronger than the uniform norm. We let $BV_r(I)$ and $BV_l(I)$ be the closed subspaces of $BV(I)$ comprised of right and left continuous functions, respectively. We write $BV_{loc}(\mathbb{R})$ for the space of functions whose restriction to any bounded interval $I \subset \mathbb{R}$ lies in $BV(I)$. Observe that

$$W_I(fg) \leq W_I(f) \sup_{x\in I} |g(x)| + W_I(g) \sup_{x\in I} |f(x)|. \tag{2.2}$$

Each $f \in BV(I)$ has a limit $f(x^-)$ (resp. $f(x^+)$) from the left (resp. right) at every $x \in I$ where the limit applies [9, sec. 1.4]. Hence, one can associate to f a finite signed Borel measure ν_f on I such that $\nu_f((a,b)) = f(b^-) - f(a^+)$, and if I is bounded on the right (resp. left) and contains its endpoint b (resp. a), then $\nu_f(\{b\}) = f(b) - f(b^-)$ (resp. $f(a^+) - f(a)$) [9, ch. 7, pp. 185–189]. Note that different f may generate the same ν_f: for example if f and f_1 coincide except at isolated interior points of I, then $\nu_f = \nu_{f_1}$. For $g : I \to \mathbb{R}$ a measurable function, summable against ν_f, the Lebesgue-Stieltjes integral $\int g df$ is defined as $\int g d\nu_f$, whence the differential element df identifies with $d\nu_f$ [9, ch. 7, pp. 190–191]. This type of integral is useful for it is suggestive of integration by parts, but caution must be used when integrating a function with respect to df over a subinterval $J \subset I$ because $\nu_{(f_{|J})}$ needs *not* coincide with the restriction $(\nu_f)_{|J}$ of ν_f to J. More precisely, if the lower bound a (resp. the upper bound b) of J belongs to J and lies interior to I, then the two measures may differ by the weight they put on $\{a\}$ (resp. $\{b\}$), and they agree only when f is left (resp. right) continuous at a (resp. b). By $\int_J g df$, we always mean that we integrate g against $\nu_{(f_{|J})}$ and *not* against $(\nu_f)_{|J}$. We often —*e.g.* in the main formula (1.6)— trade the notation $\int_J g df$ for one of the form $\int_{a^\pm}^{b^\pm} g df$, where the interval of integration J is encoded in the bounds put on the integral sign: a lower bound a^- (resp. a^+) means that J contains (resp. does not contain) its lower bound a, while an upper bound b^+ (resp. b^-) means that J contains (resp. does not contain) its upper bound b. Then, the previous word of caution applies to additive rules: for example, when splitting $\int_{a^\pm}^{b^\pm} g df$ into $\int_{a^\pm}^{c^\pm} g df + \int_{c^\pm}^{b^\pm} g df$ where $c \in (a,b)$, we must use c^+ (resp. c^-) if f is right (resp. left) continuous at c.

To a finite, signed or complex Borel measure μ on I, one can associate its *total variation measure* $|\mu|$, defined on a Borel set $B \subset I$ by $|\mu|(B) = \sup_{\mathcal{P}} \sum_{E\in\mathcal{P}} |\mu(E)|$ where $\mathcal{P}$ ranges over all partitions of B into Borel sets, see [10, sec. 6.1]; its total mass $|\mu|(I)$ is called the total variation of μ, denoted as $\|\mu\|$. Thus, the total variation is defined both for functions of bounded variation and for measures, with different meanings. When $f \in BV(I)$ is monotonic then $W_I(f) = \|\nu_f\|$, but in general it only holds that $\|\nu_f\| \leq 2W_I(f)$; this follows from the Jordan decomposition of f as a difference of two increasing functions, each of which has variation at most $W_I(f)$ on I [9, Thm. 1.4.1]. In any case, it holds that $|\int g df| \leq \int |g| d|\nu_f| \leq 2W_I(f) \sup_I |g|$.

The previous notations and definitions also apply to vector and matrix-valued functions BV-functions, replacing absolute values in (2.1) by Euclidean and matricial norms, respectively.

2.2. Volterra integral equations with kernels of type B^∞

Volterra equations for functions of a single variable have been studied extensively, see e.g. [11, 12]. *However, the specific assumption that the kernel has bounded variation seems treated somewhat tangentially. On the one hand, it is subsumed in the measure-valued case presented in* [12, Ch. 10]*, but no convenient criterion is given there for the existence of a resolvent kernel. On the other hand,* [1, Ch. 9, Sec. 1] *sketches the main arguments needed to handle kernels of bounded variation, but the exposition has issues*[4]*.*

[4] For example, the equation satisfied by $\widetilde{\rho}(t,s)$ at top of page 258 is not right.

We define a Stieltjes-Volterra kernel of type B^∞ on $[a,b]\times[a,b]$ as a measurable function $\kappa:[a,b]\times[a,b]\to\mathbb{R}^{d\times d}$, with $\kappa(t,\tau)=0$ for $\tau\geq t$, such that the partial maps $\kappa(t,.)$ lie in $BV_l([a,b])$ and $\|\kappa(t,.)\|_{BV([a,b])}$ is uniformly bounded with respect to $t\in[a,b]$. In addition, we require that $\lim_{\tau\to t^-} W_{[\tau,t)}(\kappa(t,.))=0$ uniformly with respect to t; *i.e.*, to every $\varepsilon>0$, there exists $\eta>0$ such that $W_{[\tau,t)}(\kappa(t,.))<\varepsilon$ as soon as $0<t-\tau<\eta$. Note that $W_{[\tau,t)}(\kappa(t,.))\to 0$ for fixed t as $\tau\to t^-$ whenever $\kappa(t,.)$ has bounded variation on $[a,b]$, by the very definition (2.1); so, the assumption really is that the convergence is uniform with respect to t. Hereafter, we drop the qualifier "of type B^∞" and simply speak of Stieltjes-Volterra kernels on $[a,b]$. We endow the space $\mathcal{K}_{[a,b]}$ of such kernels with the norm $\|\kappa\|_{[a,b]}:=\sup_{t\in[a,b]}\|\kappa(t,.)\|_{BV([a,b])}$. If κ_k is a Cauchy sequence in $\mathcal{K}_{[a,b]}$, then κ_k converges uniformly on $[a,b]\times[a,b]$ to a $\mathbb{R}^{d\times d}$-valued function κ because

$$|||\kappa_k(t,\tau)-\kappa_l(t,\tau)|||=|||(\kappa_k(t,\tau)-\kappa_l(t,\tau))-(\kappa_k(t,t)-\kappa_l(t,t))|||\leq\|\kappa_k(t,.)-\kappa_l(t,.)\|_{BV([a,b])}.$$

Clearly, κ is measurable and left continuous for fixed t with $\kappa(t,\tau)=0$ for $\tau\geq t$. Also, if m is so large that $\|\kappa_k-\kappa_l\|_{[a,b]}<\varepsilon$ for $k,l\geq m$ and $\eta>0$ so small that $W_{[\tau,t)}(\kappa_m)<\varepsilon$ when $t-\tau<\eta$, we get that $W_{[\tau,t)}(\kappa_l)\leq W_{[\tau,t)}(\kappa_m)+W_{[\tau,t)}(\kappa_m-\kappa_l)<2\varepsilon$, and letting $l\to\infty$ we get from [9, thm. 1.3.5] that $W_{[\tau,t)}(\kappa)\leq 2\varepsilon$. Furthermore, the same reference implies that $\|\kappa\|_{[a,b]}\leq\sup_k\|\kappa_k\|_{[a,b]}$ so that $\kappa\in\mathcal{K}_{[a,b]}$. Finally, writing that $W_{[a,b]}(\kappa_k(t,.)-\kappa_l(t,.))<\varepsilon$ and passing to the limit as $l\to\infty$, we see that $\lim_k\|\kappa_k-\kappa\|_{[a,b]}=0$, whence $\mathcal{K}_{[a,b]}$ is a Banach space. Note that a Stieltjes-Volterra kernel κ is necessarily bounded with $\sup_{[a,b]\times[a,b]}|||\kappa(t,\tau)|||\leq\|\kappa\|_{[a,b]}$.

A *resolvent* for the Stieltjes-Volterra kernel κ on $[a,b]\times[a,b]$ is a Stieltjes-Volterra kernel ρ on $[a,b]\times[a,b]$ satisfying

$$\rho(t,\beta)=-\kappa(t,\beta)+\int_{\beta^-}^{t^-}d\kappa(t,\tau)\rho(\tau,\beta),\qquad a\leq t,\beta\leq b. \tag{2.3}$$

The following two lemmata are the technical core of this paper. The first one states that any Stieltjes-Volterra kernel has a unique resolvent. The second one states existence and uniqueness of solutions of Stieltjes-Volterra equations, together with its formula in terms of the resolvent.

Lemma 2. *If κ is a Stieltjes-Volterra kernel on $[a,b]\times[a,b]$, a resolvent for κ uniquely exists.*

Lemma 3. *Let κ be a Stieltjes-Volterra kernel on $[a,b]\times[a,b]$ and ρ its resolvent. For each $\mathbb{R}^d$-valued function $g\in BV_r([a,b])$, the unique bounded measurable solution to the equation*

$$y(t)=\int_{a^-}^{t^-}d\kappa(t,\tau)y(\tau)+g(t),\qquad a\leq t\leq b, \tag{2.4}$$

is given by

$$y(t)=g(t)-\int_{a^-}^{t^-}d\rho(t,\alpha)g(\alpha),\qquad a\leq t\leq b. \tag{2.5}$$

The proofs of Lemma 2 and Lemma 3 are given in Appendix A. To achieve the proof of Theorem 1, we will frame the difference delay equations into a Stieltjes-Volterra integral equation with a B^∞ kernel.

3. Proof of Theorem 1

Observe from (1.3) that $\alpha \mapsto X(t,\alpha)$ lies in $BV_{loc}(\mathbb{R})$ for all t. In particular, $\alpha \mapsto X(t,\alpha)$ has bounded variation on $[s, s+\tau_j]$ for each j, and for $t \geq s+\tau_N$ we have that

$$d_\alpha X(t,\alpha) = \sum_{j=1}^{N} D_j(t)\, d_\alpha X(t-\tau_j,\alpha) \qquad \text{on} \quad [s, s+\tau_N]. \tag{3.1}$$

Then, substituting (3.1) in (1.6) formally yields (1.1), provided that $t \geq s+\tau_N$. Hence, by uniqueness of a solution y to (1.1) satisfying $y(s+\theta) = \phi(\theta)$ for $\theta \in [-\tau_N, 0]$, it is enough to check (1.6) for $s \leq t < s+\tau_N$. For this, we adopt the point of view of reference [1], which is to construe delay systems as Stieltjes-Volterra equations upon representing delays by measures. More precisely, we can rewrite (1.1) as a Lebesgue-Stieltjes integral:

$$y(t) = \int_{-\tau_N^-}^{0^-} d\mu(t,\theta) y(t+\theta), \qquad t \geq s, \tag{3.2}$$

with

$$\mu(t,\theta) = \sum_{j=1}^{N} D_j(t) \mathfrak{H}(\theta+\tau_j), \tag{3.3}$$

where $y(\tau)$ is understood to be $\phi(\tau - s)$ when $s-\tau_N \leq \tau \leq s$ and $\mathfrak{H}(\tau)$ is the Heaviside function which is zero for $\tau \leq 0$ and 1 for $\tau > 0$, so that the associated measure on an interval of the form $[0,a]$ or $[0,a)$ is a Dirac delta at 0. Note that $\mathfrak{H}(0) = 0$, which is not the usual convention, but if we defined $\mathfrak{H}$ so that $\mathfrak{H}(0) = 1$ then expanding (3.2) using (3.3) would not give us back (1.1) for the term $D_N(t)y(t-\tau_N)$ would be missing. Observe also, since $\tau_j > 0$ for all j, that the minus sign in the upper bound of the integral in (3.2) is immaterial and could be traded for a plus. For $s \leq t \leq s+\tau_N$, singling out the initial data in (3.2) yields

$$y(t) = \int_{(s-t)^-}^{0^-} d\mu(t,\theta) y(t+\theta) + f(t) \quad \text{with} \quad f(t) := \int_{-\tau_N^-}^{(s-t)^-} d\mu(t,\theta)\phi(t+\theta-s), \tag{3.4}$$

where we took into account, when separating the integrals, that $\theta \mapsto \mu(t,\theta)$ is left continuous, while the integral over the empty interval is understood to be zero. It will be convenient to study (3.4) for $t \in [s, s+\tau_N]$, even though in the end the values of $y(t)$ only matter to us for $t \in [s, s+\tau_N)$. Define

$$k(t,\tau) := \begin{cases} \mu(t,\tau-t) - \sum_{j=1}^{N} D_j(t) & \text{for } \tau \in [s,t], \\ 0 & \text{for } \tau > t, \end{cases} \qquad t,\tau \in [s, s+\tau_N]. \tag{3.5}$$

Note that $k(t,\tau) = 0$ when $t-\tau < \tau_1$, and $d_\tau k(t,\tau) = d_\tau \mu(t,\tau-t)$ on $[s, s+\tau_N]$ for fixed t. Hence, (3.4) becomes

$$y(t) = \int_{s^-}^{t^-} dk(t,\tau) y(\tau) + f(t), \qquad s \leq t \leq s+\tau_N. \tag{3.6}$$

Now, (3.6) is the Stieltjes-Volterra equation we shall work with.

It suffices to prove (1.6) for $t \in [s, s+\tau_N)$ under the additional assumption that ϕ and the D_j, which are continuous by hypothesis, also have bounded and locally bounded variation, on $[-\tau_N, 0]$

and $\mathbb{R}$ respectively. Indeed, functions of bounded variation are dense in C_s , for instance because C^1-functions are, and if ϕ_k converges uniformly to ϕ in C_s while $D_{j,k}$ converges uniformly to D_j in $[s, s+\tau_N]$ as $k \to \infty$, then the solution to (1.1) with initial condition ϕ_k and coefficients $D_{j,k}$ converges uniformly on $[s, s+\tau_N]$ to the solution with initial condition ϕ and coefficients D_j, as is obvious by inspection. Hence, we shall assume without loss of generality that ϕ has bounded variation and the D_j have locally bounded variation. Then, since it follows from (3.3) and (3.4) that

$$f(t) = \sum_{\tau_\ell \in (t-s, \tau_N]} D_\ell(t)\phi(t-s-\tau_\ell), \tag{3.7}$$

it is clear from (3.7) and (2.2) that $f \in BV_r([s, s+\tau_N])$.

As $\mathfrak{H}$ is left continuous and the D_j are bounded, on $[s, s+\tau_N]$, it is easy to check that $k(t,\tau)$ defined in (3.5) is a Stieltjes-Volterra kernel on $[s, s+\tau_N] \times [s, s+\tau_N]$. Thanks to Lemma 2, let ρ denote the resolvent of the Stieltjes-Volterra kernel k on $[s, s+\tau_N] \times [s, s+\tau_N]$ defined in (3.5). As f defined in (3.4) lies in $BV_r([s, s+\tau_N])$, the solution y to (3.6) is given, in view of Lemma 3, by

$$y(t) = f(t) - \int_{s^-}^{t^-} d\rho(t,\alpha) f(\alpha), \qquad s \le t \le s+\tau_N. \tag{3.8}$$

Since $\rho(t,\alpha) = 0$ when $\alpha \ge t$, the integral $\int_{s^-}^{t^-}$ can be replaced by $\int_{s^-}^{(s+\tau_N)^+}$ in (3.8). Thus, setting δ_t the Dirac delta distribution at t and $\widetilde{X}(t,\alpha) := I_d \mathcal{H}(t-\alpha) + \rho(t,\alpha)$ with $\mathcal{H}(\tau)$ the "standard" Heaviside function which is 0 for $\tau < 0$ and 1 for $\tau \ge 0$, we deduce from (3.7), since $d_\alpha \mathcal{H}(t-\alpha) = -\delta_t$ on $[s, s+\tau_N]$ for $s \le t < s+\tau_N$, that

$$y(t) = -\int_{s^-}^{(s+\tau_N)^+} d\widetilde{X}(t,\alpha) f(\alpha) = -\int_{s^-}^{(s+\tau_N)^+} d\widetilde{X}(t,\alpha) \left(\sum_{\tau_\ell \in (\alpha-s, \tau_N]} D_\ell(\alpha)\phi(\alpha-s-\tau_\ell) \right), \quad s \le t < s+\tau_N.$$

Rearranging, we get that

$$y(t) = -\sum_{j=1}^{N} \int_{s^-}^{(s+\tau_j)^-} d\widetilde{X}(t,\alpha) D_j(\alpha)\phi(\alpha-s-\tau_j), \qquad s \le t < s+\tau_N,$$

which is what we want (namely: formula (1.6) for $s \le t < s+\tau_N$) if only we can show that $\widetilde{X}(t,\alpha)$ coincides with $X(t,\alpha)$ when $\alpha \in [s, s+\tau_j)$ for each j and every $t \in [s, s+\tau_N)$; here, $X(t,\alpha)$ is defined by (1.3) where we set $s = \alpha$.

For this, we first observe that $X(t,\alpha) = \widetilde{X}(t,\alpha) = 0$ when $\alpha > t$ and that $X(t,t) = \widetilde{X}(t,t) = I_d$. Hence, we need only consider the case $\alpha \in [s,t)$ with $s < t < s+\tau_N$. For $s \le \alpha < t$, we get that

$$-k(t,\alpha) = k(t,t^-) - k(t,\alpha) = \int_{\alpha^-}^{t^-} dk(t,\tau).$$

Thus, (2.3) (where $\kappa = k$) in concert with the definition of $\widetilde{X}(t,\alpha)$ imply that

$$\begin{aligned} \widetilde{X}(t,\alpha) &= I_d\, \mathcal{H}(t-\alpha) - k(t,\alpha) + \int_{\alpha^-}^{t^-} dk(t,\tau)\rho(\tau,\alpha) \;=\; I_d + \int_{\alpha^-}^{t^-} dk(t,\tau)\big(I_d + \rho(\tau,\alpha)\big) \\ &= I_d + \int_{\alpha^-}^{t^-} dk(t,\tau)\widetilde{X}(\tau,\alpha). \end{aligned}$$

Now, on $[\alpha, t)$, we compute from (3.3) and (3.5) that $d_\tau k(t,\tau) = \sum_{t-\tau_j \geq \alpha} D_j(t)\delta_{t-\tau_j}$ and hence, since $\widetilde{X}(t-\tau_j,\alpha) = 0$ when $\alpha > t - \tau_j$, the previous equation becomes:

$$\widetilde{X}(t,\alpha) = I_d + \sum_{j=1}^{N} D_j(t)\widetilde{X}(t-\tau_j,\alpha) \qquad \text{for } s \leq \alpha < t \text{ and } s \leq t < s + \tau_N. \tag{3.9}$$

Comparing (3.9) and (1.3), we see that $\widetilde{X}(t,\alpha)$ and $X(t,\alpha)$ coincide on $[s, s+\tau_N) \times [s, s+\tau_N)$, thereby ending the proof.

References

[1] J. K. Hale, S. M. Verduyn Lunel, Introduction to functional-differential equations, Vol. 99 of Applied Mathematical Sciences, Springer-Verlag, New York, 1993. doi:10.1007/978-1-4612-4342-7.

[2] Y. Chitour, G. Mazanti, M. Sigalotti, Stability of non-autonomous difference equations with applications to transport and wave propagation on networks, Netw. Heterog. Media 11 (2016) 563–601. doi:10.3934/nhm.2016010.

[3] L. Baratchart, S. Fueyo, G. Lebeau, J.-B. Pomet, Sufficient stability conditions for time-varying networks of telegrapher's equations or difference-delay equations, SIAM J. Math. Anal. 53 (2) (2021) 1831–1856. doi:10.1137/19M1301795.
URL http://hal.inria.fr/hal-02385548/

[4] D. Henry, Linear autonomous neutral functional differential equations, Journal of Differential Equations 15 (1) (1974) 106–128. doi:10.1016/0022-0396(74)90089-8.

[5] L. Baratchart, S. Fueyo, J.-B. Pomet, Exponential stability of linear periodic difference-delay equations, preprint (Jul. 2023).
URL https://hal.inria.fr/hal-03500720

[6] G. Bastin, J.-M. Coron, Stability and boundary stabilization of 1-D hyperbolic systems, Vol. 88 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser/Springer, [Cham], 2016, subseries in Control. doi:10.1007/978-3-319-32062-5.

[7] L. Baratchart, S. Fueyo, J.-B. Pomet, Exponential stability of periodic difference delay systems and 1-D hyperbolic PDEs of conservation laws, IFAC-PapersOnLine 55 (36) (2022) 228–233, 17th IFAC Workshop on Time Delay Systems TDS 2022. doi:10.1016/j.ifacol.2022.11.362.

[8] J. K. Hale, Theory of functional differential equations, 2nd Edition, Applied mathematical sciences 3, Springer-Verlag, 1977.

[9] S. Łojasiewicz, An introduction to the theory of real functions, John Wiley & Sons Inc, 1988.

[10] W. Rudin, Real and Complex Analysis, McGraw-Hill, 1986.

[11] H. Brunner, Volterra integral equations: an introduction to theory and applications, Cambridge Monographs Appl. and Computational Math., Cambridge University Press, 2017. doi:10.1017/9781316162491.

[12] G. Gripenberg, S.-O. Londen, O. Staffans, Volterra integral and functional equations, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1990. doi: 10.1017/CBO9780511662805.

Appendix A.

This appendix is devoted to the proofs of Lemma 2 and Lemma 3.

Appendix A.1. Proof Lemma 2

Pick $r > 0$ to be adjusted later, and for $\Psi \in \mathcal{K}_{[a,b]}$ let us define

$$F_r(\Psi)(t,\beta) := \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta), \qquad a \le \beta, t \le b.$$

Then, $F_r(\Psi)(t,\beta) = 0$ for $\beta \ge t$, and for $a \le \beta_1 < \beta_2 < t$ we have that

$$F_r(\Psi)(t,\beta_2) - F_r(\Psi)(t,\beta_1) = \int_{\beta_2^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)\left(\Psi(\tau,\beta_2) - \Psi(\tau,\beta_1)\right) - \int_{\beta_1^-}^{\beta_2^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta_1),$$

where we used that $\kappa(t,.)$ is left continuous to assign the lower (resp. upper) bound β_2^- to the first (resp. second) integral in the above right hand side. Now, the first integral goes to 0 as $\beta_1 \to \beta_2$ by dominated convergence, since $\Psi(t,.)$ is left-continuous; the second integral also goes to 0, because $|\nu_{\kappa(t,.)}|([\beta_1,\beta_2)) \to 0$ when $\beta_1 \to \beta_2$, by standard properties of finite measures. Altogether, we see that $F_r(\Psi)(t,.)$ is left-continuous. Moreover, for $[c,d] \subset [a,t)$ and $c = \beta_0 < \beta_1 < \cdots < \beta_N = d$,

$$\begin{aligned}
&\sum_{i=1}^N |||F_r(\Psi)(t,\beta_i) - F_r(\Psi)(t,\beta_{i-1})||| \\
&\quad\le \sum_{i=1}^N \left|\left|\left| \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)}\, d\kappa(t,\tau)\left(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1})\right)\right|\right|\right| + \sum_{i=1}^N \left|\left|\left| \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta_{i-1})\right|\right|\right| \\
&\quad\le \sum_{i=1}^N \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)}\, d|\nu_{\kappa(t,.)}|\, |||(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1}))||| + \sum_{i=1}^N \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)}\, d|\nu_{\kappa(t,.)}|\, |||\Psi(\tau,\beta_{i-1})||| \\
&\quad\le \int_{d^-}^{t^-} d|\nu_{\kappa(t,.)}| \sum_{i=1}^N |||(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1}))||| \\
&\qquad + e^{-r(t-d)} \int_{c^-}^{d^-} d|\nu_{\kappa(t,.)}| \sum_{i=1}^N |||(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1}))||| + \sup_{[a,t]\times[a,t]} |||\Psi||| \int_{c^-}^{d^-} e^{-r(t-\tau)} d|\nu_{\kappa(t,.)}| \\
&\quad\le 2\, W_{[d,t)}(\kappa(t,.)) \sup_{\tau\in[d,t)} W_{[c,d]}(\Psi(\tau,.)) \\
&\qquad + 2\, e^{-r(t-d)} W_{[c,d)}(\kappa(t,.)) \sup_{\tau\in[c,d)} W_{[c,d]}(\Psi(\tau,.)) + 2\, e^{-r(t-d)} \sup_{[a,t]\times[a,t]} |||\Psi|||\; W_{[c,d)}(\kappa(t,.)).
\end{aligned}$$

When $d = t$, the same inequality holds but then $W_{[d,t)}(\kappa(t,.))$ is zero. Setting $c = a$ and $d = t$, we get from the above majorization that $W_{[a,t]}(F_r(\Psi)(t,.)) \le 4\|\kappa\|_{[a,b]}\|\Psi\|_{[a,b]}$, and since $F_r(\Psi)(t,\tau) = 0$ for $\tau \ge t$ we deduce that $W_{[a,b]}(F_r(\Psi)(t,.)) = W_{[a,t]}(F_r(\Psi)(t,.))$ is bounded, uniformly with

respect to t. Next, if we fix $\varepsilon > 0$ and pick $\eta > 0$ so small that $W_{[\tau,t)}(\kappa(t,.)) \leq \varepsilon$ as soon as $t - \tau \leq \eta$ (this is possible because $\kappa \in \mathcal{K}_{[a,b]}$), the same estimate yields

$$W_{[c,t)}(F_r(\Psi)(t,.)) \leq 4W_{[c,t)}(\kappa(t,.))\|\Psi\|_{[a,b]} \leq 4\,\varepsilon\,\|\Psi\|_{[a,b]}, \qquad t - c \leq \eta. \tag{A.1}$$

Altogether, we just showed that $F_r(\Psi) \in \mathcal{K}_{[a,b]}$. Moreover, if we take r so large that $e^{-r\eta} < \varepsilon$, then either $t - a \leq \eta$ and then (A.1) with $c = a$ gives us $W_{[a,t)}(F_r(\Psi)(t,.) \leq 4\varepsilon\|\Psi\|_{[a,b]}$, or else $t - \eta > a$ in which case (A.1) with $c = t - \eta$, together with our initial estimate when $c = a$ and $d = t - \eta$, team up to produce:

$$\begin{aligned} W_{[a,t)}(F_r(\Psi))(t,.) &= W_{[a,t-\eta]}(F_r(\Psi)(t,.)) + W_{[t-\eta,t)}(F_r(\Psi)(t,.)) \\ &\leq 2\varepsilon \sup_{\tau\in[t-\eta,t)} W_{[a,t-\eta]}(\Psi(\tau,.)) + 2\varepsilon W_{[a,t-\eta)}(\kappa(t,.)) \sup_{\tau\in[a,t-\tau)} W_{[a,t-\eta]}(\Psi(\tau,.)) \\ &\quad + 2\varepsilon \sup_{[a,t]\times[a,t]} |||\Psi||| W_{[a,t-\eta)}(\kappa(t,.)) + 4\varepsilon\|\Psi\|_{[a,b]} \\ &\leq 2\,\varepsilon\,\|\Psi\|_{[a,b]}\,\big(3 + 2\|\kappa\|_{[a,b]}\big)\,. \end{aligned} \tag{A.2}$$

Consequently, as $W_{[a,t)}(F_r(\Psi)(t,.)) = W_{[a,t]}(F_r(\Psi)(t,.))$ by the left continuity of $F_r(\Psi)(t,.)$, we can ensure upon choosing r sufficiently large that the operator $F_r : \mathcal{K}_{[a,b]} \to \mathcal{K}_{[a,b]}$ has arbitrary small norm. Hereafter, we fix r so that $|||F_r||| < \lambda < 1$.

Now, let $\widetilde{\rho}_0 = 0$ and define inductively:

$$\widetilde{\rho}_{k+1}(t,\beta) = -e^{-rt}\kappa(t,\beta) + F_r(\widetilde{\rho}_k)(t,\beta).$$

Clearly $(t,\beta) \mapsto e^{-rt}\kappa(t,\beta)$ lies in $\mathcal{K}_{[a,b]}$, so that $\widetilde{\rho}_k \in \mathcal{K}_{[a,b]}$ for all k. Moreover, we get from what precedes that $\|\widetilde{\rho}_{k+1} - \widetilde{\rho}_k\|_{[a,b]} \leq \lambda\|\widetilde{\rho}_k - \widetilde{\rho}_{k-1}\|_{[a,b]}$. Thus, by the shrinking lemma, $\widetilde{\rho}_k$ converges in $\mathcal{K}_{[a,b]}$ to the unique $\widetilde{\rho} \in \mathcal{K}_{[a,b]}$ such that

$$\begin{aligned} \widetilde{\rho}(t,\beta) &= -e^{-rt}\kappa(t,\beta) + F_r(\widetilde{\rho})(t,\beta) \\ &= -e^{-rt}\kappa(t,\beta) + \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)\widetilde{\rho}(\tau,\beta), \quad a \leq t,\beta \leq b. \end{aligned} \tag{A.3}$$

Putting $\rho(t,\beta) := e^{rt}\widetilde{\rho}(t,\beta)$, one can see that ρ lies in $\mathcal{K}_{[a,b]}$ if and only if $\widetilde{\rho}$ does, and that (A.3) is equivalent to (2.3). This achieves the proof. □

Appendix A.2. Proof Lemma 3

The fact that κ possesses a unique resolvent ρ comes from Lemma 2. Define y through (2.5) so that $y(a) = g(a)$, by inspection. Since $g \in BV_r([a,b])$ and $\rho(t,\cdot)$, $k(t,.)$ lie in $BV_l([a,b])$, integrating by parts [9, thm. 7.5.9][5] while using (2.3) along with Fubini's theorem and the relations

[5]This reference deals with open intervals only, and we stick to this case at the cost of a slightly lengthier computation

$\kappa(t,\alpha)=\rho(t,\alpha)=0$ for $\alpha\geq t$, we get that

$$\begin{aligned}
\textstyle\int_{a^-}^{t^-} d\kappa(t,\alpha)y(\alpha) &= (\kappa(t,a^+)-\kappa(t,a))y(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)y(\alpha)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)-\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^-}^{\alpha^-} d\rho(\alpha,\beta)g(\beta)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)-\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^+}^{\alpha^-} d\rho(\alpha,\beta)g(\beta)\\
&\qquad -\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^+}^{\alpha^-} \rho(\alpha,\beta)dg(\beta)\\
&\qquad -\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)\left[\rho(\alpha,\beta)g(\beta)\right]_{\beta=a^+}^{\beta=\alpha^-}-\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)+\int_{a^+}^{t^-}\Big(\int_{\beta^-}^{t^-} d\kappa(t,\alpha)\rho(\alpha,\beta)\Big)dg(\beta)\\
&\qquad +\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a^+)g(a)-\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)\\
&\qquad +\textstyle\int_{a^+}^{t^-}\big(\rho(t,\beta)+\kappa(t,\beta)\big)dg(\beta)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\left[\kappa(t,\alpha)g(\alpha)\right]_{\alpha=a^+}^{\alpha=t^-}+\textstyle\int_{a^+}^{t^-}\rho(t,\beta)dg(\beta)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= -\kappa(t,a)g(a)+\left[\rho(t,\beta)g(\beta)\right]_{\beta=a^+}^{\beta=t^-}-\textstyle\int_{a^+}^{t^-} d\rho(t,\beta)g(\beta)+\int_{a^-}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= -\kappa(t,a)g(a)-\rho(t,a^+)g(a)-\textstyle\int_{a^+}^{t^-} d\rho(t,\beta)g(\beta)+\big(\kappa(t,a)+\rho(t,a)\big)g(a)\\
&= -\textstyle\int_{a^-}^{t^-} d\rho(t,\beta)g(\beta)=y(t)-g(t).
\end{aligned}$$

Thus, y is a solution to (2.4). Clearly, it is measurable, and it is also bounded since $\|\rho(t,.)\|_{BV([a,b])}$ is bounded independently of t and g is bounded. If $\widetilde{y}$ is another solution to (2.4) then $\widetilde{y}(a)=y(a)=g(a)$ by inspection, so that $z:=y-\widetilde{y}$ is a bounded measurable solution to the homogeneous equation:

$$z(t)=\int_{a^+}^{t^-} d\kappa(t,\tau)z(\tau),\qquad a\leq t\leq b.$$

Pick $r>0$ to be adjusted momentarily, and set $\widetilde{z}(t):=e^{-rt}z(t)$ so that

$$\widetilde{z}(t)=\int_{a^+}^{t^-} e^{-r(t-\tau)}d\kappa(t,\tau)\widetilde{z}(\tau). \tag{A.4}$$

Let $\eta>0$ be so small that $W_{[\tau,t)}(\kappa(t,.))\leq 1/4$ as soon as $t-\tau\leq\eta$; this is possible because $\kappa\in\mathcal{K}_{[a,b]}$. Then, it follows from (A.4) that for $t-\eta>a$:

$$\begin{aligned}
|\widetilde{z}(t)| &\leq \left|\int_{a^+}^{(t-\eta)^+} e^{-r(t-\tau)^+}d\kappa(t,\tau)\widetilde{z}(\tau)\right|+\left|\int_{(t-\eta)^+}^{t^-} e^{-r(t-\tau)^-}d\kappa(t,\tau)\widetilde{z}(\tau)\right|\\
&\leq 2e^{-r\eta}W_{(a,t-\eta]}(\kappa(t,\cdot))\sup_{(a,t-\eta]}|\widetilde{z}|+\frac{1}{2}\sup_{(t-\eta,t)}|\widetilde{z}|\leq\sup_{(a,t)}|\widetilde{z}|\big(2e^{-r\eta}\|\kappa\|_{[a,b]}+\frac{1}{2}\big),
\end{aligned}$$

while for $t-\eta\leq a$ we simply get $|\widetilde{z}(t)|\leq\sup_{(a,t)}|\widetilde{z}|/2$. Hence, choosing r large enough, we may assume that $|\widetilde{z}(t)|\leq\lambda\sup_{(a,t)}|\widetilde{z}|$ for some $\lambda<1$ and all $t\in[a,b]$. Thus, if we choose $\lambda'\in(\lambda,1)$ and $t_0\in(a,b]$, we can find $t_1\in(a,t_0)$ such that $|\widetilde{z}(t_1)|\geq(1/\lambda')|\widetilde{z}(t_0)|$, and proceeding inductively we construct a sequence (t_n) in $(a,t_0]$ with $|\widetilde{z}(t_n)|\geq(1/\lambda')^n|\widetilde{z}(t_0)|$. If we had $|\widetilde{z}(t_0)|>0$, this would contradict the boundedness of $\widetilde{z}$, therefore $\widetilde{z}\equiv 0$ on $(a,b]$, whence $z\equiv 0$ so that $y=\widetilde{y}$. $\square$