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# CANONICAL COORDINATES FOR MODULI SPACES OF RANK TWO IRREGULAR CONNECTIONS ON CURVES.

ARATA KOMYO, FRANK LORAY, MASA-HIKO SAITO, AND SZILÁRD SZABÓ

ABSTRACT. In this paper, we study a geometric counterpart of the cyclic vector which allow us to put a rank 2 meromorphic connection on a curve into a “companion” normal form. This allow us to naturally identify an open set of the moduli space of  $GL_2$ -connections (with fixed generic spectral data, i.e. unramified, non resonant) with some Hilbert scheme of points on the twisted cotangent bundle of the curve. We prove that this map is symplectic, therefore providing Darboux (or canonical) coordinates on the moduli space, i.e. separation of variables. On the other hand, for  $SL_2$ -connections, we give an explicit formula for the symplectic structure for a birational model given by Matsumoto. We finally detail the case of an elliptic curve with a divisor of degree 2.

## 1. INTRODUCTION

In this paper, we introduce coordinates on the moduli spaces of rank 2 meromorphic connections on a Riemann surface, and we describe the symplectic structures on the moduli spaces by the introduced coordinates. Finally, we will have canonical coordinates on the moduli spaces. Our motivation is to give explicit descriptions of the isomonodromic deformations of meromorphic connections over a general Riemann surface. It is well-know that the isomonodromic deformations have non-autonomous Hamiltonian descriptions (in detail, see [34], [20], [14], for example). If we find explicit formulae for the isomonodromic Hamiltonians, then we have explicit descriptions of isomonodromic deformations. To find explicit formulae of Hamiltonians, it is necessary to introduce canonical coordinates (which are also called Darboux coordinates) on the moduli space of meromorphic connections. The present paper is a first step to give explicit descriptions of isomonodromic deformations.

For the isomonodromic deformations of rank 2 projective connections with regular singular points, there are some results of explicit descriptions. For example, Okamoto considered non-autonomous Hamiltonian descriptions of isomonodromic deformations on elliptic curves in [43] and [44]. Iwasaki generalized for general Riemann surfaces in [25] and [26]. Here the independent variables of the isomonodromic deformations are the position of regular singular points on the Riemann surfaces. That is, they are isomonodromic deformations of fixed Riemann surfaces. On the other hand, Kawai [28] gave explicit descriptions of the isomonodromic Hamiltonians varying the elliptic curve. Okamoto, Iwasaki, and Kawai in these papers introduced canonical coordinates on (a generic

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part of) the moduli space of rank 2 meromorphic projective connections by using apparent singularities. For our purpose, we take this strategy. That is, we will also introduce canonical coordinates on the moduli space of meromorphic connections by using apparent singularities. On the other hand, in this paper, we are interested in the isomonodromic deformations of  $GL_2$ -connections and of  $SL_2$ -connections. The coordinates using apparent singularities are an analog of the separation of variables in the Hitchin system, which is a birational map from the moduli space of stable Higgs bundles to the Hilbert scheme of points on the cotangent bundle over the underlying curve of the Higgs bundles (see [19] and [16]). Here this map is a symplectomorphism of the open dense subsets of the moduli space. The definition of the apparent singularities for general rank meromorphic connections is in [45].

**1.1. Our setting.** Let  $\nu$  be a positive integer. We set  $I := \{1, 2, \dots, \nu\}$ . Let  $C$  be a compact Riemann surface of genus  $g$  ( $g \geq 0$ ), and  $D = \sum_{i \in I} m_i [t_i]$  be an effective divisor on  $C$ . Let  $E$  be a vector bundle over  $C$  and  $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$  be a meromorphic connection acting on  $E$ . We assume that the leading term of the expansion of a connection matrix of  $\nabla$  at  $t_i$  has distinct eigenvalues. If  $m_i = 1$ , then we assume that the difference of eigenvalues of the residue matrix at  $t_i$  is not integer. That is,  $t_i$  is a generic unramified irregular singular point of  $\nabla$  or a non-resonant regular singular point of  $\nabla$ .

When  $C$  is the projective line and  $E$  is the trivial bundle, the moduli space of meromorphic connections has been studied by Boalch [7] and Hiroe–Yamakawa [18]. This moduli space has the natural symplectic structure coming from the symplectic structure on the (extended) coadjoint orbits. For general  $C$  and  $E$ , the moduli space of meromorphic connections (with quasi-parabolic structures) has been studied by Inaba–Iwasaki–Saito [22, 23], Inaba [21], and Inaba–Saito [24]. For general  $C$  and  $E$ , the moduli space has also the natural symplectic structure. In these papers, the symplectic form described by a pairing of the hypercohomologies of some complex. This description of the symplectic structure is an analog of the symplectic structure of the moduli spaces of stable Higgs bundles due to Bottacin [8]. For the case where  $\nabla$  has only regular singular points, Inaba showed that this symplectic structure coincides with the pull-back of the Goldman symplectic structure on the character variety via the Riemann–Hilbert map in [21, the proof of Proposition 7.3].

Our purpose in this paper is to introduce canonical coordinates on the moduli spaces of meromorphic connections. For this purpose, there are some strategies. First one is to consider canonical coordinates on the products of coadjoint orbits. This direction was studied by Jimbo–Miwa–Mori–Sato [27], Harnad [17], and Woodhouse [47]. Sakai–Kawakami–Nakamura [29] and Gaiur–Mazzocco–Rubtsov [15] gave some explicit formulae for the isomonodromic Hamiltonians by the coordinates of this direction. Second one is to consider the apparent singularities. As mentioned above, we take this strategies.

In this paper, we consider only the case where the rank of  $E$  is two. Let  $X$  be an irregular curve, which is described in Section 2.3. That is,  $X$  is a tuple of (i) a compact Riemann surface  $C$ , (ii) an effective divisor  $D$  on  $C$ , (iii) local coordinates around the support with  $D$ , and (iv) spectral data of meromorphic connections at the support with  $D$  (with data of residue parts). Here, the spectral data is described in Section 2.3. We fix an irregular curve  $X$ . That is, we fix spectral data of rank 2 meromorphic connections at each point of the support with  $D$ . By applying elementary transformations (which is also called Hecke modifications), we may change the degree of the underlying vector bundle of a meromorphic connection freely. So we assume that  $\deg(E) = 2g - 1$ . By this condition, the Euler characteristic of the vector bundle  $E$  is 1 by the Riemann–Roch

theorem. In this situation, for generic meromorphic connections  $(E, \nabla)$ , we have  $\dim_{\mathbb{C}} H^0(C, E) = 1$ . So the global section of  $E$  is uniquely determined up to constant. This is convenient for the definition of the apparent singularities. In this paper, we consider only meromorphic connections with  $\dim_{\mathbb{C}} H^0(C, E) = 1$ . Moreover we assume that meromorphic connections  $(E, \nabla)$  are irreducible. By this condition, the definition of apparent singularities becomes simple.

**1.2.  $GL_2$ -connections.** In the first part of this paper, we discuss on  $GL_2$ -connections. That is, we consider rank 2 meromorphic connections. We do not fix the determinant bundles of the underlying vector bundles and the traces of connections. Our purpose is to introduce canonical coordinates on the moduli space of rank 2 meromorphic connections by using apparent singularities. When  $C$  is the projective line, many people introduced canonical coordinates on the moduli space by using the apparent singularities ([43], [41], [10], [46], [33], [9], and [31]). In this paper, we consider apparent singularities for general Riemann surfaces.

Let  $X$  be the fixed irregular curve. If  $(E, \nabla)$  is a rank 2 meromorphic connection such that  $\deg(E) = 2g - 1$ ,  $\dim_{\mathbb{C}} H^0(C, E) = 1$ , and  $(E, \nabla)$  is irreducible, then we can define apparent singularities for  $(E, \nabla)$ . (In detail, see Definition 1 below). The apparent singularities are the set of points  $\{q_1, \dots, q_N\}$  on the underlying curve  $C$ . Here we set  $N := 4g - 3 + \deg(D)$ . Let  $M_X$  be the following moduli space

$$M_X := \left\{ (E, \nabla) \left| \begin{array}{l} \text{(i) } E \text{ is a rank 2 vector bundle on } C \text{ with } \deg(E) = 2g - 1 \\ \text{(ii) } \nabla: E \rightarrow E \otimes \Omega_C^1(D) \text{ is a connection} \\ \text{(iii) } (E, \nabla) \text{ is irreducible, and} \\ \text{(iv) } \nabla \text{ has the fixed spectral data in } X \end{array} \right. \right\} / \cong.$$

This moduli space  $M_X$  has a natural symplectic structure due to Inaba–Iwasaki–Saito [22], Inaba [21], and Inaba–Saito [24]. We consider a Zariski open subset  $M_X^0$  of  $M_X$  as follows:

$$M_X^0 := \left\{ (E, \nabla) \in M_X \left| \begin{array}{l} \text{(i) } \dim_{\mathbb{C}} H^0(C, E) = 1, \\ \text{(ii) } q_1 + \dots + q_N \text{ is reduced, and} \\ \text{(iii) } q_1 + \dots + q_N \text{ has disjoint support with } D \end{array} \right. \right\} / \cong$$

(in detail, see Section 3.1). The dimension of the moduli space  $M_X^0$  is  $2N$  (Proposition 10). By taking apparent singularities, we have a map

$$\begin{aligned} \text{App}: M_X^0 &\longrightarrow \text{Sym}^N(C) \\ (E, \nabla) &\longmapsto \{q_1, q_2, \dots, q_N\}. \end{aligned}$$

Remark that the dimension of  $\text{Sym}^N(C)$  is half of the dimension of  $M_X^0$ . To introduce coordinates on  $M_X^0$ , it is necessary to find further invariants of connections, that are customarily called *accessory parameters*. To find these parameters, we introduce a twist of  $\Omega_C^1(D)$  by  $c_d$ , which is the first Chern class  $c_1(\det(E)) \in H^1(C, \Omega_C^1)$  of  $E$ . (In detail, Section 3.5 below). We denote by  $\Omega_C^1(D, c_d)$  the twist of  $\Omega_C^1(D)$ . Let

$$\pi_{c_d}: \mathbf{\Omega}(D, c_d) \longrightarrow C$$

the total space of  $\Omega_C^1(D, c_d)$ . Let  $\omega_{D, c_d}$  be the rational 2-form on  $\mathbf{\Omega}(D, c_d)$  induced by the Liouville symplectic form. This rational 2-form  $\omega_{D, c_d}$  induces a symplectic structure on  $\mathbf{\Omega}(D, c_d) \setminus \pi_{c_d}^{-1}(D)$ . We consider the symmetric product  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$ . Let  $\sum_{j=1}^N \text{pr}_j^*(\omega_{D, c_d})$  be the rational 2-form on the product  $\mathbf{\Omega}(D, c_d)^N$ . Here  $\text{pr}_j: \mathbf{\Omega}(D, c_d)^N \rightarrow \mathbf{\Omega}(D, c_d)$  is the  $j$ -th projection. This rational 2-form  $\sum_{j=1}^N \text{pr}_j^*(\omega_{D, c_d})$  induces a symplectic structure on a generic part of  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$ . We will define a map from  $M_X^0$  to  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$  by the following idea.

By the theory of apparent singularities discussed in Section 2.1, we have a canonical inclusion morphism

$$\mathcal{O}_C \oplus (\Omega_C^1(D))^{-1} \longrightarrow E.$$

By this morphism, we have the connection  $\nabla_0$  on  $\mathcal{O}_C \oplus (\Omega_C^1(D))^{-1}$  induced by a connection  $\nabla$  on  $E$ . Notice that  $\nabla_0$  has simple poles at the apparent singularities. By applying automorphisms on  $\mathcal{O}_C \oplus (\Omega_C^1(D))^{-1}$ , we may normalize  $\nabla_0$  as

$$\nabla_0 = \begin{pmatrix} d & \beta \\ 1 & \delta \end{pmatrix},$$

which is called a companion normal form (in detail, see Section 2.2 below). Here  $d$  is the exterior derivative on  $C$ ,  $\beta \in H^0(C, (\Omega_C^1)^{\otimes 2}(2D + q_1 + \cdots + q_N))$ , and  $\delta$  is a connection on  $(\Omega_C^1(D))^{-1}$ , which has poles at the support of  $D$  and the apparent singularities  $q_1, \dots, q_N$ . Then we may define a map

$$(1.1) \quad \begin{aligned} f_{\text{App}}: M_X^0 &\longrightarrow \text{Sym}^N(\mathbf{\Omega}(D, c_d)) \\ (E, \nabla) &\longmapsto \{(q_j, \text{res}_{q_j}(\beta) + \text{tr}(\nabla)|_{q_j})\}_{1 \leq j \leq N}. \end{aligned}$$

Here, notice that  $\text{res}_{q_j}(\beta) \in \Omega_C^1(D)|_{q_j}$  and  $\text{tr}(\nabla)|_{q_j}$  is justified by considering the twisted cotangent bundle (in detail, see Definition 16 below). Remark that the dimension of  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$  is equal to the dimension of  $M_X^0$ . A generic part of  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$  has the natural symplectic structure induced by the symplectic structure on the product  $(\mathbf{\Omega}(D, c_d) \setminus \pi_{c_d}^{-1}(D)) \times \cdots \times (\mathbf{\Omega}(D, c_d) \setminus \pi_{c_d}^{-1}(D))$ . The first main theorem is the following:

**Theorem A** (Theorem 20 below). *The pull-back of the symplectic form on a generic part of  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$  under the map (1.1) coincides with the symplectic form on  $M_X^0$ .*

If we take canonical coordinates on  $\mathbf{\Omega}(D, c_d)$ , then we have canonical coordinate on  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$ , since the symplectic structure on  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))$  is induced by the 2-form  $\sum_{j=1}^N \text{pr}_j^*(\omega_{D, c_d})$ . Then we have canonical coordinates on  $M_X^0$  by Theorem A. Detail of construction of concrete canonical coordinates on  $M_X^0$  is discussed in the paragraph after the proof of Theorem 20 below.

In Section 5, we consider an example of this argument. We will calculate the canonical coordinates for an elliptic curve and a divisor  $D$  of length 2. The moduli space of rank 2 meromorphic connection with fixed trace connection on an elliptic curve with two simple poles was studied in [36] and [12]. In this paper, we will discuss the  $\text{GL}_2$ -connection case.

**1.3.  $\text{SL}_2$ -connections.** In the second part of this paper, we discuss on  $\text{SL}_2$ -connections. That is, we consider rank 2 meromorphic connections with fixed trace connection  $(L_0, \nabla_0)$ . Here  $L_0$  is a fixed line bundle on  $C$  of degree  $2g - 1$  and  $\nabla_0: L_0 \rightarrow L_0 \otimes \Omega_C^1(D)$  is a fixed connection. More precisely, we consider rank 2 quasi-parabolic connections  $(E, \nabla, \{l^{(i)}\})$ , defined in [24, Definition 2.1], with fixed trace connection  $(L_0, \nabla_0)$ . Here the spectral data of  $\nabla_0$  is determined by the fixed irregular curve  $X$ . The quasi-parabolic structure  $l^{(i)}$  at  $t_i$  induces a one dimensional subspace  $l_{\text{red}}^{(i)}$  of  $E|_{t_i}$ , that is the restriction of  $l^{(i)}$  to  $t_i$  (without multiplicity). Our moduli space is as follows:

$$M_X(L_0, \nabla_0)_0 := \left\{ (E, \nabla, \{l^{(i)}\}) \left| \begin{array}{l} \text{(i) } \nabla \text{ has the fixed spectral data in } X, \\ \text{(ii) } E \text{ is an extension of } L_0 \text{ by } \mathcal{O}_C, \\ \text{(iii) } \dim_{\mathbb{C}} H^0(C, E) = 1, \text{ and} \\ \text{(iv) } l_{\text{red}}^{(i)} \notin \mathcal{O}_C|_{t_i} \subset \mathbb{P}(E) \text{ for any } i \end{array} \right. \right\} / \cong,$$

which is described in Section 4.2. Here  $(E, \nabla, \{l^{(i)}\})$  are rank 2 quasi-parabolic connections on  $(C, D)$  with fixed trace connection  $(L_0, \nabla_0)$ . When  $g = 0$ , we impose one more condition (in detail, see the paragraph after the proof of Lemma 26 below). This moduli space also has a natural symplectic structure. The dimension of the moduli space  $M_X(L_0, \nabla_0)_0$  is  $2N_0$ , where  $N_0 := 3g - 3 + \deg(D)$ . For  $(E, \nabla, \{l^{(i)}\}) \in M_X(L_0, \nabla_0)_0$ , we can also define apparent singularities (Section 4.2 below). The apparent singularities give an element of  $\mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D))$ . So we have a map

$$\pi_{\text{App}}: M_X(L_0, \nabla_0)_0 \longrightarrow \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)).$$

For  $(E, \nabla, \{l^{(i)}\}) \in M(L_0, \nabla_0)_0$ , we forget the connection  $\nabla$ . So we have a quasi-parabolic bundle  $(E, \{l^{(i)}\})$ . By taking the extension class for the quasi-parabolic bundle  $(E, \{l^{(i)}\})$ , we have a map

$$\pi_{\text{Bun}}: M_X(L_0, \nabla_0)_0 \longrightarrow \mathbb{P}H^1(C, L_0^{-1}(-D)).$$

Here the extension class is described in Section 4.1 below. We consider the product

$$\pi_{\text{App}} \times \pi_{\text{Bun}}: M_X(L_0, \nabla_0)_0 \longrightarrow \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)) \times \mathbb{P}H^1(C, L_0^{-1}(-D)).$$

This map has been studied by Loray–Saito–Simpson [38], Loray–Saito [37], Fassarella–Loray [12], Fassarella–Loray–Muniz [13], and Matsumoto [39].

Notice that  $H^1(C, L_0^{-1}(-D))$  is isomorphic to the dual of  $H^0(C, L_0 \otimes \Omega_C^1(D))$ . Remark that

$$\dim_{\mathbb{C}} \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)) = \dim_{\mathbb{C}} \mathbb{P}H^1(C, L_0^{-1}(-D)) = N_0.$$

Let us introduce the homogeneous coordinates  $\mathbf{a} = (a_0 : \cdots : a_{N_0})$  on  $\mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)) \cong \mathbb{P}_{\mathbf{a}}^{N_0}$  and the dual coordinates  $\mathbf{b} = (b_0 : \cdots : b_{N_0})$  on

$$\mathbb{P}H^1(C, L_0^{-1}(-D)) \cong \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D))^{\vee} \cong \mathbb{P}_{\mathbf{b}}^{N_0}.$$

We may define a 1-form  $\eta$  on  $\mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0}$  by

$$\eta = (\text{constant}) \cdot \frac{a_0 db_0 + a_1 db_1 + \cdots + a_{N_0} db_{N_0}}{a_0 b_0 + a_1 b_1 + \cdots + a_{N_0} b_{N_0}}.$$

(In detail, see Section 4.4). The 2-form  $d\eta$  gives an symplectic structure on  $\mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0} \setminus \Sigma$ . Here we set

$$\Sigma: (a_0 b_0 + a_1 b_1 + \cdots + a_{N_0} b_{N_0} = 0) \subset \mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0}.$$

The image of  $M(L_0, \nabla_0)_0$  is contained in  $\mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0} \setminus \Sigma$ . (In detail, see Section 4.3). The second main theorem is the following:

**Theorem B** (Theorem 31 below). *We assume that the fixed spectral data satisfies the generic condition (4.8) below. The pull-back of the symplectic form  $d\eta$  on  $\mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0} \setminus \Sigma$  under the map  $\pi_{\text{App}} \times \pi_{\text{Bun}}$  coincides with the symplectic form on the moduli space  $M_X(L_0, \nabla_0)_0$ .*

**1.4. The organization of this paper.** In Section 2, the apparent singularities for a generic rank 2 meromorphic connection are defined. After the definition of the apparent singularities, we will discuss on the companion normal form of a generic rank 2 meromorphic connection. We will use this companion normal form when we will introduce canonical coordinates. In Section 3, first, we will describe our moduli space of rank 2 meromorphic connections. Second, we will discuss on tangent spaces of the moduli space of rank 2 meromorphic connections. We will recall that the tangent spaces at a meromorphic connection are isomorphic to a hypercohomology of the complex defined by the meromorphic connection. After that, we will describe a natural symplectic structure on the moduli space of rank 2 meromorphic connections. Section 3.3 and Section 3.4 are preliminaries of

the proof of the first main theorem. In Section 3.5, we will give the map from a generic part of the moduli space to  $\mathrm{Sym}^N(\Omega(D, c_d))$  and will show the first main theorem.

In Section 4, we will consider rank 2 meromorphic connections with fixed trace connection. First, to describe the bundle map  $\pi_{\mathrm{Bun}}$ , we recall the moduli space of stable quasi-parabolic bundles with fixed determinant. Second, we will describe our moduli space of rank 2 meromorphic connections with fixed trace connection. Third, we will describe the map  $\pi_{\mathrm{App}}$  defined by considering the apparent singularities. In Section 4.4, we will recall a natural symplectic structure on the moduli space of rank 2 meromorphic connections with fixed trace connection, and will show the second main theorem.

In Section 5, we will apply the argument in Section 2 and Section 3 to the case of an elliptic curve with a divisor  $D$  of length 2. When  $D$  is reduced, this amounts to two logarithmic singularities, otherwise to an irregular singularity. It is remarkable that using our approach these two cases can be studied completely similarly.

In Section 6, we will provide a method for obtaining canonical coordinates  $\tilde{p}_j \in \Omega(D, c_d)|_{q_i}$  for generic  $(E, \nabla) \in M_X^0$  by introducing a section  $s \in H^0(C, \det(E))$  and  $\gamma \in H^0(C, \Omega_C^1(D))$ . We will utilize an open set  $U_0 = C \setminus \{s = 0, \gamma = 0\}$  and the trivialization of  $E|_{U_0}$  to define  $\tilde{p}_j \in \Omega_C^1(D)|_{q_j}$ . This method can be also used for constructing a meromorphic connection  $\nabla_1: E \rightarrow E \otimes \Omega_C^1(D(s))$  for a given  $s \in H^0(C, \det(E))$ , where  $D(s)$  denotes the zero divisor of  $s$ . In Theorem 41, we will provide an alternative proof of the birationality of  $f_{\mathrm{App}}$  (cf. Proposition 17) by utilizing the Higgs fields  $\nabla - \nabla_1$  and the BNR correspondence [3]. This approach may shed new light on the relationship between the canonical coordinates of the moduli spaces of connections and the moduli spaces of Higgs bundles. (cf. [45]).

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## 2. COMPANION NORMAL FORM

Let  $C$  be a compact Riemann surface of genus  $g$  ( $g \geq 0$ ), and  $D$  be an effective divisor on  $C$ . We assume  $4g - 3 + n > 0$  where  $n = \deg(D)$ . We consider a rank 2 meromorphic connection

$$(2.1) \quad \nabla: E \rightarrow E \otimes \Omega_C^1(D)$$

on  $C$  where  $\deg(E) = 2g - 1$ .

When  $g = 0$ , Diarra–Loray have given companion normal forms of the rank 2 meromorphic connections in [9]. By the companion normal forms, we may construct a universal family of the rank 2 meromorphic connections on some generic part of the moduli space of rank 2 meromorphic connections. This universal family is useful to describe the isomonodromic deformations [31]. The purpose of this section is to give companion normal forms of rank 2 meromorphic connections when  $g \geq 0$ . For this purpose, first, we will introduce the apparent singularities for (generic) rank 2 meromorphic connections.

**2.1. Apparent singularities.** First we assume that  $\dim_{\mathbb{C}} H^0(C, E) = 1$  for the rank 2 meromorphic connection (2.1). This assumption holds for a generic vector bundle of the rank 2 meromorphic connection with  $\deg(E) = 2g - 1$ . For an element of  $H^0(C, E)$ , we define the sequence of  $\mathbb{C}$ -linear maps

$$(2.2) \quad \varphi_{\nabla}: \mathcal{O}_C \rightarrow E \xrightarrow{\nabla} E \otimes \Omega_C^1(D) \rightarrow E/\mathcal{O}_C \otimes \Omega_C^1(D).$$

This composition  $\varphi_\nabla$  is an  $\mathcal{O}_C$ -linear map. From now on we assume that  $\varphi_\nabla \neq 0$ . This assumption holds for every  $(E, \nabla)$ , provided that the eigenvalues of the residues are chosen generically (see Remark 4 below). We call the global section in  $H^0(C, E)$  in (2.2) the *cyclic vector*.

Let us now define  $E_0 \subset E$  as the rank 2 locally free subsheaf spanned by  $\mathcal{O}_C$  and

$$\mathrm{Im} \left\{ \nabla|_{\mathcal{O}_C} \otimes \mathrm{Id}_{(\Omega_C^1(D))^{-1}} : (\Omega_C^1(D))^{-1} \rightarrow E \right\}.$$

This construction gives rise to a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_0 \longrightarrow (\Omega_C^1(D))^{-1} \longrightarrow 0.$$

We claim that this sequence splits, i.e.

$$(2.3) \quad E_0 \cong \mathcal{O}_C \oplus (\Omega_C^1(D))^{-1}.$$

Indeed, equivalence classes of extensions of  $(\Omega_C^1(D))^{-1}$  by  $\mathcal{O}_C$  are classified by the group

$$\mathrm{Ext}^1((\Omega_C^1(D))^{-1}, \mathcal{O}_C) = \mathrm{Ext}^1(\mathcal{O}_C(-D), \Omega_C^1) \cong H^0(C, \mathcal{O}_C(-D))^\vee = 0,$$

where we have used Grothendieck–Serre duality. We denote by

$$(2.4) \quad \phi_\nabla : E_0 \longrightarrow E.$$

the canonical inclusion morphism, and define the meromorphic connection

$$(2.5) \quad \nabla_0 = \phi_\nabla^*(\nabla)$$

on  $E_0$ . We note that the polar divisor of  $\nabla_0$  is  $D + B$  where

$$(2.6) \quad B = \mathrm{div}(\varphi_\nabla).$$

We note that

$$(2.7) \quad \deg(B) = 4g - 3 + n.$$

From now on, moreover, we assume that  $B$  is reduced, with support disjoint from  $D$ . In different terms, in view of (2.7), we have

$$B = q_1 + \cdots + q_{4g-3+n}$$

where  $q_i \neq q_j$  once  $i \neq j$  and  $q_i \notin D$  for all  $i$ .

**Definition 1.** Assume that  $\varphi_\nabla \neq 0$  and  $\mathrm{div}(\varphi_\nabla)$  is reduced, with support disjoint from  $D$ . We call the points of the support  $\{q_1, \dots, q_{4g-3+n}\}$  of  $\mathrm{div}(\varphi_\nabla)$  the apparent singularities of  $(E, \nabla)$ .

**2.2. Companion normal form.** The desired companion normal form is a normal form of  $\nabla_0$  in (2.5). So the companion normal form is given by normalization of  $\nabla_0$  by applying automorphisms on  $\mathcal{O}_C \oplus (\Omega_C^1(D))^{-1}$ . To give the companion normal form, first, we describe a decomposition of  $\nabla_0$  relative to (2.3):

$$\nabla_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where

$$\begin{cases} \alpha & : & \mathcal{O}_C \longrightarrow \Omega_C^1(D + B) & \text{(connection)} \\ \beta & : & (\Omega_C^1(D))^{-1} \longrightarrow \Omega_C^1(D + B) & \text{(\mathcal{O}_C\text{-linear})} \\ \gamma & : & \mathcal{O}_C \longrightarrow \mathcal{O}_C(B) & \text{(\mathcal{O}_C\text{-linear})} \\ \delta & : & (\Omega_C^1(D))^{-1} \longrightarrow (\Omega_C^1(D))^{-1} \otimes \Omega_C^1(D + B) & \text{(connection)} \end{cases}$$

This form is unique only up to pre-composition by an element of the automorphism group  $\text{Aut}(E_0)$  of  $E_0$ . Elements of  $\text{Aut}(E_0)$  are described as follows:

$$\begin{pmatrix} \lambda_1 & F \\ 0 & \lambda_2 \end{pmatrix},$$

where  $\lambda_1, \lambda_2 \in \mathbb{C}^*$  and  $F \in H^0(C, \Omega_C^1(D))$ . It follows by construction that  $\nabla_0$  admits no pole in restriction to  $\mathcal{O}_C$  over the divisor  $B$ , so that actually we have

$$\begin{cases} \alpha & : \mathcal{O}_C \longrightarrow \Omega_C^1(D) & (\text{connection}) \\ \gamma & : \mathcal{O}_C \longrightarrow \mathcal{O}_C & (= \text{identity}) \end{cases}$$

The action of an automorphism of the form

$$(2.8) \quad \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}, \quad F \in H^0(C, \Omega_C^1(D))$$

transforms  $\alpha$  into  $\alpha - F\gamma$  (without affecting  $\gamma$ ). Therefore, there exists a unique choice  $F$  such that  $\alpha = d$  is the trivial connection on  $\mathcal{O}_C$ . We thus get the unique companion normal form

$$(2.9) \quad \nabla_0 = \begin{pmatrix} d & \beta \\ 1 & \delta \end{pmatrix}.$$

Notice that the same companion normal form is obtained simply by taking the generator  $\varphi_{\nabla}(1)$  for the second factor of (2.3), and the action of the automorphism (2.8) in the above argument simply amounts to switching to this particular generator.

**2.3. Spectral data.** Now we consider the polar part of the meromorphic connection (2.1) at each point of the support of  $D$ . We impose some conditions on the polar parts. To describe the conditions, we introduce the notion of irregular curves with residues. Let  $\nu$  be a positive integer. We set  $I := \{1, 2, \dots, \nu\}$ . Let  $\mathfrak{h}$  be the Cartan subalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \mid h_1, h_2 \in \mathbb{C} \right\}$$

of the Lie algebra  $\mathfrak{gl}_2(\mathbb{C})$ . Let  $\mathfrak{h}_0$  be the regular locus of  $\mathfrak{h}$ .

**Definition 2.** We say  $X = (C, D, \{z_i\}_{i \in I}, \{\theta_i\}_{i \in I}, \theta_{\text{res}})$  is an irregular curve with residues if

- (i)  $C$  is a compact Riemann surface of genus  $g$ ,
- (ii)  $D = \sum_{i \in I} m_i [t_i]$  is an effective divisor on  $C$ .
- (iii)  $z_i$  is a generator of the maximal ideal of  $\mathcal{O}_{C, t_i}$ ,
- (iv)  $\theta_i = (\theta_{i, -m_i}, (\theta_{i, -m_i+1}, \dots, \theta_{i, -2})) \in \mathfrak{h}_0 \times \mathfrak{h}^{m_i-2}$ , and
- (v)  $\theta_{\text{res}} = (\theta_{1, -1}, \theta_{2, -1}, \dots, \theta_{\nu, -1})$ , where  $\theta_{i, -1} \in \mathfrak{h}$ , such that  $\sum_{i=1}^{\nu} \text{tr}(\theta_{i, -1}) = -(2g-1)$ .

We set

$$\theta_{i, -1} = \begin{pmatrix} \theta_{i, -1}^- & 0 \\ 0 & \theta_{i, -1}^+ \end{pmatrix} \quad \text{for each } i \in I.$$

We assume that  $\sum_{i=1}^{\nu} \theta_{i, -1}^{\pm} \notin \mathbb{Z}$  whatever are the signs  $\pm$ , and, if  $m_i = 1$ , then  $\theta_{i, -1}^+ - \theta_{i, -1}^- \notin \mathbb{Z}$ .

For an irregular curve with residues  $X$ , we set

$$(2.10) \quad \omega_i(X) := \theta_{i, -m_i} \frac{dz_i}{z_i^{m_i}} + \theta_{i, -m_i+1} \frac{dz_i}{z_i^{m_i-1}} + \dots + \theta_{i, -2} \frac{dz_i}{z_i^2} + \theta_{i, -1} \frac{dz_i}{z_i}$$

and  $\mathcal{O}_{m_i[t_i]} := \mathcal{O}_{C,t_i}/(z_i^{m_i})$ . For an irregular curve with residues  $X$  and a meromorphic connection  $(E, \nabla)$  in (2.1), we set  $E|_{m_i[t_i]} := E \otimes \mathcal{O}_{m_i[t_i]}$ . Let

$$\nabla|_{m_i[t_i]}: E|_{m_i[t_i]} \longrightarrow E|_{m_i[t_i]} \otimes \Omega_C^1(D)$$

be the morphism induced by  $\nabla$ .

**Definition 3.** We call  $(E, \nabla)$  a rank 2 meromorphic connection over an irregular curve with residues  $X$  if

- (i)  $E$  is a rank 2 vector bundle of degree  $2g - 1$  on  $C$ ,
- (ii)  $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$  is a connection, and
- (iii) there exists an isomorphism  $\varphi_{m_i[t_i]}: E|_{m_i[t_i]} \rightarrow \mathcal{O}_{m_i[t_i]}^{\oplus 2}$  for each  $i \in I$  such that

$$(\varphi_{m_i[t_i]} \otimes 1) \circ \nabla|_{m_i[t_i]} \circ \varphi_{m_i[t_i]}^{-1} = d + \omega_i(X).$$

Here  $\omega_i(X)$  is defined in (2.10).

We call  $\omega_i(X)$  the spectral data of  $(E, \nabla)$  and call the submodule  $\varphi_{m_i[t_i]}^{-1}(\mathcal{O}_{m_i[t_i]} \oplus 0)$  of  $E|_{m_i[t_i]}$  the quasi-parabolic structure of  $(E, \nabla)$  at  $t_i$ .

From now on, by a connection we will mean a rank 2 meromorphic connection over a fixed irregular curve with residues  $X$ . So we impose the condition (iii) of Definition 3 on the polar parts of the meromorphic connection  $\nabla$  in (2.1) at the points of the support of  $D$ . This condition means that the polar parts of  $\nabla$  at  $t_i$  are diagonalizable with eigenvalues equal to the diagonal entries of  $\omega_i(X)$  for  $i = 1, 2, \dots, n$ .

**Remark 4.** In Definition 3, we impose the condition that  $\sum_{i=1}^{\nu} \theta_{i,-1}^{\pm} \notin \mathbb{Z}$  whatever are the signs  $\pm$ . By this assumption and the argument as in [32, Proposition 6], we have that  $(E, \nabla)$  is irreducible. Then some arguments become simple. For example,  $\varphi_{\nabla} = 0$  if and only if the free subsheaf  $\mathcal{O}_C$  of  $E$  is a proper  $\nabla$ -invariant subbundle. So we have that  $\varphi_{\nabla} \neq 0$ . Moreover,  $(E, \nabla)$  is automatically stable (described in Section 3.1 below).

**2.4. The polar parts of  $\delta$ .** We fix an irregular curve with residues  $X$ . Let  $(E, \nabla)$  be a rank 2 meromorphic connection over  $X$  and  $\nabla_0$  be the companion normal form for  $(E, \nabla)$ . We consider the  $(2, 2)$ -entry  $\delta$  of this companion normal form  $\nabla_0$ .

It immediately follows from (2.9) that the connection  $\delta$  coincides with the trace connection  $\text{tr}(\nabla_0)$  on  $\det(E_0) = (\Omega_C^1(D))^{-1}$ . It is further related to the trace connection  $\text{tr}(\nabla)$  by

$$\delta = \text{tr}(\nabla_0) = \text{tr}(\nabla) + \frac{d\varphi_{\nabla}}{\varphi_{\nabla}}.$$

- Lemma 5.**
- (1) The polar part of  $\delta$  over  $D$  is determined by the spectral data;
  - (2) The polar part of  $\delta$  over  $B$  is logarithmic with residue  $+1$ ;
  - (3)  $\delta$  is determined by the irregular curve with residues  $X$  up to adding a holomorphic 1-form of  $C$ .

*Proof.* The polar part of  $\delta$  at  $t_i$  is equal to  $\text{tr}(\omega_i)$ , showing the first assertion. In view of our assumption  $q_{j_1} \neq q_{j_2}$  for  $j_1 \neq j_2$ , the second assertion is classical. Let now  $\delta, \delta'$  be the  $(2, 2)$ -entries of companion normal forms  $\nabla_0, \nabla'_0$  of connections  $\nabla, \nabla'$  satisfying the conditions of Definition 3. By the first part,  $\delta - \delta'$  is then a global holomorphic 1-form of  $C$ .  $\square$

As a consequence of the lemma and by  $\dim_{\mathbb{C}} H^0(C, \Omega_C^1) = g$ , the possible values for  $\delta$  represent  $g$  free parameters for a meromorphic connection over  $X$ .

**2.5. The polar parts of  $\beta$ .** Next we consider the  $(1, 2)$ -entry  $\beta$  of the companion normal form  $\nabla_0$  of a meromorphic connection  $\nabla$  over the irregular curve with residues  $X$ . By the condition  $\gamma = 1$  in (2.9),  $\beta$  accounts for the determinant of the characteristic polynomial of the residues. By Definition 3, the eigenvalues of the connection matrix of  $\nabla$  are differentials (of the first kind) with a pole of order at most  $m_i$  at  $t_i$ . The same condition then holds for  $\nabla_0$  too, because it only differs from  $\nabla$  by elementary modifications at points  $q_j \neq t_i$ . As the determinant of a  $2 \times 2$  matrix is a quadratic expression of the eigenvalues, we see that  $\beta$  must be a quadratic differential with poles of order at most  $2m_i$  at  $t_i$ . Over  $B$ , a similar argument shows that  $\beta$  has poles of order at most 2.

Let us fix local coordinate charts  $z_i$  centered at the pole  $t_i$ . One may then expand  $\beta$  into Laurent series:

$$\beta = (\beta_{i,-2m_i} z_i^{-2m_i} + \cdots + \beta_{i,-2} z_i^{-2} + O(z_i^{-1})) (dz_i)^{\otimes 2}.$$

Notice that for given  $\beta$  the coefficient  $\beta_{i,-2}$  is independent of the chosen coordinate chart  $z_i$ , however the other coefficients depend on  $z_i$ . We also fix local coordinate charts  $z_j$  centered at the apparent singularity  $q_j$ , and have a similar expansion

$$\beta = (\beta_{j,-2} z_j^{-2} + \beta_{j,-1} z_j^{-1} + O(z_j^0)) (dz_j)^{\otimes 2}.$$

Analogously to Lemma 5, we therefore find

- Lemma 6.** (1) *The coefficients  $\beta_{i,-2m_i}, \dots, \beta_{i,-2}$  are uniquely determined by the irregular curve with residues  $X$  (and the holomorphic coordinate  $z_i$ );*  
 (2) *We have  $\beta_{j,-2} = 0$ .*  
 (3)  *$\beta$  is determined by the irregular curve with residues  $X$  up to adding a section of  $(\Omega_C^1)^{\otimes 2}(D)$ .*

*Proof.* The coefficients  $\beta_{i,-2m_i}, \dots, \beta_{i,-2}$  all admit homogeneous quadratic expressions in terms of the eigenvalues of  $\theta_i, \theta_{\text{res}}$ , therefore they are determined by them. Conversely, the coefficients  $\beta_{i,-2m_i}, \dots, \beta_{i,-2}$  determine the polar part of the eigenvalues. It is classical that for an apparent singularity of  $\nabla_0$ , one of the two eigenvalues of the residue must vanish. This implies that for every  $q \in B$  the product of the eigenvalues of  $\text{res}_q(\nabla_0)$  vanishes. As this latter product gives the leading (second) order term  $\beta_{j,-2}$ , we get the second assertion. The last part follows from the first two as in Lemma 5.  $\square$

As a consequence of the lemma and by  $\dim_{\mathbb{C}} H^0(C, (\Omega_C^1)^{\otimes 2}(D)) = 3g - 3 + n$ , the possible values for  $\beta$  represent  $3g - 3 + n$  free parameters for a connection  $\nabla$  on  $X$  having apparent singularities at a fixed reduced divisor  $B$  of length  $N$ .

From now on, we set  $\beta_{j,-1} = \zeta_j$ , so that we have the expansion

$$(2.11) \quad \beta = \zeta_j \frac{(dz_j)^{\otimes 2}}{z_j} + \beta^{(j)}$$

for some local holomorphic quadratic differential  $\beta^{(j)}$ . Notice that  $\zeta_j$  depends on the coordinate  $z_j$ , however the element  $\zeta_j dz_j \in \Omega_C^1|_{q_j}$  of the fiber of the holomorphic cotangent (or canonical) bundle over  $q_j$  does not depend on it. As a matter of fact, since  $\beta$  belongs to an affine space modelled over  $H^0(C, (\Omega_C^1)^{\otimes 2}(D))$  (and in order to be consistent with the decomposition (2.3)), it is even more rigorous to consider  $\zeta_j dz_j$  as elements of the fiber  $\Omega_C^1(D)|_{q_j}$ , using the inclusion  $\Omega_C^1 \subset \Omega_C^1(D)$ . In the sequel we will consider them to be such elements. It will turn out that these quantities  $\zeta_j dz_j$  are closely related to accessory parameters.

**2.6. Determination of  $\beta$  and  $\delta$  in terms of  $\zeta$ .** Fix a reduced divisor  $B$  of length  $N$  on  $C$  with support disjoint from  $D$ . In Subsections 2.4, 2.5 we have found that (normal forms of) meromorphic connections with residue on  $X$  that have apparent singularities at  $B$  can be described by an affine space of complex dimension  $g + 3g - 3 + n = N$  ( $g$  coming from the choice of  $\delta$  and  $3g - 3 + n$  from the choice of  $\beta$ ). In this section, we provide a description of such connections in terms of analogs of separated variables. Namely, it will turn out that generically the data of  $\delta, \beta$  is equivalent to the  $N$ -tuple  $(\zeta_1 dz_1, \dots, \zeta_N dz_N)$ .

The fact that singular points are apparent over  $B$  imposes further constraints on  $\beta$  and  $\delta$ . This constraint gives 1 linear condition for each point  $q_j$  and we can expect that these constraints fix  $\beta$  and  $\delta$  uniquely in terms of the data  $(q_j, \zeta_j dz_j)_{j=1}^N$ . In fact, this is true for the genus  $g = 0$  case (see [9]) and we will show in Lemma 7 that this is also true for *generic* choices of  $(q_j, \zeta_j dz_j)_{j=1}^N$  if  $g > 0$ .

In fact, the data of  $\zeta_j dz_j$  can be interpreted as a certain quasi-parabolic structure over  $B$ . Indeed, at a point  $q_j$  and with respect to the decomposition (2.3), the residue of  $\nabla_0$  reads as

$$\text{res}_{q_j} \nabla_0 = \begin{pmatrix} 0 & \zeta_j dz_j \\ 0 & 1 \end{pmatrix}.$$

So, the vector  $\begin{pmatrix} \zeta_j dz_j \\ 1 \end{pmatrix}$  is an eigenvector with respect to eigenvalue 1 and the map  $\phi_\nabla$  (see (2.4)) is just the positive elementary transformation with respect to these parabolic directions at all points  $q_j$ . In summary, the data of all values  $\zeta_j dz_j$  is equivalent to the data of a quasi-parabolic structure of  $E_0$  over  $B$  (i.e., a line in the fiber of  $E_0$  over each  $q_j$ ) distinct from the destabilizing subbundle  $\mathcal{O}_C \subset E_0$  for every  $j$ .

Let us denote by  $\Omega(D)$  the total space of the line bundle  $\Omega_C^1(D)$ .

**Lemma 7.** *For generic data  $(q_j, \zeta_j dz_j)_j \in \text{Sym}^{4g-3+n}(\Omega(D))$  there exist unique  $\beta$  and  $\delta$  as above such that the corresponding  $\nabla_0$  has apparent singular points at all the points  $q_j$  ( $1 \leq j \leq N := 4g - 3 + n$ ), and such that the Laurent expansion (2.11) is fulfilled.*

*Proof.* Let us consider  $(q_j, \zeta_j dz_j)_j$  such that  $q_j$ 's are pair-wise distinct, and do not intersect the support of  $D$ . Given one point  $(q_j, \zeta_j dz_j)$ , we can diagonalize the residue  $\text{res}_{q_j} \nabla_0$  by conjugating by a triangular matrix

$$(2.12) \quad \begin{pmatrix} 1 & \zeta_j dz_j \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \beta \\ 1 & \delta \end{pmatrix} \begin{pmatrix} 1 & \zeta_j dz_j \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & \zeta_j dz_j \\ 0 & 1 \end{pmatrix}^{-1} d \begin{pmatrix} 1 & \zeta_j dz_j \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} -\zeta_j dz_j & \beta - \zeta_j \delta \otimes dz_j - \zeta_j^2 dz_j^{\otimes 2} \\ 1 & \delta + \zeta_j dz_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{dz_j}{z_j} \end{pmatrix} + \text{holomorphic}$$

where  $z_j$  stands for a local coordinate at  $q_j$ . Then the elementary transformation  $\phi_\nabla$  is locally equivalent to the conjugacy by  $\begin{pmatrix} 1 & 0 \\ 0 & z_j^{-1} \end{pmatrix}$  yielding

$$(2.13) \quad \begin{pmatrix} -\zeta_j dz_j & \frac{\beta - \zeta_j \delta \otimes dz_j - \zeta_j^2 dz_j^{\otimes 2}}{z_j} \\ z_j & \delta + \zeta_j dz_j - \frac{dz_j}{z_j} \end{pmatrix}.$$

The apparent point condition is therefore equivalent to saying that  $\beta - \zeta_j \delta \otimes dz_j - \zeta_j^2 dz_j^{\otimes 2}$  is (holomorphic and) **vanishing** at  $q_j$ . This condition is linear on  $\beta$  and  $\delta$  and rewrites

$$(2.14) \quad \underbrace{\beta - \zeta_j \delta \otimes dz_j}_{\text{holomorphic}}|_{q_j} = \zeta_j^2 dz_j^{\otimes 2}|_{q_j},$$

where the right hand side does not involve  $\beta$  and  $\delta$ . If we assume that  $(q_1, \dots, q_N)$  lies in the image of the map  $\text{App}$  (see (1.2)), then the normal form of any  $(E, \nabla)$  in the preimage produces a solution  $(\delta_0, \beta_0)$ . Fixing such solutions, by Lemmas 5, 6 we may rewrite

$$\begin{cases} \beta &= \beta_0 + b_1 \nu_1 + \dots + b_{N-g} \nu_{N-g} \\ \delta &= \delta_0 + d_1 \omega_1 + \dots + d_g \omega_g \end{cases}$$

where  $(\omega_l)_{l=1}^g$ ,  $(\nu_k)_{k=1}^{N-g}$  are respective bases of  $H^0(C, \Omega_C^1)$  and  $H^0(C, (\Omega_C^1)^{\otimes 2}(D))$ . Using these expressions, the constraint that  $q_j$  is an apparent singularity can be rewritten as a linear system consisting of  $N$  equations in the  $N$  variables  $b_k$ ,  $d_l$ . The condition to uniquely determine  $\beta$  and  $\delta$  in terms of the data  $(q_j, \zeta_j dz_j)$  is that the following determinant does not vanish

$$(2.15) \quad \det \begin{pmatrix} \nu_1(q_1) & \cdots & \nu_{N-g}(q_1) & \zeta_1 dz_1 \omega_1(q_1) & \cdots & \zeta_1 dz_1 \omega_g(q_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \nu_1(q_N) & \cdots & \nu_{N-g}(q_N) & \zeta_N dz_N \omega_1(q_N) & \cdots & \zeta_N dz_N \omega_g(q_N) \end{pmatrix}$$

Of course, it is sufficient for our purpose to check that we can find some  $(q_j, \zeta_j dz_j)$ 's such that this determinant does not vanish, so that it will be generically non vanishing. If we set  $\zeta_1 = \dots = \zeta_{N-g} = 0$ , then the matrix has a zero block of dimension  $(N-g) \times g$  in the top right corner, and the determinant factors as

$$\zeta_{N-g+1} dz_{N-g+1} \cdots \zeta_N dz_N \cdot \det \begin{pmatrix} \nu_1(q_1) & \cdots & \nu_{N-g}(q_1) \\ \vdots & \ddots & \vdots \\ \nu_1(q_{N-g}) & \cdots & \nu_{N-g}(q_{N-g}) \end{pmatrix} \cdot \det \begin{pmatrix} \omega_1(\tilde{q}_1) & \cdots & \omega_g(\tilde{q}_1) \\ \vdots & \ddots & \vdots \\ \omega_1(\tilde{q}_g) & \cdots & \omega_g(\tilde{q}_g) \end{pmatrix}$$

where  $\tilde{q}_j = q_{j+N-g}$ . After setting  $\zeta_{N-g+1} = \dots = \zeta_N = 1$ , it is enough to find  $q_j$ 's such that the two smaller determinants are non zero. To conclude the proof, let us denote by  $L$  any of the two lines bundles  $\Omega_C^1$  or  $(\Omega_C^1)^{\otimes 2}(D)$ , and by  $\mu_1, \dots, \mu_{N'}$  a corresponding basis of  $H^0(C, L)$ . Then we want to prove that the image of the curve by the evaluation map

$$C \xrightarrow{\text{ev}} \mathbb{P}^{N'-1}; q \mapsto (\mu_1(q) : \dots : \mu_{N'}(q))$$

is not contained in some hyperplane, i.e. that we can find  $q_1, \dots, q_{N'} \in C$  such that the image is not contained in some hyperplane. But this is true, otherwise, we would have a linear relation between  $\mu_1, \dots, \mu_{N'}$  contradicting that they form a basis.  $\square$

**Remark 8.** *In the previous proof, the locus of  $q_j$ 's for which  $\det(\omega_i(\tilde{q}_j))_{i,j}$  vanishes correspond to the Brill-Noether locus for divisor  $\tilde{q}_1 + \dots + \tilde{q}_g$ .*

**Lemma 9.** *When  $g = 0$ , any data  $(q_j, \zeta_j dz_j)_j \in \text{Sym}^{n-3}(\Omega(D))$  gives rise to unique  $\beta$  and  $\delta$  such that the corresponding  $\nabla_0$  has apparent singular points at all  $q_j$ 's. However, for  $g > 0$ , there always exist data  $(q_j, \zeta_j dz_j)_j$  such that the determinant (2.15) vanishes.*

*Proof.* When  $g = 0$ , this directly follows from [9] (a consequence of Lagrange interpolation). When  $g > 0$ , fix generic  $q_j$ 's and let  $\omega \in H^0(C, \Omega_C^1(D))$ . If we set  $\zeta_j := \omega(q_j)$ , then the last column of (2.15) is just the evaluation of the section  $\omega \otimes \omega_g \in H^0(C, (\Omega_C^1)^{\otimes 2}(D))$  at  $q_1, \dots, q_{4g-3+n}$  and is therefore a linear combination of the  $3g - 3 + n$  first columns.  $\square$

## 3. SYMPLECTIC STRUCTURE AND CANONICAL COORDINATES

We fix an irregular curve with residues  $X = (C, D, \{z_i\}_{i \in I}, \{\theta_i\}_{i \in I}, \theta_{\text{res}})$ . As usual, we use the notation  $N := 4g + n - 3$ , where  $g$  is the genus of  $C$  and  $n = \deg(D)$ . We will consider the moduli space  $M_X$  of rank 2 meromorphic connections over  $X$ . This moduli space is constructed in [24, Theorem 2.1] and carries a natural symplectic structure described in [24, Proposition 4.1]. The purpose of this section is to give canonical coordinates on an open subset of  $M_X$  with respect to this symplectic structure. First we describe the moduli space  $M_X$ .

**3.1. Moduli spaces.** Let  $(E, \nabla)$  be a rank 2 meromorphic connection over  $X$ . Then, the subsheaf

$$l^{(i)} := \varphi_{m_i[t_i]}^{-1}(\mathcal{O}_{m_i[t_i]} \oplus 0) \subset E_{m_i[t_i]}.$$

equips  $(E, \nabla)$  with a canonical quasi-parabolic structure at each  $t_i$ . So we may consider  $(E, \nabla)$  as a *quasi-parabolic connection*  $(E, \nabla, \{l^{(i)}\})$  defined in [21, Definition 1.1] and [24, Definition 2.1]. A stability condition for quasi-parabolic connections is introduced in [21, Definition 2.1] and [24, Definition 2.2]. The moduli space of stable quasi-parabolic connections is constructed in [21, Theorem 2.1] and [24, Theorem 2.1]. In our situation, any rank 2 meromorphic connections over  $X$  are irreducible (see Remark 4). So our objects are automatically stable objects. We omit the stability condition of the quasi-parabolic connections.

Let  $M_X$  be the moduli space of rank 2 meromorphic connections over the irregular curve with residues  $X$ . If  $(E, \nabla) \in M_X$  satisfies  $\dim_{\mathbb{C}} H^0(C, E) = 1$ , then we have a unique  $\mathcal{O}_C$ -morphism  $\varphi_{\nabla}$  in (2.2). The  $\mathcal{O}_C$ -morphism  $\varphi_{\nabla}$  is nonzero, since  $(E, \nabla)$  is irreducible. So we may define the divisor  $\text{div}(\varphi_{\nabla})$  in (2.6) for  $(E, \nabla)$ . We set

$$M_X^0 := \left\{ (E, \nabla) \in M_X \left| \begin{array}{l} \dim_{\mathbb{C}} H^0(C, E) = 1, \\ \text{div}(\varphi_{\nabla}) \text{ is reduced, and} \\ \text{div}(\varphi_{\nabla}) \text{ has disjoint support with } D \end{array} \right. \right\}.$$

Next we recall the natural symplectic structure on  $M_X$ .

**3.2. Symplectic structure.** We will describe the natural symplectic structure on  $M_X$  via Čech cohomology. This is defined in [21, Proposition 7.2] and [24, Proposition 4.1]. This is analog of the symplectic form on the moduli space of stable Higgs bundles in [8]. This description of the symplectic structure is useful to comparing this symplectic structure with the Goldman symplectic structure on the character variety via the Riemann–Hilbert map (for example, see [21, the proof of Proposition 7.3] and [4, Theorem 3.2]). Moreover, this description of the symplectic structure is useful to describe the isomonodromic deformations (for example, see [5, Proposition 4.3], [6, Proposition 4.4], and [30, Proposition 3.8]).

First we recall the description of the tangent space of  $M_X$  at  $(E, \nabla) \in M_X$  in terms of the hypercohomology of a certain complex ([21, the proof of Theorem 2.1] and [24, the proof of Proposition 4.1]). We consider  $(E, \nabla)$  as a quasi-parabolic connection  $(E, \nabla, \{l^{(i)}\})$ . We define a complex  $\mathcal{F}^\bullet$  for  $(E, \nabla, \{l^{(i)}\})$  by

$$(3.1) \quad \begin{aligned} \mathcal{F}^0 &:= \left\{ s \in \mathcal{E}nd(E) \mid s|_{m_i t_i}(l^{(i)}) \subset l^{(i)} \text{ for any } i \right\} \\ \mathcal{F}^1 &:= \left\{ s \in \mathcal{E}nd(E) \otimes \Omega_C^1(D) \mid s|_{m_i t_i}(l^{(i)}) \subset l^{(i)} \otimes \Omega_C^1 \text{ for any } i \right\} \\ \nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 &\longrightarrow \mathcal{F}^1; \quad \nabla_{\mathcal{F}^\bullet}(s) = \nabla \circ s - s \circ \nabla. \end{aligned}$$

Then we have an isomorphism between the tangent space  $T_{(E, \nabla, \{l^{(i)}\})} M_X$  and  $\mathbf{H}^1(\mathcal{F}^\bullet)$ .

Now we recall this isomorphism. We take an analytic (or affine) open covering  $C = \bigcup_{\alpha} U_{\alpha}$  such that  $E|_{U_{\alpha}} \cong \mathcal{O}_{U_{\alpha}}^{\oplus 2}$  for any  $\alpha$ ,  $\#\{i \mid t_i \cap U_{\alpha} \neq \emptyset\} \leq 1$  for any  $\alpha$  and  $\#\{\alpha \mid t_i \cap U_{\alpha} \neq \emptyset\} \leq 1$  for any  $i$ . Take a tangent vector  $v \in T_{(E, \nabla, \{l^{(i)}\})} M_X$ . The field  $v$  corresponds to an infinitesimal deformation  $(E_{\epsilon}, \nabla_{\epsilon}, \{l_{\epsilon}^{(i)}\})$  of  $(E, \nabla, \{l^{(i)}\})$  over  $C \times \text{Spec } \mathbb{C}[\epsilon]$  such that  $(E_{\epsilon}, \nabla_{\epsilon}, \{l_{\epsilon}^{(i)}\}) \otimes \mathbb{C}[\epsilon]/(\epsilon) \cong (E, \nabla, \{l^{(i)}\})$ , where  $\mathbb{C}[\epsilon] = \mathbb{C}[t]/(t^2)$ . There is an isomorphism

$$\varphi_{\alpha} : E_{\epsilon}|_{U_{\alpha} \times \text{Spec } \mathbb{C}[\epsilon]} \xrightarrow{\sim} \mathcal{O}_{U_{\alpha} \times \text{Spec } \mathbb{C}[\epsilon]}^{\oplus 2} \xrightarrow{\sim} E|_{U_{\alpha}} \otimes \mathbb{C}[\epsilon]$$

such that  $\varphi_{\alpha} \otimes \mathbb{C}[\epsilon]/(\epsilon) : E_{\epsilon} \otimes \mathbb{C}[\epsilon]/(\epsilon)|_{U_{\alpha}} \xrightarrow{\sim} E|_{U_{\alpha}} \otimes \mathbb{C}[\epsilon]/(\epsilon) = E|_{U_{\alpha}}$  is the given isomorphism and that  $\varphi_{\alpha}|_{t_i \times \text{Spec } \mathbb{C}[\epsilon]}(l_{\epsilon}^{(i)}) = l^{(i)}|_{U_{\alpha} \times \text{Spec } \mathbb{C}[\epsilon]}$  if  $t_i \cap U_{\alpha} \neq \emptyset$ . We put

$$\begin{aligned} u_{\alpha\beta} &:= \varphi_{\alpha} \circ \varphi_{\beta}^{-1} - \text{id}_{E|_{U_{\alpha\beta} \times \text{Spec } \mathbb{C}[\epsilon]}}, \\ v_{\alpha} &:= (\varphi_{\alpha} \otimes \text{id}) \circ \nabla_{\epsilon}|_{U_{\alpha} \times \text{Spec } \mathbb{C}[\epsilon]} \circ \varphi_{\alpha}^{-1} - \nabla|_{U_{\alpha} \times \text{Spec } \mathbb{C}[\epsilon]}. \end{aligned}$$

Then  $\{u_{\alpha\beta}\} \in C^1((\epsilon) \otimes \mathcal{F}^0)$ ,  $\{v_{\alpha}\} \in C^0((\epsilon) \otimes \mathcal{F}^1)$  and we have the cocycle conditions

$$u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta} = 0 \quad \text{and} \quad \nabla \circ u_{\alpha\beta} - u_{\alpha\beta} \circ \nabla = v_{\beta} - v_{\alpha}.$$

So  $[(\{u_{\alpha\beta}\}, \{v_{\alpha}\})]$  determines an element of  $\mathbf{H}^1(\mathcal{F}^{\bullet})$ . This correspondence gives an isomorphism between the tangent space  $T_{(E, \nabla, \{l^{(i)}\})} M_X$  and  $\mathbf{H}^1(\mathcal{F}^{\bullet})$ .

We define a pairing

$$(3.2) \quad \begin{aligned} &\mathbf{H}^1(\mathcal{F}^{\bullet}) \otimes \mathbf{H}^1(\mathcal{F}^{\bullet}) \longrightarrow \mathbf{H}^2(\mathcal{O}_C \xrightarrow{d} \Omega_C^1) \cong \mathbb{C} \\ &[(\{u_{\alpha\beta}\}, \{v_{\alpha}\})] \otimes [(\{u'_{\alpha\beta}\}, \{v'_{\alpha}\})] \longmapsto [(\{\text{tr}(u_{\alpha\beta} \circ u'_{\beta\gamma})\}, -\{\text{tr}(u_{\alpha\beta} \circ v'_{\beta}) - \text{tr}(v_{\alpha} \circ u'_{\alpha\beta})\})], \end{aligned}$$

considered in Čech cohomology with respect to an open covering  $\{U_{\alpha}\}$  of  $C$ ,  $\{u_{\alpha\beta}\} \in C^1(\mathcal{F}^0)$ ,  $\{v_{\alpha}\} \in C^0(\mathcal{F}^1)$  and so on. This pairing gives a nondegenerate 2-form on the moduli space  $M_X$ . This fact follows from the Serre duality and the five lemma:

$$(3.3) \quad \begin{array}{ccccccccc} H^0(\mathcal{F}^0) & \longrightarrow & H^0(\mathcal{F}^1) & \longrightarrow & \mathbf{H}^1(\mathcal{F}^{\bullet}) & \longrightarrow & H^1(\mathcal{F}^0) & \longrightarrow & H^1(\mathcal{F}^1) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ H^1(\mathcal{F}^1)^{\vee} & \longrightarrow & H^1(\mathcal{F}^0)^{\vee} & \longrightarrow & \mathbf{H}^1(\mathcal{F}^{\bullet})^{\vee} & \longrightarrow & H^0(\mathcal{F}^1)^{\vee} & \longrightarrow & H^0(\mathcal{F}^0)^{\vee}. \end{array}$$

We denote by  $\omega$  the nondegenerate 2-form on  $M_X$ . This 2-form  $\omega$  is a symplectic structure. That is, we have  $d\omega = 0$  (see [21, Proposition 7.3] and [24, Proposition 4.2]).

We get as a consequence:

**Proposition 10.** *The dimension of  $M_X^0$  is equal to  $2N$ , where  $N = 4g - 3 + n$ .*

*Proof.* By irreducibility of  $(E, \nabla)$  and Schur's lemma we have

$$\mathbf{H}^0(\mathcal{F}^{\bullet}) \cong \mathbb{C}.$$

On a Zariski open subset of  $M_X$ , the underlying quasi-parabolic vector bundle  $(E, \{l^{(i)}\}_i)$  is irreducible, so we also have

$$H^0(\mathcal{F}^0) \cong \mathbb{C}.$$

Clearly, we have  $\deg(\mathcal{F}^0) = -\text{length}(D)$ . From Riemann–Roch we find

$$\begin{aligned} \dim_{\mathbb{C}} H^1(\mathcal{F}^0) &= \dim_{\mathbb{C}} H^0(\mathcal{F}^0) + 4(g-1) - \deg(\mathcal{F}^0) \\ &= 4g - 3 + n = N. \end{aligned}$$

By Serre duality and Euler characteristic count applied to the hypercohomology long exact sequence (3.3), we get the statement.  $\square$

**3.3. Trivializations of  $E$ .** Our purpose is to give canonical coordinates of  $M_X^0$  with respect to the symplectic form (3.2). To do it, we will calculate the Čech cohomology by taking trivializations of  $E$ . To simplify the calculation, we take trivializations of  $E$  by using

$$\phi_\nabla: E_0 \xrightarrow{\subset} E,$$

whose cokernel defines the apparent singularities. In this section, we will discuss construction of the trivializations of  $E$  by using  $\phi_\nabla$ .

We take  $(E, \nabla, \{l^{(i)}\}) \in M_X^0$ . Let  $\{(q_j, \zeta_j dz_j)\}_{j=1,2,\dots,N}$  be the point on  $\text{Sym}^N(\Omega(D))$  corresponding to  $(E, \nabla, \{l^{(i)}\})$ . We assume that the point  $\{(q_j, \zeta_j dz_j)\}_{j=1,2,\dots,N}$  is generic in the sense of Lemma 7. Let  $U_{q_j}^{\text{an}}$  be an analytic open subset of  $C$  such that  $q_j \in U_{q_j}^{\text{an}}$  and  $U_{t_i}^{\text{an}}$  be an analytic open subset of  $C$  such that  $t_i \in U_{t_i}^{\text{an}}$ . We assume that  $U_{q_j}^{\text{an}}$  and  $U_{t_i}^{\text{an}}$  are small enough. We take an analytic coordinate  $z_j$  on  $U_{q_j}^{\text{an}}$  such that it is independent of the moduli space  $M_X^0$ . We denote also by  $q_j$  the complex number so that the point  $q_j$  on  $C$  is defined by  $z_j - q_j = 0$ .

**Definition 11.** Let  $\{U_\alpha\}_\alpha$  be an analytic open covering of  $C$ :  $C = \bigcup_\alpha U_\alpha$  such that

- (i)  $\#\{i \mid t_i \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $\alpha$ , and  $\#\{\alpha \mid t_i \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $i$ ,
- (ii)  $\#\{j \mid q_j \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $\alpha$ , and  $\#\{\alpha \mid q_j \cap U_\alpha \neq \emptyset\} \leq 1$  for any  $j$ ,
- (iii)  $\Omega_C^1(D)$  is free on  $U_\alpha$  for any  $\alpha$ , that is,  $\Omega_C^1(D)|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}$ ,
- (iv)  $U_{\alpha_{t_i}} = U_{t_i}^{\text{an}}$  and  $U_{\alpha_{q_j}} = U_{q_j}^{\text{an}}$ .

Here we denote by  $\alpha_{t_i}$  the index  $\alpha$  such that  $t_i \in U_\alpha$ , and by  $\alpha_{q_j}$  the index  $\alpha$  such that  $q_j \in U_\alpha$ .

We fix trivializations  $\omega_\alpha: \mathcal{O}_{U_\alpha} \xrightarrow{\sim} \Omega_C^1(D)|_{U_\alpha}$  of  $\Omega_C^1(D)$ . We assume that  $\omega_\alpha$  is independent of the moduli space  $M_X^0$ . By using  $\omega_\alpha$ , we have  $\omega_\alpha^{-1}: \mathcal{O}_{U_\alpha} \xrightarrow{\sim} (\Omega_C^1(D))^{-1}|_{U_\alpha}$ . By the trivializations, we have trivializations  $\varphi_\alpha^{\text{norm}}: \mathcal{O}_{U_\alpha}^{\oplus 2} \xrightarrow{\sim} E_0|_{U_\alpha}$  of  $E_0$ . Assume that the connection matrices  $A_\alpha^{\text{norm}}$  of  $\nabla_0$  associated to  $\varphi_\alpha^{\text{norm}}$  are

$$(3.4) \quad A_\alpha^{\text{norm}} = \begin{pmatrix} 0 & \beta_\alpha \\ \gamma_\alpha & \delta_\alpha \end{pmatrix},$$

where  $\beta_\alpha, \delta_\alpha \in \Omega_C^1(D+B)|_{U_\alpha}$  are determined by  $\{(q_j, \text{res}_{q_j}(\beta))\}_{j=1,2,\dots,N}$  (see Lemma 7). The 1-form  $\gamma_\alpha \in \Omega_C^1(D)|_{U_\alpha}$  is the image of 1 under the composition

$$\mathcal{O}_{U_\alpha} \xrightarrow{\sim} (\Omega_C^1(D))^{-1} \otimes \Omega_C^1(D)|_{U_\alpha} \xrightarrow{\omega_\alpha \otimes 1} \mathcal{O}_{U_\alpha} \otimes \Omega_C^1(D).$$

In particular,  $\gamma_\alpha$  is independent of the moduli space  $M_X^0$  for any  $\alpha$ . The polar part of  $A_{\alpha_{t_i}}^{\text{norm}}$  at  $t_i$  is independent of the moduli space  $M_X^0$  for any  $i$ . We set

$$(3.5) \quad \zeta_j := \frac{\text{res}_{q_j}(\beta)}{\gamma_{\alpha_{q_j}}|_{q_j}} \in \mathbb{C} \quad \text{for } j = 1, 2, \dots, N.$$

Here  $\beta \in H^0(C, (\Omega_C^1)^{\otimes 2}(2D+B))$  is the (1,2)-entry of (2.9). Notice that  $\beta|_{U_\alpha} = \beta_\alpha \gamma_\alpha$ , where  $\beta_\alpha$  and  $\gamma_\alpha$  are in (3.4). So we have

$$\text{res}_{q_j}(A_{\alpha_{q_j}}^{\text{norm}}) = \begin{pmatrix} 0 & \zeta_j \\ 0 & 1 \end{pmatrix} \quad \text{for } j = 1, 2, \dots, N.$$

**Definition 12.** We define other trivializations  $\varphi_\alpha^{\text{App},0}: \mathcal{O}_{U_\alpha}^{\oplus 2} \xrightarrow{\sim} E_0|_{U_\alpha}$  of  $E_0$  for each  $\alpha$  as follows:

(i) When  $\alpha = \alpha_{q_j}$ , we take a trivialization  $\varphi_\alpha^{\text{App},0}$  as

$$\varphi_\alpha^{\text{App},0} = \varphi_\alpha^{\text{norm}} \circ \begin{pmatrix} 1 & \zeta_j \\ 0 & 1 \end{pmatrix}.$$

Note that this triangular matrix appeared in (2.12).

(ii) Otherwise, we take a trivialization  $\varphi_\alpha^{\text{App},0}$  as  $\varphi_\alpha^{\text{App},0} = \varphi_\alpha^{\text{norm}}$ .

Let  $A_\alpha^{\text{App},0}$  be the connection matrix of  $\nabla_0$  associated to  $\varphi_\alpha^{\text{App},0}$ , that is,

$$(\varphi_\alpha^{\text{App},0})^{-1} \circ (\phi_\nabla^* \nabla) \circ \varphi_\alpha^{\text{App},0} = d + A_\alpha^{\text{App},0}.$$

We have that

$$A_\alpha^{\text{App},0} = \begin{cases} \begin{pmatrix} -\zeta_j \gamma_\alpha & \beta_\alpha - \zeta_j \delta_\alpha - \zeta_j^2 \gamma_\alpha \\ \gamma_\alpha & \delta_\alpha + \zeta_j \gamma_\alpha \end{pmatrix} & \text{when } \alpha = \alpha_{q_j} \\ \begin{pmatrix} 0 & \beta_\alpha \\ \gamma_\alpha & \delta_\alpha \end{pmatrix} & \text{otherwise.} \end{cases}$$

We have

$$\text{res}_{q_j}(A_{\alpha_{q_j}}^{\text{App},0}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } j = 1, 2, \dots, N.$$

Now we define trivializations of  $E$  by using  $\phi_\nabla: E_0 \rightarrow E$  in (2.4) and the trivialization of  $E_0$  in Definition 12.

**Definition 13.** Now we define trivialization  $\varphi_\alpha^{\text{App}}: \mathcal{O}_{U_\alpha}^{\oplus 2} \xrightarrow{\sim} E|_{U_\alpha}$  of  $E$  for the open covering  $\{U_\alpha\}_\alpha$  in Definition 11 as follows.

(i) When  $\alpha = \alpha_{q_j}$ , we take a trivialization  $\varphi_\alpha^{\text{App}}$  so that

$$(\varphi_\alpha^{\text{App}})^{-1} \circ \phi_\nabla|_{U_\alpha} \circ \varphi_\alpha^{\text{App},0} = \begin{pmatrix} 1 & 0 \\ 0 & z_j - q_j \end{pmatrix}.$$

(ii) When  $\alpha = \alpha_{t_i}$ , we take  $g_\alpha^{t_i} \in \text{Aut}(\mathcal{O}_{U_\alpha}^{\oplus 2})$  so that the polar part of  $(g_\alpha^{t_i})^{-1} A_\alpha^{\text{norm}} g_\alpha^{t_i}$  is diagonal at  $m_i[t_i]$ . We take a trivialization  $\varphi_\alpha^{\text{App}}$  as

$$\varphi_\alpha^{\text{App}} = \phi_\nabla|_{U_\alpha} \circ \varphi_\alpha^{\text{norm}} \circ g_\alpha^{t_i}.$$

Here remark that  $\phi_\nabla|_{U_\alpha}$  is invertible. Since the polar part of  $A_{\alpha_{t_i}}^{\text{norm}}$  at  $t_i$  is independent of the moduli space  $M_X^0$ , we may assume that  $(g_\alpha^{t_i})_{< m_i}$  is independent of the moduli space  $M_X^0$ . Here we define  $(g_\alpha^{t_i})_{< m_i}$  so that  $g_\alpha^{t_i} = (g_\alpha^{t_i})_{< m_i} + O(z_i^{m_i})$ .

(iii) Otherwise, we take a trivialization  $\varphi_\alpha^{\text{App}}$  so that

$$(\varphi_\alpha^{\text{App}})^{-1} \circ \phi_\nabla|_{U_\alpha} \circ \varphi_\alpha^{\text{norm}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since  $\phi_\nabla|_{U_\alpha}$  is invertible in this case,  $\varphi_\alpha^{\text{App}} = \phi_\nabla|_{U_\alpha} \circ \varphi_\alpha^{\text{norm}}$ .

Let  $A_\alpha$  be the connection matrix of  $\nabla$  associated to  $\varphi_\alpha^{\text{App}}$ , that is

$$(\varphi_\alpha^{\text{App}})^{-1} \circ \nabla \circ \varphi_\alpha^{\text{App}} = d + A_\alpha.$$

We have that

$$(3.6) \quad A_\alpha = \begin{cases} \begin{pmatrix} -\zeta_j \gamma_\alpha & \frac{\beta_\alpha - \zeta_j \delta_\alpha - \zeta_j^2 \gamma_\alpha}{z_j - q_j} \\ (z_j - q_j) \gamma_\alpha & \delta_\alpha + \zeta_j \gamma_\alpha - \frac{dz_j}{z_j - q_j} \end{pmatrix} & \text{when } \alpha = \alpha_{q_j} \\ \omega_i(X) + [\text{holo. part}] & \text{when } \alpha = \alpha_{t_i} \\ \begin{pmatrix} 0 & \beta_\alpha \\ \gamma_\alpha & \delta_\alpha \end{pmatrix} & \text{otherwise.} \end{cases}$$

Here  $\omega_i(X)$  is the 1-form defined in (2.10). The connection matrix  $A_{\alpha_{q_j}}$  on  $U_{\alpha_{q_j}}$  appeared in (2.13). The connection matrix  $A_{\alpha_{q_j}}$  has no pole at  $q_j$  for any  $j = 1, 2, \dots, N$ , since  $\beta_\alpha, \delta_\alpha$  are determined by Lemma 7. We have considered diagonalization of the polar part of the connection  $(E, \nabla)$  at each  $t_i$ . The reason why we consider diagonalization of the polar parts is that we use the connection matrix (3.6) to calculate an infinitesimal deformation of  $(E, \nabla)$ . So we will calculate variations of the transition functions with respect to the trivializations in Definition 13 and variations of the connection matrices (3.6). These are elements of  $\mathcal{F}^0$  and  $\mathcal{F}^1$  of (3.1), respectively. To be elements of  $\mathcal{F}^0$  and  $\mathcal{F}^1$ , we need the compatibility with the quasi-parabolic structure. However, this compatibility follows directly from diagonalization of the polar parts.

**3.4. Descriptions of the cocycles of an infinitesimal deformation.** Let  $\Omega(D) \rightarrow C$  be the total space of  $\Omega_C^1(D)$ . By the argument as in Lemma 6, we may define a map

$$(3.7) \quad \begin{aligned} f_{\text{App},0}: M_X^0 &\longrightarrow \text{Sym}^N(\Omega(D)) \\ (E, \nabla) &\longmapsto \{(q_j, \text{res}_{q_j}(\beta))\}_{j=1,2,\dots,N}. \end{aligned}$$

Here  $\beta \in H^0(C, (\Omega_C^1)^{\otimes 2}(2D + B))$  is the (1, 2)-entry of (2.9) and  $\text{res}_{q_j}(\beta) \in \Omega_C^1(D)|_{q_j}$ . We take an analytic open subset  $V$  of  $M_X^0$ . For the analytic open subset  $V$ , we assume that we may define a composition

$$\begin{aligned} V &\longrightarrow f_{\text{App},0}(V) \longrightarrow \text{Sym}^N(\mathbb{C}_{(q,\zeta)}^2) \\ (E, \nabla) &\longmapsto \{(q_j, \text{res}_{q_j}(\beta))\}_{j=1,2,\dots,N} \longmapsto \{(q_j, \zeta_j)\}_{j=1,2,\dots,N}, \end{aligned}$$

where  $\zeta_j$  is defined in (3.5), and the image of  $V$  under the composition is isomorphic to some analytic open subset of  $\mathbb{C}_{(q,\zeta)}^{2N}$ . Let  $U_{(q,\zeta)}$  be such an analytic open subset of  $\mathbb{C}_{(q,\zeta)}^{2N}$ . So we have a map

$$(3.8) \quad \begin{aligned} M_X^0 \supset V &\longrightarrow U_{(q,\zeta)} \subset \mathbb{C}_{(q,\zeta)}^{2N} \\ (E, \nabla) &\longmapsto (q_1, \dots, q_N, \zeta_1, \dots, \zeta_N), \end{aligned}$$

which are coordinates that we will use in this subsection. We consider the family of  $(E, \nabla, \{l^{(i)}\})$  parametrized by  $U_{(q,\zeta)}$  such that this family induces the inverse map of the map  $V \rightarrow U_{(q,\zeta)}$ . Here this family is constructed by Lemma 7. By using the trivializations  $\{\varphi_\alpha^{\text{App}}\}_\alpha$  of  $E$  in Definition 13, we have transition functions and connection matrices of the family of  $(E, \nabla, \{l^{(i)}\})$  parametrized by  $U_{(q,\zeta)}$ . Indeed, the transition function is

$$(3.9) \quad B_{\alpha\beta} := (\varphi_\alpha^{\text{App}}|_{U_{\alpha\beta}})^{-1} \circ \varphi_\beta^{\text{App}}|_{U_{\alpha\beta}} : \mathcal{O}_{U_{\alpha\beta}}^{\oplus 2} \longrightarrow \mathcal{O}_{U_{\alpha\beta}}^{\oplus 2},$$

and the connection matrix is as in (3.6).

Let  $(q_j, \zeta_j)_j$  be a point on  $U_{(q, \zeta)}$ . The purpose of this subsection is to describe the tangent map

$$(3.10) \quad \begin{aligned} T_{(q_j, \zeta_j)_j} \mathbb{C}_{(q, \zeta)}^{2N} &\longrightarrow T_{(E, \nabla, \{l^{(i)}\})} M_X^0 \cong \mathbf{H}^1(\mathcal{F}^\bullet) \\ v &\longmapsto [(\{u_{\alpha\beta}(v)\}, \{v_\alpha(v)\})] \end{aligned}$$

induced by the inverse map of (3.8). For this purpose, we will calculate the variations of the transition functions and the connection matrices parametrized by  $U_{(q, \zeta)}$  with respect to the tangent vector  $v$  in  $U_{(q, \zeta)} \subset \mathbb{C}_{(q, \zeta)}^{2N}$ . By using these variations, we will calculate the cocycles  $(\{u_{\alpha\beta}(v)\}, \{v_\alpha(v)\})$  of the infinitesimal deformation of  $(E, \nabla, \{l^{(i)}\})$  with respect to  $v$ .

First, we calculate  $u_{\alpha\beta}(v) \in \mathcal{F}^1(U_{\alpha\beta})$ . We consider the variation of  $B_{\alpha\beta}$  in (3.9) by  $v$ :

$$B_{\alpha\beta}(\text{id} + \epsilon B_{\alpha\beta}^{-1} v(B_{\alpha\beta})): \mathcal{O}_{U_{\alpha\beta}}^{\oplus 2} \longrightarrow \mathcal{O}_{U_{\alpha\beta}}^{\oplus 2} \otimes \mathbb{C}[\epsilon].$$

Then  $u_{\alpha\beta}(v)$  has the following description:

$$(3.11) \quad u_{\alpha\beta}(v) = \varphi_\beta^{\text{APP}}|_{U_{\alpha\beta}} \circ \left( B_{\alpha\beta}^{-1} v(B_{\alpha\beta}) \right) \circ (\varphi_\beta^{\text{APP}}|_{U_{\alpha\beta}})^{-1}.$$

**Lemma 14.** *Let  $I_{\text{cov}}$  be the set of the indices of the open covering  $\{U_\alpha\}$  in Definition 11. We set  $I_{\text{cov}}^t = \{\alpha_{t_1}, \dots, \alpha_{t_\nu}\}$  and  $I_{\text{cov}}^q = \{\alpha_{q_1}, \dots, \alpha_{q_N}\}$ , which are subsets of  $I_{\text{cov}}$ . For  $v \in T_{(E, \nabla, \{l^{(i)}\})} M_X^0$ , we have the equality*

(3.12)

$$u_{\alpha\beta}(v) = \begin{cases} 0 & \alpha, \beta \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q) \\ \varphi_{\alpha_{q_j}}^{\text{APP}}|_{U_{\alpha_{q_j}}} \circ \begin{pmatrix} 0 & v(\zeta_j) \\ z_j^{-q_j} & v(q_j) \\ 0 & z_j^{-q_j} \end{pmatrix} \circ (\varphi_{\alpha_{q_j}}^{\text{APP}}|_{U_{\alpha_{q_j}}})^{-1} & \alpha \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q), \beta = \alpha_{q_j} \in I_{\text{cov}}^q \\ \varphi_{\alpha_{t_i}}^{\text{APP}}|_{U_{\alpha_{t_i}}} \circ \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) \right) \circ (\varphi_{\alpha_{t_i}}^{\text{APP}}|_{U_{\alpha_{t_i}}})^{-1} & \alpha \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q), \beta = \alpha_{t_i} \in I_{\text{cov}}^t \end{cases}$$

and we have that

$$(3.13) \quad (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) = O(z_i^{m_i}).$$

*Proof.* Let  $\alpha \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q)$ . If  $\beta \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q)$ , then we have the following equalities:

$$\begin{aligned} B_{\alpha\beta} &= (\varphi_\alpha^{\text{APP}}|_{U_{\alpha\beta}})^{-1} \circ \varphi_\beta^{\text{APP}}|_{U_{\alpha\beta}} \\ &= (\varphi_\alpha^{\text{norm}}|_{U_{\alpha\beta}})^{-1} \circ (\phi_\nabla|_{U_{\alpha\beta}})^{-1} \circ \phi_\nabla|_{U_{\alpha\beta}} \circ \varphi_\beta^{\text{norm}}|_{U_{\alpha\beta}} \\ &= (\varphi_\alpha^{\text{norm}}|_{U_{\alpha\beta}})^{-1} \circ \varphi_\beta^{\text{norm}}|_{U_{\alpha\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & ((\omega_\alpha^{-1})^{-1} \circ \omega_{\alpha_{q_j}}^{-1}) \end{pmatrix}. \end{aligned}$$

Here  $\omega_\alpha^{-1}$  is a trivialization  $\mathcal{O}_{U_\alpha} \xrightarrow{\cong} (\Omega_C^1(D))^{-1}|_{U_\alpha}$  for any  $\alpha$ . Since  $((\omega_\alpha^{-1})^{-1} \circ \omega_{\alpha_{q_j}}^{-1})$  is independent of the moduli space  $M_X^0$ , we have  $v(B_{\alpha\beta}) = 0$ . So  $u_{\alpha\beta}(v) = 0$ .

If  $\beta = \alpha_{q_j}$ , then we have the following equalities:

$$\begin{aligned}
(3.14) \quad B_{\alpha\alpha_{q_j}} &= (\varphi_{\alpha_{q_j}}^{\text{APP}}|_{U_{\alpha\alpha_{q_j}}})^{-1} \circ \varphi_{\alpha_{q_j}}^{\text{APP}}|_{U_{\alpha\alpha_{q_j}}} \\
&= (\varphi_{\alpha}^{\text{APP},0}|_{U_{\alpha\alpha_{q_j}}})^{-1} \circ (\phi_{\nabla}|_{U_{\alpha\alpha_{q_j}}})^{-1} \circ \phi_{\nabla}|_{U_{\alpha\alpha_{q_j}}} \circ \varphi_{\alpha_{q_j}}^{\text{APP},0}|_{U_{\alpha\alpha_{q_j}}} \circ \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z_j - q_j} \end{pmatrix} \\
&= (\varphi_{\alpha}^{\text{APP},0}|_{U_{\alpha\alpha_{q_j}}})^{-1} \circ \varphi_{\alpha_{q_j}}^{\text{APP},0}|_{U_{\alpha\alpha_{q_j}}} \circ \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z_j - q_j} \end{pmatrix} \\
&= (\varphi_{\alpha}^{\text{norm}}|_{U_{\alpha\alpha_{q_j}}})^{-1} \circ \varphi_{\alpha_{q_j}}^{\text{norm}}|_{U_{\alpha\alpha_{q_j}}} \circ \begin{pmatrix} 1 & \zeta_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z_j - q_j} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & ((\omega_{\alpha}^{-1})^{-1} \circ \omega_{\alpha_{q_j}}^{-1}) \end{pmatrix} \begin{pmatrix} 1 & \frac{\zeta_j}{z_j - q_j} \\ 0 & \frac{1}{z_j - q_j} \end{pmatrix}.
\end{aligned}$$

So we have

$$B_{\alpha\alpha_{q_j}}^{-1} v(B_{\alpha\alpha_{q_j}}) = \begin{pmatrix} 1 & -\zeta_j \\ 0 & z_j - q_j \end{pmatrix} \begin{pmatrix} 0 & \frac{v(\zeta_j)(z_j - q_j) + \zeta_j v(q_j)}{(z_j - q_j)^2} \\ 0 & -\frac{v(q_j)}{(z_j - q_j)^2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{v(\zeta_j)}{z_j - q_j} \\ 0 & \frac{v(q_j)}{z_j - q_j} \end{pmatrix}.$$

If  $\beta = \alpha_{t_i}$ , then we have the following equalities:

$$\begin{aligned}
B_{\alpha\alpha_{t_i}} &= (\varphi_{\alpha_{t_i}}^{\text{APP}}|_{U_{\alpha\alpha_{t_i}}})^{-1} \circ \varphi_{\alpha_{t_i}}^{\text{APP}}|_{U_{\alpha\alpha_{t_i}}} \\
&= (\varphi_{\alpha}^{\text{norm}}|_{U_{\alpha\alpha_{t_i}}})^{-1} \circ (\phi_{\nabla}|_{U_{\alpha\alpha_{t_i}}})^{-1} \circ \phi_{\nabla}|_{U_{\alpha\alpha_{t_i}}} \circ \varphi_{\alpha_{t_i}}^{\text{norm}}|_{U_{\alpha\alpha_{t_i}}} \circ g_{\alpha_{t_i}}^{t_i} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & ((\omega_{\alpha}^{-1})^{-1} \circ \omega_{\alpha_{q_j}}^{-1}) \end{pmatrix} \circ g_{\alpha_{t_i}}^{t_i}.
\end{aligned}$$

So we have  $B_{\alpha\alpha_{t_i}}^{-1} v(B_{\alpha\alpha_{t_i}}) = (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i})$ . Since  $(g_{\alpha}^{t_i})_{< m_i}$  is independent of the moduli space  $M_X^0$ , we have that  $v(g_{\alpha_{t_i}}^{t_i}) = O(z_i^{m_i})$ . Finally, we have the statement of the lemma.  $\square$

Next we calculate  $v_{\alpha}(v) \in \mathcal{F}^1(U_{\alpha})$  for  $v \in T_{(E, \nabla, \{l^{(i)}\})} M_X^0$ . This is given by calculating the variation of the connection matrix  $A_{\alpha}$  in (3.6) with respect to  $v$ . So we have

$$(3.15) \quad v_{\alpha}(v) = \begin{cases} \varphi_{\alpha}^{\text{APP}} \circ \begin{pmatrix} -v(\zeta_j)\gamma_{\alpha} & v\left(\frac{\beta_{\alpha} - \zeta_j \delta_{\alpha} - \zeta_j^2 \gamma_{\alpha}}{z_j - q_j}\right) \\ -v(q_j)\gamma_{\alpha} & v(\text{tr}(A_{\alpha_{q_j}})) + v(\zeta_j)\gamma_{\alpha} \end{pmatrix} \circ (\varphi_{\alpha}^{\text{APP}})^{-1} & \text{when } \alpha = \alpha_{q_j} \\ \varphi_{\alpha}^{\text{APP}} \circ \begin{pmatrix} 0 & v(\beta_{\alpha}) \\ 0 & v(\text{tr}(A_{\alpha})) \end{pmatrix} \circ (\varphi_{\alpha}^{\text{APP}})^{-1} & \text{when } \alpha \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q) \end{cases}.$$

Here remark that  $\gamma_{\alpha}$  is independent of the moduli space  $M_X^0$  for any  $\alpha$ . When  $\alpha = \alpha_{t_i}$ , we have that  $v_{\alpha}(v)$  is holomorphic at  $t_i$ .

**3.5. Canonical coordinates.** Now we introduce canonical coordinates on  $M_X^0$  with respect to the symplectic form (3.2). We recall that we have set  $N := 4g + n - 3$ .

Let  $\pi: \Omega(D) \rightarrow C$  and  $\pi_0: \Omega \rightarrow C$  be the total spaces of  $\Omega_C^1(D)$  and  $\Omega_C^1$ , respectively. The total space  $\Omega$  has the Liouville symplectic form  $\omega_{\text{Liouv}}$ . Since we have an isomorphism

$$\pi_0^{-1}(C \setminus \text{Supp}(D)) \xrightarrow{\sim} \pi^{-1}(C \setminus \text{Supp}(D)),$$

the Liouville symplectic form induces a symplectic form  $\pi^{-1}(C \setminus \text{Supp}(D))$ . Let  $\pi_N: \text{Sym}^N(\Omega(D)) \rightarrow \text{Sym}^N(C)$  be the map induced by the map  $\pi: \Omega(D) \rightarrow C$ . We set

$$\text{Sym}^N(\Omega(D))_0 := \{ \{q_1, \dots, q_N\} \in \pi_N^{-1}(\text{Sym}^N(C \setminus \text{Supp}(D))) \mid q_{j_1} \neq q_{j_2} (j_1 \neq j_2) \}.$$

Then  $\text{Sym}^N(\Omega(D))_0$  has the induced symplectic form from the Liouville symplectic form.

**Remark 15.** *We have a map  $f_{\text{App},0}: M_X^0 \rightarrow \text{Sym}^N(\Omega(D))_0$ , which is described in (3.7). Notice that  $M_X^0$  and  $\text{Sym}^N(\Omega(D))_0$  have symplectic forms. But by the explicit calculation as below, we realize that this map  $f_{\text{App},0}$  does not preserve these symplectic structures. So  $f_{\text{App},0}$  does not give canonical coordinates directly. To give canonical coordinates, we have to modify the map  $f_{\text{App},0}$  as follows.*

We twist  $\Omega(D)$  by a class in  $H^1(C, \Omega_C^1)$  as follows. Let  $c_d$  be the image of the line bundle  $\det(E)$  under the morphism

$$H^1(C, \mathcal{O}_C^*) \xrightarrow{\text{dlog}} H^1(C, \Omega_C^1) \cong \text{Ext}_C^1(T_C, \mathcal{O}_C).$$

Let  $\mathcal{A}_C(c_d)$  be the sheaf produced by the Atiyah sequence on  $C$  with respect to  $c_d$ , that is,  $\mathcal{A}_C(c_d)$  is given by the extension

$$(3.16) \quad 0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{A}_C(c_d) \longrightarrow T_C \longrightarrow 0$$

with respect to  $c_d \in H^1(C, \Omega_C^1)$ . Then,  $\mathcal{A}_C(c_d)$  is naturally a Lie-algebroid, called the Atiyah algebroid of the  $\mathbb{G}_m$ -principal bundle  $\text{Tot}(T_C) \setminus 0$ , where 0 stands for the 0-section; for details, see [35, Section 3.1.2]. We denote by  $\text{symb}_1: \mathcal{A}_C(c_d) \rightarrow T_C$  the morphism in (3.16). We consider the subsheaf  $T_C(-D) \subset T_C$ . We set  $\mathcal{A}_C(c_d, D) := \text{symb}_1^{-1}T_C(-D)$ , which is an extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{A}_C(c_d, D) \longrightarrow T_C(-D) \longrightarrow 0.$$

Let  $\Omega_C^1(D, c_d)$  be the twisted cotangent bundle over  $C$  with respect to  $\mathcal{A}_C(c_d, D)$ , that is,

$$\Omega_C^1(D, c_d) = \{ \phi \in \mathcal{A}_C(c_d, D)^\vee \mid \langle \phi, 1_{\mathcal{A}_C(c_d, D)} \rangle = 1 \}.$$

We denote by

$$\pi_{c_d}: \Omega(D, c_d) \longrightarrow C$$

the total space of the twisted cotangent bundle  $\Omega_C^1(D, c_d)$ , and a generic element of this affine bundle by  $(q, \tilde{p})$  in analogy with classical notation  $(q, p)$  for points of  $\Omega(D)$ . For each  $(E, \nabla, \{l^{(i)}\}) \in M_X^0$ , we have  $(\det(E), \text{tr}(\nabla))$ . The connection  $\text{tr}(\nabla)$  on the line bundle  $\det(E)$  is considered as a *global* section of  $\Omega(D, c_d) \rightarrow C$ , which is the total space of the twisted cotangent bundle with respect to  $\det(E)$ . The global section  $\text{tr}(\nabla)$  gives a diffeomorphism

$$\Omega(D) \longrightarrow \Omega(D, c_d); \quad (q, p) \longmapsto (q, p + \text{tr}(\nabla)).$$

Notice that  $\text{tr}(\nabla)$  does depend on  $M_X^0$ . So this morphism depends on  $M_X^0$ . Moreover, it is not a morphism of vector bundles.

**Definition 16.** *We define the accessory parameter associated to  $(E, \nabla)$  at  $q_j$  by*

$$\tilde{p}_j = \text{res}_{q_j}(\beta) + \text{tr}(\nabla)|_{q_j},$$

where  $\beta \in H^0(C, (\Omega_C^1)^{\otimes 2}(2D + B))$  is the (1, 2)-entry of (2.9) and  $\text{res}_{q_j}(\beta) \in \Omega_C^1(D)|_{q_j}$ . The  $N$ -tuple  $\{(q_j, \tilde{p}_j)\}_{j=1,2,\dots,N}$  will be called canonical coordinates of  $(E, \nabla)$ . We let  $f_{\text{App}}$  be the map

$$\begin{aligned} f_{\text{App}}: M_X^0 &\longrightarrow \text{Sym}^N(\Omega(D, c_d)) \\ (E, \nabla, \{l^{(i)}\}) &\longmapsto \{(q_j, \tilde{p}_j)\}_{j=1,2,\dots,N}. \end{aligned}$$

Notice that the map  $f_{\text{App},0}$  in (3.7) is defined by using only  $\text{res}_{q_j}(\beta)$ . The reason why we consider the twisted cotangent bundle  $\Omega(D, c_d)$  is to justify  $\text{tr}(\nabla)|_{q_j}$ . The next proposition shows that the quantities introduced in the definition may indeed be called coordinates.

**Proposition 17.** *The map  $f_{\text{App}}$  introduced in Definition 16 is birational.*

*Proof.* It follows from Proposition 10 that the dimensions of the source and target of  $f_{\text{App}}$  agree. We therefore need to show two things: first, that  $f_{\text{App}}$  is rational, and second, that it admits an inverse over a Zariski open subset of  $\text{Sym}^N(\Omega(D, c_d))$ .

The first assertion is trivial, because the construction of the apparent singularities  $q_j$  and their accessory parameters  $\tilde{p}_j$  follow from algebraic arguments on certain Zariski open subsets.

The key statement is existence of a generic inverse. This is now a variant of Lemma 7. Namely, fixing generic  $\{(q_j, \tilde{p}_j)\}_{j=1,2,\dots,N}$ , we must find a unique  $(\delta, \beta)$ . Since we have  $\delta = \text{tr}(\nabla_0)$ , we get the expression

$$\tilde{p}_j = \zeta_j dz_j + \delta - \frac{dz_j}{z_j}.$$

An algebraic manipulation shows that the constraint (2.14) expressing that the singularity at  $q_j$  be apparent is equivalent to the holomorphicity and vanishing of the expression

$$(3.17) \quad \beta + \delta \left( \tilde{p}_j + \frac{dz_j}{z_j} \right) - \left( \tilde{p}_j + \frac{dz_j}{z_j} \right)^2.$$

We now study these conditions by taking the Laurent expansion of this expression with respect to  $z_j$ . We first observe that it clearly admits a pole of order at most 2 at  $q_j$ , because  $q_j \neq t_i$ . Since  $\delta$  has a simple pole with residue 1, the term of degree  $-2$  is

$$(dz_j)^{\otimes 2} - (dz_j)^{\otimes 2} = 0.$$

So the pole is automatically at most simple.

For the study of the residue, we need to introduce some notation: let us write

$$\begin{aligned} \delta_0 &= \frac{dz_j}{z_j} + \delta_0^{(j)} \\ \beta_0 &= \zeta_j \frac{(dz_j)^{\otimes 2}}{z_j} + \beta_0^{(j)} \end{aligned}$$

for a holomorphic rank 1 connection  $\delta_0^{(j)}$  and a holomorphic quadratic differential  $\beta_0^{(j)}$  on  $U_{q_j}$ . Then, the degree  $-1$  part of (3.17) is (up to a global factor  $dz_j$ )

$$\zeta_j dz_j + \tilde{p}_j + \left( \delta - \frac{dz_j}{z_j} \right) - 2\tilde{p}_j = 0$$

by the definition of  $\tilde{p}_j$ .

Finally, to deal with the vanishing constraint, we make use of the same basis expansions for  $\delta$  and  $\beta$  as in Lemma 7. Then, the conditions read as

$$\sum_{k=1}^{N-g} b_k \nu_k(q_j) + \tilde{p}_j \sum_{l=1}^g d_l \omega_l(q_j) = (\tilde{p}_j)^{\otimes 2} - \delta_0^{(j)}(q_j) \tilde{p}_j - \beta_0^{(j)}(q_j).$$

Now, the determinant of this linear system of  $N$  equations (for  $1 \leq j \leq N$ ) in  $N$  variables  $b_1, \dots, b_{N-g}, d_1, \dots, d_g$  agrees with the determinant studied in Lemma 7, up to replacing each occurrence of  $\zeta_j dz_j$  by  $\tilde{p}_j$ . The end of the proof then follows word by word the method of Lemma 7.  $\square$

**Remark 18.** *The expression (3.17) has variables  $\tilde{p}_j$  in the twisted cotangent sheaf rather than the ordinary cotangent sheaf. The quadratic polynomial of  $\tilde{p}_j$  can be viewed as the characteristic polynomial of the connection matrix of  $\nabla_0$ . Thus, in a sense the vanishing condition on (3.17) may be interpreted as the requirement that  $\tilde{p}_j$  lie on the quantum spectral curve of  $\nabla_0$ , see e.g. [11].*

By taking a local trivialization of  $\det(E)$ , we have a concrete description of the map  $f_{\text{App}}$ . Now we will discuss on such a description of  $f_{\text{App}}$ . The description discussed below is useful for the proof of Theorem 20 below. Let  $(E, \nabla, \{l^{(i)}\}) \in M_X^0$ . As a local trivialization of  $\det(E)$ , we take the isomorphism

$$(3.18) \quad \det(\varphi_{\alpha_{q_j}}^{\text{App}}): \mathcal{O}_{U_{\alpha_{q_j}}} \longrightarrow \det(E)|_{U_{\alpha_{q_j}}},$$

which is the determinant of the trivialization in Definition 13. Notice that the composition

$$\mathcal{O}_{U_{\alpha_{q_j}}} \xrightarrow{\omega_{\alpha_{q_j}}^{-1}} (\Omega_C^1(D))^{-1}|_{U_{\alpha_{q_j}}} \xrightarrow{\det(\phi_{\nabla})|_{U_{\alpha_{q_j}}}} \det(E)|_{U_{\alpha_{q_j}}} \xrightarrow{\det(\varphi_{\alpha_{q_j}}^{\text{App}})^{-1}} \mathcal{O}_{U_{\alpha_{q_j}}}$$

coincides with  $(z_j - q_j): \mathcal{O}_{U_{\alpha_{q_j}}} \rightarrow \mathcal{O}_{U_{\alpha_{q_j}}}$ . Let  $\text{tr}(A_{\alpha_{q_j}}) \in \Omega_C^1(D)|_{U_{\alpha_{q_j}}}$  be the connection matrix of  $(\det(E), \text{tr}(\nabla))$  on  $U_{\alpha_{q_j}}$  with respect to the local trivialization  $\det(\varphi_{\alpha_{q_j}}^{\text{App}})$ . Then, by using (3.5), the map  $f_{\text{App}}$  has the following description:

$$f_{\text{App}}: (E, \nabla, \{l^{(i)}\}) \longmapsto \left\{ \left( q_j, \zeta_j \gamma_{\alpha_{q_j}}|_{q_j} + \text{tr}(A_{\alpha_{q_j}})|_{q_j} \right) \right\}_{j=1,2,\dots,N},$$

Here  $\zeta_j \gamma_{\alpha_{q_j}}|_{q_j} + \text{tr}(A_{\alpha_{q_j}})|_{q_j}$  is an element of  $\Omega_C^1(D)|_{q_j}$ . We set

$$(3.19) \quad p_j := \text{res}_{q_j} \left( \frac{\zeta_j \gamma_{\alpha_{q_j}}}{z_j - q_j} \right) + \text{res}_{q_j} \left( \frac{\text{tr}(A_{\alpha_{q_j}})}{z_j - q_j} \right),$$

which is the image of  $\zeta_j \gamma_{\alpha_{q_j}}|_{q_j} + \text{tr}(A_{\alpha_{q_j}})|_{q_j}$  under the isomorphism  $\Omega_C^1(D)|_{q_j} \cong \mathbb{C}$ .

**Remark 19.** *This  $p_j$  is just the evaluation of the (2, 2)-entry of the connection matrix  $A_{\alpha_{q_j}}$  in (3.6) at  $q_j$ . Note that the (2, 1)-entry of this connection matrix  $A_{\alpha_{q_j}}$  at  $q_j$  vanishes. So  $p_j$  is an ‘‘eigenvalue’’ of  $\nabla$  at  $q_j$ . (On the other hand,  $\zeta_j$  is an ‘‘eigenvector’’ of  $\nabla_0$  at  $q_j$ ). This fact means that the coordinates  $(q_j, p_j)_j$  are an analog of the coordinates on the moduli space of (parabolic) Higgs bundles given as in [16] and [19]. The coordinates on the moduli space of (parabolic) Higgs bundles are by using the BNR correspondence [3]. (See Section 6).*

Let  $\pi_{c_d, N}: \text{Sym}^N(\mathbf{\Omega}(D, c_d)) \rightarrow \text{Sym}^N(C)$  be the map induced by the map  $\pi_{c_d}: \mathbf{\Omega}(D, c_d) \rightarrow C$ . We set

$$\text{Sym}^N(\mathbf{\Omega}(D, c_d))_0 := \left\{ \{(q_j, \tilde{p}_j)\}_{j=1}^N \in \pi_{c_d, N}^{-1}(\text{Sym}^N(C \setminus \text{Supp}(D))) \mid q_{j_1} \neq q_{j_2} (j_1 \neq j_2) \right\}.$$

Then  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))_0$  has the induced symplectic form from the Liouville symplectic form. Notice that by construction the image of  $M_X^0$  under the map  $f_{\text{App}}$  is contained in  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))_0$ .

**Theorem 20.** *Let  $\omega$  be the symplectic form on  $M_X^0$  defined by (3.2). The pull-back of the symplectic form on  $\text{Sym}^N(\mathbf{\Omega}(D, c_d))_0$  under the map*

$$f_{\text{App}}: M_X^0 \longrightarrow \text{Sym}^N(\mathbf{\Omega}(D, c_d))_0$$

*in Definition 16 coincides with  $\omega$ .*

*Proof.* Let  $V$  be an analytic open subset of  $M_X^0$  as in Section 3.4. Moreover, we assume that we may define a composition

$$\begin{aligned} V &\longrightarrow f_{\text{App}}(V) \longrightarrow \text{Sym}^N(\mathbb{C}_{(q,p)}^2) \\ (E, \nabla) &\longmapsto f_{\text{App}}(E, \nabla) \longmapsto \{(q_j, p_j)\}_{j=1,2,\dots,N}, \end{aligned}$$

where  $p_j$  is defined in (3.19), and the image of  $V$  under the composition is isomorphic to some analytic open subset of  $\mathbb{C}_{(q,p)}^{2N}$ . Let  $U_{(q,p)}$  be such an analytic open subset of  $\mathbb{C}_{(q,p)}^{2N}$ . We denote by  $f_2$  the map

$$\begin{aligned} M_X^0 \supset V &\longrightarrow U_{(q,p)} \subset \mathbb{C}_{(q,\zeta)}^{2N} \\ (E, \nabla) &\longmapsto (q_1, \dots, q_N, p_1, \dots, p_N). \end{aligned}$$

We consider the following maps

$$U_{(q,\zeta)} \xleftarrow[\sim]{f_1} V \xrightarrow{f_2} U_{(q,p)}.$$

Here  $f_1: V \xrightarrow{\sim} U_{(q,\zeta)}$  is the isomorphism (3.8). The symplectic structure on  $U_{(q,p)}$  induced by the symplectic structure on  $\text{Sym}^N(\Omega(D, c_d))$  is  $\sum_{j=1}^N dp_j \wedge dq_j$ . We will show that

$$(f_1^{-1})^*(\omega|_V) = (f_2 \circ f_1^{-1})^* \left( \sum_{j=1}^N dp_j \wedge dq_j \right).$$

Let  $v, v'$  be elements of  $T_{(q_j, \zeta_j)} U_{(q,\zeta)}$  for  $(q_j, \zeta_j)_j \in U_{(q,\zeta)}$ . We will use the description of the tangent map (3.10) of  $f_1^{-1}: U_{(q,\zeta)} \rightarrow V$ . That is, we calculate  $(f_1^{-1})^*(\omega|_V)$  by applying the descriptions (3.12) and (3.15) of  $u_{\alpha\beta}(v)$  and  $v_\alpha(v)$ , respectively.

First we consider  $\{u_{\alpha\beta}(v)u_{\beta\gamma}(v')\}_{\alpha\beta\gamma}$ . Remark that  $U_{\alpha_{q_{j_1}}} \cap U_{\alpha_{q_{j_2}}} = \emptyset$  for any  $j_1$  and  $j_2$ ,  $U_{\alpha_{i_1}} \cap U_{\alpha_{i_2}} = \emptyset$  for any  $i_1$  and  $i_2$ , and  $U_{\alpha_{q_j}} \cap U_{\alpha_{i_j}} = \emptyset$  for any  $j$  and  $i$ . Then we have  $u_{\alpha\beta}u_{\beta\gamma} = 0$  by Lemma 14. So we may take a representative of the class in the pairing (3.2) so that

$$[-\{\text{tr}(u_{\alpha\beta}(v) \circ v_\beta(v')) - \text{tr}(v_\alpha(v) \circ u_{\alpha\beta}(v'))\}_{\alpha\beta}] \in H^1(C, \Omega_C^1) \cong \mathbb{C}.$$

Now we calculate  $\text{tr}(u_{\alpha\beta}(v) \circ v_\beta(v')) - \text{tr}(v_\alpha(v) \circ u_{\alpha\beta}(v'))$ . If  $\alpha \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q)$  and  $\beta = \alpha_{q_j}$ , then, by applying (3.12) and (3.15), we have the following equalities

$$\begin{aligned} &\text{tr}(u_{\alpha\alpha_{q_j}}(v)v_{\alpha_{q_j}}(v')) - \text{tr}(v_\alpha(v)u_{\alpha\alpha_{q_j}}(v')) \\ &= \text{tr} \left( \begin{pmatrix} 0 & \frac{v(\zeta_j)}{z_j - q_j} \\ 0 & \frac{v(q_j)}{z_j - q_j} \end{pmatrix} \begin{pmatrix} * & * \\ -v'(q_j)\gamma_{\alpha_{q_j}} & v'(\text{tr}(A_{\alpha_{q_j}})) + v'(\zeta_j)\gamma_{\alpha_{q_j}} \end{pmatrix} \right) \\ &\quad - \text{tr} \left( \begin{pmatrix} * & * \\ 0 & v(\text{tr}(A_\alpha)) \end{pmatrix} \begin{pmatrix} 0 & \frac{v'(\zeta_j)}{z_j - q_j} \\ 0 & \frac{v'(q_j)}{z_j - q_j} \end{pmatrix} \right) \\ (3.20) \quad &= -\frac{v(\zeta_j)v'(q_j)\gamma_{\alpha_{q_j}}}{z_j - q_j} + \frac{v(q_j)(v'(\text{tr}(A_{\alpha_{q_j}})) + v'(\zeta_j)\gamma_{\alpha_{q_j}})}{z_j - q_j} - \frac{v'(q_j)(v(\text{tr}(A_\alpha)))}{z_j - q_j} \\ &= -\frac{(v(\zeta_j)\gamma_{\alpha_{q_j}} + v(\text{tr}(A_\alpha)))v'(q_j)}{z_j - q_j} + \frac{v(q_j)(v'(\text{tr}(A_{\alpha_{q_j}})) + v'(\zeta_j)\gamma_{\alpha_{q_j}})}{z_j - q_j}. \end{aligned}$$

Now we consider the difference between  $v(\text{tr}(A_{\alpha_{q_j}}))$  and  $v(\text{tr}(A_\alpha))$ . So we consider infinitesimal deformation of  $(\det(\nabla), \text{tr}(\nabla))$ . We have that

$$\det(B_{\alpha_{\alpha_{q_j}}}) = \det \left( \begin{pmatrix} 1 & 0 \\ 0 & ((\omega_\alpha^{-1})^{-1} \circ \omega_{\alpha_{q_j}}^{-1}) \end{pmatrix} \begin{pmatrix} 1 & \frac{\zeta_j}{z_j - q_j} \\ 0 & \frac{1}{z_j - q_j} \end{pmatrix} \right) = \frac{((\omega_\alpha^{-1})^{-1} \circ \omega_{\alpha_{q_j}}^{-1})}{z_j - q_j}.$$

Here  $B_{\alpha_{\alpha_{q_j}}}$  is calculated in (3.14). Set

$$(3.21) \quad u_{\alpha_{\alpha_{q_j}}}^{\det}(v) := \det(B_{\alpha_{\alpha_{q_j}}})^{-1} v(\det(B_{\alpha_{\alpha_{q_j}}})) = \frac{v(q_j)}{z_j - q_j}.$$

Here remark that  $((\omega_\alpha^{-1})^{-1} \circ \omega_{\alpha_{q_j}}^{-1})$  is independent of the moduli space  $M_X^0$ . We have a cocycle condition

$$v(\text{tr}(A_{\alpha_{q_j}})) - v(\text{tr}(A_\alpha)) = \text{tr}(\nabla) \circ u_{\alpha_{\alpha_{q_j}}}^{\det} - u_{\alpha_{\alpha_{q_j}}}^{\det} \circ \text{tr}(\nabla).$$

So we have

$$v(\text{tr}(A_{\alpha_{q_j}})) - v(\text{tr}(A_\alpha)) = d \left( \frac{v(q_j)}{z_j - q_j} \right) = - \frac{v(q_j) dz_j}{(z_j - q_j)^2}.$$

By applying this difference to (3.20), we have that

$$(3.22) \quad \begin{aligned} & \text{tr}(u_{\alpha_{\alpha_{q_j}}}(v)v_{\alpha_{q_j}}(v')) - \text{tr}(v_\alpha(v)u_{\alpha_{\alpha_{q_j}}}(v')) \\ &= - \frac{\left( v(\zeta_j)\gamma_{\alpha_{q_j}} + v(\text{tr}(A_{\alpha_{q_j}})) \right) v'(q_j)}{z_j - q_j} + \frac{v(q_j) \left( v'(\text{tr}(A_{\alpha_{q_j}})) + v'(\zeta_j)\gamma_{\alpha_{q_j}} \right)}{z_j - q_j} - \frac{v(q_j)v'(q_j) dz_j}{(z_j - q_j)^3}. \end{aligned}$$

So we may extend the 1-form

$$\text{tr}(u_{\alpha_{\alpha_{q_j}}}(v)v_{\alpha_{q_j}}(v')) - \text{tr}(v_\alpha(v)u_{\alpha_{\alpha_{q_j}}}(v'))$$

from  $U_{\alpha_{\alpha_{q_j}}}$  to  $U_{\alpha_{q_j}}$  by (3.22). Then we have a meromorphic 1-form defined on  $U_{\alpha_{q_j}}$ , which has a pole at  $q_j$ . We denote by  $\omega_{\alpha_{q_j}}(v, v')$  the meromorphic 1-form defined on  $U_{\alpha_{q_j}}$ .

Next we consider the case where  $\alpha \in I_{\text{cov}} \setminus (I_{\text{cov}}^t \cup I_{\text{cov}}^q)$  and  $\beta = \alpha_{t_i}$ . We have the following equalities

$$\begin{aligned} & \text{tr}(u_{\alpha_{\alpha_{t_i}}}(v)v_{\alpha_{t_i}}(v')) - \text{tr}(v_\alpha(v)u_{\alpha_{\alpha_{t_i}}}(v')) \\ &= \text{tr} \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) v'(A_{\alpha_{t_i}}) \right) - \text{tr} \left( \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(A_\alpha) g_{\alpha_{t_i}}^{t_i} \right) (g_{\alpha_{t_i}}^{t_i})^{-1} v'(g_{\alpha_{t_i}}^{t_i}) \right) \end{aligned}$$

We have the cocycle condition

$$\begin{aligned} & v(A_{\alpha_{t_i}}) - (g_{\alpha_{t_i}}^{t_i})^{-1} v(A_\alpha) g_{\alpha_{t_i}}^{t_i} \\ &= (d + A_{\alpha_{t_i}}) \circ \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) \right) - \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) \right) \circ (d + A_{\alpha_{t_i}}) \\ &= d \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) \right) + \left[ A_{\alpha_{t_i}}, \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) \right) \right]. \end{aligned}$$

By this condition, we have

$$(3.23) \quad \begin{aligned} & \text{tr}(u_{\alpha_{\alpha_{t_i}}}(v)v_{\alpha_{t_i}}(v')) - \text{tr}(v_\alpha(v)u_{\alpha_{\alpha_{t_i}}}(v')) \\ &= \text{tr} \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) v'(A_{\alpha_{t_i}}) \right) - \text{tr} \left( v(A_{\alpha_{t_i}}) (g_{\alpha_{t_i}}^{t_i})^{-1} v'(g_{\alpha_{t_i}}^{t_i}) \right) \\ & \quad + \text{tr} \left( \left( d \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) \right) + \left[ A_{\alpha_{t_i}}, \left( (g_{\alpha_{t_i}}^{t_i})^{-1} v(g_{\alpha_{t_i}}^{t_i}) \right) \right] \right) (g_{\alpha_{t_i}}^{t_i})^{-1} v'(g_{\alpha_{t_i}}^{t_i}) \right) \end{aligned}$$

So we may extend the 1-form

$$\mathrm{tr}(u_{\alpha\alpha_{t_i}}(v)v_{\alpha_{t_i}}(v')) - \mathrm{tr}(v_{\alpha}(v)u_{\alpha\alpha_{t_i}}(v'))$$

from  $U_{\alpha\alpha_{t_i}}$  to  $U_{\alpha_{t_i}}$  by (3.23). Since we have the vanishing of the lower terms (3.13), the extended 1-form defined on  $U_{\alpha_{t_i}}$  is holomorphic. We denote by  $\omega_{\alpha_{t_i}}(v, v')$  the holomorphic 1-form defined on  $U_{\alpha_{t_i}}$ .

For  $\alpha \in I_{\mathrm{cov}} \setminus (I_{\mathrm{cov}}^t \cup I_{\mathrm{cov}}^q)$ , we set  $\omega_{\alpha}(v, v') = 0$ . By (3.22) and (3.23), we have a meromorphic coboundary  $\{\omega_{\alpha}(v, v')\}_{\alpha}$  of

$$\{\mathrm{tr}(u_{\alpha\beta}(v) \circ v_{\beta}(v')) - \mathrm{tr}(v_{\alpha}(v) \circ u_{\alpha\beta}(v'))\}_{\alpha\beta}.$$

So we have

$$\begin{aligned} H^1(C, \Omega_C^1) &\xrightarrow{\cong} \mathbb{C} \\ [-\{\mathrm{tr}(u_{\alpha\beta}(v) \circ v_{\beta}(v')) - \mathrm{tr}(v_{\alpha}(v) \circ u_{\alpha\beta}(v'))\}_{\alpha\beta}] &\longmapsto \sum_{x \in C} -\mathrm{res}_x(\omega_{\alpha}(v, v')). \end{aligned}$$

By taking the residues of the right hand sides of (3.22) and (3.23), we have that

$$\begin{aligned} -\sum_{x \in C} \mathrm{res}_x(\omega_{\alpha}(v, v')) &= \sum_{j=1}^N \mathrm{res}_{q_j} \left( \frac{\left( v(\zeta_j)\gamma_{\alpha_{q_j}} + v(\mathrm{tr}(A_{\alpha_{q_j}})) \right) v'(q_j)}{z_j - q_j} \right) \\ &\quad - \sum_{j=1}^N \mathrm{res}_{q_j} \left( \frac{v(q_j) \left( v'(\mathrm{tr}(A_{\alpha_{q_j}})) + v'(\zeta_i)\gamma_{\alpha_{q_j}} \right)}{z_j - q_j} \right) \\ &= \sum_{j=1}^N (v(p_j)v'(q_j) - v(q_j)v'(p_j)) = \left( \sum_{j=1}^N dp_j \wedge dq_j \right) (v, v'). \end{aligned}$$

Here remark that  $\gamma_{\alpha}$  is independent of the moduli space  $M_X^0$  for any  $\alpha$ .  $\square$

By the map  $f_{\mathrm{App}}$ , we have concrete canonical coordinates as follows. We take an analytic open subset  $V$  of  $M_X^0$  at a point  $(E, \nabla)$ , which is small enough. We define functions  $q_j$  and  $p_j$  ( $j = 1, 2, \dots, N$ ) on  $V$  as follows. (So, here, the notation  $q_j$  has a double meaning). Let  $U_{\alpha_{q_j}}$  be an analytic open subset of  $C$  such that  $U_{\alpha_{q_j}}$  contains the apparent singularity  $q_j$  of the point  $(E, \nabla)$  and is small enough. Let  $q'_j$  be the apparent singularity of each  $(E', \nabla') \in V$ , where  $q'_j \in U_{\alpha_{q_j}}$ . First we take a local coordinate  $z_j$  on  $U_{\alpha_{q_j}}$ . By evaluating the apparent singularity  $q'_j$  by the local coordinate  $z_j$  for each  $(E', \nabla') \in V$ , we have a function  $q_j: V \rightarrow \mathbb{C}$ . Second, let  $(E_V, \nabla_V)$  be a vector bundles on  $C \times V$ , which is a family of vector bundles on  $C$  parametrized by  $V$ . We take a trivialization of  $\det(E_V)$  on  $U_{\alpha_{q_j}} \times V$  which depends on only  $q_j: V \rightarrow \mathbb{C}$  (which is described in (3.18)). We take the connection matrix of  $\mathrm{tr}(\nabla_V)$  with respect to the local trivialization. Let  $\Omega(D, c_d)_V \rightarrow C \times V$  be the relative twisted cotangent bundle over  $V$  with respect to the family of line bundles  $\det(E_V)$  on  $C \times V$ . We have an identification between  $\Omega(D, c_d)_V$  and  $\Omega(D) \times V$  on  $U_{\alpha_{q_j}} \times V$  that depends only on  $q_j: V \rightarrow \mathbb{C}$ . By evaluating  $\mathrm{res}_{q'_j}(\beta') + \mathrm{tr}(\nabla')|_{q'_j}$  by the identification  $\Omega(D, c_d)|_{q'_j} \cong \Omega(D)|_{q'_j} \cong \mathbb{C}$  for each  $(E', \nabla') \in V$ , we have a function  $p_j: V \rightarrow \mathbb{C}$ . This is just

(3.19). That is, this is the following composition:

$$\begin{aligned} V &\longrightarrow U_{\alpha_{q_j}} \times V \longrightarrow \Omega(D, c_d)_V|_{U_{\alpha_{q_j}} \times V} \longrightarrow \Omega(D)|_{U_{\alpha_{q_j}}} \longrightarrow \mathbb{C} \\ (E', \nabla') &\longmapsto (q'_j, (E', \nabla')) \longmapsto \left( (\zeta_j \gamma_{\alpha_{q_j}})_V + \text{tr}(\nabla V) \right) |_{(q'_j, (E', \nabla'))} \\ &\longmapsto \left( \zeta'_j \gamma_{\alpha_{q_j}} + \text{tr}(A'_{\alpha_{q_j}}) \right) |_{q'_j} \longmapsto \text{res}_{q'_j} \left( \frac{\zeta'_j \gamma_{\alpha_{q_j}} + \text{tr}(A'_{\alpha_{q_j}})}{z_j - q'_j} \right). \end{aligned}$$

By Theorem 20, the symplectic structure on  $V$  has the following description:  $\sum_{j=1}^N dp_j \wedge dq_j$ .

**Remark 21.** *We set*

$$p_j^0 := \text{res}_{q_j} \left( \frac{\zeta_j \gamma_{\alpha_{q_j}}}{z_j - q_j} \right) \in \mathbb{C}.$$

If  $g = 0$ , then  $\text{res}_{q_j} \left( \frac{\text{tr}(A_{\alpha_{q_j}})}{z_j - q_j} \right) dp_j \wedge dq_j$  depends on only  $q_j$ . So we have  $\sum_{j=1}^N dp_j \wedge dq_j = \sum_{j=1}^N dp_j^0 \wedge dq_j$ . Here the symplectic form  $\sum_{j=1}^N dp_j^0 \wedge dq_j$  is induced by the symplectic form on  $\text{Sym}^N(\Omega(D))_0$ .

**Remark 22.** *In general,  $\sum_{j=1}^N dp_j \wedge dq_j \neq \sum_{j=1}^N dp_j^0 \wedge dq_j$ , that is,*

$$(3.24) \quad \sum_j d \left( \text{res}_{q_j} \left( \frac{\text{tr}(A_{\alpha_{q_j}})}{z_j - q_j} \right) \right) \wedge dq_j$$

*does not vanish. This is related to the determinant map*

$$\begin{aligned} M_X^0 &\longrightarrow M_X^{\text{rk}=1}(\boldsymbol{\nu}_{\text{res}}) \\ (E, \nabla, \{l^{(i)}\}) &\longmapsto (\det(E), \text{tr}(\nabla)). \end{aligned}$$

*The 2-form (3.24) comes from*

$$\left\{ \{u_{\alpha\beta}^{\det}(v)u_{\beta\gamma}^{\det}(v')\}, -\{u_{\alpha\beta}^{\det}(v)v'(\text{tr}(A_{\beta})) - v(\text{tr}(A_{\alpha}))u_{\alpha\beta}^{\det}(v')\} \right\} \in \mathbf{H}^2(\mathcal{O}_C \rightarrow \Omega_C^1).$$

*Here  $u_{\alpha\beta}^{\det}(v)$  is defined as in (3.21). This class gives rise to the 2-form on  $M_X^0$  which is just the pull-back of the natural symplectic form on  $M_X^{\text{rk}=1}(\boldsymbol{\theta}_{\text{res}})$  under the determinant map. The determinant map is not degenerate in general. So the class (3.24) does not vanish in general.*

#### 4. SYMPLECTIC STRUCTURE ON THE MODULI SPACE WITH FIXED TRACE CONNECTION

In this section, we consider the moduli spaces of rank 2 quasi-parabolic connections *with fixed trace connection*. When the effective divisor  $D$  is reduced, this moduli space is detailed in [1], [37] (when  $g = 0$ ), [12], [13] (when  $g = 1$ ), and [39] (when  $g \geq 1$ ). The moduli spaces of rank 2 quasi-parabolic connections with fixed trace connection has a natural symplectic structure described as in Section 3.2. The purpose of this section is to give coordinates on some generic part of the moduli space and to describe the natural symplectic structure by using the coordinates. As in the case where the effective divisor  $D$  is reduced ([37], [12], [13], [39]), we may define the map forgetting connections and the apparent map. These maps are from a generic part of the moduli space to projective spaces. These maps will give our coordinates on the generic part of the moduli space. First we describe these maps.

**4.1. Moduli space of quasi-parabolic bundles with fixed determinant.** To describe the map forgetting connections, we recall the moduli space of quasi-parabolic bundles. The moduli space of (quasi-)parabolic bundles was introduced in Mehta–Seshadri [40]. Yokogawa generalized this notion to (quasi-)parabolic sheaves and studied their moduli [48].

Let  $\nu$  be a positive integer. Set  $I := \{1, 2, \dots, \nu\}$ . Let  $C$  be a compact Riemann surface of genus  $g$ , and  $D = \sum_{i \in I} m_i [t_i]$  be an effective divisor on  $C$ . We assume  $3g - 3 + n > 0$  where  $n = \text{length}(D)$ . Let  $z_i$  be a generator of the maximal ideal of  $\mathcal{O}_{C, t_i}$ . We fix a line bundle  $L_0$  with  $\deg(L_0) = 2g - 1$ .

**Definition 23.** We say  $(E, \{l^{(i)}\})$  a rank 2 quasi-parabolic bundle with determinant  $L_0$  over  $(C, D)$  if

- (i)  $E$  is a rank 2 vector bundle of degree  $2g - 1$  on  $C$  with  $\det(E) \cong L_0$ , and
- (ii)  $E|_{m_i[t_i]} \supset l^{(i)} \supset 0$  is a filtration by free  $\mathcal{O}_{m_i[t_i]}$ -modules such that  $E|_{m_i[t_i]}/l^{(i)} \cong \mathcal{O}_{m_i[t_i]}$  and  $l^{(i)} \cong \mathcal{O}_{m_i[t_i]}$  for any  $i \in I$ .

We fix weights  $\mathbf{w} = (w_1, \dots, w_\nu)$  such that  $w_i \in [0, 1]$  for any  $i \in I$ . When  $g = 0$ , we assume that  $(w_i)_{i \in I}$  satisfies

$$(4.1) \quad w_1 = \dots = w_\nu \quad \text{and} \quad \frac{1}{\deg(D)} < w_i < \frac{1}{\deg(D) - 2}.$$

When  $g \geq 1$ , we assume that  $(w_i)_{i \in I}$  satisfy

$$(4.2) \quad 0 < w_i \ll 1.$$

**Definition 24.** Let  $(E, \{l^{(i)}\})$  be a rank 2 quasi-parabolic bundle with determinant  $L_0$ . Let  $L$  be a line subbundle of  $E$ . We define the  $\mathbf{w}$ -stability index of  $L$  to be the real number

$$\text{Stab}_{\mathbf{w}}(L) := \deg(E) - 2 \deg(L) + \sum_{i \in I} w_i (m_i - 2 \text{length}(l_i \cap L|_{m_i[t_i]})).$$

**Definition 25.** A rank 2 quasi-parabolic bundle  $(E, \{l^{(i)}\})$  is  $\mathbf{w}$ -stable if for any subbundle  $L \subset E$ , the inequality  $\text{Stab}_{\mathbf{w}}(L) > 0$  holds.

We say that a quasi-parabolic bundle  $(E, \{l^{(i)}\})$  is decomposable if there exists a decomposition  $E = L_1 \oplus L_2$  such that  $l^{(i)} = l_1^{(i)}$  or  $l^{(i)} = l_2^{(i)}$  for any  $i \in I$ , where we set  $l_1^{(i)} := l^{(i)} \cap (L_1|_{m_i[t_i]})$  and  $l_2^{(i)} := l^{(i)} \cap (L_2|_{m_i[t_i]})$ . We say that  $(E, \{l^{(i)}\})$  is undecomposable if  $(E, \{l^{(i)}\})$  is not decomposable. A free  $\mathcal{O}_{m_i[t_i]}$ -submodule  $l^{(i)}$  of  $E|_{m_i[t_i]}$  induces a one dimensional subspace  $l_{\text{red}}^{(i)}$  of  $E|_{t_i}$ , that is the restriction of  $l^{(i)}$  to  $t_i$  (without multiplicity).

**Lemma 26.** Let  $(E, \{l^{(i)}\})$  be a rank 2 quasi-parabolic bundle with determinant  $L_0$ . If

- (i)  $E$  is an extension of  $L_0$  by  $\mathcal{O}_C$  (when  $g = 0$ , moreover we assume that  $(E, \{l^{(i)}\})$  is undecomposable)
- (ii)  $\dim_{\mathbb{C}} H^1(C, E) = 0$
- (iii)  $l_{\text{red}}^{(i)} \not\subset \mathcal{O}_C|_{t_i} \subset \mathbb{P}(E)$  for any  $i$ ,

then  $(E, \{l^{(i)}\})$  is  $\mathbf{w}$ -stable.

*Proof.* When  $g = 0$ , we have this statement from [32, Proposition 46] by the condition (4.1). When  $g \geq 1$ , we have that  $E$  is stable, that is,  $\deg(E) - 2 \deg(L)$  is a positive integer for any line subbundle  $L \subset E$ . This claim follows from the same argument as in [39, Lemma 4.2]. Since  $0 < w_i \ll 1$  in (4.2), we have that  $\text{Stab}_{\mathbf{w}}(L) > 0$ .  $\square$

Let  $P_{(C,D)}^w$  be a moduli space of  $w$ -stable quasi-parabolic bundles constructed in [48]. Let  $P_{(C,D)}^w(L_0)$  be the fiber of  $L_0$  under the determinant map

$$P_{(C,D)}^w \longrightarrow \text{Pic}_C^{2g-1}; \quad (E, \{l^{(i)}\}) \longmapsto \det(E).$$

We set

$$P_{(C,D)}(L_0)_0 := \left\{ (E, \{l^{(i)}\}) \left| \begin{array}{l} (E, \{l^{(i)}\}) \text{ is rank 2 quasi-parabolic bundle over } (C, D) \text{ such that} \\ \text{(i) } \det(E) \cong L_0, \quad \text{(ii) } E \text{ is an extension of } L_0 \text{ by } \mathcal{O}_C, \\ \text{(iii) } \dim_{\mathbb{C}} H^1(C, E) = 0, \quad \text{(iv) } l_{\text{red}}^{(i)} \notin \mathcal{O}_C|_{t_i} \subset \mathbb{P}(E) \text{ for any } i, \\ \text{(v) } (E, \{l^{(i)}\}) \text{ is undecomposable (when } g = 0) \end{array} \right. \right\}.$$

By Lemma 26, we have an inclusion

$$P_{(C,D)}(L_0)_0 \subset P_{(C,D)}^w(L_0).$$

For  $(E, \{l^{(i)}\}) \in P_{(C,D)}(L_0)_0$ , we have an extension

$$(4.3) \quad 0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L_0 \longrightarrow 0.$$

Since  $\dim_{\mathbb{C}} H^1(C, E) = 0$ , we have that  $\dim_{\mathbb{C}} H^0(C, E) = 1$ . So the injection  $\mathcal{O}_C \xrightarrow{\subset} E$  in (4.3) is unique up to a constant.

**Definition 27.** Let  $(E, \{l^{(i)}\}) \in P_{(C,D)}(L_0)_0$ . We take an affine open covering  $\{U_\alpha\}_\alpha$  of  $C$ , i.e.  $C = \bigcup_\alpha U_\alpha$ . Let  $\{\varphi_\alpha^{\text{Ext}}\}_\alpha$  be trivializations  $\varphi_\alpha^{\text{Ext}}: \mathcal{O}_{U_\alpha}^{\oplus 2} \rightarrow E|_{U_\alpha}$  of the underlying vector bundle  $E$  such that

(i) the composition

$$\begin{array}{c} \mathcal{O}_{U_\alpha} \longrightarrow \mathcal{O}_{U_\alpha}^{\oplus 2} \xrightarrow{\varphi_\alpha^{\text{Ext}}} E|_{U_\alpha} \\ f \longmapsto (f, 0) \end{array}$$

is just the inclusion  $\mathcal{O}_C \subset E$  of the extension (4.3) for any  $\alpha$ , and

(ii) the image of the composition

$$\begin{array}{c} \mathcal{O}_{U_\alpha} \longrightarrow \mathcal{O}_{U_\alpha}^{\oplus 2} \xrightarrow{\varphi_\alpha^{\text{Ext}}} E|_{U_\alpha} \longrightarrow E|_{m_i[t_i]} \\ f \longmapsto (0, f) \end{array}$$

generates the submodule  $l^{(i)} \subset E|_{m_i[t_i]}$  for each  $i$  and  $\alpha$  where  $t_i \in U_\alpha$ .

Notice that the claim that we may take  $\varphi_\alpha^{\text{Ext}}$  which satisfies the condition (ii) of Definition 27 follows from the condition that  $l_{\text{red}}^{(i)} \notin \mathcal{O}_C|_{t_i} \subset \mathbb{P}(E)$  for any  $i$ .

Now we define a map

$$(4.4) \quad P_{(C,D)}(L_0)_0 \longrightarrow \mathbb{P}H^1(C, L_0^{-1}(-D))$$

as follows. Let  $\{\varphi_\alpha^{\text{Ext}}\}_\alpha$  be the trivializations in Definition 27. We have the transition matrices

$$B_{\alpha\beta} := (\varphi_\alpha^{\text{Ext}}|_{U_{\alpha\beta}})^{-1} \circ \varphi_\beta^{\text{Ext}}|_{U_{\alpha\beta}}: \mathcal{O}_{U_{\alpha\beta}}^{\oplus 2} \longrightarrow \mathcal{O}_{U_{\alpha\beta}}^{\oplus 2}.$$

We represent  $B_{\alpha\beta}$  as a matrix:

$$(4.5) \quad B_{\alpha\beta} = \begin{pmatrix} 1 & b_{\alpha\beta}^{12} \\ 0 & b_{\alpha\beta}^{22} \end{pmatrix}.$$

Remark that  $\{b_{\alpha\beta}^{22}\}_{\alpha\beta}$  is a multiplicative cocycle which defines the fixed line bundle  $L_0$ . We take a meromorphic coboundary

$$(4.6) \quad \{b_{\alpha}^{22}\}_{\alpha} \quad \left( \text{where } b_{\alpha\beta}^{22} = \frac{b_{\alpha}^{22}}{b_{\beta}^{22}} \right)$$

of the multiplicative cocycle  $\{b_{\alpha\beta}^{22}\}_{\alpha\beta}$ . By using the coboundary  $\{b_{\alpha}^{22}\}_{\alpha}$ , we define a cocycle

$$(4.7) \quad b_{\alpha\beta}^{\text{Bun}} := b_{\alpha\beta}^{12} b_{\alpha}^{22},$$

which gives a class  $[\{b_{\alpha\beta}^{\text{Bun}}\}] \in H^1(C, L_0^{-1}(-D))$ . Then we have a map (4.4):

$$(E, \{l^{(i)}\}) \longmapsto \overline{[\{b_{\alpha\beta}^{\text{Bun}}\}]}.$$

**4.2. Moduli space of quasi-parabolic connections with fixed trace connection.** Now we recall the moduli space of quasi-parabolic connections. We fix an irregular curve with residues  $X = (C, D, \{z_i\}, \{\theta_i\}, \theta_{\text{res}})$  defined in Definition 2. Moreover we assume that

$$(4.8) \quad \sum_{i \in I} \theta_{i,-1}^- \neq 0.$$

Let  $(L_0, \nabla_{L_0}: L_0 \rightarrow L_0 \otimes \Omega_C^1(D))$  be a rank 1 connection on  $C$  with degree  $2g - 1$  such that the polar part of  $\nabla_{L_0}$  at  $t_i$  is  $\text{tr}(\omega_i(X))$ .

**Definition 28.** We say  $(E, \nabla, \lambda, \{l^{(i)}\})$  a rank 2 quasi-parabolic  $\lambda$ -connection over  $X$  with fixed trace connection  $(L_0, \nabla_{L_0})$  if

- (i)  $E$  is a rank 2 vector bundle on  $C$  with  $\det(E) \cong L_0$ ,
- (ii)  $\lambda \in \mathbb{C}$  and  $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$  is a  $\lambda$ -connection that is,  $\nabla(fs) = \lambda s \otimes df + f \nabla(s)$  for any  $f \in \mathcal{O}_C$  and  $s \in E$ , and
- (iii)  $\nabla(s_1) \wedge s_2 + s_1 \wedge \nabla(s_2) = \lambda \nabla_L(s_1 \wedge s_2)$  for  $s_1, s_2 \in E$ ,
- (iv)  $E|_{m_i[t_i]} \supset l^{(i)} \supset 0$  is a filtration by free  $\mathcal{O}_{m_i[t_i]}$ -modules such that, for any  $i \in I$ ,
  - $E|_{m_i[t_i]}/l^{(i)} \cong \mathcal{O}_{m_i[t_i]}$  and  $l^{(i)} \cong \mathcal{O}_{m_i[t_i]}$ ,
  - $\nabla|_{m_i[t_i]}(l^{(i)}) \subset l^{(i)} \otimes \Omega_C^1(D)$ , and
  - the image of  $(E|_{m_i[t_i]}/l^{(i)}) \oplus l^{(i)}$  under  $\text{Gr}_i(\nabla) - \lambda \cdot \omega_i(X)$  is contained in

$$\left( (E|_{m_i[t_i]}/l^{(i)}) \oplus l^{(i)} \right) \otimes \Omega_C^1.$$

Here  $\text{Gr}_i(\nabla)$  is the induced morphism

$$\text{Gr}_i(\nabla): (E|_{m_i[t_i]}/l^{(i)}) \oplus l^{(i)} \longrightarrow \left( (E|_{m_i[t_i]}/l^{(i)}) \oplus l^{(i)} \right) \otimes \Omega_C^1(D).$$

Notice that, if  $\lambda = 0$ , then  $\nabla$  is an  $\mathcal{O}_C$ -morphism, which is called a Higgs field. So  $(E, \nabla, \lambda, \{l^{(i)}\})$  is called a (trace free) quasi-parabolic Higgs bundle when  $\lambda = 0$ . We consider only rank 2 quasi-parabolic  $\lambda$ -connections  $(E, \nabla, \lambda, \{l^{(i)}\})$  over  $X$  with  $(L_0, \nabla_{L_0})$  such that the underlying quasi-parabolic bundle  $(E, \{l^{(i)}\})$  is in the moduli space  $P_{(C,D)}(L_0)_0$ .

We define the moduli spaces  $\widetilde{M}_X(L_0, \nabla_{L_0})_0$  and  $M_X(L_0, \nabla_{L_0})_0$  as follows:

$$\widetilde{M}_X(L_0, \nabla_{L_0})_0 = \left\{ \begin{array}{l} (E, \nabla, \lambda, \{l^{(i)}\}) \\ \text{quasi-parabolic } \lambda\text{-connection} \\ \text{over } X \text{ with trace } (L_0, \nabla_{L_0}) \end{array} \middle| (E, \{l^{(i)}\}) \in P_{(C,D)}(L_0)_0 \right\} / \cong$$

and

$$M_X(L_0, \nabla_{L_0})_0 = \left\{ \begin{array}{l} (E, \nabla, \lambda, \{l^{(i)}\}) \\ \text{quasi-parabolic } \lambda\text{-connection} \\ \text{over } X \text{ with trace } (L_0, \nabla_{L_0}) \end{array} \middle| \begin{array}{l} (E, \{l^{(i)}\}) \in P_{(C,D)}(L_0)_0 \\ \text{and } \lambda \neq 0 \end{array} \right\} / \cong.$$

**4.3. Maps from the moduli space.** Now we describe two maps: the forgetful map  $\pi_{\text{Bun}}$  forgetting connections and the apparent map  $\pi_{\text{App}}$ . First we consider the composition

$$\widetilde{M}_X(L_0, \nabla_{L_0})_0 \longrightarrow P_{(C,D)}(L_0)_0 \longrightarrow \mathbb{P}H^1(C, L_0^{-1}(-D)).$$

Here the first map is the forgetful map, and the second map is (4.4). We denote by

$$\pi_{\text{Bun}}: \widetilde{M}_X(L_0, \nabla_{L_0})_0 \longrightarrow \mathbb{P}H^1(C, L_0^{-1}(-D)).$$

the composition.

Second we define a map

$$(4.9) \quad \pi_{\text{App}}: \widetilde{M}_X(L_0, \nabla_{L_0})_0 \longrightarrow \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D))$$

as follows. Let  $(E, \nabla, \lambda, \{l^{(i)}\})$  be a point on  $\widetilde{M}_X(L_0, \nabla_{L_0})_0$ . Let  $\{\varphi_\alpha^{\text{Ext}}\}_\alpha$  be the trivializations in Definition 27. Let  $A_\alpha$  be the connection matrix of the  $\lambda$ -connection  $\nabla$  with respect to  $\varphi_\alpha^{\text{Ext}}$ , that is,

$$\lambda d + A_\alpha := (\varphi_\alpha^{\text{Ext}})^{-1} \circ \nabla \circ \varphi_\alpha^{\text{Ext}}: \mathcal{O}_{U_\alpha}^{\oplus 2} \longrightarrow (\Omega_{U_\alpha}^1(D))^{\oplus 2}.$$

We denote the matrix  $A_\alpha$  as follows:

$$(4.10) \quad A_\alpha = \begin{pmatrix} a_\alpha^{11} & a_\alpha^{12} \\ a_\alpha^{21} & a_\alpha^{22} \end{pmatrix}.$$

By the condition (ii) in Definition 27 and the condition (iv) in Definition 28, the polar part of the connection matrix  $A_\alpha$  at  $t_i$  is a lower triangular matrix, that is, the Laurent expansion of  $A_\alpha$  at  $t_i$  is as follows:

$$(4.11) \quad A_\alpha = \begin{pmatrix} \lambda \nu_i^- & 0 \\ * & \lambda \nu_i^+ \end{pmatrix} \frac{1}{z_i^{m_i}} + [\text{holo. part}].$$

Here  $\nu_i^-, \nu_i^+ \in \Omega_C^1(D)|_{m_i[t_i]}$  are defined so that

$$\lambda \cdot \omega_i(X) = \begin{pmatrix} \lambda \nu_i^- & 0 \\ 0 & \lambda \nu_i^+ \end{pmatrix}.$$

By using the coboundary  $\{b_\alpha^{22}\}_\alpha$  in (4.6), we define cocycles

$$(4.12) \quad a_\alpha^{\text{APP}} := a_\alpha^{21} (b_\alpha^{22})^{-1},$$

which give a class  $[\{a_\alpha^{\text{APP}}\}] \in H^0(C, L_0 \otimes \Omega_C^1(D))$ . Then we have a map (4.9):

$$(E, \nabla, \lambda, \{l^{(i)}\}) \longmapsto \overline{[\{a_\alpha^{\text{APP}}\}]}$$

Finally, we have a map

$$(4.13) \quad (\pi_{\text{App}}, \pi_{\text{Bun}}): \widetilde{M}_X(L_0, \nabla_{L_0})_0 \longrightarrow \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)) \times \mathbb{P}H^1(C, L_0^{-1}(-D)).$$

We consider the natural pairing

$$(4.14) \quad H^0(C, L_0 \otimes \Omega_C^1(D)) \times H^1(C, L_0^{-1}(-D)) \longrightarrow H^1(C, \Omega_C^1) \cong \mathbb{C}.$$

**Lemma 29.** *Let  $(E, \nabla, \lambda, \{l^{(i)}\}) \in \widetilde{M}_X(L_0, \nabla_{L_0})_0$ . Let  $a_\alpha^{\text{App}}$  and  $b_{\alpha\beta}^{\text{Bun}}$  be the cocycles in (4.7) and in (4.12), respectively. Then we have*

$$[\{b_{\alpha\beta}^{\text{Bun}} \cdot a_\beta^{\text{App}}\}] = \lambda \cdot \sum_{i \in I} \theta_{i,-1}^-.$$

Here the left hand side is the pairing (4.14).

*Proof.* Let  $B_{\alpha\beta}$  be the transition function in (4.5). Let  $A_\alpha$  be the connection matrix in (4.10). Then we have

$$\lambda \cdot dB_{\alpha\beta} + A_\alpha B_{\alpha\beta} = B_{\alpha\beta} A_\beta.$$

By comparing the (1,1)-entries of the both hand sides, we have

$$a_\alpha^{11} - a_\beta^{11} = b_{\alpha\beta}^{\text{Bun}} \cdot a_\beta^{\text{App}}.$$

By (4.11) and the isomorphism  $H^1(C, \Omega_C^1) \cong \mathbb{C}$ , we have  $[\{b_{\alpha\beta}^{\text{Bun}} \cdot a_\beta^{\text{App}}\}] = \lambda \cdot \sum_i \theta_{i,-1}^-$ .  $\square$

Set

$$N_0 := \dim_{\mathbb{C}} \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)) = 3g + n - 3.$$

Let us introduce the homogeneous coordinates  $\mathbf{a} = (a_0 : \cdots : a_{N_0})$  on  $\mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)) \cong \mathbb{P}_{\mathbf{a}}^{N_0}$  and the dual coordinates  $\mathbf{b} = (b_0 : \cdots : b_{N_0})$  on

$$\mathbb{P}H^1(C, L_0^{-1}(-D)) \cong \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D))^\vee \cong \mathbb{P}_{\mathbf{b}}^{N_0}.$$

Let  $\Sigma \subset \mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0}$  be the incidence variety whose defining equation is given by  $\sum_j a_j b_j = 0$ . By Lemma 29, we have that

$$\widetilde{M}_X(L_0, \nabla_{L_0})_0 \setminus M_X(L_0, \nabla_{L_0})_0 \xrightarrow{(\pi_{\text{App}}, \pi_{\text{Bun}})} \Sigma.$$

**Remark 30.** *Loray–Saito (for  $g = 0$ ) and Matsumoto (for  $g \geq 1$ ) discussed on the birationality of the map (4.13). They showed the birationality of the map (4.13) when  $D$  is a reduced effective divisor ([37, Theorem 4.3] for  $g = 0$  and [39, Theorem 4.5] for  $g \geq 1$ ). In these cases, quasi-parabolic connections have only simple poles. But we may apply the arguments in [37, Theorem 4.3] and in [39, Theorem 4.5] to our cases where quasi-parabolic connections admit generic unramified irregular singular points. So we can reconstruct  $(E, \nabla, \lambda, \{l^{(i)}\}) \in \widetilde{M}_X(L_0, \nabla_{L_0})_0$  from an element of*

$$\mathbb{P}H^1(C, L_0^{-1}(-D))_0 \times \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)).$$

Here we set

$$\mathbb{P}H^1(C, L_0^{-1}(-D))_0 := \left\{ b \in \mathbb{P}H^1(C, L_0^{-1}(-D)) \mid \begin{array}{l} \text{The extension } E \text{ corresponding to } b \\ \text{satisfies } \dim_{\mathbb{C}} H^1(C, E) = 0 \end{array} \right\}.$$

Then we have isomorphisms

$$\widetilde{M}_X(L_0, \nabla_{L_0})_0 \cong \mathbb{P}H^1(C, L_0^{-1}(-D))_0 \times \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D))$$

and

$$M_X(L_0, \nabla_{L_0})_0 \cong \mathbb{P}H^1(C, L_0^{-1}(-D))_0 \times \mathbb{P}H^0(C, L_0 \otimes \Omega_C^1(D)) \setminus \Sigma.$$

**4.4. Symplectic structure and explicit description.** Now we recall the natural symplectic structure on  $M_X(L_0, \nabla_{L_0})_0$ . We define a complex  $\mathcal{F}_0^\bullet$  for  $(E, \frac{1}{\lambda}\nabla, \{l^{(i)}\})$  by

$$\begin{aligned}\mathcal{F}_0^0 &:= \left\{ s \in \mathcal{E}nd(E) \mid \text{tr}(s) = 0, s|_{m_{it_i}(l^{(i)})} \subset l^{(i)} \text{ for any } i \right\} \\ \mathcal{F}_0^1 &:= \left\{ s \in \mathcal{E}nd(E) \otimes \Omega_C^1(D) \mid \text{tr}(s) = 0, s|_{m_{it_i}(l^{(i)})} \subset l^{(i)} \otimes \Omega_C^1 \text{ for any } i \right\} \\ \nabla_{\mathcal{F}_0^\bullet} : \mathcal{F}_0^0 &\longrightarrow \mathcal{F}_0^1; \quad \nabla_{\mathcal{F}_0^\bullet}(s) = \left(\frac{1}{\lambda}\nabla\right) \circ s - s \circ \left(\frac{1}{\lambda}\nabla\right).\end{aligned}$$

We define the following morphism

$$(4.15) \quad \mathbf{H}^1(\mathcal{F}_0^\bullet) \otimes \mathbf{H}^1(\mathcal{F}_0^\bullet) \longrightarrow \mathbf{H}^2(\mathcal{O}_C \xrightarrow{d} \Omega_C^1) \cong \mathbb{C}$$

as in (3.2). This pairing gives the symplectic form on  $M_X(L_0, \nabla_{L_0})_0$ . We denote by  $\omega_0$  the symplectic form.

The maps  $\pi_{\text{App}}$  and  $\pi_{\text{Bun}}$  give coordinates on  $M_X(L_0, \nabla_{L_0})_0$  (see Remark 30). Now we describe the symplectic structure (4.15) by using the coordinates on  $M_X(L_0, \nabla_{L_0})_0$ . We define a 1-form  $\eta$  on  $\mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0}$  as follows:

$$\eta := \left( - \sum_i \theta_{i,-1}^- \right) \cdot \frac{a_0 db_0 + a_1 db_1 + \cdots + a_{N_0} db_{N_0}}{a_0 b_0 + a_1 b_1 + \cdots + a_{N_0} b_{N_0}}.$$

**Theorem 31.** *Assume that  $\sum_{i \in I} \theta_{i,-1}^- \neq 0$ . Let  $\omega_{\mathbf{a}, \mathbf{b}}$  be the 2-form on  $\mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0}$  defined by  $\omega_{\mathbf{a}, \mathbf{b}} = d\eta$ . The pull-back of  $\omega_{\mathbf{a}, \mathbf{b}}$  under the map*

$$M_X(L_0, \nabla_{L_0})_0 \xrightarrow{(\pi_{\text{App}}, \pi_{\text{Bun}})} \mathbb{P}_{\mathbf{a}}^{N_0} \times \mathbb{P}_{\mathbf{b}}^{N_0}$$

*coincides with the symplectic form  $\omega_0$  on  $M_X(L_0, \nabla_{L_0})_0$ .*

*Proof.* Let  $v, v' \in T_{(E, \frac{1}{\lambda}\nabla, \{l^{(i)}\})} M_X(L_0, \nabla_{L_0})_0$ . We have the isomorphism

$$T_{(E, \frac{1}{\lambda}\nabla, \{l^{(i)}\})} M_X(L_0, \nabla_{L_0})_0 \xrightarrow{\cong} \mathbf{H}^1(\mathcal{F}_0^\bullet).$$

Let  $u_{\alpha\beta}(v)$  and  $v_\alpha(v)$  be cocycles such that the class  $[\{u_{\alpha\beta}(v)\}_{\alpha\beta}, \{v_\alpha(v)\}_\alpha]$  is the image of  $v$  under the isomorphism. We calculate  $u_{\alpha\beta}(v)$  and  $v_\alpha(v)$  by using the trivialization  $\{\varphi_\alpha^{\text{Ext}}\}_\alpha$  as follows:

$$\begin{aligned}(4.16) \quad u_{\alpha\beta}(v) &= \varphi_\beta^{\text{Ext}}|_{U_{\alpha\beta}} \circ \left( B_{\alpha\beta}^{-1} v(B_{\alpha\beta}) \right) \circ (\varphi_\beta^{\text{Ext}}|_{U_{\alpha\beta}})^{-1} \\ &= \varphi_\beta^{\text{Ext}}|_{U_{\alpha\beta}} \circ \begin{pmatrix} 0 & v(b_{\alpha\beta}^{12}) \\ 0 & 0 \end{pmatrix} \circ (\varphi_\beta^{\text{Ext}}|_{U_{\alpha\beta}})^{-1} \\ &= \varphi_\beta^{\text{Ext}}|_{U_{\alpha\beta}} \circ \begin{pmatrix} 0 & \frac{v(b_{\alpha\beta}^{\text{Bun}})}{b_\alpha^{22}} \\ 0 & 0 \end{pmatrix} \circ (\varphi_\beta^{\text{Ext}}|_{U_{\alpha\beta}})^{-1}\end{aligned}$$

and

$$\begin{aligned}(4.17) \quad v_\alpha(v) &= \varphi_\alpha^{\text{Ext}} \circ v \left( \frac{1}{\lambda} A_\alpha \right) \circ (\varphi_\alpha^{\text{Ext}})^{-1} \\ &= \varphi_\alpha^{\text{Ext}} \circ \begin{pmatrix} v(a_\alpha^{11}/\lambda) & v(a_\alpha^{12}/\lambda) \\ v(a_\alpha^{21}/\lambda) & v(a_\alpha^{22}/\lambda) \end{pmatrix} \circ (\varphi_\alpha^{\text{Ext}})^{-1} \\ &= \varphi_\alpha^{\text{Ext}} \circ \begin{pmatrix} v(a_\alpha^{11}/\lambda) & v(a_\alpha^{12}/\lambda) \\ v(a_\alpha^{\text{App}}/\lambda) b_\alpha^{22} & v(a_\alpha^{22}/\lambda) \end{pmatrix} \circ (\varphi_\alpha^{\text{Ext}})^{-1}.\end{aligned}$$

Here  $\{b_\alpha^{22}\}_\alpha$  is the coboundary in (4.6). Since we fix the determinant bundle  $L_0$ , we may assume that the coboundary  $\{b_\alpha^{22}\}_\alpha$  is independent of the moduli space  $M_X(L_0, \nabla_{L_0})_0$ .

Now we calculate the class

$$(4.18) \quad [(\{\operatorname{tr}(u_{\alpha\beta}(v)u_{\beta\gamma}(v'))\}), -\{\operatorname{tr}(u_{\alpha\beta}(v)v_\beta(v')) - \operatorname{tr}(v_\alpha(v)u_{\alpha\beta}(v'))\})]$$

in  $\mathbf{H}^2(\mathcal{O}_C \xrightarrow{d} \Omega_C^1) \cong \mathbb{C}$ . First we calculate  $u_{\alpha\beta}(v)u_{\beta\gamma}(v')$  as follows:

$$\begin{aligned} & u_{\alpha\beta}(v)u_{\beta\gamma}(v') \\ &= \varphi_\beta^{\operatorname{Ext}}|_{U_{\alpha\beta}} \circ \begin{pmatrix} 0 & \frac{v(b_{\alpha\beta}^{\operatorname{Bun}})}{b_\alpha^{22}} \\ 0 & 0 \end{pmatrix} \circ (\varphi_\beta^{\operatorname{Ext}}|_{U_{\alpha\beta}})^{-1} \circ \varphi_\gamma^{\operatorname{Ext}}|_{U_{\alpha\beta}} \circ \begin{pmatrix} 0 & \frac{v(b_{\beta\gamma}^{\operatorname{Bun}})}{b_\alpha^{22}} \\ 0 & 0 \end{pmatrix} \circ (\varphi_\gamma^{\operatorname{Ext}}|_{U_{\alpha\beta}})^{-1} \\ &= \varphi_\beta^{\operatorname{Ext}}|_{U_{\alpha\beta}} \circ \begin{pmatrix} 0 & \frac{v(b_{\alpha\beta}^{\operatorname{Bun}})}{b_\alpha^{22}} \\ 0 & 0 \end{pmatrix} B_{\beta\gamma} \begin{pmatrix} 0 & \frac{v(b_{\beta\gamma}^{\operatorname{Bun}})}{b_\alpha^{22}} \\ 0 & 0 \end{pmatrix} \circ (\varphi_\gamma^{\operatorname{Ext}}|_{U_{\alpha\beta}})^{-1} \\ &= \varphi_\beta^{\operatorname{Ext}}|_{U_{\alpha\beta}} \circ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \circ (\varphi_\gamma^{\operatorname{Ext}}|_{U_{\alpha\beta}})^{-1} = 0. \end{aligned}$$

So we may take a representative of the class (4.18) so that

$$[-\{\operatorname{tr}(u_{\alpha\beta}(v)v_\beta(v')) - \operatorname{tr}(v_\alpha(v)u_{\alpha\beta}(v'))\}]$$

is in  $H^1(C, \Omega_C^1)$ . By using equalities (4.16) and (4.17), we have the following equality

$$(4.19) \quad \operatorname{tr}(u_{\alpha\beta}(v)v_\beta(v')) - \operatorname{tr}(v_\alpha(v)u_{\alpha\beta}(v')) = v(b_{\alpha\beta}^{\operatorname{Bun}})v' \left( \frac{a_\beta^{\operatorname{App}}}{\lambda} \right) - v \left( \frac{a_\alpha^{\operatorname{App}}}{\lambda} \right) v'(b_{\alpha\beta}^{\operatorname{Bun}}).$$

We take bases

$$a^{\operatorname{App}(0)}, a^{\operatorname{App}(1)}, \dots, a^{\operatorname{App}(N_0)} \in H^0(C, L_0 \otimes \Omega_C^1(D))$$

of  $H^0(C, L_0 \otimes \Omega_C^1(D))$  and

$$[\{b_{\alpha\beta}^{\operatorname{App}(0)}\}], [\{b_{\alpha\beta}^{\operatorname{App}(1)}\}], \dots, [\{b_{\alpha\beta}^{\operatorname{App}(N_0)}\}]$$

of  $H^1(C, L_0^{-1}(-D))$  so that these bases give the homogeneous coordinates  $(a_0 : \dots : a_{N_0})$  on  $\mathbb{P}_a^{N_0}$  and  $(b_0 : \dots : b_{N_0})$  on  $\mathbb{P}_b^{N_0}$ . We may assume that these bases are independent of the moduli space  $M_X(L_0, \nabla_{L_0})_0$ . We set

$$a_\alpha^{\operatorname{App}} = a_0 a^{\operatorname{App}(0)}|_{U_\alpha} + a_1 a^{\operatorname{App}(1)}|_{U_\alpha} + \dots + a_{N_0} a^{\operatorname{App}(N_0)}|_{U_\alpha}$$

and

$$b_{\alpha\beta}^{\operatorname{App}} = b_0 b_{\alpha\beta}^{\operatorname{App}(0)} + b_1 b_{\alpha\beta}^{\operatorname{App}(1)} + \dots + b_{N_0} b_{\alpha\beta}^{\operatorname{App}(N_0)}.$$

By (4.19), we have that

$$\begin{aligned}
\mathrm{tr}(u_{\alpha\beta}(v)v_{\beta}(v')) - \mathrm{tr}(v_{\alpha}(v)u_{\alpha\beta}(v')) &= v \left( \sum_{k=0}^{N_0} b_k b_{\alpha\beta}^{\mathrm{App}(k)} \right) v' \left( \frac{\sum_{k=0}^{N_0} a_k a^{\mathrm{App}(k)}|_{U_{\alpha}}}{\lambda} \right) \\
&\quad - v \left( \frac{\sum_{k=0}^{N_0} a_k a^{\mathrm{App}(k)}|_{U_{\alpha}}}{\lambda} \right) v' \left( \sum_{k=0}^{N_0} b_k b_{\alpha\beta}^{\mathrm{App}(k)} \right) \\
&= \left( \sum_{k=0}^{N_0} v(b_k) b_{\alpha\beta}^{\mathrm{App}(k)} \right) \left( \sum_{k=0}^{N_0} v' \left( \frac{a_k}{\lambda} \right) a^{\mathrm{App}(k)}|_{U_{\alpha}} \right) \\
&\quad - \left( \sum_{k=0}^{N_0} v \left( \frac{a_k}{\lambda} \right) a^{\mathrm{App}(k)}|_{U_{\alpha}} \right) \left( \sum_{k=0}^{N_0} v'(b_k) b_{\alpha\beta}^{\mathrm{App}(k)} \right).
\end{aligned}$$

Since  $(b_0 : \cdots : b_{N_0})$  is dual of  $(a_0 : \cdots : a_{N_0})$  with respect to the natural pairing

$$H^0(C, L_0 \otimes \Omega_C^1(D)) \times H^1(C, L_0^{-1}(-D)) \longrightarrow H^1(C, \Omega_C^1) \cong \mathbb{C},$$

we have that

$$\begin{aligned}
&\mathrm{tr}(u_{\alpha\beta}(v)v_{\beta}(v')) - \mathrm{tr}(v_{\alpha}(v)u_{\alpha\beta}(v')) \\
&= \sum_{k=0}^{N_0} v(b_k) v' \left( \frac{a_k}{\lambda} \right) - \sum_{k=0}^{N_0} v' (b_k) v \left( \frac{a_k}{\lambda} \right).
\end{aligned}$$

On the other hand, we have that

$$\lambda = \frac{\langle [\{a_{\alpha}^{\mathrm{App}}\}, \{b_{\alpha\beta}^{\mathrm{Bun}}\}] \rangle}{-\sum_i \theta_{i,-1}^-} = \frac{a_0 b_0 + a_1 b_1 + \cdots + a_{N_0} b_{N_0}}{-\sum_i \theta_{i,-1}^-}.$$

Then we have

$$\begin{aligned}
&H^1(C, \Omega_C^1) \xrightarrow{\cong} \mathbb{C} \\
&[-\{\mathrm{tr}(u_{\alpha\beta}(v)v_{\beta}(v')) - \mathrm{tr}(v_{\alpha}(v)u_{\alpha\beta}(v'))\}] \longmapsto d\eta(v, v').
\end{aligned}$$

This means the statement.  $\square$

## 5. COMPANION NORMAL FORMS FOR AN ELLIPTIC CURVE WITH TWO POLES

In Section 2, we introduced the companion normal form of a rank 2 meromorphic connection with some assumption. The purpose of the present section is to detail the case of an elliptic curve with two simple poles, or with an unramified irregular singularity of order 2. The latter case arises by confluence from the first one, up to some modification in the arguments. We will give explicit description of the companion normal form for an elliptic curve in these cases. Moreover, we will calculate the canonical coordinates introduced in Section 3.5. First we start from construction of the companion normal form  $(\mathcal{O}_C \oplus (\Omega_C^1(D))^{-1}, \nabla_0)$ . Next we will construct a rank 2 meromorphic connection  $(E, \nabla)$  by transforming the companion normal form.

Let  $C$  be the elliptic curve constructed by gluing affine cubic curves

$$U_0 := (y_1^2 - x_1(x_1 - 1)(x_1 - \lambda) = 0) \quad \text{and} \quad U_{\infty} := (y_2^2 - x_2(1 - x_2)(1 - \lambda x_2) = 0)$$

with the relations  $x_1 = x_2^{-1}$  and  $y_1 = y_2 x_2^{-2}$ . We fix some  $t \in \mathbb{C}$  and set  $D = t_1 + t_2$  where  $t_1 = (t, s)$  and  $t_2 = (t, -s)$ , so that  $D$  is the positive part of  $\mathrm{div}(x - t)$ . Let  $q_1, q_2, q_3$  be points on  $C$ :

$$q_j : (x_1, y_1) = (u_j, v_j)$$

for each  $j = 1, 2, 3$ . Now we assume that  $u_j \notin \{0, 1, \lambda, \infty, t\}$  for any  $j$ .

We take trivialization of the line bundle  $(\Omega_C^1(D))^{-1}$  over  $C$  as follows:

$$(5.1) \quad \mathcal{O}_{U_0} \xrightarrow{\sim} (\Omega_C^1(D))^{-1}|_{U_0}; \quad 1 \mapsto \left( \frac{dx_1}{(x_1 - t)y_1} \right)^{-1}$$

and

$$(5.2) \quad \mathcal{O}_{U_\infty} \xrightarrow{\sim} (\Omega_C^1(D))^{-1}|_{U_\infty}; \quad 1 \mapsto \left( \frac{dx_2}{(1 - tx_2)y_2} \right)^{-1}.$$

Then the corresponding transition function  $f_{0\infty}$  is as follows:

$$(5.3) \quad \begin{aligned} f_{\infty 0}: \mathcal{O}_{U_0}|_{U_0 \cap U_\infty} &\xrightarrow{\sim} \mathcal{O}_{U_\infty}|_{U_0 \cap U_\infty} \\ 1 &\mapsto -\frac{1}{x_2}. \end{aligned}$$

**5.1. Definition of a connection  $\nabla_0$  on  $\mathcal{O}_C \oplus (\Omega_C^1(D))^{-1}$ .** For  $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$ , we define 1-forms  $\omega_{12}$ ,  $\omega_{21}$ , and  $\omega_{22}$  as follows:

$$(5.4) \quad \begin{aligned} \omega_{12} &= \sum_{j=1}^3 \frac{\zeta_j}{2} \cdot \frac{y_1 + v_j}{x_1 - u_j} \cdot \frac{dx_1}{y_1} + \left( \frac{A_1 + A_2 y_1}{x_1 - t} + A_3 + A_4 x_1 \right) \frac{dx_1}{y_1} \\ \omega_{21} &:= \frac{1}{x_1 - t} \frac{dx_1}{y_1} \\ \omega_{22} &:= \sum_{j=1}^3 \frac{1}{2} \cdot \frac{y_1 + v_j}{x_1 - u_j} \cdot \frac{dx_1}{y_1} + \left( \frac{B_1 + B_2 y_1}{x_1 - t} + B_3 \right) \frac{dx_1}{y_1}. \end{aligned}$$

Here  $A_1, \dots, A_4 \in \mathbb{C}$  and  $B_1, \dots, B_3 \in \mathbb{C}$  are parameters. Notice that  $\omega_{12} \otimes \omega_{21}$  is a global section of  $(\Omega_C^1)^{\otimes 2}(2D + B)$  and  $\omega_{22}$  is a global section of  $\Omega_C^1(D + B + \infty)$ .

**5.1.1. Fixing the polar parts in the logarithmic case.** We start by analyzing the case where  $t \notin \{0, 1, \lambda, \infty\}$ . In this case, we have  $s \neq 0$ , so  $t_1 \neq t_2$ . We fix complex numbers  $\theta_1^\pm, \theta_2^\pm$  such that  $\sum_{i=1}^2 (\theta_i^+ + \theta_i^-) = -1$ , which is called Fuchs' relation. Now we assume that the eigenvalues of the matrix

$$\text{res}_{t_1} \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

are given by  $\theta_1^+, \theta_1^-$  and the eigenvalues of the matrix

$$\text{res}_{t_2} \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

are given by  $\theta_2^+, \theta_2^-$ . (To be coherent with Definition 2, we should write  $\theta_{1,-1}$  and  $\theta_{2,-1}$  for elements of the Cartan subalgebra, and  $\theta_{1,-1}^\pm$  and  $\theta_{2,-1}^\pm$  for their eigenvalues; however, we drop the subscript  $-1$  for ease of notation, because there are only poles of order 1, so no confusion is possible.) Specifically, these conditions read as

$$(5.5) \quad \text{res}_{(t,s)} \omega_{12} \cdot \text{res}_{(t,s)} \omega_{21} = \theta_1^+ \cdot \theta_1^-, \quad \text{res}_{(t,-s)} \omega_{12} \cdot \text{res}_{(t,-s)} \omega_{21} = \theta_2^+ \cdot \theta_2^-,$$

and

$$(5.6) \quad \text{res}_{(t,s)} \omega_{22} = \theta_1^+ + \theta_1^-, \quad \text{res}_{(t,-s)} \omega_{22} = \theta_2^+ + \theta_2^-.$$

Notice that  $\text{res}_{(u_j, v_j)} \omega_{22} = 1$  for each  $j$ . By the residue theorem,  $\text{res}_\infty \omega_{22} = -2$ . By the assumption (5.5) and (5.6), we may determine the parameters  $A_1, A_2, B_1$ , and  $B_2$ .

**Lemma 32.** *Let complex numbers  $\theta_1^\pm, \theta_2^\pm$  satisfying Fuchs' relation be given. Then, there exist unique values of the parameters  $A_1, A_2, B_1$ , and  $B_2$  such that (5.5) and (5.6) are fulfilled. Moreover, these parameter values are independent of  $u_1, u_2, u_3, \zeta_1, \zeta_2$ , and  $\zeta_3$ . So the polar parts of  $\omega_{12}, \omega_{21}$ , and  $\omega_{22}$  at  $t_i$  are independent of  $u_1, u_2, u_3, \zeta_1, \zeta_2$ , and  $\zeta_3$ .*

*Proof.* By the equalities (5.5), we have

$$\frac{A_1 + A_2 s}{s} \cdot \frac{1}{s} = \theta_1^+ \cdot \theta_1^- \quad \text{and} \quad \frac{A_1 - A_2 s}{-s} \cdot \frac{1}{-s} = \theta_2^+ \cdot \theta_2^-.$$

By the equalities in (5.6), we have

$$\frac{B_1 + B_2 s}{s} = \theta_1^+ + \theta_1^- \quad \text{and} \quad \frac{B_1 - B_2 s}{-s} = \theta_2^+ + \theta_2^-.$$

By these equalities,  $A_1, A_2, B_1$ , and  $B_2$  are determined, and  $A_1, A_2, B_1$ , and  $B_2$  are independent of  $u_1, u_2, u_3, \zeta_1, \zeta_2$ , and  $\zeta_3$ . It is clear that the polar parts of  $\omega_{12}, \omega_{21}$ , and  $\omega_{22}$  at  $t_i$  are independent of  $u_1, u_2, u_3, \zeta_1, \zeta_2$ , and  $\zeta_3$ .  $\square$

5.1.2. *Fixing the polar part in the irregular case.* We now study the situation  $t \in \{0, 1, \lambda, \infty\}$ . For sake of concreteness, we let  $t = 0$ , the other cases being similar. Then,  $s = 0$  and  $t_1 = t_2$ , so the divisor  $D$  is reduced of length 2. A local holomorphic coordinate of the elliptic curve  $C$  in a neighbourhood of  $t_1$  is given by  $y_1$ .

We fix  $\theta_{-2}^\pm, \theta_{-1}^\pm \in \mathbb{C}$  so that  $\theta_{-2}^+ \neq \theta_{-2}^-$  and set  $\theta_{-1}^- = -1 - \theta_{-1}^+$ . (To be coherent with Definition 2, we should write  $\theta_{1,-2}$  and  $\theta_{1,-1}$  for elements of the Cartan subalgebra, and  $\theta_{1,-2}^\pm$  and  $\theta_{1,-1}^\pm$  for their eigenvalues; however, we omit the subscript 1 for ease of notation, because there is only one singular point, so no confusion is possible.)

**Lemma 33.** *Fix  $\theta_{-2}^\pm, \theta_{-1}^\pm$  as above. Then, there exist unique values  $A_1, A_2, B_1, B_2 \in \mathbb{C}$  such that the eigenvalues of*

$$\text{res} \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

*admit Laurent expansions of the form*

$$\left( \theta_{-2}^\pm \frac{1}{y_1^2} + \theta_{-1}^\pm \frac{1}{y_1} + O(1) \right) \otimes dy_1.$$

*Moreover, the values of the solutions are independent of  $u_i, \zeta_i$ .*

*Proof.* By the inverse function theorem, there exists an analytic open subset  $U \subset \mathbb{C}$  and a holomorphic function  $h: U \rightarrow \mathbb{C}$  satisfying  $h(0) = 0$  such that  $C$  is given by the explicit equation  $x_1 = h(y_1^2)$ . It is obvious that this function  $h$  is independent of the choice of  $u_i, \zeta_i$ , and it is easy to see that  $h'(0) = \frac{1}{\lambda} \neq 0$ . From the defining equation of  $C$  we get

$$\frac{dx_1}{y_1} = \frac{2 dy_1}{3x_1^2 - 2(1 + \lambda)x_1 + \lambda},$$

so  $\frac{dx_1}{y_1}$  is a holomorphic 1-form around  $t_1$ . Moreover,

$$\frac{dx_1}{x_1 y_1} = \frac{dy_1}{y_1^2} g(y_1^2)$$

for some holomorphic function  $g: U \rightarrow \mathbb{C}$  satisfying  $g(0) = 2$ . The polar parts of the coefficients can be separated as

$$\begin{aligned}\omega_{12} &= (A_1 + A_2 y_1) \frac{dx_1}{x_1 y_1} + O(1) = 2(A_1 + A_2 y_1) \frac{dy_1}{y_1^2} + O(1) \\ \omega_{21} &= 2 \frac{dy_1}{y_1^2} + O(1) \\ \omega_{22} &= (B_1 + B_2 y_1) \frac{dx_1}{x_1 y_1} + O(1) = 2(B_1 + B_2 y_1) \frac{dy_1}{y_1^2} + O(1).\end{aligned}$$

Now, the sum of the eigenvalues must be

$$(\theta_{-2}^+ + \theta_{-2}^-) \frac{1}{y_1^2} + (\theta_{-1}^+ + \theta_{-1}^-) \frac{1}{y_1}.$$

These conditions determine

$$B_1 = \frac{1}{2}(\theta_{-2}^+ + \theta_{-2}^-), \quad B_2 = \frac{1}{2}(\theta_{-1}^+ + \theta_{-1}^-) = -\frac{1}{2}.$$

Moreover, we have

$$-\omega_{12}\omega_{21} = -4(A_1 + A_2 y_1) \frac{(dy_1)^{\otimes 2}}{y_1^4} + O\left(\frac{1}{y_1^2}\right).$$

On the other hand, the product of the eigenvalues must have the expansion (up to a global factor  $(dy_1)^{\otimes 2}$ )

$$\theta_{-2}^+ \theta_{-2}^- \frac{1}{y_1^4} + (\theta_{-2}^+ \theta_{-1}^- + \theta_{-2}^- \theta_{-1}^+) \frac{1}{y_1^3}.$$

These condition then determine

$$A_1 = -\frac{1}{4}\theta_{-2}^+ \theta_{-2}^-, \quad A_3 = -\frac{1}{4}(\theta_{-2}^+ \theta_{-1}^- + \theta_{-2}^- \theta_{-1}^+).$$

This finishes the proof.  $\square$

5.1.3. *Construction of the connection.* We define

$$\begin{aligned}\beta: (\Omega_C^1(D))^{-1} &\longrightarrow \Omega_C^1(D+B) && (\mathcal{O}_C\text{-morphism}) \\ \delta: (\Omega_C^1(D))^{-1} &\longrightarrow (\Omega_C^1(D))^{-1} \otimes \Omega_C^1(D+B) && (\text{connection}) \\ \gamma: \mathcal{O}_C &\longrightarrow (\Omega_C^1(D))^{-1} \otimes \Omega_C^1(D) && (\mathcal{O}_C\text{-morphism})\end{aligned}$$

by using the trivializations (5.1) and (5.2) of  $(\Omega_C^1(D))^{-1}$  as follows:

$$\begin{aligned}\beta &= \begin{cases} \omega_{12}: \mathcal{O}_{U_0} \rightarrow \mathcal{O}_{U_0} \otimes \Omega_C^1(D+B)|_{U_0} \\ \text{id} \circ \omega_{12} \circ f_{\infty 0}^{-1}: \mathcal{O}_{U_\infty} \rightarrow \mathcal{O}_{U_\infty} \otimes \Omega_C^1(D+B)|_{U_\infty}, \end{cases} \\ \delta &= \begin{cases} d + \omega_{22}: \mathcal{O}_{U_0} \rightarrow \mathcal{O}_{U_0} \otimes \Omega_C^1(D+B)|_{U_0} \\ d + f_{\infty 0} \circ \omega_{22} \circ f_{\infty 0}^{-1} + f_{\infty 0} \circ d f_{\infty 0}^{-1}: \mathcal{O}_{U_\infty} \rightarrow \mathcal{O}_{U_\infty} \otimes \Omega_C^1(D+B)|_{U_\infty}, \end{cases} \\ \gamma &:= \begin{cases} \omega_{21}: \mathcal{O}_{U_0} \rightarrow \mathcal{O}_{U_0} \otimes \Omega_C^1(D+B)|_{U_0} \\ f_{\infty 0} \circ \omega_{21} \circ \text{id}: \mathcal{O}_{U_\infty} \rightarrow \mathcal{O}_{U_\infty} \otimes \Omega_C^1(D+B)|_{U_\infty}. \end{cases}\end{aligned}$$

Here  $f_{\infty 0}$  is the transition function of  $(\Omega_C^1(D))^{-1}$  described in (5.3). Notice that

$$f_{\infty 0} \circ \omega_{22} \circ f_{\infty 0}^{-1} + f_{\infty 0} \circ d f_{\infty 0}^{-1} = \omega_{22} + \frac{dx_2}{x_2},$$

which is holomorphic at  $\infty \in C$ , since we have  $\text{res}_\infty \omega_{22} = -2$ . We define a connection as follows:

$$(5.7) \quad \nabla_0 := d + \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} : \mathcal{O}_C \oplus (\Omega_C^1(D))^{-1} \longrightarrow (\mathcal{O}_C \oplus (\Omega_C^1(D))^{-1}) \otimes \Omega_C^1(D+B),$$

which is the companion normal form. Remark that

$$\text{res}_{q_j}(\nabla_0) = \begin{pmatrix} 0 & \zeta_j \\ 0 & 1 \end{pmatrix}$$

for  $j = 1, 2, 3$ .

**Lemma 34.** *The fact that  $\nabla_0$  has apparent singular points at  $q_1, q_2, q_3$  imposes 3 linear conditions on  $A_3, A_4, B_3$  in terms of spectral data, and  $((u_j, v_j), \zeta_j)$ 's; we can uniquely determine  $A_3, A_4, B_3$  from these conditions if, and only if, we have*

$$(5.8) \quad \det \begin{pmatrix} 1 & u_1 & \zeta_1 \\ 1 & u_2 & \zeta_2 \\ 1 & u_3 & \zeta_3 \end{pmatrix} \neq 0.$$

*Proof.* It is just Lemma 7 specified to the present elliptic case with 2 poles. We set

$$(5.9) \quad C_j = \sum_{j' \in \{1,2,3\} \setminus \{j\}} \frac{\zeta_{j'} - \zeta_j}{2} \cdot \frac{v_j + v_{j'}}{u_j - u_{j'}} + \frac{A_1 + A_2 v_j - \zeta_j(B_1 + B_2 v_j) - \zeta_j^2}{u_j - t}.$$

We denote by  $((a_j)_j, (b_j)_j, (c_j)_j)$  the  $3 \times 3$ -matrix

$$((a_j)_j, (b_j)_j, (c_j)_j) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

The condition where  $q_1, q_2, q_3$  are apparent singularities means that

$$(5.10) \quad ((1)_j, (u_j)_j, (-\zeta_j)_j) \begin{pmatrix} A_3 \\ A_4 \\ B_3 \end{pmatrix} = - \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

By Cramer's rule, the parameters  $A_3, A_4, B_3$  of the family of connections  $\nabla_0$  are uniquely determined

$$A_3 = - \frac{\det((C_j)_j, (u_j)_j, (\zeta_j)_j)}{\det(((1)_j, (u_j)_j, (\zeta_j)_j))} \quad A_4 = - \frac{\det(((1)_j, (C_j)_j, (\zeta_j)_j))}{\det(((1)_j, (u_j)_j, (\zeta_j)_j))}$$

$$B_3 = \frac{\det(((1)_j, (u_j)_j, (C_j)_j))}{\det(((1)_j, (u_j)_j, (\zeta_j)_j)},$$

if and only if (5.8). □

**Lemma 35.** *We have:*

$$\det \begin{pmatrix} 1 & u_1 & \zeta_1 \\ 1 & u_2 & \zeta_2 \\ 1 & u_3 & \zeta_3 \end{pmatrix} = 0$$

*if, and only if,  $E$  is not stable.*

*Proof.* The vanishing of the determinant gives that  $\zeta_j = \sigma(q_j)$  for a global section  $\sigma \in H^0(C, \Omega_C^1(D))$ . In other words, the quasi-parabolic structure on  $E_0$  given over each  $q_j$  by the eigenvectors corresponding to eigenvalue 1 lie on a subbundle  $(\Omega_C^1(D))^{-1} \subset E_0$ . After elementary transformations at each  $q_j$ , we get  $L \subset E$  with  $\deg(L) = 1$  (in fact  $L = \det(E)$ ). □

5.2. **Definition of a rank 2 vector bundle  $E$ .** We set

$$\tilde{U}_0 := U_0 \setminus \{q_1, q_2, q_3\} \quad \text{and} \quad \tilde{U}_\infty := U_\infty \setminus \{q_1, q_2, q_3\}.$$

We take an analytic open subsets  $\tilde{U}_{q_j}$  ( $j = 1, 2, 3$ ) of  $C$  such that  $q_j \in \tilde{U}_{q_j}$  and  $\tilde{U}_{q_j}$  are small enough. In particular,  $(u_j, -v_j) \notin \tilde{U}_{q_j}$ . On  $\tilde{U}_{q_j}$ , the apparent singular point  $q_j$  is defined by  $x_1 - u_j = 0$ . We have an open covering  $(\tilde{U}_k)_{k \in \{0, 1, q_1, q_2, q_3\}}$  of  $C$ . We define transition functions  $B_{k_1 k_2}$  ( $k_1, k_2 \in \{0, 1, q_1, q_2, q_3\}$ ) as follows:

$$B_{0q_j} := \begin{pmatrix} 1 & \frac{\zeta_j}{x_1 - u_j} \\ 0 & \frac{1}{x_1 - u_j} \end{pmatrix} : \mathcal{O}_{\tilde{U}_{q_j}}^{\oplus 2} |_{\tilde{U}_0 \cap \tilde{U}_{q_j}} \xrightarrow{\sim} \mathcal{O}_{\tilde{U}_0}^{\oplus 2} |_{\tilde{U}_0 \cap \tilde{U}_{q_j}};$$

$$B_{0\infty} := \begin{pmatrix} 1 & 0 \\ 0 & -x_2 \end{pmatrix} : \mathcal{O}_{\tilde{U}_\infty}^{\oplus 2} |_{\tilde{U}_0 \cap \tilde{U}_\infty} \xrightarrow{\sim} \mathcal{O}_{\tilde{U}_0}^{\oplus 2} |_{\tilde{U}_0 \cap \tilde{U}_\infty}.$$

Then we have a vector bundle

$$E = \left( (\tilde{U}_k)_{k \in \{0, 1, q_1, q_2, q_3\}}, (B_{k_1 k_2})_{k_1, k_2 \in \{0, 1, q_1, q_2, q_3\}} \right),$$

where  $E$  is trivial on each  $\tilde{U}_k$  and the transition function from  $\tilde{U}_{k_2}$  to  $\tilde{U}_{k_1}$  is  $B_{k_1 k_2}$ .

5.3. **Definition of a connection  $\nabla$  on  $E$ .** We define matrices  $A_0, A_{q_j}, A_\infty$  as follows:

$$A_0 := \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}, \quad A_\infty := \begin{pmatrix} 0 & -x_2 \omega_{12} \\ -\frac{\omega_{21}}{x_2} & \omega_{22} + \frac{dx_2}{x_2} \end{pmatrix},$$

$$A_{q_j} := \begin{pmatrix} \omega_{11}^{(j)} & \frac{\omega_{12}^{(j)}}{x_1 - u_j} \\ (x_1 - u_j) \omega_{21} & \omega_{22}^{(j)} \end{pmatrix}.$$

The 1-form  $\omega_{12}$ ,  $\omega_{21}$ , and  $\omega_{22}$  are defined in (5.4). The 1-form  $\omega_{12}^{(j)}$ ,  $\omega_{21}^{(j)}$ , and  $\omega_{22}^{(j)}$  are defined as follows:

$$\omega_{11}^{(j)} = -\frac{\zeta_j}{x_1 - t} \cdot \frac{dx_1}{y_1},$$

$$\omega_{12}^{(j)} = \sum_{j' \in \{1, 2, 3\} \setminus \{j\}} \frac{\zeta_{j'} - \zeta_j}{2} \cdot \frac{y_1 + v_{j'}}{x_1 - u_{j'}} \cdot \frac{dx_1}{y_1}$$

$$+ \left( \frac{A_1 + A_2 y_1 - \zeta_j (B_1 + B_2 y_1) - \zeta_j^2}{x_1 - t} + A_3 + A_4 x_1 - \zeta_j B_3 \right) \frac{dx_1}{y_1},$$

$$\omega_{22}^{(j)} = \frac{1}{2} \cdot \frac{-y_1 + v_j}{x_1 - u_j} \cdot \frac{dx_1}{y_1} + \sum_{j' \in \{1, 2, 3\} \setminus \{j\}} \frac{1}{2} \cdot \frac{y_1 + v_{j'}}{x_1 - u_{j'}} \cdot \frac{dx_1}{y_1}$$

$$+ \left( \frac{B_1 + B_2 y_1}{x_1 - t} + B_3 + \frac{\zeta_j}{x_1 - t} \right) \frac{dx_1}{y_1}.$$

**Proposition 36.** • The  $(1, 2)$ -entry of  $A_{q_j}$  is a section of  $\Omega_C^1(D)|_{\tilde{U}_{q_j}}$  for each  $j = 1, 2, 3$ .

• We define a local connection on each  $\tilde{U}_k$  ( $k \in \{0, 1, q_1, q_2, q_3\}$ ) by

$$\begin{cases} d + A_0 : \mathcal{O}_{\tilde{U}_0}^{\oplus 2} \longrightarrow \mathcal{O}_{\tilde{U}_0}^{\oplus 2} \otimes \Omega_C^1(D)|_{\tilde{U}_0} & \text{on } \tilde{U}_0 \\ d + A_{q_j} : \mathcal{O}_{\tilde{U}_{q_j}}^{\oplus 2} \longrightarrow \mathcal{O}_{\tilde{U}_{q_j}}^{\oplus 2} \otimes \Omega_C^1(D)|_{\tilde{U}_{q_j}} & \text{on } \tilde{U}_{q_j} \\ d + A_\infty : \mathcal{O}_{\tilde{U}_\infty}^{\oplus 2} \longrightarrow \mathcal{O}_{\tilde{U}_\infty}^{\oplus 2} \otimes \Omega_C^1(D)|_{\tilde{U}_\infty} & \text{on } \tilde{U}_\infty. \end{cases}$$

Then we can glue these local connections. So we have a global connection  $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$  on  $E$ .

*Proof.* Since  $A_3, A_4, B_3$  are determined so that these parameters satisfy the condition (5.10), we have

$$\omega_{12}^{(j)}|_{q_j} = (C_j + A_3 + A_4 u_j - \zeta_j B_3) \frac{dx_1|_{q_j}}{v_j} = 0.$$

Here,  $C_j$  is in (5.9). So  $\frac{\omega_{12}^{(j)}}{x_1 - u_j}$  has no pole at  $q_j$  for each  $j = 1, 2, 3$ . Since we have

$$B_{k_1 k_2}^{-1} A_{k_1} B_{k_1 k_2} + B_{k_1 k_2}^{-1} dB_{k_1 k_2} = A_{k_2}$$

for each  $k_1, k_2 \in \{0, \infty, q_1, q_2, q_3\}$ , the connection  $\nabla$  acting on  $E$  is defined globally.  $\square$

**Remark 37.** By Definition 13 in Section 3.3, we have trivializations of  $E$ . On  $C \setminus \{t_1, t_2\}$ , the trivializations in Definition 13 coincide with the trivializations described in the present section. We have defined the trivialization in Definition 13 at  $t_i$  ( $i = 1, 2$ ) so that the residue matrix (respectively, the polar part in the reduced case) is a diagonal matrix. On the other hand, by the trivializations described in the present section, the residue matrix at  $t_i$  ( $i = 1, 2$ ) (respectively, the polar part) is not a diagonal matrix. The reason why the residue matrix at  $t_i$  ( $i = 1, 2$ ) is a diagonal matrix is that the corresponding description of the variation (3.11) satisfies the compatibility conditions of the quasi-parabolic structure in  $\mathcal{F}^0$  and  $\mathcal{F}^1$  of (3.1). On the other hand, now we are interested in behavior of the connection  $\nabla$  around  $q_j$  ( $j = 1, 2, 3$ ). So now we do not consider the diagonalization of the residue matrices at  $t_i$  ( $i = 1, 2$ ) (respectively, of the polar part when  $D$  is reduced).

**5.4. Canonical coordinates.** We will calculate the canonical coordinates introduced in Section 3.5. For the transition functions  $B_{k_1 k_2}$  ( $k_1, k_2 \in \{0, 1, q_1, q_2, q_3\}$ ) of  $E$ , we have transition functions of  $\det(E)$  as follows:

$$\det(B_{0q_j}) = \frac{1}{x_1 - u_j} : \mathcal{O}_{\tilde{U}_{q_j}}|_{\tilde{U}_0 \cap \tilde{U}_{q_j}} \xrightarrow{\sim} \mathcal{O}_{\tilde{U}_0}|_{\tilde{U}_0 \cap \tilde{U}_{q_j}};$$

$$\det(B_{0\infty}) = -x_2 : \mathcal{O}_{\tilde{U}_\infty}|_{\tilde{U}_0 \cap \tilde{U}_\infty} \xrightarrow{\sim} \mathcal{O}_{\tilde{U}_0}|_{\tilde{U}_0 \cap \tilde{U}_\infty}.$$

So we have a cocycle  $(\det(B_{k_1 k_2}))_{k_1, k_2 \in \{0, 1, q_1, q_2, q_3\}}$ , which gives a class of  $H^1(C, \mathcal{O}_C^*)$ . We have

$$d \log(\det(B_{0q_j})) = -\frac{dx_1}{x_1 - u_j} \quad \text{and} \quad d \log(\det(B_{0\infty})) = \frac{dx_2}{x_2},$$

and these 1-forms give a class of  $H^1(C, \Omega_C^1)$ . We denote by  $c_1$  and  $\mathbf{\Omega}(D, c_1)$  the class of  $H^1(C, \Omega_C^1)$  and the total space of the twisted cotangent bundle corresponding to  $c_1$ , respectively. We have the following description of  $\text{tr}(\nabla)$ :

$$\text{tr}(\nabla) = \begin{cases} d + \omega_{22} : \mathcal{O}_{\tilde{U}_0} \longrightarrow \mathcal{O}_{\tilde{U}_0} \otimes \Omega_C^1(D)|_{\tilde{U}_0} & \text{on } \tilde{U}_0 \\ d + \omega_{11}^{(j)} + \omega_{22}^{(j)} : \mathcal{O}_{\tilde{U}_{q_j}} \longrightarrow \mathcal{O}_{\tilde{U}_{q_j}}^{\oplus 2} \otimes \Omega_C^1(D)|_{\tilde{U}_{q_j}} & \text{on } \tilde{U}_{q_j} \\ d + \omega_{22} + \frac{dx_2}{x_2} : \mathcal{O}_{\tilde{U}_\infty} \longrightarrow \mathcal{O}_{\tilde{U}_\infty}^{\oplus 2} \otimes \Omega_C^1(D)|_{\tilde{U}_\infty} & \text{on } \tilde{U}_\infty. \end{cases}$$

Notice that we have

$$\begin{aligned} \omega_{11}^{(j)} + \omega_{22}^{(j)} &= \omega_{22} + d \log(\det(B_{0q_j})), \quad \text{and} \\ \omega_{22} + \frac{dx_2}{x_2} &= \omega_{22} + d \log(\det(B_{0\infty})). \end{aligned}$$

So these connection matrices of  $\text{tr}(\nabla)$  give an explicit global section of  $\Omega(D, c_1) \rightarrow C$ . We consider a section of  $\Omega(D, c_1)|_{\tilde{U}_{q_j}} \rightarrow \tilde{U}_{q_j}$

$$\frac{\zeta_j dx_1}{(x_1 - t)y_1} + \omega_{11}^{(j)} + \omega_{22}^{(j)}.$$

For this section on  $\tilde{U}_{q_j}$ , we define  $p_j$  ( $j = 1, 2, 3$ ) by

$$p_j = \text{res}_{q_j} \left( \frac{\zeta_j}{x_1 - u_j} \cdot \frac{dx_1}{(x_1 - t)y_1} \right) + \text{res}_{q_j} \left( \frac{\omega_{11}^{(j)} + \omega_{22}^{(j)}}{x_1 - u_j} \right).$$

Then we have a map

$$(E, \nabla) \mapsto (u_1, u_2, u_3, \zeta_1, \zeta_2, \zeta_3) \mapsto (u_1, u_2, u_3, p_1, p_2, p_3),$$

where

$$p_j = \frac{\zeta_j}{(u_j - t)v_j} - \frac{K'(u_j)}{4v_j^2} + \sum_{j' \in \{1, 2, 3\} \setminus \{j\}} \frac{1}{2} \cdot \frac{v_j + v_{j'}}{u_j - u_{j'}} \cdot \frac{1}{v_j} + \left( \frac{B_1 + B_2 v_j}{u_j - t} + B_3 \right) \frac{1}{v_j}$$

Here we set  $K(x_1) := x_1(x_1 - 1)(x_1 - \lambda)$ . Notice that  $B_1$  and  $B_2$  are determined by Lemma 32 and  $B_3$  is determined by Lemma 34. Notice that  $B_3$  depends on  $\zeta_1, \zeta_2$  and  $\zeta_3$ . The symplectic structure is  $\sum_{j=1}^3 dp_j \wedge du_j$  by Theorem 20.

## 6. CANONICAL COORDINATES REVISED AND ANOTHER PROOF FOR BIRATIONALITY

In this section, we will give another proof of Proposition 17. For simplicity, we will consider the cases where  $D$  is a reduced effective divisor. Let  $(E, \nabla) \in M_X^0$  be a connection on a fixed irregular curve  $X = (C, D, \{z_i\}_{i \in I}, \{\theta_i\}_{i \in I}, \theta_{res})$  with genericity conditions as before.

We set  $D = t_1 + \cdots + t_n$  and the connection is given by

$$\nabla: E \longrightarrow E \otimes \Omega_C^1(D).$$

In this section, we assume that  $g = g(C) \geq 1$  and  $n \geq 1$  as in the previous sections. Moreover if  $g = g(C) = 1$ , we assume that  $n \geq 2$ .

Note that we have the unique extension

$$(6.1) \quad 0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L_0 \longrightarrow 0$$

with  $L_0 = \det(E)$ . Moreover for  $(E, \nabla) \in M_X^0$  we have  $\deg L_0 = 2g - 1$  and  $\dim_{\mathbb{C}} H^0(C, E) = 1$ . Then we can define apparent singularities  $q_1, \dots, q_N \in C$  where  $N = 4g - 3 + n$ . Since  $\deg D = 2g - 2 + n \geq 1$  and  $\deg L_0 = 2g - 1 \geq 1$ , we see that  $\dim_{\mathbb{C}} H^0(C, \Omega_C^1(D)) = g - 1 + n \geq 2$ . We can choose  $\gamma \in H^0(C, \Omega_C^1(D))$  and  $s \in H^0(C, L_0)$  whose zeros are given by

$$\{\gamma = 0\} = \{c_1, \dots, c_{2g-2+n}\} \quad \text{and} \quad \{s = 0\} = \{u_1, \dots, u_{2g-1}\}.$$

We assume the following genericity conditions:

- (1)  $u_{i_1} \neq u_{i_2}$  (for  $i_1 \neq i_2$ ), and  $c_{k_1} \neq c_{k_2}$  (for  $k_1 \neq k_2$ );
- (2)  $\{u_1, \dots, u_{2g-1}\} \cap \{c_1, \dots, c_{2g-2+n}\} = \emptyset$ ;
- (3)  $\{q_1, \dots, q_N\} \cap \{u_1, \dots, u_{2g-1}, c_1, \dots, c_{2g-2+n}\} = \emptyset$ .

Set

$$U_0 = C \setminus \{u_1, \dots, u_{2g-1}, c_1, \dots, c_{2g-2+n}\}.$$

Moreover we take small an analytic neighborhood  $U_i$  of  $u_i$  for  $1 \leq i \leq 2g-1$  and  $U_{2g-1+k}$  of  $c_k$  for  $1 \leq k \leq 2g-2+n$ . For  $i = 1, \dots, 4g-3+n$ , we can identify  $U_i$  with a unit disc  $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$  with the origin corresponding to  $u_i$  ( $1 \leq i \leq 2g-1$ ) and  $c_{i-2g+1}$  ( $2g \leq i \leq 4g-3+n$ ). We can assume that  $U_{i_1} \cap U_{i_2} = \emptyset$  for  $i_1 \neq i_2$ ,  $i_1, i_2 \geq 1$ . Note that since  $U_0$  is an affine variety and  $U_0 \cap U_i \cong \Delta \setminus \{0\}$  for  $i = 1, \dots, 4g-3+n$ , the covering  $C = U_0 \cup U_1 \cup \dots \cup U_{4g-3+n}$  gives a Stein covering of  $C$ . For  $0 \leq i \leq 4g-3+n$ , we have nonzero sections  $e_1^{(i)} \in \mathcal{O}_{U_i}$ ,  $e_2^{(i)} \in (L_0)|_{U_i}$  giving trivializations of  $E$  on  $U_i$  respectively:

$$E|_{U_i} \simeq \mathcal{O}|_{U_i} e_1^{(i)} \oplus \mathcal{O}|_{U_i} e_2^{(i)}.$$

Moreover we have a transition matrix  $H_{0i}$  on  $U_0 \cap U_i$  of the form

$$(6.2) \quad H_{0i} = \begin{pmatrix} 1 & h_{0i} \\ 0 & g_{0i} \end{pmatrix}$$

satisfying

$$(6.3) \quad (e_1^{(i)}, e_2^{(i)}) = (e_1^{(0)}, e_2^{(0)}) H_{0i} = (e_1^{(0)}, h_{0i} e_1^{(0)} + g_{0i} e_2^{(0)}).$$

Here  $\{h_{0i}\}_i \in \text{Ext}^1(L_0, \mathcal{O}_C) \cong H^1(C, L_0^{-1})$  corresponds to the extension class of (6.1) and  $\{g_{0i}\}_i \in H^1(C, \mathcal{O}_C^*)$  gives the transition function of  $L_0 = \det(E)$ . With these trivializations we have connection matrices  $A^{(i)}$ :

$$(6.4) \quad \nabla(e_1^{(i)}, e_2^{(i)}) = (e_1^{(i)}, e_2^{(i)}) A^{(i)}$$

of the form

$$(6.5) \quad A^{(i)} = \begin{pmatrix} a_{11}^{(i)} \gamma_i & a_{12}^{(i)} \gamma_i \\ a_{21}^{(i)} \gamma_i & a_{22}^{(i)} \gamma_i \end{pmatrix}.$$

Here  $a_{kl}^{(i)} \in \Gamma(U_i, \mathcal{O}_{U_i})$  and  $\gamma_i \in \Gamma(U_i, \Omega_{U_i}^1(D))$ . We set  $\gamma_0 = \gamma|_{U_0}$  as above.

From (6.3) and (6.4), we can verify the following

**Lemma 38.** *For  $1 \leq i \leq 4g-3+n$ , on  $U_0 \cap U_i$ , we gave*

$$(6.6) \quad A^{(i)} = H_{0i}^{-1} A^{(0)} H_{0i} + H_{0i}^{-1} dH_{0i}.$$

*Specifically, we have the following identities:*

$$(6.7) \quad a_{21}^{(i)} \gamma_i = a_{21}^{(0)} \gamma_0 g_{0i}^{-1}; \quad \text{and}$$

$$(6.8) \quad a_{22}^{(i)} \gamma_i = a_{22}^{(0)} \gamma_0 + a_{21}^{(0)} \gamma_0 h_{0i} g_{0i}^{-1} + \frac{dg_{0i}}{g_{0i}}.$$

The identity (6.7) shows that  $a_{21}^{(i)} \gamma_i$  defines a section of  $H^0(C, \Omega_1(D) \otimes L_0)$  and the zeros of this section are nothing but the apparent singularities  $q_1, \dots, q_N$ . Evaluating the identity (6.8) at  $q_j$  ( $j = 1, \dots, N$ ), we then have

$$(6.9) \quad (a_{22}^{(i)} \gamma_i)_{q_j} = (a_{22}^{(0)} \gamma_0)_{q_j} + \left( \frac{dg_{0i}}{g_{0i}} \right)_{q_j}$$

Noting that the cohomology class of the cocycle  $\left\{ \frac{dg_{0i}}{g_{0i}} \right\}_i$  corresponds to  $c_d = c_1(L_0)$ , from (6.9), we have the following

**Proposition 39.** *For each  $0 \leq j \leq N$ , the data  $(E, \nabla) \in M_X^0$  defines  $N$  points  $(q_j, \tilde{p}_j)$  on the total space of  $\Omega(D, c_d)$  by the formula*

$$(6.10) \quad \tilde{p}_j = (a_{22}^{(0)} \gamma_0)_{q_j} \in \Omega_C^1(D, c_d)|_{q_j}$$

The above definition of  $\tilde{p}_j$  does not depend on the choice of the sections  $s \in H^0(C, L_0)$  and  $\gamma \in H^0(C, \Omega_C^1(D))$  and defines the same map as in Definition 16:

$$(6.11) \quad f_{\text{App}}: M_X^0 \longrightarrow \text{Sym}^N(\Omega(D, c_d)).$$

Now we consider  $q_j$  as a local coordinate near  $q_j$  and we write  $\gamma = c(q_j) dq_j$  for some local holomorphic function  $c(q_j)$ . Then we have

$$\tilde{p}_j = p_j dq_j$$

with

$$p_j = a_{22}^{(0)}(q_j) c(q_j).$$

As we have proved in Theorem 20, the map  $f_{\text{App}}$  is symplectic.

**6.1. From a connection to a Higgs field.** Keeping the notation, let us consider the section  $s \in H^0(C, L_0)$  as before, and set  $s^{(0)} = s$ . Take trivialization of  $L_{0|U_i}$  over  $U_i$  we have a holomorphic function  $s^{(i)} \in \Gamma(U_i, \mathcal{O}_{U_i})$  such that

$$s^{(0)} = g_{0i} s^{(i)}.$$

Note that  $s^{(i)}$  has zeros at  $u_i \in U_i$  for  $1 \leq i \leq 2g-1$ . Set  $D(s) = u_1 + \cdots + u_{2g-1}$ . We can show that

**Lemma 40.** *There exists a connection*

$$\nabla_1: E \longrightarrow E \otimes \Omega^1(D(s))$$

such that for each  $0 \leq i \leq N = 4g-3+n$ , on  $U_i$  it has the form

$$\nabla_1^{(i)} = d + S^{(i)} = d + \begin{pmatrix} 0 & \frac{\beta_i}{s^{(i)}} \\ 0 & \frac{ds^{(i)}}{s^{(i)}} \end{pmatrix}$$

with respect to the trivialization  $(e_1^{(i)}, e_2^{(i)})$ . Here  $\beta_i \in \Gamma(U_i, \Omega_{U_i}^1)$ .

*Proof.* Since  $s^{(0)} = g_{0i} s^{(i)}$ , one has

$$\frac{ds^{(0)}}{s^{(0)}} = \frac{dg_{0i}}{g_{0i}} + \frac{ds^{(i)}}{s^{(i)}}$$

in  $U_{0i} = U_0 \cap U_i$ . The compatibility condition for connection matrices  $S^{(i)}$  is

$$(6.12) \quad S^{(i)} = H_{0i}^{-1} S^{(0)} H_{0i} + H_{0i}^{-1} dH_{0i}.$$

The right hand side of (6.12) is

$$(6.13) \quad \begin{pmatrix} 0 & g_{0i} \frac{\beta_0}{s^{(0)}} - h_{0i} \left( \frac{ds^{(0)}}{s^{(0)}} - \frac{dg_{0i}}{g_{0i}} \right) - dh_{0i} \\ 0 & \frac{ds^{(0)}}{s^{(0)}} - \frac{dg_{0i}}{g_{0i}} \end{pmatrix}$$

Since  $\{h_{0i}\}_i$  is a class in  $H^1(C, L_0^{-1})$  and  $s \in H^0(C, L_0)$ , the class  $\{s^{(i)} h_{0i}\}_i$  defines a class in  $H^1(C, \mathcal{O}_C)$ . Then, by the Hodge theory, the derivative  $\{d(s^{(i)} h_{0i})\}_i \in H^1(C, \Omega_C^1)$  vanishes, so there exist  $\beta_i \in \Gamma(U_i, \Omega_{U_i}^1)$  such that

$$d(s^{(i)} h_{0i}) = \beta_0 - \beta_i.$$

Choose such  $\beta_i$ 's for the formula. Then we have

$$dh_{0i} = -h_{0i} \frac{ds^{(i)}}{s^{(i)}} + g_{0i} \frac{\beta_0}{s^{(0)}} - \frac{\beta_i}{s^{(i)}}.$$

Then the right hand side of (6.13) becomes

$$\begin{pmatrix} 0 & \frac{\beta_i}{s^{(i)}} \\ 0 & \frac{ds^{(i)}}{s^{(i)}} \end{pmatrix}$$

as desired.  $\square$

For any  $(E, \nabla) \in M_X^0$ , the difference

$$\nabla - \nabla_1: E \longrightarrow E \otimes \Omega_C^1(D + D(s))$$

defines an  $\mathcal{O}_C$ -homomorphism, that is a rational Higgs fields on  $E$ . We reprove Proposition 17.

**Theorem 41.** *For generic  $(E, \nabla) \in M_X^0$ , the point  $(q_j, \tilde{p}_j)_{j=1, \dots, N} \in \text{Sym}^N(\Omega(D, c_d))$  determines  $(E, \nabla)$ . So the map  $f_{\text{App}}$  is birational.*

*Proof.* Consider the Higgs field

$$\Phi = \Phi_\nabla = \nabla - \nabla_1: E \longrightarrow E \otimes \Omega_C^1(D + D(s))$$

where  $D = t_1 + \dots + t_n$  and  $D(s) = u_1 + \dots + u_{2g-1}$  as in the notation above. We assume that the set of apparent singularities  $q_1, \dots, q_N$  of  $(E, \nabla)$  is disjoint from  $D$  and  $D(s)$ . We will consider the characteristic curve of  $\Phi$ . On  $U_i$ , we have

$$\Phi_i = A^{(i)} - S^{(i)} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} - \frac{\beta_i}{s^{(i)}} \\ \tilde{a}_{21} & \tilde{a}_{22} - \frac{ds^{(i)}}{s^{(i)}} \end{pmatrix}.$$

The characteristic curve  $C_s$  can be defined in the total space of  $\Omega(D + D(s))$  of the line bundle  $\Omega_C^1(D + D(s))$  by

$$C_s : x^2 - b_1 x - b_2 = 0$$

with  $b_i \in H^0(C, (\Omega_C^1(D + D(s)))^{\otimes i})$ , and  $x$  the canonical section. The dimension of the family of spectral curves is thus given by

$$\begin{aligned} \dim H^0(C, \Omega_C^1(D + D(s))) + \dim H^0(C, (\Omega_C^1(D + D(s)))^{\otimes 2}) &= N + 1 - g + 2N + 1 - g \\ &= 3N + 2 - 2g = 3(4g - 3 + n) + 2 - 2g \\ &= 10g - 7 + 3n. \end{aligned}$$

Then  $\Phi$  is constrained the following conditions.

- (1) At  $t_i, i = 1, \dots, n$ ,  $\Phi$  has eigenvalues fixed by data  $X$ . These impose  $2n - 1$  conditions because of the Fuchs relation.
- (2) At  $u_k, k = 1, \dots, 2g - 1$ , take a local coordinate  $z_k$  such that  $z_k(u_k) = 0$ . Then  $\Phi$  has the following form near  $z_k = 0$

$$\Phi = \begin{pmatrix} 0 & \frac{\beta_i(0)}{z_k} \\ 0 & -\frac{dz_k}{z_k} \end{pmatrix} + \text{holomorphic}.$$

Then eigenvalues of the residue matrix are 0,  $-1$  and the  $\beta_i(0)$  gives a restriction on  $C_s$ . Then totally we have  $3 \times (2g - 1)$  conditions.

- (3) At  $q_j, j = 1, \dots, N$ , the points  $\tilde{a}_{22}(q_j) - \frac{ds^{(i)}}{s^{(i)}}(q_j) = \tilde{p}_j - c_j \in \Omega(D + D(s))$  lie on the characteristic curve  $C_s$ . These give  $N = 4g - 3 + n$  conditions.

For generic choice of  $q_1, \dots, q_N$  and  $s \in H^0(C, L_0)$ , we can see using the method of Lemma 7 and Proposition 17 that these conditions are independent, so we obtain a total of

$$2n - 1 + 3(2g - 1) + (4g - 3 + n) = 10g - 7 + 3n$$

conditions, so these determine the spectral curve  $C_s$ . Now the divisor  $\mu = \sum_{j=1}^N (\tilde{p}_j - c_j) + \sum_{k=1}^{2g-1} (1_k)$  determines the rank 1 sheaf  $\mathcal{O}_{C_s}(\mu)$  where  $(1_k) \in C_s$  denotes the point over  $u_k$  corresponding to the eigenvalue  $-1$  of the residue of  $\Phi$  at  $u_k$ . Then  $(\pi: C_s \rightarrow C, \mathcal{O}_{C_s}(\mu))$  determines  $(E, \Phi)$  uniquely by [3, Proposition 3.6]. Hence  $E$  and  $\nabla = \Phi + \nabla_1$  is determined uniquely.  $\square$

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DEPARTMENT OF MATERIAL SCIENCE, GRADUATE SCHOOL OF SCIENCE, UNIVERSITY OF HYOGO, 2167 SHOSHA,  
HIMEJI, HYOGO 671-2280, JAPAN

*Email address:* akomyo@sci.u-hyogo.ac.jp

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE

*Email address:* frank.loray@univ-rennes1.fr

DEPARTMENT OF DATA SCIENCE, FACULTY OF BUSINESS ADMINISTRATION, KOBEGAKUIN UNIVERSITY, MINA-  
TOJIMA, CHUOU-KU, KOBE, 650-8586, JAPAN

*Email address:* mhsaito@ba.kobegakuin.ac.jp

DEPARTMENT OF ALGEBRA AND GEOMETRY, INSITUTE OF MATHEMATICS, FACULTY OF NATURAL SCIENCES,  
BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, MŰEGYETEM RKP. 3., H-1111 BUDAPEST, HUNGARY

*Email address:* szabosz@math.bme.hu