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A Triadic Generalisation of the Boolean Concept Lattice

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Abstract. Boolean concept lattices are fundamental structures in formal concept analysis, both from a theoretical and an applied point of view. There are multiple ways to generalise them in the triadic concept analysis framework and one of them, the so-called powerset trilattice, has already been proposed by Biedermann in 1998. However, it lacks some interesting properties such as extremality in the number of triconcepts for tricontexts of a given size. In this paper, we discuss another generalisation of Boolean concept lattices that exhibit such properties. We argue that those structures form equivalence classes and should be studied as such, and investigate the minimum number of objects required to produce them.

Keywords: Formal Concept Analysis · Triadic Concept Analysis · Boolean lattice.

1 Introduction

Formal Concept Analysis (FCA [9]) is a formalism that establishes a connection between classical binary data (crosstables) and the structure of concepts and rules that can be found in said data. It is very powerful, if underutilised, as it offers well-studied mathematical structures to be exploited by algorithms.

As crosstables are a rather limiting way of representing data, various extensions of the formalism have been proposed to deal with more complex data, such as Pattern Structures [7], Relational Concept Analysis [12], fuzzy FCA [11] or graph FCA [6]. Just like FCA, they are based on lattice theory [5].

Triadic Concept Analysis [10] is another such formalism that aims to extend FCA to data in the form of ternary relations (*i.e.* tridimensional crosstables), and has the peculiarity of involving trilattices instead of lattices. Such structures are much less studied and many questions remain open (or waiting to be opened).

In this paper, we are interested in the tridimensional generalisation of the well known Boolean concept lattices. There are multiple ways to generalise these structures to the tridimensional case, one of them having been studied by Biedermann [3]. While interesting in many aspects, these so-called powerset trilattices lack properties that are of interest in pattern mining. For instance, Boolean concept lattices contain all possible subsets of attributes as intents, *i.e.* all possible descriptions, making it the fundamental search space for anything itemset-related. This is not the case of powerset trilattices as its triconcepts do not contain all possible combinations of attributes and conditions. Powerset trilattices are also not extremal in the number of triconcepts for tricontexts of a given size.

We thus discuss here another generalisation of Boolean concept lattices that exhibits the properties we are looking for. In Section 3, we begin by arguing that the absence of duality between the quasi-orders of a trilattice opens a new way to study these structures: as classes of trilattices sharing the same structure of “descriptions” of triconcepts. In Section 4, we define the classes of trilattices generalising Boolean concept lattices, and in Section 5 we discuss the minimum number of objects required to produce such trilattices.

2 Formal and Triadic Concept Analysis

2.1 Formal Concept Analysis

We begin by presenting the essential notions of formal concept analysis. For an indepth introduction to FCA, we refer the interested reader to the book [9]. A formal context is a triple $(\mathcal{O}, \mathcal{A}, \mathcal{I})$ where \mathcal{O} is a set of *objects*, \mathcal{A} is a set of *attributes* and $\mathcal{I} \subseteq \mathcal{O} \times \mathcal{A}$ is a binary relation between objects and attributes. We say that object o is *described* by attribute a when $(o, a) \in \mathcal{I}$. The *description* of the object o is thus a set of attributes. Formal contexts can be represented as crosstables, as depicted in Fig. 1. A formal context is said to be *reduced* when no row (resp. column) is empty, full of crosses or equal to the intersection of other rows (resp. columns). We suppose in this paper that all contexts are reduced.

	a_1	a_2	a_3	a_4	a_5
o_1	×	×			
o_2		×	×	×	
o_3	×		×	×	
o_4	×		×		
o_5			×	×	

Fig. 1: A crosstable representing a formal context with five objects $(\{o_1, o_2, o_3, o_4, o_5\})$ and five attributes $(\{a_1, a_2, a_3, a_4, a_5\})$.

From a formal context, two derivation operators \cdot' are defined such that

- $\cdot' : 2^{\mathcal{O}} \rightarrow 2^{\mathcal{A}}, O \mapsto O' = \{a \in \mathcal{A} \mid \forall o \in O, (o, a) \in \mathcal{I}\}$, and
- $\cdot' : 2^{\mathcal{A}} \rightarrow 2^{\mathcal{O}}, A \mapsto A' = \{o \in \mathcal{O} \mid \forall a \in A, (o, a) \in \mathcal{I}\}$.

A *formal concept* is a pair $(E, I) \in 2^{\mathcal{O}} \times 2^{\mathcal{A}}$ such that $E = I'$ and $I = E'$. We call E the *extent* of the concept while I is its *intent*. Concepts correspond to maximal rectangles of crosses in the formal context, up to permutation of rows and columns. Hence, another definition is that a formal concept is a pair (E, I) such that $E \times I \in \mathcal{I}$ and both components are maximal for this property. According to the basic theorem of formal concept analysis, the set of formal concepts in a formal context ordered by the inclusion relation on either their intents or extents forms a complete lattice called the *concept lattice* of the context.

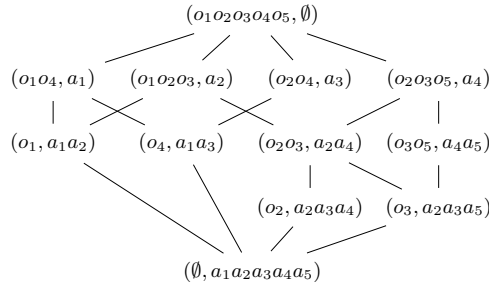


Fig. 2: Concept lattice of the formal context depicted in Fig. 1.

The Boolean concept lattices are the concept lattices of contexts of the form (S, S, \neq) called *contranominal scales*. Fig. 3 depicts such a Boolean concept lattice for $|S| = 3$. In a Boolean concept lattice, all subsets of attributes (resp. objects) are intents (resp. extents) of concepts. As such, it contains $2^{|S|}$ concepts.

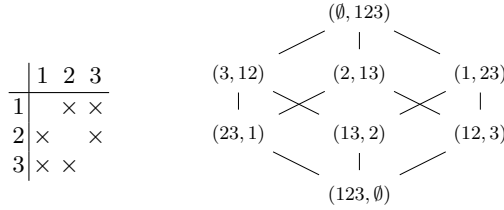


Fig. 3: A Boolean concept lattice

2.2 Triadic Concept Analysis

Triadic concept analysis is the tridimensional generalisation of FCA introduced by Lehman and Wille [10]. It has since then been further generalised to the n -dimensional case by Voutsadakis [13] but this is outside the scope of this paper. In this setting, a triadic context (or *tricontext*, or *3-context*) is a triple $(\mathcal{O}, \mathcal{A}, \mathcal{C}, \mathcal{I})$ where the \mathcal{O} is a set of object, \mathcal{A} is a set of attributes, \mathcal{C} is a set of conditions and $\mathcal{I} \subseteq \mathcal{O} \times \mathcal{A} \times \mathcal{C}$ is a ternary relation between elements of the three dimensions. In this paper, we shall say that the pairs (a, c) such that $(o, a, c) \in \mathcal{I}$ form the *description* of the object o . An example of a triadic context is depicted in Fig. 4. For practical reasons that will be clarified later, we represent the object dimension on the bottom of the figure so that the descriptions of objects are visually represented as classical formal contexts.

A triadic concept (or *triconcept*, or *3-concept*) is then a maximal tridimensional box full of crosses, *i.e.* a triple (X_1, X_2, X_3) such that $X_1 \times X_2 \times X_3 \subseteq \mathcal{I}$ and the X_i are maximal for this property. For instance, $(\{o_1, o_2\}, \{a_1, a_2\}, \{c_1\})$ is a triconcept in

	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3
a_1	×			×		×			×
a_2	×			×					
a_3	×	×			×			×	×
		o_1			o_2			o_3	

Fig. 4: A triadic context $(\{o_1, o_2, o_3\}, \{a_1, a_2, a_3\}, \{c_1, c_2, c_3\}, \mathcal{I})$.

Fig. 4. We shall say that, in a triconcept (O, A, C) , O is the extent while (A, C) is the description of the triconcept. The set \mathcal{S} of all triconcepts in a tricontext together with the three quasi-orders

$$(X_1, X_2, X_3) \lesssim_i (Y_1, Y_2, Y_3) \Leftrightarrow X_i \subseteq Y_i, i \in \{1, 2, 3\},$$

forms a triadic lattice (or trilattice, or 3-lattice) $\mathcal{L} = (\mathcal{S}, \lesssim_1, \lesssim_2, \lesssim_3)$ [4] called the concept trilattice of the tricontext. A trilattice is a triordered set, *i.e.*:

- if $\forall i \in \{1, 2, 3\} \setminus \{j\}$, $A \lesssim_i B$ then $A \gtrsim_j B$ (antiordinal dependency) and
- if $\forall i \in \{1, 2, 3\}$, $A \sim_i B$, then $A = B$ (uniqueness condition).

The duality of the two partial orders on extents and intents in the dyadic case is thus lost in the triadic case. This has consequences that are discussed in the next section.

In [3], Biedermann proposed a first triadic generalisation of Boolean lattices called *powerset trilattices*. They are the concept trilattices of tricontexts of the form

$$(S, S, S, S^3 \setminus \{(a, a, a) \mid a \in S\}).$$

They contain $3^{|S|}$ triconcepts (X_1, X_2, X_3) that are such that $X_1 \cap X_2 \cap X_3 = \emptyset$ and $X_i \cup X_j = S$ for distinct $i, j \in \{1, 2, 3\}$. Fig 5 depicts such a powerset trilattice for $|S| = 3$.

	1 2 3	1 2 3	1 2 3
1	×	×	×
2	×	×	×
3	×	×	×
	1	2	3

$(\{1\}, \{1, 2, 3\}, \{2, 3\})$	$(\{1\}, \{2, 3\}, \{1, 2, 3\})$	$(\{2\}, \{1, 2, 3\}, \{1, 3\})$
$(\{2\}, \{1, 3\}, \{1, 2, 3\})$	$(\{3\}, \{1, 2\}, \{1, 2, 3\})$	$(\{3\}, \{1, 2, 3\}, \{1, 2\})$
$(\{1, 3\}, \{1, 2\}, \{2, 3\})$	$(\{1, 3\}, \{2, 3\}, \{1, 2\})$	$(\{2, 3\}, \{1, 2\}, \{1, 3\})$
$(\{1, 2\}, \{2, 3\}, \{1, 3\})$	$(\{2, 3\}, \{1, 3\}, \{1, 2\})$	$(\{1, 2\}, \{1, 3\}, \{2, 3\})$
$(\{2, 3\}, \{1, 2, 3\}, \{1\})$	$(\{1, 2\}, \{1, 2, 3\}, \{2\})$	$(\{1, 2\}, \{1, 2, 3\}, \{3\})$
$(\{2, 3\}, \{1\}, \{1, 2, 3\})$	$(\{1, 3\}, \{2\}, \{1, 2, 3\})$	$(\{1, 2\}, \{3\}, \{1, 2, 3\})$
$(\{1, 2, 3\}, \{1\}, \{2, 3\})$	$(\{1, 2, 3\}, \{2\}, \{1, 3\})$	$(\{1, 2, 3\}, \{3\}, \{1, 2\})$
$(\{1, 2, 3\}, \{2, 3\}, \{1\})$	$(\{1, 2, 3\}, \{1, 3\}, \{2\})$	$(\{1, 2, 3\}, \{1, 2\}, \{3\})$
$(\emptyset, \{1, 2, 3\}, \{1, 2, 3\})$	$(\{1, 2, 3\}, \emptyset, \{1, 2, 3\})$	$(\{1, 2, 3\}, \{1, 2, 3\}, \emptyset)$

Fig. 5: The tricontext and triconcepts of a powerset trilattice on three elements.

3 The Loss of Duality in Triadic Concept Analysis

The biggest difference between the bidimensional and the tridimensional cases is that, in a concept lattice, the partial order on extents is dual to the partial order on intents whereas fixing two quasi-orders in a concept trilattice does not determine the third quasi-order. This is exemplified in Fig 6 where two different tricontexts produce two sets of triconcepts that have the same descriptions but different extents. The second and third quasi-orders of both trilattices are thus isomorphic while the two first quasi-orders are not.

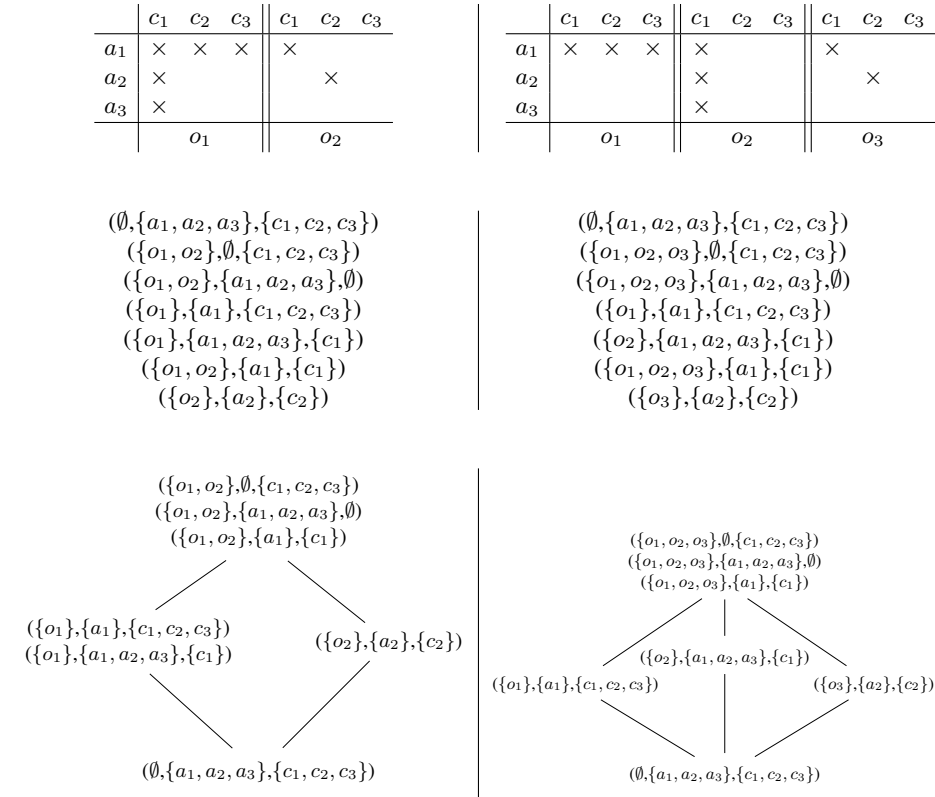


Fig. 6: Two tricontexts ($\{o_1, o_2, o_3\}, \{1, 2, 3\}, \{a, b, c\}, \mathcal{T}_1$) (left) and ($\{o_1, o_2\}, \{1, 2, 3\}, \{a, b, c\}, \mathcal{T}_2$) (right), their associated triconcepts and the two first quasi-orders.

For a trilattice $\mathcal{L} = (\mathcal{S}, \lesssim_1^{\mathcal{L}}, \lesssim_2^{\mathcal{L}}, \lesssim_3^{\mathcal{L}})$, we call \mathcal{L}^{\equiv} the equivalence class comprised of all trilattices $\mathcal{L}_2 = (\mathcal{S}_2, \lesssim_1^{\mathcal{L}_2}, \lesssim_2^{\mathcal{L}_2}, \lesssim_3^{\mathcal{L}_2})$ such that $\lesssim_i^{\mathcal{L}}$ is isomorphic to $\lesssim_i^{\mathcal{L}_2}$ for $i \in \{2, 3\}$. We argue that the study of such equivalence classes in triadic and polyadic concept analysis opens up new and interesting questions, even beyond the purely mathematical aspects:

- what does it mean from the point of view of knowledge representation when there are different ways objects can belong to formal concepts as defined by their descriptions?
- if the size of a dimension can be changed without modifying the structure, can it lead to new data augmentation/reduction techniques?
- if a tricontext produces a trilattice with the minimum number of objects among its equivalence class, does it mean anything?

Brief divagation These equivalence classes impact the way implications [8,2] should be considered in the multidimensional case as different tricontexts can produce different implications sets that all allow for the reconstruction of the exact same descriptions. We propose that all implications common to all the tricontexts/trilattices of an equivalence class be called *structural implications* while implications that only hold in some tricontexts/trilattices be called *contextual implications*. Further study of such objects is outside the scope of this paper and shall be the subject of future work.

4 A Triadic Generalisation of the Boolean Concept Lattice

A Boolean concept lattice has useful features that are not present in the powerset trilattice and that justify our proposition of another triadic generalisation. Firstly, the lattice of its intents is the powerset lattice of attributes, *i.e.* all subsets of attributes are present as the intent (description) of a concept. Secondly, it is maximal in the number of concepts for a given context size, *i.e.* given an $n \times n$ context, the biggest possible corresponding concept lattice is the Boolean concept lattice with 2^n elements ($2^{\min(n,m)}$ elements in a non-reduced $n \times m$ formal context). By contrast, the powerset trilattice does not have all possible descriptions (there are no $(X, \{2, 3\}, \{2, 3\})$ triconcept in Fig. 5) and it is not maximal in the number of triconcepts for its tricontext's size (the powerset trilattice on a $3 \times 3 \times 3$ tricontext has 27 triconcepts while the trilattice of the tricontext in Fig 7 has 30).

	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3
a_1	×	×			×	×	×		×
a_2	×		×	×	×			×	×
a_3		×	×	×		×	×		×
		o_1			o_2			o_3	

Fig. 7: A $3 \times 3 \times 3$ tricontext that contains more triconcepts than the $3 \times 3 \times 3$ powerset trilattice.

Having all possible descriptions in a concept lattice is useful as a basic search space for pattern mining. Most itemset mining algorithms can be seen as cleverly searching the powerset of items, so having the multidimensional equivalent of this search space would allow for similar new multidimensional description mining algorithms. As for the

maximal number of concepts in fixed contexts, its knowledge is required in the rigorous analysis of algorithms. We also expect these structures to play a role in the study of “big” trilattices, similarly to Boolean concept lattices [1]. Therefore, we propose the following triadic generalisations of the Boolean concept lattice.

Let \mathcal{A} and \mathcal{C} be an attribute and a condition set and $\mathcal{L} = (S, \lesssim_1, \lesssim_2, \lesssim_3)$ be a concept trilattice such that

$$(X_1, X_2, X_3) \in S \Leftrightarrow (X_2, X_3) \in ((2^{\mathcal{A}} \setminus \emptyset) \times (2^{\mathcal{C}} \setminus \emptyset)) \cup \{(\emptyset, \mathcal{C}), (\mathcal{A}, \emptyset)\}.$$

This trilattice is a generalisation of the Boolean concept lattice as it contains all possible concept description and, as two components of a concept uniquely define the third, it is thus maximal in the number of concepts among all trilattices on tricontexts of the form $(\mathcal{O}, \mathcal{A}, \mathcal{C}, \mathcal{I})$. However, as discussed in the previous section, there are multiple such trilattices for a given \mathcal{A} and \mathcal{C} . Hence, for $n = |\mathcal{A}|$ and $m = |\mathcal{C}|$, we use $\mathcal{B}_{n,m} = \mathcal{L}^{\equiv}$ to denote the equivalence class of trilattices that contain all possible descriptions, *i.e.* all possible rectangles, in an $n \times m$ crosstable. More formally, the concept trilattices $(S, \lesssim_1, \lesssim_2, \lesssim_3)$ in $\mathcal{B}_{n,m}$ are the concept trilattices of tricontexts $(\mathcal{O}, \mathcal{A}, \mathcal{C}, \mathcal{I})$ (we still suppose that all contexts are reduced) such that $|\mathcal{A}| = n$, $|\mathcal{C}| = m$ and, for all $(Y, Z) \in (2^{\mathcal{A}} \setminus \emptyset) \times (2^{\mathcal{C}} \setminus \emptyset)$, there is a triconcept $(X, Y, Z) \in S$. Clearly, the powerset trilattice depicted in Fig 5 does not belong to $\mathcal{B}_{3,3}$ as it has no triconcept $(X, \{3\}, \{2, 3\})$.

Fig. 8 depicts the tricontexts and triconcepts of two elements of $\mathcal{B}_{2,2}$. We can see that the 9 non-empty rectangles in a 2×2 crosstable, as well as two occurrences of empty rectangles, are present as descriptions of triconcepts.

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<p style="margin: 0;"> $(\emptyset, \{a_1, a_2\}, \{c_1, c_2\})$ $(\{o_1\}, \{a_1\}, \{c_1, c_2\})$ $(\{o_2\}, \{a_2\}, \{c_1, c_2\})$ $(\{o_3\}, \{a_1, a_2\}, \{c_1\})$ $(\{o_4\}, \{a_1, a_2\}, \{c_2\})$ $(\{o_1, o_3\}, \{a_1\}, \{c_1\})$ $(\{o_1, o_4\}, \{a_1\}, \{c_2\})$ $(\{o_2, o_3\}, \{a_2\}, \{c_1\})$ $(\{o_2, o_4\}, \{a_2\}, \{c_2\})$ $(\{o_1, o_2, o_3, o_4\}, \emptyset, \{c_1, c_2\})$ $(\{o_1, o_2, o_3, o_4\}, \{a_1, a_2\}, \emptyset)$ </p>		<p style="margin: 0;"> $(\emptyset, \{a_1, a_2\}, \{c_1, c_2\})$ $(\{o_1\}, \{a_1\}, \{c_1, c_2\})$ $(\{o_2\}, \{a_2\}, \{c_1, c_2\})$ $(\{o_2\}, \{a_1, a_2\}, \{c_1\})$ $(\{o_1\}, \{a_1, a_2\}, \{c_2\})$ $(\{o_1, o_2\}, \{a_1\}, \{c_1\})$ $(\{o_1, o_3\}, \{a_1\}, \{c_2\})$ $(\{o_2, o_3\}, \{a_2\}, \{c_1\})$ $(\{o_1, o_2\}, \{a_2\}, \{c_2\})$ $(\{o_1, o_2, o_3\}, \emptyset, \{c_1, c_2\})$ $(\{o_1, o_2, o_3\}, \{a_1, a_2\}, \emptyset)$ </p>																																																																																

Fig. 8: Two tricontexts producing trilattices in $\mathcal{B}_{2,2}$.

A trilattice in $\mathcal{B}_{n,m}$ contains $(2^n - 1) \times (2^m - 1) + 2$ triconcepts, corresponding to the $(2^n - 1) \times (2^m - 1)$ descriptions that are non-empty rectangles plus two times the empty rectangle in the two triconcepts of the form $(\mathcal{O}, \emptyset, \mathcal{C})$ and $(\mathcal{O}, \mathcal{A}, \emptyset)$.

5 On the Minimum Number of Objects in a $\mathcal{B}_{n,m}$ Tricontext

In Fig. 8, we see two $\mathcal{B}_{2,2}$ trilattices, one built on 3 objects and the other on 4. A question then naturally arises: what is the minimum number of objects required to produce a $\mathcal{B}_{n,m}$ trilattice? In the bidimensional case, Boolean concept lattices are produced by contranominal scales and require as many objects as attributes. In the triadic case, it is easy to see that any $\mathcal{B}_{n,m}$ trilattice can be produced on $n + m$ objects. Indeed, the rectangles missing only a single row or column are *irreducible*, *i.e.* they cannot be obtained by intersecting two other rectangles, so they have to appear as maximal rectangles in the description of an object. Having all these rectangles on different objects produces a $\mathcal{B}_{n,m}$ trilattice on $n + m$ objects, as exemplified on the left hand side of Fig. 8 and in Fig. 9. Can we do it with fewer objects? In the case of $\mathcal{B}_{2,2}$ yes, as shown in Fig. 8 with 3 objects, but the answer is not so simple for other sizes.

First of all, we know that if (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are triconcepts such that $X_2 \subseteq Y_2$ and $X_3 \subseteq Y_3$, then $Y_1 \subseteq X_1$. As the height of the poset of rectangles in an $n \times m$ crosstable ordered by inclusion is $n + m$, we have that a $\mathcal{B}_{n,m}$ trilattice cannot be produced on fewer than $n + m - 1$ objects. The minimum number of objects thus lies in $[n + m - 1, n + m]$. We now inelegantly show that it is $n + m$ when either $n > 2$ or $m > 2$.

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	× × ×	× × ×		× ×	× ×	× ×
a_2	× × ×		× × ×	× ×	× ×	× ×
a_3		× × ×	× × ×	× ×	× ×	× ×
	o_1	o_2	o_3	o_4	o_5	o_6

Fig. 9: A $6 \times 3 \times 3$ $\mathcal{B}_{3,3}$ tricontext.

Let us start with the case $n = m$ and illustrate with $n = m = 3$ without loss of generality. As the $n + m$ irreducible rectangles have to appear somewhere, if we want to construct a $\mathcal{B}_{n,m}$ tricontext on $n + m - 1$ objects, we have to put two irreducible rectangles on the same object. For instance, $(\{a_1, a_2\}, \{c_1, c_2, c_3\})$ and $(\{a_1, a_2, a_3\}, \{c_2, c_3\})$.

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	× × ×	× × ×		× ×	× ×	
a_2	× × ×		× × ×	× ×	× ×	
a_3	× ×	× × ×	× × ×	× ×	× ×	
	o_1	o_2	o_3	o_4	o_5	o_6

However, if we do this, their intersection $(\{a_1, a_2\}, \{c_2, c_3\})$ is not the description of a triconcept anymore. The $(\{a_1, a_2\}, \{c_2, c_3\})$ rectangle must thus appear as a maximal rectangle in the description of another object. The only option is to put it on its own object and then we go back to having $n + m$ objects. From there, being able to construct a $\mathcal{B}_{n,m}$ tricontext on $n + m - 1$ objects implies being able to construct a $\mathcal{B}_{n-1,m-1}$ tricontext on $n + m - 3$ objects (in the greyed area) as two objects are already taken by the two irreducible rectangles and their intersection.

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	× × ×	× ×			
a_2	× × ×	× ×			
a_3	× ×				
	o_1	o_2	o_3	o_4	o_5

It is possible to construct a $\mathcal{B}_{2,2}$ tricontext with 3 objects (see Fig. 8): two pairs of irreducible rectangles are put on two objects and both their intersections coexist on a third object. To show that this is only one way to do so, up to permutation of rows and columns, is left as an exercise for the reader. The construction of a hypothetical $\mathcal{B}_{3,3}$ tricontext on 5 objects would thus proceed as follows:

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	× × ×	× ×	× ×	×	×
a_2	× × ×	× ×	×	× ×	×
a_3	× ×				
	o_1	o_2	o_3	o_4	o_5

As the irreducible rectangles still need to appear somewhere, the only way to have them appear without destroying the $\mathcal{B}_{2,2}$ subcontext is thus as follows:

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	× × ×	× ×	× × ×	× ×	×
a_2	× × ×	× ×	× ×	× × ×	×
a_3	× ×		× × ×	× × ×	
	o_1	o_2	o_3	o_4	o_5

The coexistence of irreducible rectangles on the same object again causes their intersection to stop being descriptions of triconcepts, so we have to add $(\{a_1, a_3\}, \{c_1, c_3\})$ and $(\{a_2, a_3\}, \{c_1, c_2\})$ as maximal rectangles in the description of an object. One of them can be added to the description of o_5 but it can be seen that none of them can be added to the description of o_2 . For example, adding $(\{a_2, a_3\}, \{c_1, c_2\})$ would cause $(\{a_2\}, \{c_2, c_3\})$ to stop being the description of a triconcept, creating the need for a sixth object. We are thus stuck, and conclude that $\mathcal{B}_{3,3}$ trilattices require at least 6 objects. From this we deduce that $\mathcal{B}_{n,n}$ trilattices require at least $2n$ objects.

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	× × ×	× ×	× × ×	× ×	× ×	
a_2	× × ×	× ×	× ×	× × ×	×	× ×
a_3	× ×		× × ×	× × ×	× ×	× ×
	o_1	o_2	o_3	o_4	o_5	o_6

Fig. 10: Another way to build a $\mathcal{B}_{3,3}$ trilattice on 6 objects.

Now, we show analogously that $\mathcal{B}_{2,m}$ trilattices require at least $2 + m$ objects. We illustrate on a $\mathcal{B}_{2,3}$ trilattice but assert that the same construction and reasoning apply

to $m > 3$. As for the 3×3 case, all irreducible rectangles in a $2 \times m$ crosstable have to appear in the description of an object.

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	$\times \times$	$\times \times$	$\times \times$	$\times \times \times$	
a_2	$\times \times$	$\times \times$	$\times \times$		$\times \times \times$
	o_1	o_2	o_3	o_4	o_5

In order to reduce the number of objects, two such rectangles have to be put on the same object. Their intersection ceases to be the description of a triconcept so we have to put it on its own object.

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	$\times \times \times$	$\times \times$		
a_2	$\times \times$			
	o_1	o_2	o_3	o_4

If it is possible to construct a $\mathcal{B}_{2,m}$ trilattice on $2+m-1$ objects, then it is possible to construct a $\mathcal{B}_{1,m-1}$ trilattice on $2+m-3$ objects. It is the case here. In order to complete the tricontext so that all irreducible rectangles appear as maximal rectangles in the description of an object, the unused horizontal irreducible rectangle ($(\{a_2\}, \{c_1, c_2, c_3\})$ here) has to coexist with another irreducible rectangle. Their intersection thus has to appear as a maximal rectangle in the description of another object and it easy to see, on such a small example, that it is impossible without having rectangles cease to be descriptions of triconcepts (adding it to o_2 would cause $(\{a_1\}, \{c_1\})$ to stop being the description of a triconcept).

	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3	c_1 c_2 c_3
a_1	$\times \times \times$	$\times \times$	$\times \times$	$\times \times$	
a_2	$\times \times$		$\times \times \times$	$\times \times$	$\times \times$
	o_1	o_2	o_3	o_4	o_5

Hence, $\mathcal{B}_{n,m}$ trilattices require at least $n + m$ objects when either $n > 2$ or $m > 2$.

6 Discussion and Conclusion

The generalisation of Boolean concept lattices proposed in this paper is only one of many but we feel that its properties are interesting for both pattern mining and the study of the maximal size of concept trilattices. The latter is still an open question as, for instance, powerset trilattices are not the biggest trilattices that can be built in a $n \times n \times n$ tricontexts and do not fit. The next step would be to further generalise these structures to the n -dimensional case. The definition is straightforward but our proof of the minimum number of objects required to build the trilattice is hardly scalable as it relies on handmade contexts that would be too big and numerous in higher dimensions. We also plan on further studying the structure of the classes of trilattices/ n -lattices defined in Section 3 and the consequences of that loss of duality on implications.

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References

1. Albano, A., Chornomaz, B.: Why Concept Lattices Are Large: Extremal Theory for Generators, Concepts, and VC-Dimension. *International Journal of General Systems* **46**(5), 440–457 (2017)
2. Bazin, A.: On Implication Bases in n-Lattices. *Discrete Applied Mathematics* **273**, 21–29 (2020)
3. Biedermann, K.: Powerset Trilattices. In: *International Conference on Conceptual Structures*. pp. 209–221. Springer (1998)
4. Biedermann, K.: An equational theory for trilattices. *Algebra Universalis* **42**, 253–268 (1999)
5. Birkhoff, G.: *Lattice Theory*, vol. 25. American Mathematical Soc. (1940)
6. Ferré, S., Cellier, P.: Graph-fca: An extension of formal concept analysis to knowledge graphs. *Discrete applied mathematics* **273**, 81–102 (2020)
7. Ganter, B., Kuznetsov, S.O.: Pattern Structures and Their Projections. In: *International conference on conceptual structures*. pp. 129–142. Springer (2001)
8. Ganter, B., Obiedkov, S.: Implications in Triadic Formal Contexts. In: *International Conference on Conceptual Structures*. pp. 186–195. Springer (2004)
9. Ganter, B., Wille, R.: *Formal Concept Analysis: Mathematical Foundations*. Springer Science & Business Media (2012)
10. Lehmann, F., Wille, R.: A Triadic Approach to Formal Concept Analysis. In: *International conference on conceptual structures*. pp. 32–43. Springer (1995)
11. Poelmans, J., Ignatov, D.I., Kuznetsov, S.O., Dedene, G.: Fuzzy and Rough Formal Concept Analysis: a Survey. *International Journal of General Systems* **43**(2), 105–134 (2014)
12. Rouane-Hacene, M., Huchard, M., Napoli, A., Valtchev, P.: Relational Concept Analysis: Mining Concept Lattices from Multi-Relational Data. *Annals of Mathematics and Artificial Intelligence* **67**(1), 81–108 (2013)
13. Voutsadakis, G.: Polyadic Concept Analysis. *Order* **19**(3), 295–304 (2002)