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# HEAT FLOW OF $p$ -HARMONIC MAPS FROM COMPLETE MANIFOLDS INTO GENERALISED REGULAR BALLS

ZEINA AL DAWOUD

ABSTRACT. We study the heat flow of  $p$ -harmonic maps between complete Riemannian manifolds. We prove the global existence of the flow when the initial datum has values in a generalised regular ball. In particular, if the target manifold has nonpositive sectional curvature, we obtain the global existence of the flow for any initial datum with finite  $p$ -energy. If, in addition, the target manifold is compact, the flow converges to a  $p$ -harmonic map. This gives an extension of the results of Liao-Tam [12] concerning the harmonic heat flow ( $p = 2$ ) to the case  $p \geq 2$ . We also derive a Liouville type theorem for  $p$ -harmonic maps between complete Riemannian manifolds.

## 1. INTRODUCTION

Let  $(M^m, g)$  and  $(N^n, h)$  be two Riemannian manifolds, with  $M$  compact. For  $p > 1$ , the  $p$ -energy of a map  $u \in C^1(M, N)$  is defined by

$$E_p(u) = \frac{1}{p} \int_M |du(x)|^p dx, \quad (1.1)$$

where  $|du(x)|$  is the Hilbert-Schmidt norm of  $du(x)$ , and  $dx$  stands for the Riemannian volume element of  $M$ .

$p$ -harmonic maps are critical points of the functional (1.1), that is, they are solutions of the Euler-Lagrange equation associated to (1.1)

$$\tau_p(u) = 0, \quad (1.2)$$

where  $\tau_p(u) = \text{Trace}_g(\nabla(|du|^{p-2} du))$  denotes the  $p$ -tension field of  $u$ . More precisely, let  $(x^1, \dots, x^m)$  and  $(y^1, \dots, y^n)$  be local coordinates on  $M$  and  $N$  respectively, then denoting  $\partial_j = \frac{\partial}{\partial x^j}$  and  $u^\alpha = y^\alpha(u)$ , equation (1.2) takes the form of the following system

$$-\frac{1}{\sqrt{|g|}} \partial_i \left( |du|^{p-2} \sqrt{|g|} g^{ij} \partial_j u^k \right) = |du|^{p-2} g^{ij} \Gamma_{\alpha\beta}^k(u) \partial_i u^\alpha \partial_j u^\beta, \quad 1 \leq k \leq n, \quad (1.3)$$

where  $\Gamma_{\alpha\beta}^l$  are the Christoffel symbols of  $N$ , and the Einstein summation convention on repeated indices is used.

In order to consider more general class of solutions of system (1.3), we shall write it in an equivalent form which is independent of the choice of coordinates. By Nash's embedding theorem, we can embed  $N$  isometrically in an Euclidean space  $\mathbb{R}^L$  using an isometric embedding  $i : N \rightarrow \mathbb{R}^L$ . If we note again  $u = i \circ u$ , the energy functional (1.1) becomes

$$E_p(u) = \frac{1}{p} \int_M |\nabla u(x)|^p dx,$$

where  $|\nabla u|^2 = g^{ij} \langle \partial_i u, \partial_j u \rangle$ , and  $\langle \cdot, \cdot \rangle$  denotes the Euclidean inner product of  $\mathbb{R}^L$ .

Equation (1.3) takes the form

$$-\Delta_p u = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u), \quad (1.4)$$

where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplacian, that is,

$$\Delta_p u = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} |\nabla u|^{p-2} g^{ij} \partial_j u \right),$$

and  $A$  is the second fundamental form of  $N$  in  $\mathbb{R}^L$  with the notation

$$A(u)(\nabla u, \nabla u) = g^{ij} A(u)(\partial_i u, \partial_j u).$$

In this case,  $p$ -harmonic maps are defined to be solutions of equation (1.4). We note that equation (1.4) makes sense even if  $M$  is not compact.

We define the Sobolev space  $W^{1,p}(M, N) = \{u \in W^{1,p}(M, \mathbb{R}^L); u(x) \in N \text{ a.e.}\}$ . We say that a map  $u \in W^{1,p}(M, N) \cap L^\infty(M, \mathbb{R}^L)$  is a weakly  $p$ -harmonic map if it is a weak solution of (1.4). The study of the regularity of weakly  $p$ -harmonic maps is a delicate question due to the fact that (1.4) is a degenerate quasilinear elliptic system. In general, the optimal regularity for weak solutions to systems involving the  $p$ -Laplacian is  $C^{1+\beta}$  as shown by [19], and  $C^\infty$  out off the vanishing set of  $\nabla u$ . Solutions which are  $p$ -energy-minimizing are in  $C^{1+\beta}(M \setminus S, N)$  ( $0 < \beta < 1$ ) where  $S$  is a singular set of Hausdorff dimension at most  $m - [p] - 1$  (see [6] [14]).

In this paper we are interested in heat flow of  $p$ -harmonic maps which is the gradient flow associated with the  $p$ -energy functional. Namely,

$$\begin{cases} \partial_t u - \Delta_p u = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u), \\ u(x, 0) = u_0(x), \end{cases} \quad (1.5)$$

where  $u_0 : M \rightarrow N$  is the initial datum of the flow.

Throughout this paper, what we mean by a solution  $u$  of (1.5) on  $M \times [0, T)$  is a map  $u \in C_{loc}^{1+\beta, \beta/p}(M \times [0, T), N)$  (for some  $0 < \beta < 1$ ) which is a weak solution (in the distributional sense) of (1.5). Indeed, as stated above, the optimal regularity one could expect for  $p$ -harmonic type equations is  $C^{1+\beta}$  (and  $C^{1+\beta, \beta/p}$  for parabolic equations).

When  $p = 2$ , Eells and Sampson [3] were the first who studied the heat flow problem of harmonic maps. They proved that if  $M$  and  $N$  are compact Riemannian manifolds and  $N$  has nonpositive sectional curvature, then (1.5) admits a global solution which converges at infinity to a harmonic map. Li and Tam [11] considered the case when both  $M$  and  $N$  are complete noncompact Riemannian manifolds. They proved the existence of a global solution when the Ricci curvature of  $M$  is bounded from below,  $N$  has nonpositive sectional curvature and if the initial datum  $u_0$  is bounded as well as its energy density. Later on, Liao and Tam [12] showed that if  $M$  is a complete non-compact manifold,  $N$  is compact with nonpositive sectional curvature, and if the initial map has finite total energy, then (1.5) admits a global solution which converges on compact subsets of  $M$  to a harmonic map from  $M$  into  $N$ . It is well known that if the sectional curvature of  $N$  is nonnegative, a blow

up phenomenon may occur. Without this condition, one has to impose some assumptions on  $N$  in order to prevent the blow up of the solution. We refer the reader to the work of Struwe ([17], [18]) and Chen-Struwe [1] concerning the singularities of the harmonic heat flow. When  $M$  is complete, Li and Wang [13] proved that there exists a global solution of the harmonic heat flow from  $M$  into a generalised regular ball of the target manifold  $N$  which converges at infinity to a harmonic map when the initial datum has finite energy (we refer to [13] for the notion of generalised regular balls.)

When  $p \geq 2$  Fardoun-Regbaoui [4] and Misawa [15] proved the global existence and the convergence of the  $p$ -harmonic heat flow when  $M$  and  $N$  are compact and  $N$  has nonpositive sectional curvature generalising the result Eells and Sampson [3] to the case  $p \geq 2$ . When  $N$  has arbitrary sectional curvature, Hungerbuhler [9] proved the existence of weak solutions in the conformal case  $p = m$ . See also Hungerbuhler [8] when the target manifold is a homogeneous space. For small initial data, Fardoun-Regbaoui [5] obtained the existence and convergence of the flow. We also mention the recent work of Misawa [16] concerning the regularity of the  $p$ -harmonic heat flow.

Our goal in this paper is to extend the results of Liao-Tam above to the  $p$ -harmonic heat flow for  $p \geq 2$ . Accordingly, we will introduce the notion of regular sets inspired by the work of Li and Wang [13] concerning the harmonic heat flow.

**Definition 1.1.** *Let  $\Omega$  be an open subset of a Riemannian manifold  $N$  and let  $\delta > 0$ . We say that  $\Omega$  is a  $\delta$ -regular set if there exist a positive function  $f \in C^2(\Omega)$  and a constant  $C > 0$  such that, for all  $y \in \Omega$ , we have*

$$\begin{cases} -\nabla^2 f(y) - K_2(y)f(y)h(y) \geq \delta \frac{|\nabla f(y)|^2}{f(y)} h(y) \\ C^{-1} \leq f(y) \leq C, \end{cases} \quad (1.6)$$

where  $h$  is the metric of  $N$  and  $K_2(y) = \sup\{K(y, \pi), 0\}$ , with  $K(y, \pi)$  being the sectional curvature of a 2-plane  $\pi \subset T_y N$ .

According to our definition of  $\delta$ -regular sets, any Riemannian manifold with nonpositive sectional curvature is a  $\delta$ -regular set for any  $\delta > 0$ . Indeed, if  $N$  has nonpositive sectional curvature, then condition (1.6) above is automatically satisfied by taking  $f = 1$ .

**Definition 1.2.** *We say that  $\Omega \subset N$  is a  $\delta$ -generalised regular ball if it is a  $\delta$ -regular set and if there exists a positive function  $f^* \in C^2(N)$  which is convex on  $\Omega$  such that*

$$\Omega = (f^*)^{-1}([0, a]) \quad (1.7)$$

for some  $a > 0$ .

**Example 1.1.** *If  $N$  is a Riemannian manifold with nonpositive sectional curvature, then  $N$  is a  $\delta$ -generalised regular ball for any  $\delta > 0$  by taking  $f = f^* = 1$  and  $a > 1$ .*

**Example 1.2.** *On the sphere  $\mathbb{S}^n$  any geodesic ball  $B(y, r)$ , with  $0 < r < \frac{\pi}{2}$ , is a  $\delta$ -generalised regular ball with  $\delta = \frac{(\cos r - \cos r_1) \cos r_1}{\sin^2 r}$ , where  $r_1$  is any real number such that  $r < r_1 < \frac{\pi}{2}$ . Indeed, in polar coordinates  $(\rho, \theta)$  centered at  $y$ , if we set  $f(\rho, \theta) = \cos \rho - \cos r_1$  and  $f^*$  any smooth function on  $\mathbb{S}^n$  such that  $f^*(\rho, \theta) = \rho^2$  on  $B(y, r)$ , then one can check that  $\nabla^2 f = -(\cos \rho)h$ , where  $h = d\rho^2 + \sin^2 \rho d\theta$  is the standard metric on  $\mathbb{S}^n$ , and that  $f$  satisfies condition (1.6) with  $\delta = \frac{(\cos r - \cos r_1) \cos r_1}{\sin^2 r}$ , and  $f^*$  satisfies condition (1.7). More generally, by using the Hessian comparison theorem on a Riemannian manifold  $N$ , one can see that any regular geodesic ball  $B(y, r)$  in the sense of Hildebrandt [7] is a  $\delta$ -generalised regular ball for some  $\delta > 0$  depending on  $r$ .*

Troughout this paper we suppose  $p \geq 2$ . We state our first main result :

**Theorem 1.** *Let  $(M^m, g)$  and  $(N^n, h)$  be two Riemannian manifolds such that  $M$  is compact and  $N$  is complete. Let  $\Omega \subset N$  be a  $\delta$ -generalised regular ball with*

$$\delta > \delta_p := 3(p-2)^2 (\sqrt{m} + 2p + 6)^2 + 3.$$

*Then for any  $u_0 \in C^\infty(M, \Omega)$ , there exists a unique global solution  $u$  of (1.5) such that  $u \in C_{loc}^{1+\beta, \beta/p}(M \times [0, \infty), \Omega)$  for some  $\beta \in (0, 1)$ . Moreover, we have  $\partial_t u \in L^2(M \times [0, +\infty))$  and satisfies the energy inequality*

$$\int_0^T \int_M |\partial_t u(x, t)|^2 dx dt + E_p(u(\cdot, T)) \leq E_p(u_0) \quad (1.8)$$

*for all  $T > 0$ . If we assume in addition that  $N$  is compact, then there exists a sequence  $t_k \rightarrow \infty$  such that  $u(\cdot, t_k)$  converges in  $C^{1+\beta'}(M, \Omega)$  (for all  $\beta' < \beta$ ) to a  $p$ -harmonic map  $u_\infty \in C^{1+\beta}(M, \Omega)$  satisfying  $E_p(u_\infty) \leq E_p(u_0)$ .*

Theorem 1 allows us to prove the following result between complete Riemannian manifolds which is, to our knowledge, the first result concerning the existence of  $p$ -harmonic heat flow from complete noncompact Riemannian manifolds.

**Theorem 2.** *Let  $(M^m, g)$  and  $(N^n, h)$  be two complete Riemannian manifolds. Let  $\Omega \subset N$  be a  $\delta$ -generalised regular ball with*

$$\delta > \delta_p := 3(p-2)^2 (\sqrt{m} + 2p + 6)^2 + 3.$$

*Then for any  $u_0 \in C^\infty(M, \Omega)$  with  $E_p(u_0) < +\infty$ , there exists a global solution  $u$  of (1.5) such that  $u \in C_{loc}^{1+\beta, \beta/p}(M \times [0, \infty), \Omega)$  for some  $\beta \in (0, 1)$ . Moreover,  $\partial_t u \in L^2(M \times [0, +\infty))$  and satisfies the energy inequality*

$$\int_0^T \int_M |\partial_t u(x, t)|^2 dx dt + E_p(u(\cdot, T)) \leq E_p(u_0) \quad (1.9)$$

*for all  $T > 0$ . If we assume in addition that  $N$  is compact, then there exists a sequence  $t_k \rightarrow \infty$  such that  $u(\cdot, t_k)$  converges in  $C_{loc}^{1+\beta'}(M, \Omega)$  (for all  $\beta' < \beta$ ) to a  $p$ -harmonic map  $u_\infty \in C_{loc}^{1+\beta}(M, \Omega)$  satisfying  $E_p(u_\infty) \leq E_p(u_0)$ .*

As a consequence of Theorem 2, we have the following theorem concerning target manifolds with negative sectional curvature. It can be considered as a natural generalisation to the case  $p \geq 2$  of the work of Liao-Tam [12] concerning the heat flow of harmonic maps ( $p = 2$ ).

**Theorem 3.** *Let  $M$  and  $N$  be two complete Riemannian manifolds such that  $N$  has nonpositive sectional curvature. Then for any  $u_0 \in C^\infty(M, N)$  with  $E_p(u_0) < +\infty$ , there exists a global solution  $u$  of (1.5) such that  $u \in C_{loc}^{1+\beta, \beta/p}(M \times [0, \infty), \Omega)$  for some  $\beta \in (0, 1)$ . Moreover,  $\partial_t u \in L^2(M \times [0, +\infty))$  and satisfies the energy inequality*

$$\int_0^T \int_M |\partial_t u(x, t)|^2 dx dt + E_p(u(\cdot, T)) \leq E_p(u_0) \quad (1.10)$$

for all  $T > 0$ . If we assume in addition that  $N$  is compact, then there exists a sequence  $t_k \rightarrow \infty$  such that  $u(\cdot, t_k)$  converges in  $C_{loc}^{1+\beta'}(M, \Omega)$  (for all  $\beta' < \beta$ ) to a  $p$ -harmonic map  $u_\infty \in C_{loc}^{1+\beta}(M, \Omega)$  satisfying  $E_p(u_\infty) \leq E_p(u_0)$ .

Our method allows us to prove the following Liouville Theorem for  $p$ -harmonic maps.

**Theorem 4.** *Let  $M$  be a complete Riemannian manifold with nonnegative Ricci curvature and  $N$  be a complete Riemannian manifold. Suppose that  $\Omega \subset N$  is a  $\delta$ -regular set with  $\delta > \delta_p$ , where  $\delta_p$  is as in Theorem 1. If  $u \in C_{loc}^1(M, \Omega)$  is a  $p$ -harmonic map from  $M$  into  $\Omega$  with finite  $p$ -energy, then  $u$  is constant. In particular, if  $N$  has nonpositive sectional curvature, then any  $p$ -harmonic map  $u \in C_{loc}^1(M, N)$  with finite  $p$ -energy from  $M$  to  $N$  is constant.*

The paper is organised as follows. The heat flow equation being a degenerate parabolic problem, we first establish in Section 2 the existence of a global solution to the regularised equation of (1.5). We then prove uniform *a priori* gradient estimates on the solutions of the regularised equation in Section 3. Section 4 is devoted to the proof of our main results.

## 2. THE REGULARISED HEAT FLOW

Since (1.5) is a degenerate parabolic system, one can not apply directly the existence theory for parabolic equations. To overcome this difficulty we introduce the regularised  $p$ -harmonic heat flow equation. Namely, for  $0 < \varepsilon < 1$ , the regularised  $p$ -energy of  $u$  is defined by

$$E_{p,\varepsilon}(u) = \frac{1}{p} \int_M (|\nabla u|^2 + \varepsilon)^{\frac{p}{2}} dx,$$

and the gradient flow associated to  $E_{p,\varepsilon}$  is given by the following second order parabolic system

$$\begin{cases} \partial_t u - \Delta_{p,\varepsilon} = (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} A(u)(\nabla u, \nabla u), \\ u(x, 0) = u_0(x) \end{cases} \quad (2.1)$$

where

$$\Delta_{p,\varepsilon} = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} g^{ij} \partial_j u \right)$$

is the regularised  $p$ -Laplacian of  $M$ .

Since (2.1) is a parabolic system, then it follows from the classical theory of parabolic equations that (2.1) admits a unique smooth solution  $u_\varepsilon$  defined on a maximum interval  $[0, T_\varepsilon)$ . For the sake of simplicity we denote  $u$  our solution instead of  $u_\varepsilon$  and  $T$  instead of  $T_\varepsilon$ . We have the following proposition.

**Proposition 2.1.** *Let  $p \geq 2$  and let  $\Omega \subset N$  be a  $\delta$ -generalised regular ball for some  $\delta > 0$ . Let  $u_0 \in C^\infty(M, \Omega)$  and let  $u$  be the solution of the regularised problem (2.1) defined on a maximal interval  $[0, T)$ . Then we have for any  $(x, t) \in M \times [0, T)$*

$$u(x, t) \in \Omega. \quad (2.2)$$

Moreover,  $u$  satisfies the energy formula

$$\frac{d}{dt} E_{p,\varepsilon}(u(\cdot, t)) = - \int_M |\partial_t u(x, t)|^2 dx. \quad (2.3)$$

In particular the energy  $E_{p,\varepsilon}$  is nonincreasing along the flow.

*Proof.* Since  $\Omega$  is a generalised regular ball, then there exist a positive function  $f^* \in C^2(N)$  which is convex on  $\Omega$  and  $a > 0$  such that  $\Omega = (f^*)^{-1}([0, a])$ . Let

$$T_1 = \sup \{t \in [0, T) : u(M \times [0, t]) \subset \Omega\}$$

and suppose by contradiction that  $T_1 < T$ . Since  $u_0(M) \subset \Omega$  and  $M$  is compact, then by continuity of  $u$  we have that  $T_1 > 0$ . Then we compute on  $M \times [0, T_1)$

$$\begin{aligned} \partial_t(f^* \circ u) - \operatorname{div} \left( (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla f^* \circ u \right) &= \left\langle (\nabla f^*) \circ u, (\partial_t u - \Delta_{p,\varepsilon} u) \right\rangle \\ &\quad - (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} (\nabla^2 f^*) \circ u (\nabla u, \nabla u) \\ &= (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} \left\langle (\nabla f^*) \circ u, A(u)(\nabla u, \nabla u) \right\rangle \\ &\quad - (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} (\nabla^2 f^*) \circ u (\nabla u, \nabla u), \end{aligned}$$

which implies, since  $\nabla f^*$  is orthogonal to  $A(u)(\nabla u, \nabla u)$  and  $f^*$  is convex on  $\Omega$ , that

$$\partial_t(f^*(u)) - \operatorname{div} \left( (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla f^*(u) \right) \leq 0.$$

Hence it follows from the maximum principle for parabolic equations that for all  $t \in [0, T_1)$ , we have

$$\max_{x \in M} f^*(u(x, t)) \leq \max_{x \in M} f^*(u_0(x))$$

which implies by continuity of  $f^*$  and  $u$  that

$$\max_{x \in M} f^*(u(x, T_1)) \leq \max_{x \in M} f^*(u_0(x)).$$

Since  $u_0(M) \subset \Omega$ , then  $\max_{x \in M} f^*(u_0(x)) < a$ , so

$$\max_{x \in M} f^*(u(x, T_1)) < a.$$

It follows by continuity of  $f^*(u)$  that there exists  $\alpha > 0$  such  $\max_{x \in M} f^*(u(x, t)) < a$  for all  $t \in [0, T_1 + \alpha]$ , that is,  $u(M, t) \subset \Omega$  for all  $t \in [0, T_1 + \alpha]$  contradicting the definition of  $T_1$ .

Now to prove (2.3) it suffices to take the inner product (in  $\mathbb{R}^L$ ) of equation (2.1) with  $\partial_t u$  and integrate on  $M$  to get

$$\int_M |\partial_t u(x, t)|^2 dx = \int_M (|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} \langle \partial_t u, A(u)(\nabla u, \nabla u) \rangle dx - \frac{d}{dt} E_{p,\varepsilon}(u(\cdot, t)).$$

This achieves the proof of the proposition since  $\langle \partial_t u, A(u)(\nabla u, \nabla u) \rangle = 0$ .

□

In order to prove uniform gradient estimates on the solution of the regularised equation (2.1) we need a Bochner-type formula on  $u$ . To this end let us introduce the following notations. We set for all  $0 < \varepsilon < 1$ :

$$F = |\nabla u|^2 + \varepsilon,$$

and let  $L_p$  be the operator defined for  $\varphi \in C^2(M)$  by

$$L_p(\varphi) = \operatorname{div} \left( F^{\frac{p-2}{2}} \nabla \varphi \right). \quad (2.4)$$

We define the symmetric contravariant 2-tensor  $B$  on  $M$  by setting in local coordinates on  $M$ :

$$B_{ij} = \frac{\langle \partial_k u, \partial_l u \rangle}{F} g^{ik} g^{lj}. \quad (2.5)$$

One checks immediately that for any  $x \in M$  and covectors  $X, Y \in T_x^*M$ , we have

$$\begin{cases} B(X, X) \geq 0 \\ B(X, Y) \leq |X||Y|. \end{cases} \quad (2.6)$$

If  $X = X_i dx^i \in T_x^*M$ , we denote by  $B(X, \cdot)$  the vector in  $T_x M$  defined by  $B(X, \cdot) = B_{ij} X_i \partial_j$ .

Then we have the following Bochner-type formula

$$\begin{aligned} \partial_t F - L_p F &= (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(dF, \cdot) \right) - 2F^{\frac{p-2}{2}} |\nabla^2 u|^2 - \frac{(p-2)}{2} F^{\frac{p-4}{2}} |\nabla F|^2 \\ &\quad - 2F^{\frac{p-2}{2}} \operatorname{Ric}^M(\nabla u, \nabla u) + 2F^{\frac{p-2}{2}} \langle \operatorname{Riem}^N(\nabla u, \nabla u) \nabla u, \nabla u \rangle, \end{aligned} \quad (2.7)$$

where  $\operatorname{Ric}^M$  is the Ricci tensor of  $M$  and  $\operatorname{Riem}^N$  is the Riemann curvature tensor of  $N$  with the following notations in an orthonormal frame  $\{e_1, \dots, e_m\}$  of  $T_x M$ :

$$\operatorname{Ric}^M(\nabla u, \nabla u) = \sum_{k=1}^L \sum_{i=1}^m \operatorname{Ric}^M(\nabla_{e_i} u^k, \nabla_{e_i} u^k)$$

and

$$\langle \operatorname{Riem}^N(\nabla u, \nabla u) \nabla u, \nabla u \rangle = \sum_{i,j=1}^m \langle \operatorname{Riem}^N(\nabla_{e_i} u, \nabla_{e_j} u) \nabla_{e_i} u, \nabla_{e_j} u \rangle$$

### 3. GRADIENT ESTIMATES

In this section, we derive uniform gradient estimates on the solution  $u$  of the regularised equation (2.1). We first need the following useful inequality.

**Proposition 3.1.** *Let  $(M^m, g)$ ,  $(N^n, h)$  be two Riemannian manifolds and let  $\Omega \subset N$  be a  $\delta$ -regular set. Let  $u : M \times [0, T] \rightarrow N$  be a smooth solution of (2.1) having its image in  $\Omega$  and set*

$$\varphi(x, t) = \frac{F(x, t)}{f^2(u(x, t))},$$

where  $F(x, t) = |\nabla u(x, t)|^2 + \varepsilon$  and  $f$  is the function satisfying condition (1.6). Then we have at any point  $(x, t) \in M \times [0, T]$

$$\begin{aligned} \partial_t \varphi - L_p \varphi &\leq (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) - \frac{1}{40} F^{\frac{p-4}{2}} (f \circ u)^2 |\nabla \varphi|^2 \\ &\quad - 2(\delta - \delta_p) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2 |\nabla u|^2 + 2K_1 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla u|^2, \end{aligned} \quad (3.1)$$

where  $\delta_p = 3(p-2)^2 (\sqrt{m} + 2p + 2)^2 + 3$  and  $-K_1 \leq 0$  is a lower bound of the Ricci curvature of  $M$  at  $x$ . The operator  $L_p$  and the tensor  $B$  are defined by (2.4) and (2.5) above.

*Proof.* Fix a point  $x_0 \in M$ , then in normal coordinates at  $x_0$ , a basic computation gives,

$$\begin{aligned} \partial_t \varphi - L_p \varphi &= \frac{1}{(f \circ u)^2} \left( \partial_t F - L_p F \right) - 2 \frac{F}{(f \circ u)^3} \left( \partial_t (f \circ u) - L_p (f \circ u) \right) \\ &\quad + 4 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \nabla F \cdot \nabla (f \circ u) - 6 \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla (f \circ u)|^2, \end{aligned} \quad (3.2)$$

where the dot  $\cdot$  denotes the Riemannian inner product on  $M$ . By using Bochner's formula (2.7), the first term in the right hand side of (3.2) can be bounded as :

$$\begin{aligned} \frac{1}{(f \circ u)^2} \left( \partial_t F - L_p F \right) &\leq \frac{(p-2)}{(f \circ u)^2} \operatorname{div} \left( F^{\frac{p-2}{2}} B(dF, \cdot) \right) + 2K_1 \frac{F^{\frac{p-2}{2}} |\nabla u|^2}{(f \circ u)^2} \\ &\quad + 2K_2 \frac{F^{\frac{p}{2}} |\nabla u|^2}{(f \circ u)^2} - 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla^2 u|^2 - \frac{1}{2} (p-2) \frac{F^{\frac{p-4}{2}}}{(f \circ u)^2} |\nabla F|^2, \end{aligned} \quad (3.3)$$

where  $-K_1 \leq 0$  is a lower bound of the Ricci curvature of  $M$ , and  $K_2 \geq 0$  is an upper bound of the sectional curvature of  $N$ . To bound the second term in the right hand side of (3.2), a direct computation gives

$$\partial_t (f \circ u) - L_p (f \circ u) = \left\langle (\nabla f) \circ u, (\partial_t u - L_p u) \right\rangle - F^{\frac{p-2}{2}} (\nabla^2 f) \circ u (\nabla u, \nabla u)$$

where in local coordinates :

$$(\nabla^2 f) \circ u (\nabla u, \nabla u) = \sum_{i,j=1}^m g^{ij} (\nabla^2 f) \circ u (\partial_i u, \partial_j u)$$

and where  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $\mathbb{R}^L$  (we recall here that  $N$  is isometrically embedded in  $\mathbb{R}^L$ ). Since  $\langle (\nabla f) \circ u, (\partial_t u - L_p u) \rangle = 0$  by equation (2.1) ( $\partial_t u - \Delta_p u$  being orthogonal to  $T_u N$  and  $(\nabla f) \circ u \in T_u N$ ), then we obtain

$$\partial_t (f \circ u) - L_p (f \circ u) = -F^{\frac{p-2}{2}} (\nabla^2 f) \circ u (\nabla u, \nabla u). \quad (3.4)$$

Substituting (3.4) and (3.3) in (3.2) gives

$$\begin{aligned}
\partial_t \varphi - L_p \varphi &\leq \frac{(p-2)}{(f \circ u)^2} \operatorname{div} \left( F^{\frac{p-2}{2}} B(dF, \cdot) \right) - 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla^2 u|^2 - \frac{1}{2} (p-2) \frac{F^{\frac{p-4}{2}}}{(f \circ u)^2} |\nabla F|^2 \\
&\quad + 2K_1 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla u|^2 + 2 \frac{F^{\frac{p}{2}}}{(f \circ u)^3} \left( K_2 (f \circ u) |\nabla u|^2 + (\nabla^2 f) \circ u (\nabla u, \nabla u) \right) \\
&\quad + 4 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \nabla F \cdot \nabla (f \circ u) - 6 \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla (f \circ u)|^2.
\end{aligned} \tag{3.5}$$

Since  $f$  satisfies condition (1.6), then we have

$$K_2 (f \circ u) |\nabla u|^2 + (\nabla^2 f) \circ u (\nabla u, \nabla u) \leq -\delta \frac{|(\nabla f) \circ u|^2}{(f \circ u)} |\nabla u|^2,$$

so it follows from (3.5)

$$\begin{aligned}
\partial_t \varphi - L_p \varphi &\leq \frac{(p-2)}{(f \circ u)^2} \operatorname{div} \left( F^{\frac{p-2}{2}} B(dF, \cdot) \right) - 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla^2 u|^2 - \frac{1}{2} (p-2) \frac{F^{\frac{p-4}{2}}}{(f \circ u)^2} |\nabla F|^2 \\
&\quad + 2K_1 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla u|^2 - 2\delta \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2 |\nabla u|^2 \\
&\quad + 4 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \nabla F \cdot \nabla (f \circ u) - 6 \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2.
\end{aligned} \tag{3.6}$$

To estimate the first term in the right hand side of (3.6) we compute, by using the fact that  $F = (f \circ u)^2 \varphi$  and that we are working in normal coordinates at  $x_0$ ,

$$\begin{aligned}
\frac{1}{(f \circ u)^2} \operatorname{div} \left( F^{\frac{p-2}{2}} B(dF, \cdot) \right) &= \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + 2 \operatorname{div} \left( \frac{F^{\frac{p}{2}}}{(f \circ u)^3} B(d(f \circ u), \cdot) \right) \\
&+ 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} B(dF, d(f \circ u)) \\
&= \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + 2 \partial_i \left( \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \langle \partial_i u, \partial_j u \rangle \partial_j (f \circ u) \right) \\
&+ 4 \frac{F^{\frac{p-4}{2}}}{(f \circ u)^3} \langle \partial_i u, \partial_j u \rangle \partial_i F \partial_j (f \circ u) \\
&= \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + (p+2) \frac{F^{\frac{p-4}{2}}}{(f \circ u)^3} \langle \partial_i u, \partial_j u \rangle \partial_i F \partial_j (f \circ u) \quad (3.7) \\
&+ 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \langle \partial_{ii}^2 u, \partial_j u \rangle \partial_j (f \circ u) + 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \langle \partial_i u, \partial_{ij}^2 u \rangle \partial_j (f \circ u) \\
&+ 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \langle \partial_i u, \partial_j u \rangle (\nabla^2 f) \circ u (\partial_i u, \partial_j u) \\
&+ 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \langle \partial_i u, \partial_j u \rangle \langle (\nabla f) \circ u, \partial_{ij}^2 u \rangle \\
&- 6 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^4} \langle \partial_i u, \partial_j u \rangle \partial_i (f \circ u) \partial_j (f \circ u)
\end{aligned}$$

Since by condition (1.6)  $f$  is concave, then we have

$$\langle \partial_i u, \partial_j u \rangle (\nabla^2 f) \circ u (\partial_i u, \partial_j u) \leq 0. \quad (3.8)$$

To bound the other terms in (3.7) observe that the last term is nonpositive and the other terms can be bounded by using the Cauchy-Schwarz inequality. So we obtain from (3.7) and (3.8)

$$\begin{aligned}
\frac{1}{(f \circ u)^2} \operatorname{div} \left( F^{\frac{p-2}{2}} B(dF, \cdot) \right) &\leq \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + (p+2) \frac{F^{\frac{p-4}{2}}}{(f \circ u)^3} |\nabla(f \circ u)| |\nabla u|^2 |\nabla F| \\
&+ 2(\sqrt{m}+1) \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} |\nabla(f \circ u)| |\nabla u| |\nabla^2 u| + 2 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} |(\nabla f) \circ u| |\nabla u|^2 |\nabla^2 u|
\end{aligned}$$

which gives, by Young's inequality for  $\alpha > 0$  (to be chosen later),

$$\begin{aligned}
\frac{(p-2)}{(f \circ u)^2} \operatorname{div} \left( F^{\frac{p-2}{2}} B(dF, \cdot) \right) &\leq (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + \frac{\alpha}{2} (p+2)(p-2) \frac{F^{\frac{p-4}{2}}}{(f \circ u)^2} |\nabla F|^2 \\
&+ \frac{1}{2\alpha} (p+2)(p-2) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla(f \circ u)|^2 + \alpha (p-2) (\sqrt{m}+2) \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla^2 u|^2 \quad (3.9) \\
&+ \frac{1}{\alpha} (p-2) (\sqrt{m}+1) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla(f \circ u)|^2 + \frac{p-2}{\alpha} \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2 |\nabla u|^2,
\end{aligned}$$

where we have used the fact that  $F = |\nabla u|^2 + \varepsilon \geq |\nabla u|^2$ .

Now, to bound the terms  $4\frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \nabla F \cdot \nabla(f \circ u) - 6\frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla(f \circ u)|^2$  in (3.6), we use again Young's inequality for any  $\beta > 0$  :

$$4\frac{F^{\frac{p-2}{2}}}{(f \circ u)^3} \nabla F \cdot \nabla(f \circ u) - 6\frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla(f \circ u)|^2 \leq 2\beta\frac{F^{\frac{p-4}{2}}}{(f \circ u)^2} |\nabla F|^2 + \left(\frac{2}{\beta} - 6\right) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla(f \circ u)|^2. \quad (3.10)$$

By combining (3.6), (3.9) and (3.10), we obtain

$$\begin{aligned} \partial_t \varphi - L_p \varphi &\leq (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + \left( -2 + \alpha(p-2)(\sqrt{m}+2) \right) \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla^2 u|^2 \\ &+ \left( -\frac{1}{2}(p-2) + \frac{\alpha}{2}(p+2)(p-2) + 2\beta \right) \frac{F^{\frac{p-4}{2}}}{(f \circ u)^2} |\nabla F|^2 + 2K_1 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla u|^2 \\ &+ \left( -6 + \frac{2}{\beta} + \frac{1}{\alpha}(p-2) \left( \frac{p}{2} + \sqrt{m} + 2 \right) \right) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla(f \circ u)|^2 \\ &+ \left( -2\delta + \frac{p-2}{\alpha} \right) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2 |\nabla u|^2. \end{aligned} \quad (3.11)$$

Observe that  $|\nabla F|^2 = |\nabla(|\nabla u|^2)|^2 \leq 4|\nabla^2 u|^2 |\nabla u|^2$ . Hence if we choose  $\beta = \frac{p-1}{5}$  and fix  $\alpha > 0$  such that

$$-2 + \alpha(p-2)(\sqrt{m}+2) \leq 0, \quad (3.12)$$

then it follows from (3.11) that

$$\begin{aligned} \partial_t \varphi - L_p \varphi &\leq (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + 2K_1 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla u|^2 \\ &- \left( \frac{p-1}{10} - \frac{1}{4}\alpha(p-2)(\sqrt{m}+2p+6) \right) \frac{F^{\frac{p}{2}-2}}{(f \circ u)^2} |\nabla F|^2 \\ &\left( -6 + \frac{10}{p-1} + \frac{1}{\alpha}(p-2) \left( \frac{p}{2} + \sqrt{m} + 2 \right) \right) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |\nabla(f \circ u)|^2 \\ &+ \left( -2\delta + \frac{p-2}{\alpha} \right) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2 |\nabla u|^2. \end{aligned} \quad (3.13)$$

If we choose

$$\alpha = \begin{cases} \frac{1}{5(p-2)(\sqrt{m}+2p+2)} & \text{if } p > 2 \\ 1 & \text{if } p = 2, \end{cases}$$

then it is clear that (3.12) is satisfied, and observe that  $|\nabla(f \circ u)| \leq |(\nabla f) \circ u| |\nabla u|$ , so it follows from (3.13) that

$$\begin{aligned} \partial_t \varphi - L_p \varphi &\leq (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) + 2K_1 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla u|^2 \\ &\quad - \frac{1}{20} \frac{F^{\frac{p-4}{2}}}{(f \circ u)^2} |\nabla F|^2 + (-2\delta + c_p) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2 |\nabla u|^2, \end{aligned} \quad (3.14)$$

where

$$c_p = -6 + \frac{10}{p-1} + 5(p-2)^2 (\sqrt{m} + 2p + 6) \left( \sqrt{m} + \frac{p}{2} + 3 \right).$$

On the other hand we have

$$\partial_i F = (f \circ u)^2 \partial_i \phi + 2 \frac{\langle (\nabla f) \circ u, \partial_i u \rangle}{f \circ u},$$

so

$$|\nabla F|^2 \geq \frac{1}{2} (f \circ u)^4 |\nabla \phi|^2 - 4 \frac{F}{(f \circ u)^2} |\nabla(f \circ u)|^2.$$

Hence it follows from (3.14)

$$\begin{aligned} \partial_t \varphi - L_p \varphi &\leq (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right) - \frac{1}{40} F^{\frac{p-4}{2}} (f \circ u)^2 |\nabla \varphi|^2 \\ &\quad + (-2\delta + c_p + \frac{1}{5}) \frac{F^{\frac{p}{2}}}{(f \circ u)^4} |(\nabla f) \circ u|^2 |\nabla u|^2 + 2K_1 \frac{F^{\frac{p-2}{2}}}{(f \circ u)^2} |\nabla u|^2, \end{aligned}$$

This proves the proposition since  $\frac{1}{2}c_p + \frac{1}{10} \leq \delta_p := 3(p-2)^2 (\sqrt{m} + 2p + 6)^2 + 3$ . □

Proposition 3.1 allows to prove the following important gradient estimates :

**Proposition 3.2.** *Let  $(M^m, g)$  and  $(N^n, h)$  be two complete Riemannian manifolds. Let  $u : M \times [0, T) \rightarrow \Omega$  a smooth solution of (2.1), where  $\Omega$  is a  $\delta$ -regular set with*

$$\delta > \delta_p := 3(p-2)^2 (\sqrt{m} + 2p + 2)^2 + 3.$$

Let  $(x_0, t_0) \in M \times (0, T)$  and  $R > 0$ . Then if  $t_0 > R$ , we have

$$\|\nabla u\|_{L^\infty(B(x_0, R/2) \times [t_0 - R/2, t_0])} \leq C_R \left( \int_{B(x_0, R) \times [t_0 - R, t_0]} |\nabla u|^p dx dt + 1 \right),$$

and if  $t_0 \leq R$ , we have

$$\|\nabla u\|_{L^\infty(B(x_0, R/2) \times [0, t_0])} \leq C_R \left( \int_{B(x_0, R) \times [0, t_0]} |\nabla u|^p dx dt + \|\nabla u_0\|_{L^\infty(B(x_0, R))} + 1 \right),$$

where  $C_R$  is a positive constant depending on  $R, p, M$  and  $\Omega$ .

*Proof.* Fix  $(x_0, t_0) \in M \times (0, T)$  and let  $R > 0$ . In what follows  $C_R$  is a positive constant that depends on  $R, p, M$  and  $\Omega$  and its value may change from line to line. In this proof we suppose that  $t_0 > R$ , as the case  $t_0 \leq R$  is easier to handle, and therefore, we omit it. For  $0 < r < R$ , we set  $Q_r = B(x_0, r) \times (t_0 - r, t_0)$  where  $B(x_0, r)$  is the geodesic ball of radius  $r$ . Let  $0 < \rho < r < R$  and let  $\phi \in C_0^1(B(x_0, r) \times (t_0 - r, \infty))$  such that  $\phi = 1$  on  $Q_\rho$  with

$$0 \leq \phi \leq 1, \quad |\nabla \phi| \leq \frac{C_m}{r - \rho}, \quad |\partial_t \phi| \leq \frac{C_m}{r - \rho}, \quad (3.15)$$

where  $C_m$  is a positive constant depending only on the dimension  $m$  of  $M$ .

As in Proposition 3.1, let  $\varphi = \frac{F}{f^2(u)}$ , where  $F = |\nabla u|^2 + \varepsilon$  and  $f$  satisfies condition (1.6). If we multiply inequality (3.1) by  $\varphi^\gamma \phi^2$ , where  $\gamma \geq 0$ , and we integrate on  $Q_r$ , by using the hypothesis that  $\delta \geq \delta_p$ , we get

$$\begin{aligned} & \frac{1}{\gamma + 1} \sup_{t \leq t_0} \int_{B(x_0, r)} \varphi^{\gamma+1} \phi^2 dx - \frac{2}{\gamma + 1} \int_{Q_r} \varphi^{\gamma+1} \phi \partial_t \phi dx dt \\ & + \gamma \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma - 2} |\nabla \varphi|^2 dx dt - 2 \int_{Q_r} f^{p-2}(u) \varphi^{\frac{p}{2} + \gamma - 1} \phi |\nabla \phi| |\nabla \varphi| dx dt \\ & \leq -\frac{1}{40} \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma - 2} |\nabla \varphi|^2 dx dt + 2K_R \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma} dx dt \\ & - (p-2)\gamma \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma - 2} B(d\varphi, d\varphi) dx dt \\ & - 2(p-2) \int_{Q_r} f^{p-2}(u) \phi \varphi^{\frac{p}{2} + \gamma - 1} B(d\varphi, d\phi) dx dt, \end{aligned} \quad (3.16)$$

where  $-K_R \leq 0$  is a lower bound of the Ricci curvature of  $M$  on  $B(x_0, R)$  and  $B$  is given by (2.5). We have by (2.6) that

$$B(d\varphi, d\varphi) \geq 0 \quad \text{and} \quad |B(d\varphi, d\phi)| \leq |\nabla \varphi| |\nabla \phi|. \quad (3.17)$$

It follows from (3.15), (3.16) and (3.17) that

$$\begin{aligned} & \frac{1}{\gamma + 1} \sup_{t \leq t_0} \int_{B(x_0, r)} \varphi^{\gamma+1} \phi^2 dx + \left( \gamma + \frac{1}{40} \right) \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma - 2} |\nabla \varphi|^2 dx dt \leq \\ & \frac{2}{(\gamma + 1)(r - \rho)} \int_{Q_r} \varphi^{\gamma+1} \phi dx dt + 2K_R \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma} dx dt \\ & + 2(p-1) \frac{1}{r - \rho} \int_{Q_r} f^{p-2}(u) \varphi^{\frac{p}{2} + \gamma - 1} \phi |\nabla \varphi| dx dt \end{aligned}$$

which gives, by applying Young's inequality to the last term,

$$\begin{aligned} & \frac{1}{\gamma + 1} \sup_{t \leq t_0} \int_{B(x_0, r)} \varphi^{\gamma+1} \phi^2 dx + \frac{1}{2} \left( \gamma + \frac{1}{40} \right) \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma - 2} |\nabla \varphi|^2 dx dt \leq \\ & \frac{2}{(\gamma + 1)(r - \rho)} \int_{Q_r} \varphi^{\gamma+1} \phi dx dt + 2K_R \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\frac{p}{2} + \gamma} dx dt \\ & 2 \left( \gamma + \frac{1}{40} \right)^{-1} \frac{(p-1)^2}{(r - \rho)^2} \int_{Q_r} f^{p-2}(u) \varphi^{\frac{p}{2} + \gamma} dx dt. \end{aligned} \quad (3.18)$$

It is easy to see that

$$\begin{aligned} \int_{Q_r} f^{p-2}(u) \phi^2 \varphi^{\gamma+\frac{p}{2}-2} |\nabla \varphi|^2 dxdt &\geq \frac{8}{(p+2\gamma)^2} \int_{Q_r} |\nabla(\varphi^{\frac{\gamma}{2}+\frac{p}{4}} \phi)|^2 f^{p-2}(u) dxdt \\ &\quad - \frac{16}{(p+2\gamma)^2} \int_{Q_r} \varphi^{\gamma+\frac{p}{2}} f^{p-2}(u) |\nabla \phi|^2 dxdt. \end{aligned} \quad (3.19)$$

Thus, if we multiply (3.19) by  $\gamma + 1$  and combine it with (3.18) and (3.15), using the fact that  $C^{-1} \leq f \leq C$ , we get

$$\begin{aligned} \sup_{t \leq t_0} \int_{B(x_0, r)} \varphi^{\gamma+1} \phi^2 dx + \int_{Q_r} |\nabla(\varphi^{\frac{\gamma}{2}+\frac{p}{4}} \phi)|^2 dxdt &\leq \\ C_R \left( \left( \gamma + 1 + \frac{1}{(r-\rho)^2} \right) \int_{Q_r} \varphi^{\frac{p}{2}+\gamma} dxdt + \frac{1}{r-\rho} \int_{Q_r} \varphi^{\gamma+1} dxdt \right). \end{aligned} \quad (3.20)$$

We recall the following Sobolev inequality for  $m > 2$  and  $V \in C_0^\infty(B(x_0, R))$

$$\left( \int_{B(x_0, R)} V^{\frac{2m}{m-2}} dx \right)^{\frac{m-2}{2m}} \leq C_R \left( \int_{B(x_0, R)} |\nabla V|^2 dx \right)^{\frac{1}{2}}.$$

For  $m = 2$  we have for any  $s \geq 1$ ,

$$\left( \int_{B(x_0, R)} V^s dx \right)^{\frac{1}{s}} \leq C_{R,s} \left( \int_{B(x_0, R)} |\nabla V|^2 dx \right)^{\frac{1}{2}}.$$

In this proof we shall consider only the case  $m > 2$ . The case  $m = 2$  can be handled in the same way. Applying Sobolev inequality to  $V = \varphi^{\frac{\gamma}{2}+\frac{p}{4}} \phi$  we get

$$\int_{t_0-r}^{t_0} \left( \int_{B(x_0, r)} (\varphi^{\frac{\gamma}{2}+\frac{p}{4}} \phi)^{\frac{2m}{m-2}} dx \right)^{\frac{m-2}{m}} dt \leq C_R \int_{Q_r} |\nabla(\varphi^{\frac{\gamma}{2}+\frac{p}{4}} \phi)|^2 dxdt. \quad (3.21)$$

On the other hand if we set  $q_\gamma = (1 + \frac{2}{m})\gamma + \frac{p}{2} + \frac{2}{m}$ , then by using the fact that  $\phi = 1$  on  $Q_\rho$ , we infer from Holder's inequality that

$$\begin{aligned} \int_{Q_\rho} \varphi^{q_\gamma} dxdt &= \int_{Q_\rho} \phi^{\frac{4}{m}} \varphi^{\frac{2\gamma}{m} + \frac{2}{m}} \phi^2 \varphi^{\gamma+\frac{p}{2}} dxdt \leq \\ \sup_{t \leq t_0} \left( \int_{B(x_0, r)} \varphi^{\gamma+1} \phi^2 dx \right)^{\frac{2}{m}} &\times \int_{t_0-r}^{t_0} \left( \int_{B(x_0, r)} (\varphi^{\frac{\gamma}{2}+\frac{p}{4}} \phi)^{\frac{2m}{m-2}} dx \right)^{\frac{m-2}{m}} dt. \end{aligned} \quad (3.22)$$

Hence by combining (3.20), (3.21) and (3.22) we obtain

$$\begin{aligned} \int_{Q_\rho} \varphi^{q_\gamma} dxdt &\leq \\ C_R \left( \left( \gamma + 1 + \frac{1}{(r-\rho)^2} \right) \int_{Q_r} \varphi^{\frac{p}{2}+\gamma} dxdt + \frac{1}{r-\rho} \int_{Q_r} \varphi^{\gamma+1} dxdt \right)^{1+\frac{2}{m}}. \end{aligned} \quad (3.23)$$

By Hölder's inequality and Young' inequality we have

$$\frac{1}{r-\rho} \int_{Q_r} \varphi^{\gamma+1} dx \leq |Q_r| + (r-\rho)^{-\frac{2\gamma+p}{2\gamma+2}} \int_{Q_r} \varphi^{\frac{p}{2}+\gamma} dxdt,$$

where  $|Q_r|$  is the volume of  $Q_r$ . Since  $|Q_r| \leq |Q_R| \leq C_R$ , then it follows from (3.23) that

$$\begin{aligned} & \int_{Q_\rho} \varphi^{a_j} dxdt \leq \\ & C_R \left( (\gamma + 1 + (r - \rho)^{-2} + (r - \rho)^{-\frac{2\gamma+p}{2\gamma+2}}) \int_{Q_r} \varphi^{\frac{p}{2}+\gamma} dxdt + 1 \right)^{1+\frac{2}{m}}. \end{aligned} \quad (3.24)$$

We apply now Moser iteration process. For  $j \in \mathbb{N}$ , let  $R_j = \frac{R(1+2^{-j})}{2}$  and  $\theta = 1 + \frac{2}{m}$ . We define  $\gamma_j = \theta^j - 1$  and  $a_j = \gamma_j + \frac{p}{2}$ . Then we have

$$a_{j+1} = \theta \gamma_j + \frac{p}{2} + \frac{2}{m} = \theta a_j - \frac{p-2}{m}.$$

If we set  $\gamma = \gamma_j$ ,  $r = R_j$ ,  $\rho = R_{j+1}$ , then it is easy to check that

$$\gamma + 1 + (r - \rho)^{-2} + (r - \rho)^{-\frac{2\gamma+p}{2\gamma+2}} \leq C_R 4^{pj}.$$

Thus it follows from (3.24) that

$$\int_{Q_{R_{j+1}}} \varphi^{a_{j+1}} dxdt \leq C_R \left( 4^{pj} \int_{Q_{R_j}} \varphi^{a_j} dxdt + 1 \right)^\theta$$

which gives by setting  $I_j = \left( \int_{Q_{R_j}} \varphi^{a_j} dxdt + 1 \right)^{\theta^{-j}}$ ,

$$I_{j+1} \leq C_R^{\theta^{-j-1}} 4^{pj\theta^{-j}} I_j.$$

Since  $\sum_{j=0}^{\infty} j\theta^{-j} \leq C$ , then by iterating we get

$$I_{j+1} \leq C_R I_0 \quad (3.25)$$

Now observing that

$$\left( \int_{Q_{R/2}} \varphi^{a_j} dxdt \right)^{\frac{1}{a_j}} \leq I_j^{\frac{\theta^j}{a_j}}$$

and using the fact that  $\lim_{j \rightarrow +\infty} \frac{\theta^j}{a_j} = 1$ , then it follows from (3.25) that

$$\|\varphi\|_{L^\infty(Q_{R/2})} \leq C_R I_0 = C_R \left( \int_{Q_R} \varphi^{\frac{p}{2}} dxdt + 1 \right).$$

This proves the proposition since  $\varphi = \frac{|\nabla u|^2 + \varepsilon}{f(u)}$  and  $C^{-1} \leq f \leq C$ .

□

## 4. GLOBAL EXISTENCE AND CONVERGENCE

In this section we make use of our gradient estimates on the solution of the regularised  $p$ -harmonic flow obtained in Section 3 to prove our main results.

**Proof of theorem 1.** In this proof  $C$  denotes a positive constant depending on  $M, p, \Omega$  and the initial datum  $u_0$ , and whose value may change from line to line. Let  $u_\varepsilon$  be the solution of the regularised equation (2.1) and let  $[0, T_\varepsilon)$  be its maximal existence interval. Since by Proposition 2.1,  $u_\varepsilon$  has its image in  $\Omega$ , then if we apply Proposition 3.2 by taking  $R = 1$ , using the compactness of  $M$  and the fact that the  $p$ -energy functional  $E_{p,\varepsilon}$  is nonincreasing along the flow (formula (2.3) in Proposition 2.1), we get

$$\|\nabla u_\varepsilon\|_{L^\infty(M \times [0, T_\varepsilon))} \leq C. \quad (4.1)$$

Suppose by contradiction that  $T_\varepsilon < +\infty$ . Then by integrating formula (2.3) in Proposition 2.1, we have

$$\int_0^{T_\varepsilon} \int_M |\partial_t u_\varepsilon(x, t)|^2 dx dt \leq E_{p,\varepsilon}(u_0) \leq E_{p,1}(u_0). \quad (4.2)$$

On the other hand, we have for all  $t \in [0, T_\varepsilon)$  the following bound on the mean value  $\bar{u}_\varepsilon(t)$  of  $u_\varepsilon(\cdot, t)$ :

$$|\bar{u}_\varepsilon(t)| = \frac{1}{|M|} \left| \int_M u_\varepsilon(x, t) dx \right| \leq |\bar{u}_0| + \frac{1}{|M|} \int_0^{T_\varepsilon} \int_M |\partial_t u_\varepsilon(x, t)| dx dt,$$

which implies by using the Cauchy-Schwarz inequality and (4.2)

$$|\bar{u}_\varepsilon(t)| \leq |\bar{u}_0| + C\sqrt{T_\varepsilon}. \quad (4.3)$$

We have by the mean-value Theorem that

$$\sup_{x \in M} |u_\varepsilon(x, t) - \bar{u}_\varepsilon(t)| \leq \text{diam}(M) \|\nabla u_\varepsilon(\cdot, t)\|_{L^\infty(M)}, \quad (4.4)$$

where  $\text{diam}(M)$  is the diameter of  $M$ . Hence it follows from (4.1), (4.3) and (4.4) that

$$\|u_\varepsilon\|_{L^\infty(M \times [0, T_\varepsilon))} + \|\nabla u_\varepsilon\|_{L^\infty(M \times [0, T_\varepsilon))} \leq C + C\sqrt{T_\varepsilon}. \quad (4.5)$$

Using (4.5) and the results of Dibenedetto [2], we have for some  $\beta \in (0, 1)$ ,

$$\|u_\varepsilon\|_{C^{1+\beta, \beta/p}(M \times [0, T_\varepsilon))} \leq C_{T_\varepsilon}, \quad (4.6)$$

where the constant  $C_{T_\varepsilon}$  depends also on  $T_\varepsilon$ . The theory of linear parabolic equations (see [10]) together with (4.6) give, for some  $0 < \alpha < 1$ ,

$$\|u_\varepsilon\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T_\varepsilon))} \leq C_{T_\varepsilon} \quad (4.7)$$

where  $C_{T_\varepsilon}$  is a new constant that also depends on the modulus of ellipticity  $\varepsilon$ . Estimate (4.7) implies that  $u_\varepsilon$  can be extended beyond  $T_\varepsilon$  contradicting thus the maximality of  $T_\varepsilon$ . Hence we have  $T_\varepsilon = +\infty$ .

Now we are in position to prove Theorem 1. By the result above,  $u_\varepsilon$  is defined on  $[0, +\infty)$  and we have by (4.6) for all  $T > 0$

$$\|u_\varepsilon\|_{C^{1+\beta, \beta/p}(M \times [0, T])} \leq C_T, \quad (4.8)$$

where  $C_T$  is a positive constant depending on  $T, u_0, p, M$  and  $\Omega$  but not on  $\varepsilon$ .

It follows from estimate (4.8) that there exist a sequence  $\varepsilon_k \rightarrow 0$  and a map  $u \in C_{loc}^{1+\beta, \beta/p}(M \times [0, +\infty), \Omega)$  such that  $u_{\varepsilon_k} \rightarrow u$  in  $C_{loc}^{1+\beta', \beta'/p}(M \times [0, +\infty), \Omega)$  for all  $\beta' < \beta$ . In addition, the energy formula (2.3) in Proposition 2.1 gives for all  $T > 0$ ,

$$\int_0^T \int_M |\partial_t u_{\varepsilon_k}(x, t)|^2 dx dt + E_{p, \varepsilon_k}(u_{\varepsilon_k}(\cdot, T)) \leq E_{p, \varepsilon_k}(u_0) \quad (4.9)$$

which implies that  $\partial_t u_{\varepsilon_k} \rightarrow \partial_t u$  weakly in  $L^2(M \times [0, +\infty))$  and we have the energy inequality for the limit  $u$

$$\int_0^T \int_M |\partial_t u(x, t)|^2 dx dt + E_p(u(\cdot, T)) \leq E_p(u_0). \quad (4.10)$$

Passing to the limit in (2.1) when  $\varepsilon_k \rightarrow 0$ , one can easily check that  $u$  is a solution of (1.5).

In order to prove the uniqueness of solutions, we recall the following well known inequality valid for any  $a, b \in \mathbb{R}^L$  and  $p \geq 2$  :

$$\langle |a|^{p-2}a - |b|^{p-2}b, a - b \rangle \geq |a - b|^p, \quad (4.11)$$

where  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  denotes Euclidean inner product and the corresponding Euclidean norm in  $\mathbb{R}^L$ .

Let  $T > 0$  and let  $u_1, u_2 \in C^{1+\beta}(M \times [0, T])$  be two solutions of (1.5) such that  $u_1(\cdot, 0) = u_2(\cdot, 0)$ . If we set  $w = u_1 - u_2$ , then taking the difference of the equations satisfied by  $u_1$  and  $u_2$  (the same as equation (1.5)), multiplying it by  $w$ , integrating on  $M \times [0, T]$ , and using (4.11) along with the fact that  $\nabla u_1$  and  $\nabla u_2$  are bounded in  $L^\infty(M \times [0, T])$ , one can easily check that for any  $t \in [0, T]$ ,

$$\int_M |w(x, t)| dx \leq C \int_0^t \int_M |w(x, s)| dx ds.$$

The right hand side of the above inequality is increasing in  $t$ , therefore

$$\sup_{t' \in [0, t]} \int_M |w(\cdot, t')|^2 dx \leq Ct \sup_{t' \in [0, t]} \int_M |w(\cdot, t')| dx,$$

thus, for  $t < \frac{1}{C}$  we get  $w \equiv 0$  for  $t' \in [0, t]$ . Iterating the argument proves the assertion.

Now let us prove the convergence of the flow at infinity when the target manifold  $N$  is compact. First we observe that by the energy inequality (4.10) we have

$$\int_0^{+\infty} \int_M |\partial_t u(x, t)|^2 dx dt \leq E_p(u_0),$$

which implies the existence of a sequence  $t_k \rightarrow +\infty$  such that

$$\int_M |\partial_t u(x, t_k)|^2 dx \rightarrow 0 \quad \text{as } t_k \rightarrow +\infty. \quad (4.12)$$

On the other hand, it follows from estimate (4.1) and the fact that  $N$  is compact

$$\|u\|_{L^\infty(M \times [0, +\infty))} + \|\nabla u\|_{L^\infty(M \times [0, +\infty))} \leq C$$

and the results of Dibenedetto [2] imply that

$$\|u\|_{C^{1+\beta, \beta/p}(M \times [0, +\infty))} \leq C. \quad (4.13)$$

Hence by passing to a subsequence if necessary, we deduce from (4.13) that  $(u(\cdot, t_k))_k$  converges in  $C^{1+\beta'}(M, \Omega)$  for all  $\beta' < \beta$  to a map  $u_\infty \in C^{1+\beta}(M, \Omega)$ . By passing to the limit in equation (1.5) and using (4.12) we have that  $u_\infty$  is a  $p$ -harmonic map satisfying  $E_p(u_\infty) \leq E_p(u_0)$ . The proof of Theorem 1 is complete.  $\square$

The proof of Theorem 2 relies on Theorem 1 by using an exhaustion of  $M$  by a sequence of compact manifolds and the following proposition.

**Proposition 4.1.** *Let  $u$  be the solution of problem (1.5) given by Theorem 1. Then for any ball  $B(x_0, R) \subset M$ , there exists a constant  $C_R > 0$  depending on  $B(x_0, R), p$  and  $\Omega$  such that*

$$\sup_{t \geq 0} \|\nabla u(\cdot, t)\|_{L^\infty(B(x_0, R/2))} \leq C_R \left( \int_M |\nabla u_0|^p dx + \|\nabla u_0\|_{L^\infty(B(x_0, R))} + 1 \right).$$

*Proof.* As in the proof of Theorem 1,  $u$  is the limit of sequence  $(u_{\varepsilon_k})_k$  of solutions to the regularised problem 2.1 such that  $\varepsilon_k \rightarrow 0$ . Then if we apply Proposition 3.2 to  $u_{\varepsilon_k}$  and pass to the limit when  $k \rightarrow \infty$ , we obtain easily the desired result.  $\square$

**Proof of Theorem 2 .** Suppose that  $(M, g)$  is a complete noncompact Riemannian manifold. Let  $(U_i)_{i \geq 1}$  be an exhaustion of  $M$  by compact manifolds with smooth boundaries. More precisely, each  $U_i$  is an open set of  $M$  such that  $\bar{U}_i$  is a compact manifold with smooth boundary  $\partial U_i$  and

$$\begin{cases} \bar{U}_i \subset U_{i+1} \\ \bigcup_{i \geq 1} U_i = M. \end{cases} \quad (4.14)$$

In order to apply Theorem 1 it is necessary to consider manifolds without boundary. To this end, we consider for each  $i \geq 1$ , the double manifold of  $U_i$  that we denote by  $\tilde{U}_i$ . Thus  $\tilde{U}_i$  is a compact manifold without boundary such that  $\bar{U}_i \subset \tilde{U}_i$  and the metric  $g$  on  $\bar{U}_i$  extends to a  $C^1$ -metric  $\tilde{g}_i$  on  $\tilde{U}_i$ . We smooth out  $\tilde{g}_i$  on a neighborhood of  $\partial U_i$ . More precisely, for a fixed  $0 < \varepsilon < \frac{1}{4} \text{diam}(U_1)$ , where  $\text{diam}(U_1)$  is the diameter of  $U_1$ , we let

$$U_i^\varepsilon = \{x \in U_i : d(x, \partial U_i) > \varepsilon\}.$$

Then the new metric, that we denote still  $\tilde{g}_i$  for simplicity, is  $C^\infty$  on  $\tilde{U}_i$ , it can be chosen arbitrary close to  $g$  in the  $C^1$ -norm and it satisfies

$$\tilde{g}_i = g \quad \text{on } U_i^\varepsilon. \quad (4.15)$$

In the same way, we extend the initial datum  $u_0$  to a map  $\tilde{u}_{0,i}$  on  $\tilde{U}_i$  that we smooth out on a neighborhood of  $\partial U_i$ , and one can choose  $\tilde{u}_{0,i}$  arbitrary close to  $u_0$  in the  $C^1$ -norm. Thus we have  $\tilde{u}_{0,i} \in C^\infty(\tilde{U}_i, \Omega)$  with

$$\tilde{u}_{0,i} = u_0 \text{ on } U_i^\varepsilon, \quad (4.16)$$

and we may suppose without loss of generality that

$$\int_{\tilde{U}_i} |\nabla \tilde{u}_{0,i}|^p d\tilde{g}_i \leq 2 \int_{U_i} |\nabla u_0|^p dx + 1, \quad (4.17)$$

where  $d\tilde{g}_i$  is the volume element with respect to  $\tilde{g}_i$  and  $dx$  is the volume element with respect to  $g$ .

Then we consider on each  $\tilde{U}_i$  the  $p$ -harmonic heat flow problem

$$\begin{cases} \partial_t u - \tilde{\Delta}_{p,i} u = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u), \\ u(x, 0) = \tilde{u}_{0,i}(x), \end{cases} \quad (4.18)$$

where  $\tilde{\Delta}_{p,i}$  is the  $p$ -Laplacian with respect to the metric  $\tilde{g}_i$ .

Thanks to Theorem 1, problem (4.18) admits a global solution  $u_i \in C_{loc}^{1+\beta, \beta/p}(\tilde{U}_i \times [0, +\infty), \Omega)$  satisfying the  $p$ -energy inequality

$$\int_0^T \int_{\tilde{U}_i} |\partial_t u_i|^2 d\tilde{g}_i dt + \frac{1}{p} \int_{\tilde{U}_i} |\nabla u_i(\cdot, T)|^p d\tilde{g}_i \leq \frac{1}{p} \int_{\tilde{U}_i} |\nabla \tilde{u}_{0,i}|^p d\tilde{g}_i \quad (4.19)$$

for all  $T > 0$ .

Since  $\tilde{g}_i = g$  on  $U_i^\varepsilon$ , then  $u_i$  is a solution of equation (1.5) in  $U_i^\varepsilon$ . We shall prove uniform gradient estimates on  $u_i$  on fixed balls of  $M$ . For each fixed  $R > 0$ , we denote by  $C_R$  a positive constant that depends on  $R, M, p, \Omega$  and the initial datum  $u_0$ , and whose value may change from line to line. Fix  $x_0 \in M$  and  $R > 0$ , then we have from (4.14) and the definition of  $U_i^\varepsilon$ , for  $i$  large enough, that  $B(x_0, R) \subset U_i^\varepsilon$ . It follows from Proposition 4.1 that

$$\sup_{t \geq 0} \|\nabla u_i(\cdot, t)\|_{L^\infty(B(x_0, R/2))} \leq C_R \left( \int_{\tilde{U}_i} |\nabla \tilde{u}_{0,i}|^p dx + \|\nabla \tilde{u}_{0,i}\|_{L^\infty(B(x_0, R))} + 1 \right)$$

and by (4.17) we get

$$\begin{aligned} \sup_{t \geq 0} \|\nabla u_i(\cdot, t)\|_{L^\infty(B(x_0, R/2))} &\leq C_R \left( \int_M |\nabla u_0|^p dx + \|\nabla u_0\|_{L^\infty(B(x_0, R))} + 1 \right) \\ &\leq C_R. \end{aligned} \quad (4.20)$$

As in the proof of Theorem 1, by using (4.17), (4.19), (4.20) and the mean value Theorem, we have for any  $T > 0$ ,

$$\|u_i\|_{L^\infty(B(x_0, R/2) \times [0, T])} \leq C_R + C_R \sqrt{T}. \quad (4.21)$$

It follows from (4.20), (4.21) and the results of Dibenedetto [2] on degenerate parabolic equations that

$$\|u_i\|_{C^{1+\beta, \beta/p}(B(x_0, R/2) \times [0, T])} \leq C_{R, T}, \quad (4.22)$$

for some constant  $\beta \in (0, 1)$ , where the constant  $C_{R, T}$  depends also on  $T$ .

Since  $B(x_0, R) \subset U_i^\varepsilon$  for  $i$  large enough and  $\tilde{g}_i = g$  on  $U_i^\varepsilon$ , then we have from (4.19) and (4.17) that

$$\int_0^T \int_{B(x_0, R)} |\partial_t u_i(x, t)|^2 dx dt \leq E_p(u_0) + 1. \quad (4.23)$$

Thus if we set  $T = R = R_k$ , where  $(R_k)_k$  is a sequence such that  $R_k \rightarrow +\infty$ , then by using the Cantor diagonal argument, it follows from (4.22) and (4.23) that there exists a subsequence  $(u_{i_k})_k$  and a map  $u \in C_{loc}^{1+\beta, \beta/p}(M \times [0, +\infty), \Omega)$  such that

$$u_{i_k} \longrightarrow u \text{ in } C^{1+\beta', \beta'/p}(B(x_0, R) \times [0, T], \Omega) \text{ for all } R, T > 0, 0 < \beta' < \beta,$$

and

$$\partial_t u_{i_k} \longrightarrow \partial_t u \text{ weakly in } L^2(B(x_0, R) \times [0, T], \Omega) \text{ for all } R, T > 0.$$

It is easy to check that by passing to the limit in (4.18) and using (4.15) and (4.16),  $u$  is a solution of (1.5). By passing to the limit in (4.19), one obtains formula (1.9) in Theorem 2.

When  $N$  is compact, the convergence of the flow can be proved in the same way as in the proof of Theorem 1. Indeed, if we take  $i = i_k$  in (4.23) and pass to the limit when  $k \rightarrow +\infty$ , we obtain for any  $R, T > 0$

$$\int_0^T \int_{B(x_0, R)} |\partial_t u(x, t)|^2 dx dt \leq E_p(u_0) + 1.$$

which implies that by letting  $T \rightarrow +\infty$  and  $R \rightarrow +\infty$ ,

$$\int_0^\infty \int_M |\partial_t u(x, t)|^2 dx dt \leq E_p(u_0) + 1. \quad (4.24)$$

It follows from (4.24) that there exists a sequence  $t_j \rightarrow +\infty$  such that

$$\int_M |\partial_t u(x, t_j)|^2 dx \rightarrow 0 \text{ as } t_j \rightarrow +\infty. \quad (4.25)$$

On the other hand, if we take  $i = i_k$  in (4.20) and pass to the limit when  $k \rightarrow +\infty$ , we obtain for any  $R > 0$

$$\|\nabla u\|_{L^\infty(B(x_0, R) \times [0, +\infty))} \leq C_R$$

which together with the results of Dibenedetto [2] imply, since  $N$  is compact,

$$\|u\|_{C^{1+\beta, \beta/p}(B(x_0, R) \times [0, +\infty))} \leq C_R. \quad (4.26)$$

Hence by taking a sequence  $R_j \rightarrow \infty$ , it follows from (4.25), (4.26) and the Cantor Diagonal argument that the sequence  $u(\cdot, t_j)$  admits a subsequence that converges in  $C^{1+\beta'}(M, \Omega)$  for all  $\beta' < \beta$  to a map  $u_\infty \in C^{1+\beta}(M, \Omega)$ . By passing to the limit in equation (1.5) and using (4.25) we deduce that  $u_\infty$  is a  $p$ -harmonic map satisfying  $E_p(u_\infty) \leq E_p(u_0)$ . The proof of Theorem 2 is complete. □

**Proof of Theorem 3 .** Theorem 3 is a direct consequence of Theorem 2 since a manifold  $N$  with nonpositive sectional curvature is a  $\delta$ -generalised regular ball for any  $\delta > 0$  (see Example 1.1 ). In this case, it suffices to apply Theorem 2 by taking any  $\delta > \delta_p$ . □

For the proof of Theorem 4 we need a modified version of Proposition 3.1 concerning solutions of the  $p$ -harmonic equation (1.4).

**Proposition 4.2.** *Let  $(M^m, g)$ ,  $(N^n, h)$  be two Riemannian manifolds and let  $\Omega \subset N$  be a  $\delta$ -regular set. Let  $u \in C^1(M, \Omega)$  a  $p$ -harmonic map and set*

$$\varphi(x) = \frac{F(x, t)}{f^2(u(x))},$$

where  $F(x) = |\nabla u(x)|^2$  and  $f$  is the function satisfying condition (1.6). Then we have on the set  $\{x \in M : \nabla u(x) \neq 0\}$ ,

$$\begin{aligned} -\operatorname{div}\left(F^{\frac{p-2}{2}}\nabla\varphi\right) &\leq (p-2)\operatorname{div}\left(F^{\frac{p-2}{2}}B(d\varphi, \cdot)\right) - \frac{1}{40}F^{\frac{p-4}{2}}(f \circ u)^2|\nabla\varphi|^2 \\ &- 2(\delta - \delta_p)\frac{F^{\frac{p}{2}}}{(f \circ u)^4}|(\nabla f) \circ u|^2|\nabla u|^2 + 2K_1\frac{F^{\frac{p-2}{2}}}{(f \circ u)^2}|\nabla u|^2, \end{aligned} \quad (4.27)$$

where  $\delta_p = 3(p-2)^2(\sqrt{m} + 2p + 2)^2 + 3$  and  $-K_1 \leq 0$  is a lower bound of the Ricci curvature of  $M$ . The tensor  $B$  is defined in Section 2 by (2.5) (by taking  $\varepsilon = 0$ ).

*Proof.* The proof is exactly the same as that of Proposition 3.1. It is even more easier since the parabolic term  $\partial_t u$  is not present. Nevertheless, we have to consider only points  $x \in M$  such that  $\nabla u(x) \neq 0$ . The reason is that our  $p$ -harmonic map is sufficiently smooth at such points to apply the elliptic version of Bochner formula (2.7). □

**Proof of Theorem 4 .** In this proof  $C$  denotes a positive constant depending only on  $M, \Omega$  and  $p$ , and whose value may change from line to line. Define the set  $E$  by

$$E = \{x \in M : \nabla u(x) \neq 0\}$$

which is an open set of  $M$  since  $u \in C^1(M)$ . By the regularity theory of elliptic equations we have  $u \in C^\infty(E)$ .

As in Proposition 4.2, we set  $\varphi = \frac{F}{f^2(u)}$ , where  $F = |\nabla u|^2$ , and  $f$  is as in (1.6). For  $\varepsilon > 0$ , let  $\varphi_\varepsilon = (\varphi - \varepsilon)^+$ . Then  $\varphi_\varepsilon$  is a locally Lipschitz function on  $M$  with support in  $E$  and satisfies

$$\begin{cases} (i) & \varphi_\varepsilon(x) = 0 & \text{if } \varphi(x) < \varepsilon \\ (ii) & \varphi_\varepsilon(x) = \varphi(x) - \varepsilon & \text{if } \varphi(x) \geq \varepsilon \\ (iii) & \nabla\varphi_\varepsilon(x) = \nabla\varphi(x) & \text{if } \varphi(x) \geq \varepsilon \\ (iv) & \nabla\varphi_\varepsilon(x) = 0 & \text{if } \varphi(x) < \varepsilon. \end{cases} \quad (4.28)$$

Fix a point  $x_0 \in E$  and  $R > 0$ , and let  $\phi_R \in C_0^1(B(x_0, 2R))$  such that

$$\begin{cases} 0 \leq \phi_R \leq 1 \\ \phi_R = 1 \text{ on } B(x_0, R) \\ |\nabla\phi_R| \leq CR^{-1}. \end{cases} \quad (4.29)$$

We have from Proposition 4.2, since by hypothesis we have  $K_1 = 0$  ( $M$  is supposed to be of nonnegative Ricci curvature) and  $\delta > \delta_p$ ,

$$-\operatorname{div} \left( F^{\frac{p-2}{2}} \nabla \varphi \right) + \frac{1}{40} F^{\frac{p-4}{2}} f^2(u) |\nabla \varphi|^2 \leq (p-2) \operatorname{div} \left( F^{\frac{p-2}{2}} B(d\varphi, \cdot) \right). \quad (4.30)$$

If we multiply (4.30) by  $\phi_R^2 \varphi_\varepsilon \varphi^{-1}$  and integrate on  $E$  by using (4.28) we have

$$\begin{aligned} & \varepsilon \int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \phi_R^2 \varphi^{-2} |\nabla \varphi_\varepsilon|^2 dx + \frac{1}{40} \int_{E \cap B(x_0, 2R)} F^{\frac{p-4}{2}} f^2(u) \phi_R^2 \varphi_\varepsilon \varphi^{-1} |\nabla \varphi_\varepsilon|^2 dx \leq \\ & - 2 \int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \varphi_\varepsilon \phi_R \varphi^{-1} \nabla \varphi_\varepsilon \cdot \nabla \phi_R dx - \varepsilon(p-2) \int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \phi_R^2 \varphi^{-2} B(d\varphi_\varepsilon, d\varphi_\varepsilon) dx \\ & - 2(p-2) \int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \varphi_\varepsilon \phi_R \varphi^{-1} B(d\varphi_\varepsilon, d\phi_R) dx. \end{aligned} \quad (4.31)$$

Since by (2.6) we have  $B(d\varphi_\varepsilon, d\varphi_\varepsilon) \geq 0$  and  $|B(d\varphi_\varepsilon, d\phi_R)| \leq |\nabla \varphi_\varepsilon| |\nabla \phi_R|$ , then it follows from (4.31) by using (4.29) and the fact that  $F = \frac{\varphi}{f^2(u)}$ :

$$\begin{aligned} & \frac{39}{40} \varepsilon \int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \phi_R^2 \varphi^{-2} |\nabla \varphi_\varepsilon|^2 dx + \frac{1}{40} \int_{E \cap B(x_0, 2R)} F^{\frac{p-4}{2}} f^2(u) \phi_R^2 |\nabla \varphi_\varepsilon|^2 dx \leq \\ & CR^{-1} \int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \phi_R |\nabla \varphi_\varepsilon| dx \end{aligned}$$

which implies, since  $C^{-1} \leq f \leq C$ ,

$$\int_{E \cap B(x_0, 2R)} F^{\frac{p-4}{2}} \phi_R^2 |\nabla \varphi_\varepsilon|^2 dx \leq CR^{-1} \int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \phi_R |\nabla \varphi_\varepsilon| dx. \quad (4.32)$$

On the other hand, we have by the Cauchy-Schwarz inequality

$$\int_{E \cap B(x_0, 2R)} F^{\frac{p-2}{2}} \phi_R |\nabla \varphi_\varepsilon| dx \leq \left( \int_{E \cap B(x_0, 2R)} F^{\frac{p-4}{2}} \phi_R^2 |\nabla \varphi_\varepsilon|^2 dx \right)^{\frac{1}{2}} \left( \int_{E \cap B(x_0, 2R)} F^{\frac{p}{2}} dx \right)^{\frac{1}{2}}.$$

Hence it follows from (4.32) that

$$\int_{E \cap B(x_0, 2R)} F^{\frac{p-4}{2}} \phi_R^2 |\nabla \varphi_\varepsilon|^2 dx \leq CR^{-2} \int_{E \cap B(x_0, 2R)} F^{\frac{p}{2}} dx$$

and since  $\phi_R = 1$  on  $B(x_0, R)$ , then we obtain

$$\int_{E \cap B(x_0, R)} F^{\frac{p-4}{2}} |\nabla \varphi_\varepsilon|^2 dx \leq CR^{-2} \int_{E \cap B(x_0, 2R)} F^{\frac{p}{2}} dx \leq CR^{-2} E_p(u). \quad (4.33)$$

Thus by letting  $R \rightarrow +\infty$  in (4.33) we obtain  $F^{\frac{p-2}{2}} |\nabla \varphi_\varepsilon|^2 = 0$  on  $E$ , and then  $\nabla \varphi_\varepsilon = 0$  on  $E$  since  $F > 0$  on  $E$ . The constant  $\varepsilon > 0$  being arbitrary, we have then

$$\nabla \varphi = 0 \text{ on } E. \quad (4.34)$$

Our objective is to prove that  $F = 0$  on  $M$ . Suppose by contradiction that there exist  $x_0 \in M$  such that  $F(x_0) \neq 0$ , that is,  $x_0 \in E$ . Let  $\mathcal{C}_0 \subset E$  be the connected component of  $x_0$  in  $E$ . Since  $\mathcal{C}_0$  is open and connected, then by (4.34)  $\varphi$  is constant on  $\mathcal{C}_0$ , that is,  $F = \lambda_0 f(u)$  on  $\mathcal{C}_0$  for some constant  $\lambda_0 \geq 0$ . But by (1.6), we have  $f \geq C^{-1}$ . Hence we have

$$F \geq \lambda_0 C^{-1} \quad \text{on } \mathcal{C}_0. \quad (4.35)$$

We distinguish two cases :  $E = M$  and  $E \neq M$ .

**Case 1 :  $E = M$ .** Since  $M$  is connected, then  $\mathcal{C}_0 = M$  in this case. We have by hypothesis

$$\int_M F^{\frac{p}{2}} dx = \int_M |\nabla u|^p dx < +\infty$$

which implies by using (4.35) that  $\lambda_0 = 0$ . Hence  $F = 0$  on  $M$  contradicting the fact that  $F(x_0) \neq 0$ .

**Case 2 :  $E \neq M$ .** In this case we have  $\partial\mathcal{C}_0 \neq \emptyset$ , where  $\partial\mathcal{C}_0$  is the topological boundary of  $\mathcal{C}_0$ . It follows from (4.35) by continuity of  $F$  that  $F \geq \lambda_0 C^{-1}$  on  $\partial\mathcal{C}_0$ . On the other hand, by the definition of a connected component and since  $E$  is open, we have  $\partial\mathcal{C}_0 \subset M \setminus E$ . This implies that  $F = 0$  on  $\partial\mathcal{C}_0$ , and then  $\lambda_0 = 0$ . Thus we have  $F = 0$  on  $\mathcal{C}_0$  contradicting  $F(x_0) \neq 0$ .

Therefore, we have proved that  $F = 0$ , which implies that  $u$  is constant on  $M$  since  $M$  is connected. The proof of Theorem 4 is complete. □

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