

Equality-Based Weighted Residual Formulation for Periodic Solutions of Dry Friction Damped Systems

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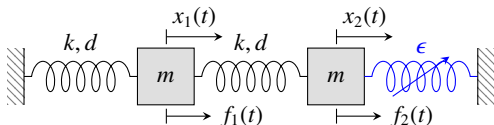
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- System of interest: 2-dof oscillator with cubic spring



- Dynamics: unknowns $(x_1(t), x_2(t), f_{nl}(t))$

$$m\ddot{x}_1 + d(2\dot{x}_1 - \dot{x}_2) + k(2x_1 - x_2) - f_1 = 0 \quad \text{[dynamics of mass 1]} \quad (1)$$

$$m\ddot{x}_2 + d(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) - f_2 - f_{nl} = 0 \quad \text{[dynamics of mass 2]} \quad (2)$$

$$\Psi_{nl}(x_2, f_{nl}) = f_{nl} + \epsilon x_2^3 = 0 \quad \text{[cubic spring law]} \quad (3)$$

- Common solution method

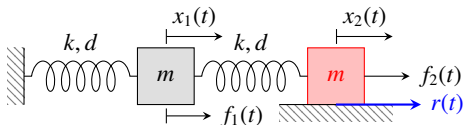
- ▶ Invert $\Psi_{nl}(x_2, f_{nl})$ in Equation (3) $\rightarrow f_{nl}(x_2) = -\epsilon x_2^3$
- ▶ Plug into Equation (2)
- ▶ Solve new Equations (1) and (2) with Harmonic Balance (for instance)

- What if $\Psi_{nl}(x_2, f_{nl})$ cannot be inverted?

- ▶ Simplify Equations (1) and (2) using Harmonic Balance (for instance)
- ▶ Plug into Equation (3)
- ▶ Solve new Equation (3)



- System of interest



- Dynamics: unknowns $(x_1(t), x_2(t), r(t))$

$$m\ddot{x}_1 + d(2\dot{x}_1 - \dot{x}_2) + k(2x_1 - x_2) - f_1 = 0 \quad \text{[dynamics of mass 1]} \quad (4)$$

$$m\ddot{x}_2 + d(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) - f_2 - r = 0 \quad \text{[dynamics of mass 2]} \quad (5)$$

$$\Psi_r(\dot{x}_2, r) = 0 \quad \text{[Coulomb's friction law]} \quad (6)$$

- Questions

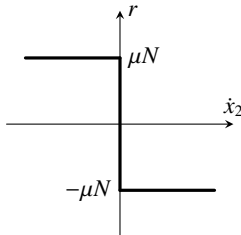
- ▶ Expression of $\Psi_r(\dot{x}_2, r)$?
- ▶ Invertibility of $\Psi_r(\dot{x}_2, r)$?



- Formulation^[1]

$$\begin{cases} \dot{x}_2 = 0 & \implies & |r| \leq \mu N & \text{[sticking]} \\ \dot{x}_2 \neq 0 & \implies & r = -\mu N \dot{x}_2 / |\dot{x}_2| & \text{[sliding]} \end{cases} \quad (7)$$

- Admissible set



- Problem: set-valued map r in terms of \dot{x}_2 (or vice-versa)

- ▶ Single-valued map $r(\dot{x}_2)$ does not seem to exist
- ▶ Zero level-set function $\Psi_r(\dot{x}_2, r) = 0$ exists^[2]
- ▶ Common strategies: regularize the friction law and/or suppress the frictional mass \rightarrow transforms the set-valued map into a simpler single-valued map

[1] Acary, Brémond, and Huber. "On solving contact problems with Coulomb friction: formulations and numerical comparisons". *Transactions of the European Network for Nonsmooth Dynamics*. Springer, 2018.

[2] Alart and Curnier. "A mixed formulation for frictional contact problems prone to Newton like solution methods". *Computer Methods in Applied Mechanics and Engineering* (1991).

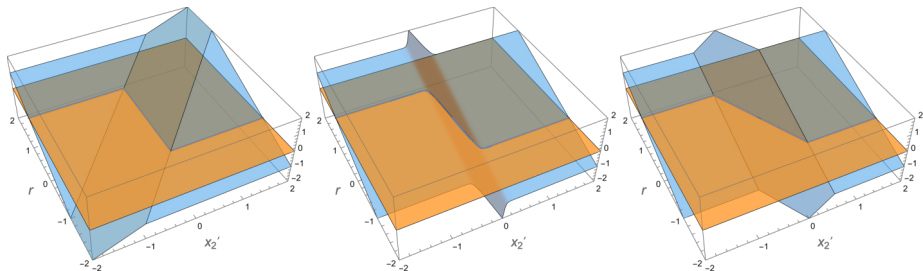


Figure 1: Function $\Psi_r(\dot{x}_2, r)$ (blue) and zero plane (orange). Intersecting level-set is the admissible set

From left to right

- $\Psi_r(\dot{x}_2, r) = \dot{x}_2 + \min(0, \rho(r + \mu N) - \dot{x}_2) + \max(0, \rho(r - \mu N) - \dot{x}_2)$ $[\rho > 0]$ [exact]^{[3],[4],[5]}
- $\Psi_r(\dot{x}_2, r) = r + \mu N \tanh(\beta \dot{x}_2)$ [smooth regularization]
- $\Psi_r(\dot{x}_2, r) = r + \mu N (\beta \dot{x}_2 + \min(1 - \beta \dot{x}_2, 0) + \max(-1 - \beta \dot{x}_2, 0))$ [piecewise regularization]

[3] Stadler. "Semismooth Newton and augmented Lagrangian methods for a simplified friction problem". *SIAM Journal on Optimization* (2004).

[4] Hüeber, Stadler, and Wohlmuth. "A primal-dual active set algorithm for three-dimensional contact problems with Coulomb friction". *SIAM Journal on Scientific Computing* (2008).

[5] Xuewen, Soh, and Wanji. "A new non-smooth model for three dimensional frictional contact problems". *Computational Mechanics* (2000).



- Dynamics

$$\Psi_1(x_1, x_2, r) = m\ddot{x}_1 + d(2\dot{x}_1 - \dot{x}_2) + k(2x_1 - x_2) - f_1 \cos \omega t = 0 \quad (8)$$

$$\Psi_2(x_1, x_2, r) = m\ddot{x}_2 + d(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) - r = 0 \quad (9)$$

$$\Psi_r(\dot{x}_2, r) = \dot{x}_2 + \min(0, \rho(r + \mu N) - \dot{x}_2) + \max(0, \rho(r - \mu N) - \dot{x}_2) = 0 \quad (10)$$

- Approximated solution

$$x_1(t) \approx x_{1h}(t) = \sum_{k=1}^{N_h} x_{1k} \phi_k(t)$$

$$x_2(t) \approx x_{2h}(t) = \sum_{k=1}^{N_h} x_{2k} \phi_k(t) \quad (11)$$

$$r(t) \approx r_h(t) = \sum_{k=1}^{N_h} r_k \phi_k(t)$$

- ▶ $\phi_k(t)$: family of pre-selected shape functions (for instance, Fourier functions \rightarrow HBM)
- ▶ notation: $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{r}) \equiv (x_{1k}, x_{2k}, r_k)_{k=1, \dots, N_h}$
- ▶ $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{r})$: unknown participations of the shape functions in the solution



- Termwise differentiation in time (for x_1 and x_2)

$$\dot{x}_1(t) \approx \dot{x}_{1h}(t) = \sum_k x_{1k} \dot{\phi}_k(t), \quad \ddot{x}_1(t) \approx \ddot{x}_{1h}(t) = \sum_k x_{1k} \ddot{\phi}_k(t) \quad (12)$$

- Ritz-Galerkin projection/weighted residuals^[6] → quadrature schemes

$$\forall k, \int_0^T \phi_k(t) \Psi_1(x_{1h}(t), x_{2h}(t), r_h(t)) dt = 0 \quad [\text{dyn. of mass 1: linear in } (\mathbf{x}_1, \mathbf{x}_2, \mathbf{r})] \quad (13)$$

$$\forall k, \int_0^T \phi_k(t) \Psi_2(x_{1h}(t), x_{2h}(t), r_h(t)) dt = 0 \quad [\text{dyn. of mass 2: linear in } (\mathbf{x}_1, \mathbf{x}_2, \mathbf{r})] \quad (14)$$

$$\forall k, \int_0^T \phi_k(t) \Psi_r(\dot{x}_{2h}(t), r_h(t)) dt = 0 \quad [\text{friction: nonlinear in } (\mathbf{x}_2, \mathbf{r})] \quad (15)$$

- Nonsmooth system of equations $\mathbf{g}(\mathbf{x}) = \mathbf{0}$

$$\mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 - \mathbf{f}_1 = \mathbf{0} \quad (16)$$

$$\mathbf{g}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{r}) = \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 - \mathbf{A}_{2r}\mathbf{r} - \mathbf{f}_2 = \mathbf{0} \quad (17)$$

$$\mathbf{g}_r(\mathbf{x}_2, \mathbf{r}) = \mathbf{0} \quad (18)$$

- Solvers

- ▶ Nonsmooth Newton solver or alike
- ▶ fsolve of Matlab® → dogleg trust region procedure



- Equations (16) and (17) solved via substitution \rightarrow affine relationship $\mathbf{x}_2 = \mathbf{B}\mathbf{r} + \mathbf{c}$ inserted into Equation (18) $\rightarrow \mathbf{g}_r(\mathbf{x}_2(\mathbf{r}), \mathbf{r}) = \mathbf{0}$
- Jacobian

$$\nabla_{\mathbf{r}} \mathbf{g}_r = \frac{\partial \mathbf{g}_{ri}}{\partial r_j} = \int_0^T \phi_k(t) \frac{\partial}{\partial r_j} \Psi_{ri}(\dot{x}_{2h}(t), r_h(t)) dt \quad (19)$$

with $\Psi_r(\dot{x}_2, r) = \dot{x}_2 + \min(0, \rho(r + \mu N) - \dot{x}_2) + \max(0, \rho(r - \mu N) - \dot{x}_2) = 0$

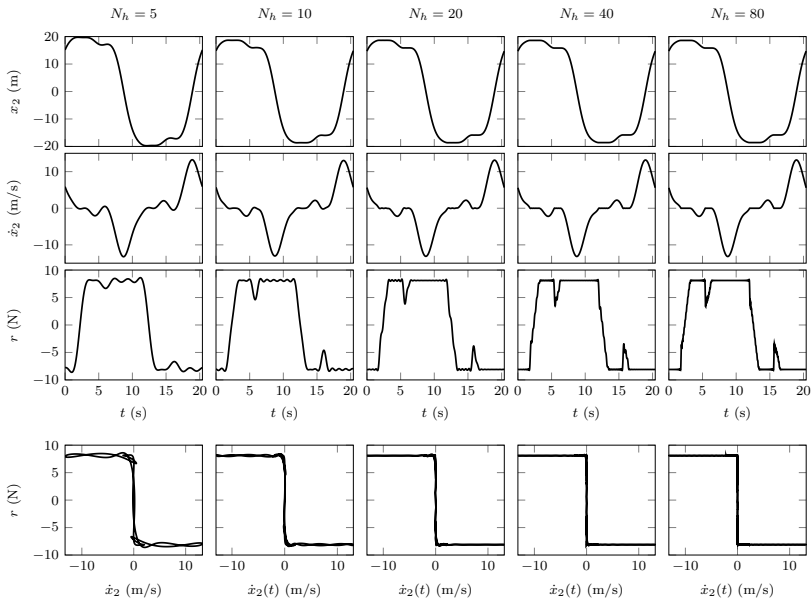
$$= \begin{cases} \rho(r - \mu N) & \text{if } \rho(r + \mu N) - \dot{x}_2 > 0 \text{ and } \rho(r - \mu N) - \dot{x}_2 > 0 \\ \dot{x}_2 & \text{if } \rho(r + \mu N) - \dot{x}_2 > 0 \text{ and } \rho(r - \mu N) - \dot{x}_2 < 0 \\ \rho(r + \mu N) & \text{if } \rho(r + \mu N) - \dot{x}_2 < 0 \text{ and } \rho(r - \mu N) - \dot{x}_2 < 0 \end{cases} \quad (20)$$

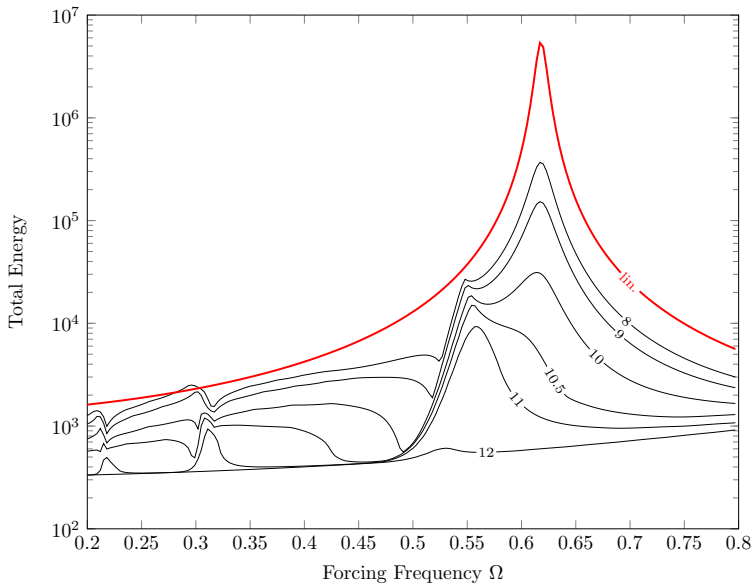
and

$$\frac{\partial \Psi_{ri}}{\partial r_j} = \frac{\partial \Psi_{ri}}{\partial \dot{x}_2} \frac{\partial \dot{x}_2}{\partial r_j} + \frac{\partial \Psi_{ri}}{\partial r} \frac{\partial r}{\partial r_j} \quad (21)$$

and

$$\frac{\partial \Psi_{ri}}{\partial \dot{x}_2} = \begin{cases} 0 \\ 1 \\ 0 \end{cases} \quad \text{and} \quad \frac{\partial \Psi_{ri}}{\partial r} = \begin{cases} \rho & \text{if } \rho(r + \mu N) - \dot{x}_2 > 0 \text{ and } \rho(r - \mu N) - \dot{x}_2 > 0 \\ 0 & \text{if } \rho(r + \mu N) - \dot{x}_2 > 0 \text{ and } \rho(r - \mu N) - \dot{x}_2 < 0 \\ \rho & \text{if } \rho(r + \mu N) - \dot{x}_2 < 0 \text{ and } \rho(r - \mu N) - \dot{x}_2 < 0 \end{cases} \quad (22)$$







- HBM with $N_h = 80$ [red]
- Implicit Euler ($h = 10^{-4}$) [dashed black]
- $\omega = 0.308$ and $N = 9$

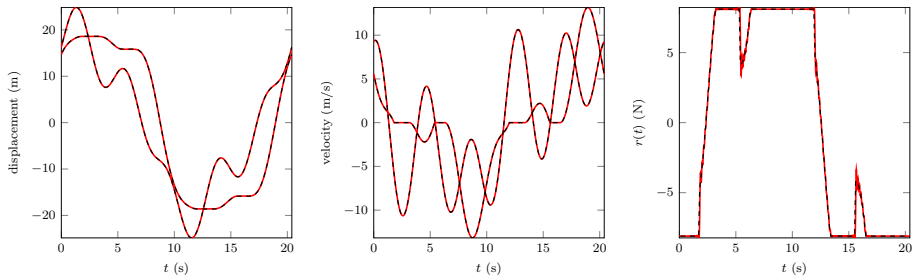
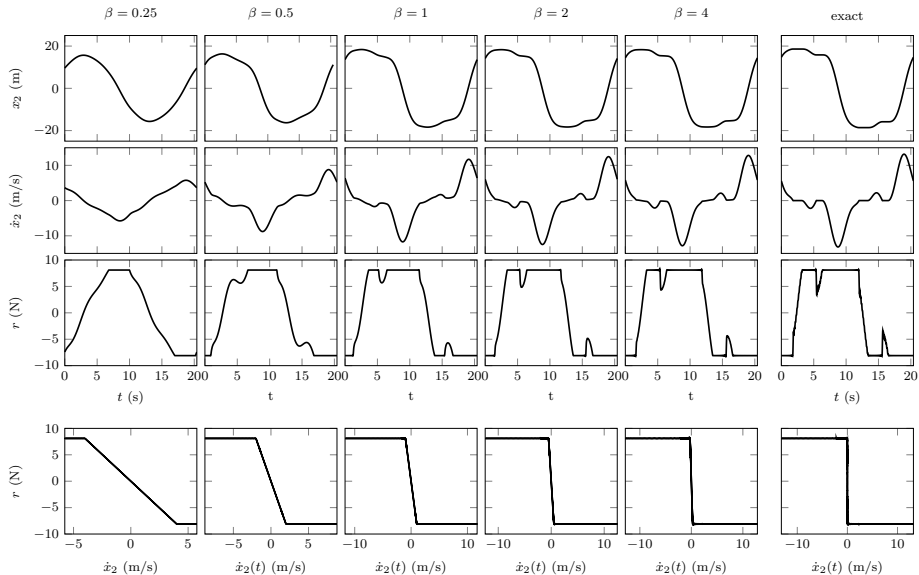


Figure 2: Frequency and time domain (steady-state) solutions:

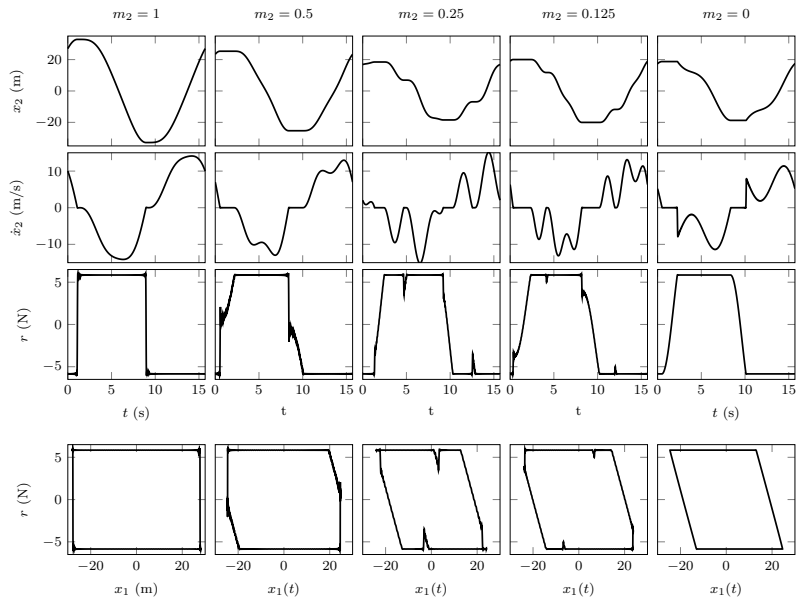
[left] Displacements x_1 and x_2

[middle] Velocities \dot{x}_1 and \dot{x}_2

[right] Friction force r



Case $m \rightarrow 0$ (massless damper)





- Solution method
 - ▶ Easy to implement and numerically robust
 - ▶ Solution accuracy solely dictated by the selected $\{\phi_k\}$ family
 - ▶ No regularization/smoothing procedure needed
 - ▶ Massless damper \rightarrow useful assumption?^[7]
 - ▶ Parameter $\rho = 1$ in all simulations but could be adapted to the magnitudes of r , μN and \dot{x}_2
- Future work
 - ▶ Investigation with other shape functions $\{\phi_k(t)\}$
 - ▶ Replacement of quadrature by FFT
 - ▶ Extension to surface frictional framework
 - ▶ Extension to multiple (surface) frictional dampers
 - ▶ Extension to more advanced friction models
 - ▶ Application to industrial structures with dry friction dampers (e.g., bladed disks)
 - ▶ Extension to nonlinear modal analysis within the invariant manifold framework
- Open questions
 - ▶ Existence of true periodic solutions?^[8]
 - ▶ Stability analysis without regularization? via Hill/Koopman analysis?^[9]

[7] Baek and Epureanu. "Contact model identification for friction ring dampers in blisks with reduced order modeling". *International Journal of Non-Linear Mechanics* (2020).

[8] Kogelbauer and Breunung. "When does the method of harmonic balance give a correct prediction for mechanical systems?" *Applicable Analysis* (2023).

[9] Bayer and Leine. "Sorting-free Hill-based stability analysis of periodic solutions through Koopman analysis". *Nonlinear Dynamics* (2023).