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# Compact weighted residual formulation for periodic solutions of systems undergoing frictional occurrences

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**Abstract.** A very compact weighted residual formulation is proposed for the construction of periodic solutions of oscillators subject to frictional occurrences. Coulomb's friction is commonly expressed as a differential inclusion which can be cast into the complementarity formalism. When targeting periodic solutions, existing algorithms rely on a procedure alternating between the frequency domain, where the dynamics is solved, and the time domain, where friction is dealt with. In contrast, the key idea of the present work is to express all governing equations including friction as equalities, which are then satisfied in a weak integral sense through a weighted residual formulation. The resulting algebraic nonlinear equations are numerically solved using an adapted trust-region nonlinear solver. The shape functions considered in this work are the classical Fourier functions. It is shown that periodic solutions with clear sticking and sliding phases can be found with an improved accuracy when the number of harmonics is increased.

## 1 Introduction

Non-smooth nonlinearities due to unilateral contact and dry friction are ubiquitous in structural engineering systems. Turbomachinery rotors are a prime example of industrial systems that are subject to intermittent contact and feature dry friction dampers to mitigate adverse vibrations. While predicting the dynamic response of non-smooth systems is of great importance and has been the subject of much research over the years, their equations of motion can be remarkably challenging to solve, because Signorini unilateral contact and Coulomb friction conditions involve a system of equalities and inequalities. Existing methods commonly rely on the penalization of the friction force by introducing a finite stiffness or on the smoothing of the contact force [2]. Frequency-domain formulations require the calculation of the contact conditions in the time domain at each iteration of the nonlinear solver via an FFT [6], while time-domain methods mandate advanced time-stepping or event-driven schemes [1]. The present work suggests a simpler methodology where dry friction is expressed as an equality enforced in a weighted residual sense.

## 2 Theory

**Governing equations** The considered academic system is shown in Figure 1. The system is subject to an external harmonic force  $f \cos \omega t$  of period  $T = 2\pi/\omega$  at the first degree-of-freedom along with a frictional force  $r$  at the second degree-of-freedom. Denoting the displacements of the two degrees-of-freedom by  $x_1(t)$  and  $x_2(t)$  and their respective masses by  $m_1$  and  $m_2$  and stiffnesses by  $k_1$  and  $k_2$ , the equations of motion governing the dynamics are:

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 - f \cos \omega t = 0 \quad (1a)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 - r = 0. \quad (1b)$$

The friction condition on mass  $m_2$  is characterized by the friction coefficient  $\mu$ . It is assumed that there is no vertical separation such that the corresponding normal force  $N$  can be specified.

**Equality-based Coulomb's friction** Assuming a closed contact in the normal direction, Coulomb's friction classically says the following:

$$\begin{cases} \dot{x}_2 = 0 & \implies & |r| \leq \mu N \\ \dot{x}_2 \neq 0 & \implies & |r| = \mu |N| \text{ and } \exists \alpha \geq 0 \mid r = -\alpha \dot{x}_2. \end{cases} \quad (2)$$

Among others, Equation (2) can equivalently be recast into the nonsmooth equality [1, 8, 3]

$$\Psi(r, \dot{x}_2) = \dot{x}_2 + \min(0, \rho(r + \mu N) - \dot{x}_2) + \max(0, \rho(r - \mu N) - \dot{x}_2) = 0 \quad (3)$$

where  $\rho$  is any strictly positive real number. Equations (1) and (3) in the unknowns functions of time  $x_1$ ,  $x_2$ , and  $r$  collectively describe the dynamics of the system. The equivalent Coulomb's friction equality is depicted in Figure 2.

**Weighted-residual formulation** It is proposed to search for periodic solutions by solving the above formulation in a weighted residual sense. All unknowns of the problem are expanded on an appropriate truncated basis of  $T$ -periodic functions with  $n$  members  $\phi_k(t)$ ,  $k = 1, \dots, n$ , commonly the Fourier basis in the Harmonic Balance Method, as follows:

$$x_1(t) = \sum_k x_{1k} \phi_k(t), \quad x_2(t) = \sum_k x_{2k} \phi_k(t), \quad r(t) = \sum_k r_k \phi_k(t). \quad (4)$$

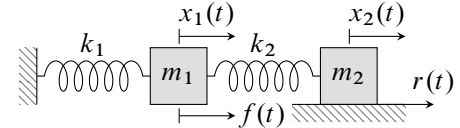


Figure 1: Two-degree of freedom mass-spring system with Coulomb friction at the second degree of freedom.

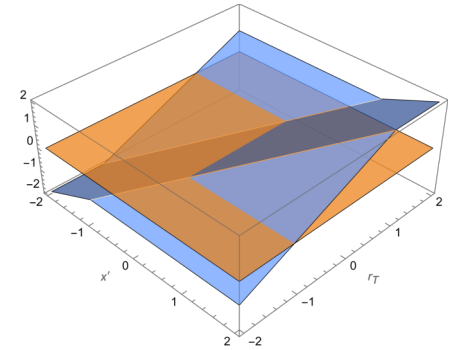


Figure 2: Function  $\Psi(r, \dot{x}_2)$  (blue) in Equation (3) showing that its intersection with the zero plane (orange) is the classical friction condition (2).

Depending on the smoothness of the selected basis functions, time derivatives might either be obtained by pointwise differentiation in time or expanded on a less smooth basis and related to the differentiated quantity in a weak sense. Once the expressions in Equation (4) and their time-derivatives are inserted in Equations (1) and (3), the corresponding *residuals*  $R_{(1a)}(t)$ ,  $R_{(1b)}(t)$  and  $R_{(3)}(t)$  are built and the goal is to find the  $3n$  unknowns  $x_{1k}$ ,  $x_{2k}$ , and  $r_k$  satisfying the  $3n$  equations

$$\int_0^T \phi_k(t) R_{(1a)}(t) dt = \int_0^T \phi_k(t) R_{(1b)}(t) dt = \int_0^T \phi_k(t) R_{(3)}(t) dt = 0, \quad \forall k = 1, \dots, n. \quad (5)$$

The above integrals can be numerically computed using appropriate quadrature schemes. The resulting system of nonlinear equations can be solved using a classical trust-region nonlinear solver [4]. The proposed strategy can be seen as a very compact form of the AFT methodology [2] without regularization, and it shares similarities with the DLFT technique [6] which also relies on the AFT. The proposed formulation is very compact and involves simple implementations such as basic integral quadrature schemes and existing nonlinear solvers. Its engineering value lies in its capability to generate reasonable approximations without difficulty in contrast to much more advanced time-stepping or event-driven schemes [1, 5, 7].

### 3 Results and discussion

Periodic responses are sought for the following dimensionless values of the system parameters:  $m_1 = 1$ ,  $k_1 = 1$ ,  $m_2 = 1$ ,  $k_2 = 1$ ,  $\mu = 0.45$ , and  $f = 10$ . The normal force is  $N = 1$ . Responses and comparison with a first-order time-marching Implicit Euler scheme are shown in Figure 3. Associated *linear* systems can be considered in the limiting cases of sliding or

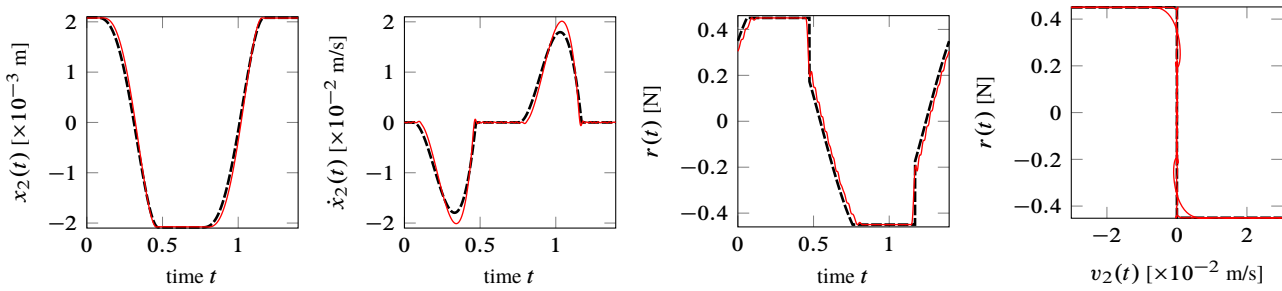


Figure 3: Solution over one period of motion for  $n = 30$ ,  $\rho = 0.2$  and  $\omega = 4.5$  rad/s: Frequency domain [solid line], time domain [dashed line].

sticking over an entire period of the motion. In the case of sliding for the mass  $m_2$  ( $\mu = 0$ ), linear natural frequencies are  $\omega_1 = 0.618$  rad/s and  $\omega_2 = 1.618$  rad/s. In the case of sticking for the mass  $m_2$  ( $\mu = \infty$ ), then the associated linear system has a single degree of freedom and its free vibration natural frequency is  $\omega_1 = 1.414$  rad/s. For the specified parameters  $\mu$  and  $N$ , sliding only is observed when the forcing frequency lies in the vicinity of the (linear) natural frequencies, with large velocities  $\dot{x}_1$  and  $\dot{x}_2$ . However, at  $\omega = 4.5$  rad/s, the computed motion over one period features stick-slip transitions at the friction point, with two stick-slip phases per cycle. The sticking phases are characterized by a constant displacement  $x_2$ . The number of harmonics is set to  $n = 30$  and the method has the ability to capture sticking behavior accurately. The graph of friction force versus velocity is such that the velocity at the friction point remains near zero as the friction force varies in the interval  $[-\mu N, \mu N] = [-0.45, 0.45]$  N, in agreement with Coulomb's friction law. The slipping phases of the motion are also well captured. Again, the friction force remains nearly constant at  $\pm 0.45$  N as a function of the slipping velocity, in accordance with Coulomb friction law. Overall, the results in the time and frequency domains agree well, at least for the considered set of parameters.

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