Convergence analysis for the wave equation discretized with hybrid methods in space (HHO, HDG and WG) and the leapfrog scheme in time

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Abstract

We prove the optimal convergence in space and time for the linear acoustic wave equation in its second-order formulation in time, using the hybrid high-order method for space discretization and the leapfrog (central finite difference) scheme for time discretization. The proof hinges on energy arguments similar to those classically deployed in the context of continuous finite elements or discontinuous Galerkin methods, but some novel ideas need to be introduced to handle the static coupling between cell and face unknowns. Because of the close ties between the methods, the present proof can be readily extended to cover space semi-disretization using the hybridizable discontinuous Galerkin method and the weak Galerkin method.

1 Introduction

The wave equation is encountered in the modeling of various physical phenomena, such as seismic, sound or water waves. We focus here on the linear acoustic wave equation written using the second-order formulation in time. A vast literature is available on the numerical simulation of this equation. Concerning the semi-discretization in space, we mention continuous Galerkin (cG) finite element methods with specific quadratures to lump the mass matrix [3, 14, 25, 24], interior penalty discontinuous Galerkin (dG) methods in either nonsymmetric [33] or symmetric [26] form, local discontinuous Galerkin (LDG) methods [9], hybridizable discontinuous Galerkin (HDG) methods [11], and weak Galerkin (WG) methods [27].

In this work, we consider the hybrid high-order (HHO) method for space semi-discretization. The HHO method was introduced in [20] for linear diffusion and in [19] for locking-free linear elasticity. It was developed extensively over the past years; for brevity, we only mention the two textbooks [16, 10]. The HHO method for the wave equation was developed in [4], and extended to unfitted meshes in [5]. The discrete unknowns in the HHO method consist of a pair, namely piecewise polynomials of order $l \geq 0$ approximating the solution in the mesh cells and piecewise polynomials of order $k \geq 0$ approximating the solution trace on the mesh faces. The HHO method is formulated by means of a local (cellwise) gradient reconstruction and a stabilization operator whose role is to ensure in a weak sense the matching of the trace of the cell unknowns with the face unknowns. The HHO method offers numerous advantages: support of polyhedral meshes, optimal convergence rates, local conservation principles, and computational efficiency. In particular, the support of polyhedral meshes allows for a natural use of mesh refinement with hanging nodes.

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HHO and WG methods share the same devising principles. The reconstructed gradient in the HHO method is called weak gradient in the WG method. The equal-order ($l = k$) HHO stabilization has not yet been considered in the WG method, where the stabilization is either a plain least-squares penalty or, in the mixed-order case ($l = k + 1$), involves the L^2-projection of the cell unknowns onto polynomials of order k on the faces, as in the Lehrenfeld–Schöberl HDG stabilization [29, 30]. Notice that the latter indeed leads to optimal error estimates, whereas one order of convergence is lost with the plain least-squares stabilization. A detailed comparison of HHO and WG methods for the biharmonic problem can be found in [21]. Furthermore, as shown in [12], the HHO and HDG methods are closely related. Indeed, the HDG method approximates a triple which is composed of the pair of unknowns considered in the HHO method together with vector-valued piecewise polynomials for the dual variable (typically, the gradient). In the HDG method, the discrete dual variable can be expressed locally in terms of the other two variables, and this formula turns out to correspond to the HHO gradient reconstruction. Moreover, the numerical flux trace, which is one of the cornerstones in the devising of the HDG method, can be explicitly related to the normal component of the HHO reconstructed gradient and the stabilization used in the HHO method. We also notice that, in the equal-order case ($l = k$), the HHO stabilization has not yet been considered in the HDG method and that, in the mixed-order case ($l = k + 1$), the HHO stabilization is the Lehrenfeld–Schöberl HDG stabilization. One interesting difference concerns the convergence analysis since the HHO method solely relies on (local) L^2-orthogonal projections, whereas in the HDG method, one usually invokes a special projection operator. Finally, as discussed in [12, 17, 10], the HHO method with the choice $l = k - 1$, $k \geq 1$, is closely related to the nonconforming virtual element method (ncVEM); see also [8] in the context of multiscale problems.

The goal of the present work is to derive optimal energy-error estimates for the wave equation discretized by the HHO method in space and the leapfrog (central finite difference) scheme in time. Owing to the above discussion, the present results are readily applicable to HDG or WG space semi-discretizations, and with some adaptations to ncVEM. The leapfrog scheme is an explicit, conditionally stable, second-order scheme that is very popular to discretize in time the wave equation combined with either cG or dG space semi-discretization. To the best of our knowledge, a convergence analysis of the wave equation using the leapfrog scheme has not yet been performed with either HDG, WG or HHO space semi-discretization. The space convergence analysis in the time-continuous case is performed in [13] for the HDG method and in [6] for the HHO method using different arguments. Fully discrete schemes have been so far considered only in the context of implicit schemes in time, as the fourth-order Störmer–Numerov scheme combined with the HDG method in [11] and the second-order backward differentiation formula with the WG scheme in [27]. Thus, the convergence analysis of HHO, HDG or WG methods with an explicit time-marching scheme under a CFL stability condition appears to be a novel contribution to the literature. One practical difficulty with the leapfrog scheme combined with any hybrid method is the static coupling between cell and face unknowns at each time step. This drawback has been recently lifted in [36], which proposes to handle the static coupling by means of an iterative scheme. The key point is that weighting the stabilization by a large enough coefficient ensures a fast convergence of the scheme while mildly impacting the accuracy and the CFL stability condition. We refer the reader to [36] for the mathematical analysis in the linear case and for various numerical results on linear and nonlinear wave equations.

Our main result is Theorem 5.1 below, stating the optimal convergence in space and in time of the discrete wave equation approximated using the HHO method in space and the leapfrog scheme in time. The convergence proof follows the general pattern that is already known for the analysis of the leapfrog scheme combined with cG or dG space semi-discretization: derivation of error equations where the consistency errors in space and time appear as source terms, derivation of an energy identity on the errors, and bound on the consistency terms. There are, however, two nontrivial differences. The first one is that the naive expression of the energy error fails to deliver a strongly convex functional on the time- and space-derivatives of the error, because there is a nonzero consistency error related to the mesh faces. This difficulty is lifted by

introducing a modified energy which satisfies a time-discrete energy identity with an additional source term (see Lemma 5.5 below). The second difference is the need to bound this additional source term. Fortunately, this can be achieved by an additional control gained by the modified discrete energy on the face velocity errors; see (69).

We close this discussion by a short literature review on the wave equation with the first-order formulation in time. One can consider either the Hamiltonian-type formulation in which the unknowns are the primal variable and its velocity or the mixed (or Friedrichs-type) formulation in which the unknowns are the velocity and the dual variable (typically, the gradient of the primal variable). Both formulations lead, after space semi-discretization, to a set of coupled first-order ODEs that can be discretized using, e.g., Runge–Kutta (RK) methods in various flavors. The first formulation is considered in the context of dG methods in [2], of HDG methods in [34], and of WG methods (with Crank–Nicolson time-stepping) in [28]. For the second formulation, we mention HDG methods with either implicit [32, 31] or explicit [35] RK schemes and HHO methods with either implicit or explicit RK schemes [4].

The rest of the paper is organized as follows. In Section 2, we present the continuous, the space semi-discrete and the fully discrete wave equation. In Section 3, we collect from the literature various results on the HHO method that are useful to perform the analysis. In Section 4, we address energy stability to highlight how the static coupling between cell and face unknowns impacts the energy balance. In Section 5, we perform the convergence analysis on the energy error. Finally, in Section 6, we present some numerical experiments.

2 Continuous, semi-discrete and fully discrete wave equations

In this section, we present the continuous, the space semi-discrete, and the fully discrete wave equation. The space discretization hinges on the HHO method and the time discretization on the leapfrog (central finite difference) scheme.

2.1 Model problem

The acoustic wave equation is posed on the space domain $\Omega \subset \mathbb{R}^d$ and the time interval $J := [0, \mathfrak{T}]$, with $\mathfrak{T} > 0$. Standard notation is used for Lebesgue, Sobolev and Bochner spaces. Let $(\cdot,\cdot)_\Omega$ denote the L^2-inner product on Ω and $\|\cdot\|_\Omega$ the associated norm. Boldface notation is used for $\mathbb{R}^d$-valued vectors and vector-valued fields, as well as for $\mathbb{R}^{d\times d}$-valued matrices and matrix-valued fields. For dimensional consistency, we consider a length scale ℓ_Ω representative of Ω, e.g. its diameter.

We consider a source term $f \in C^0(\bar{J}; L^2(\Omega))$ with $\bar{J} = [0, \mathfrak{T}]$ and a coefficient $\mu \in L^\infty(\Omega)$ representing the speed of sound and taking values uniformly bounded from below away from zero. For simplicity, we enforce homogeneous Dirichlet boundary conditions and we prescribe initial data $u_0, v_0 \in H_0^1(\Omega)$ satisfying these conditions. Focusing on the strong solution, the acoustic wave equation consists of finding $u \in C^2(\bar{J}; L^2(\Omega)) \cap C^1(\bar{J}; H_0^1(\Omega))$ such that

$$(\partial_t^2 u(t), w)_\Omega + \big(\mu^2 \nabla u(t), \nabla w\big)_\Omega = (f, w)_\Omega, \quad \forall t \in \bar{J},\ \forall w \in H_0^1(\Omega), \tag{1}$$

with the initial conditions

$$u(0) = u_0, \quad \partial_t u(0) = v_0. \tag{2}$$

The notion of energy is central to the analysis of the paper. We set, for all $t \in \bar{J}$,

$$E(t) := \frac{1}{2}\|\partial_t u(t)\|_\Omega^2 + \frac{1}{2}\|\nabla u(t)\|_\Omega^2. \tag{3}$$

This energy verifies

$$E(t) = E(0) + \int_0^t (f(s), \partial_t u(s))_\Omega \,\mathrm{d}s, \tag{4}$$

as readily follows by testing (1) with $w := \partial_t u(t)$ for all $t \in \bar{J}$ and integrating in time.

2.2 HHO space semi-discretization

The space semi-discretization of (1)-(2) is performed using the HHO method. To avoid technicalities, we assume that Ω is a Lipschitz polyhedron and that μ is piecewise constant on a polyhedral partition of Ω.

Discrete setting Let $(\mathcal{T}_h)_{h>0}$ be a sequence of polyhedral meshes of Ω, such that each mesh $\mathcal{T}_h$ covers exactly Ω. For all $h > 0$, let T denote a generic mesh cell in $\mathcal{T}_h$, h_T its diameter and $\boldsymbol{n}_T$ its unit outward normal. We set $h := \max_{T\in\mathcal{T}_h} h_T$ for the mesh size. We say that the $(d-1)$-dimensional set F is a mesh face if there is a hyperplane H_F such that either $F = H_F \cap \partial T_- \cap T_+$ for two distinct mesh cells T_- and T_+ (and F is called mesh interface) or $F = H_F \cap \partial T_- \cap \partial\Omega$ (and F is called mesh boundary face). The collection of all the mesh faces is denoted $\mathcal{F}_h$. For all $T \in \mathcal{T}_h$, we denote by $\mathcal{F}_T$ the collection of the mesh faces composing the boundary ∂T. The mesh sequence is assumed to be shape-regular; see, e.g., [18, 7, 16, 10]. Moreover, each mesh $\mathcal{T}_h$ is assumed to be compatible with the partition on which the coefficient μ is piecewise constant. Hence, μ takes a constant value denoted by μ_T in each mesh cell $T \in \mathcal{T}_h$. Thus, there are $0 < \mu_\flat \le \mu_\sharp < \infty$ such that $\mu_\flat \le \mu_T \le \mu_\sharp$ for all $T \in \mathcal{T}_h$. We assume that the ratio $\frac{\mu_\sharp}{\mu_\flat}$ is not too large, so that it can be hidden in the generic constants used in the error analysis.

Recall that the HHO method is formulated using face and cell unknowns which are polynomials attached to the mesh faces and to the mesh cells, respectively. Let the integer $k \ge 0$ be the polynomial order of the face unknowns and let $l \in \{k, k+1\}$ be the order of the cell unknowns. The setting is said to be of equal-order if $l = k$ and of mixed-order if $l = k+1$. Let $\mathbb{P}_d^l(T)$ (resp. $\mathbb{P}_{d-1}^k(F)$) denote the set of d-variate (resp. $(d-1)$-variate) polynomials of degree at most l (resp. k) restricted to the cell $T \in \mathcal{T}_h$ (resp. to the face $F \in \mathcal{F}_h$). The linear space composed of all the cell degrees of freedom is denoted $\mathcal{U}_\mathcal{T}^l$, and the linear space composed of all the face degrees of freedom is denoted $\mathcal{U}_\mathcal{F}^k$. These spaces are defined as Cartesian products in the form

$$\mathcal{U}_\mathcal{T}^l := \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}_d^l(T), \qquad \mathcal{U}_\mathcal{F}^k := \bigtimes_{F\in\mathcal{F}_h} \mathbb{P}_{d-1}^k(F), \tag{5}$$

and we slightly abuse the notation by viewing an element $w_\mathcal{T} = (w_T)_{T\in\mathcal{T}_h} \in \mathcal{U}_\mathcal{T}^l$ as a function defined a.e. over Ω such that $w_\mathcal{T}|_T := w_T$ for all $T \in \mathcal{T}_h$. The collection of all the cell and face degrees of freedom is the hybrid space

$$\widehat{\mathcal{U}}_h^{l,k} := \mathcal{U}_\mathcal{T}^l \times \mathcal{U}_\mathcal{F}^k. \tag{6}$$

A generic element of $\widehat{\mathcal{U}}_h^{l,k}$ is denoted $\hat{w}_h := (w_\mathcal{T}, w_\mathcal{F}) \in \mathcal{U}_\mathcal{T}^l \times \mathcal{U}_\mathcal{F}^k$ and, in what follows, variables with hats refer to hybrid variables. For a given cell $T \in \mathcal{T}_h$, we also define a local hybrid space of degrees of freedom

$$\widehat{\mathcal{U}}_T^{l,k} := \mathbb{P}_d^l(T) \times \mathcal{U}_{\partial T}^k, \qquad \mathcal{U}_{\partial T}^k := \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}_{d-1}^k(F). \tag{7}$$

Then $\hat{w}_T := (w_T, w_{\partial T} = (w_F)_{F\in\mathcal{F}_T}) \in \widehat{\mathcal{U}}_T^{l,k}$ denotes a generic local hybrid unknown in T, composed of one cell unknown and the collection of the face unknowns for all the faces in $\mathcal{F}_T$. As above, we slightly abuse the notation by viewing an element $w_{\partial T} = (w_F)_{F\in\mathcal{F}_T} \in \mathcal{U}_{\partial T}^k$ as a function defined a.e. over ∂T such that $w_{\partial T}|_F := w_F$ for all $F \in \mathcal{F}_T$. Let $\mathcal{U}_{\mathcal{F},0}^k := \{v_\mathcal{F} \in \mathcal{U}_\mathcal{F}^k, \text{ s.t. } v_F = 0,\ \forall F \subset \partial\Omega\}$ be the subspace of face

unknowns respecting the homogeneous Dirichlet conditions. The subspace of hybrid unknowns respecting the homogeneous Dirichlet conditions is denoted

$$\widehat{\mathcal{U}}_{h,0}^{l,k} := \mathcal{U}_{\mathcal{T}}^{l} \times \mathcal{U}_{\mathcal{F},0}^{k}. \tag{8}$$

L^2-orthogonal projections onto polynomial spaces are denoted with the symbol Π. For instance, for all $T \in \mathcal{T}_h$, Π_T^l is the projection onto $\mathbb{P}_d^l(T)$, $\Pi_{\partial T}^k$ the projection onto $\mathcal{U}_{\partial T}^k$, and for all $F \in \mathcal{F}_h$, Π_F^k the projection onto $\mathbb{P}_{d-1}^k(F)$. The L^2-orthogonal projection onto the broken polynomial spaces $\mathcal{U}_{\mathcal{T}}^l$ and $\mathcal{U}_{\mathcal{F}}^k$ is denoted by $\Pi_{\mathcal{T}}^l$ and $\Pi_{\mathcal{F}}^k$ respectively. Let $(\cdot,\cdot)_T$, $(\cdot,\cdot)_{\partial T}$ and $(\cdot,\cdot)_F$ respectively denote the L^2-inner product in the cell $T \in \mathcal{T}_h$, its boundary ∂T and the face $F \in \mathcal{F}_h$. Let $\|\cdot\|_T$, $\|\cdot\|_{\partial T}$, $\|\cdot\|_F$ denote the norms associated respectively with the L^2-inner products $(\cdot,\cdot)_T$, $(\cdot,\cdot)_{\partial T}$ and $(\cdot,\cdot)_F$.

Gradient and potential reconstruction. The local gradient reconstruction operator builds a gradient in the cell $T \in \mathcal{T}_h$ from the local cell and face unknowns in $\widehat{\mathcal{U}}_T^{l,k}$. This operator $\boldsymbol{G}_T^k : \widehat{\mathcal{U}}_T^{l,k} \to \mathbb{P}_d^k(T;\mathbb{R}^d)$ is evaluated by solving the following problem: For all $\hat{v}_T \in \widehat{\mathcal{U}}_T^{l,k}$,

$$(\boldsymbol{G}_T^k(\hat{v}_T), \boldsymbol{q})_T = (\nabla v_T, \boldsymbol{q})_T + (v_{\partial T} - v_T, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{\partial T}, \quad \forall \boldsymbol{q} \in \mathbb{P}_d^k(T;\mathbb{R}^d), \tag{9}$$

where $\mathbb{P}_d^k(T;\mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued d-variate polynomials of degree k in the cell T. In practice, each component of the reconstructed gradient is found independently by inverting the mass matrix associated with a chosen scalar-valued basis of $\mathbb{P}_d^k(T)$.

One can also build a potential reconstruction operator $R_T^{k+1} : \widehat{\mathcal{U}}_T^{l,k} \to \mathbb{P}_d^{k+1}(T)$ by solving, for all $\hat{v}_T \in \widehat{\mathcal{U}}_T^{l,k}$, the following Neumann problem:

$$(\nabla R_T^{k+1}(\hat{v}_T), \nabla q)_T = (\nabla v_T, \nabla q)_T + (v_{\partial T} - v_T, \nabla q\cdot\boldsymbol{n}_T)_{\partial T}, \quad \forall q \in \mathbb{P}_{d,*}^{k+1}(T), \tag{10}$$

with $\mathbb{P}_{d,*}^{k+1}(T) := \{q \in \mathbb{P}_d^{k+1}(T) \mid (q,1)_T = 0\}$, with the mean-value condition $(R_T^{k+1}(\hat{v}_T),1)_T = (v_T,1)_T$.

Remark 2.1 (Gradient reconstruction)**.** Notice that we have $\nabla R_T^{k+1}(\hat{v}_T) = \Pi_{\nabla\mathbb{P}_d^{k+1}(T)}\boldsymbol{G}_T^k(\hat{v}_T)$ for all $\hat{v}_T \in \widehat{\mathcal{U}}_T^{l,k}$. For nonlinear problems, it is preferable to consider the gradient reconstruction operator $\boldsymbol{G}_T^k$ rather than ∇R_T^{k+1}, as discussed in [15, 1]. In the present linear setting, the stiffness part of the semi-discrete problem can be evaluated using ∇R_T^{k+1}. Notice though that even when working with the operator $\boldsymbol{G}_T^k$, it is necessary to evaluate the operator R_T^{k+1} to compute the stabilization in the equal-order setting (see (12) below).

Stabilization. The role of the stabilization is to weakly enforce the matching between the cell and the face unknowns at each mesh face. Let $T \in \mathcal{T}_h$. For all $\hat{w}_T \in \widehat{\mathcal{U}}_T^{l,k}$, set $\delta_{\partial T}(\hat{w}_T) := w_{\partial T} - w_T|_{\partial T}$. In the mixed-order setting, the local stabilization operator $S_{\partial T}$ is defined as

$$S_{\partial T}(\hat{w}_T) := \Pi_{\partial T}^k(\delta_{\partial T}(\hat{w}_T)), \quad \forall \hat{w}_T \in \widehat{\mathcal{U}}_T^{l,k}, \tag{11}$$

which corresponds to the Lehrenfeld–Schöberl HDG stabilization (see, e.g. [29, 30]). In the equal-order setting, the definition of $S_{\partial T}$ requires the computation of R_T^{k+1} and writes

$$S_{\partial T}(\hat{w}_T) := \Pi_{\partial T}^k\big(\delta_{\partial T}(\hat{w}_T) + (I - \Pi_T^k)R_T^{k+1}(0,\delta_{\partial T}(\hat{w}_T))|_{\partial T}\big), \quad \forall \hat{w}_T \in \widehat{\mathcal{U}}_T^{k,k} =: \widehat{\mathcal{U}}_T^k. \tag{12}$$

HHO space semi-discrete wave equation. The global bilinear form a_h is defined, for all $\hat{v}_h, \hat{w}_h \in \widehat{\mathcal{U}}_h^{l,k}$, as

$$a_h(\hat{v}_h, \hat{w}_h) := \sum_{T\in\mathcal{T}_h} \mu_T^2 \Big\{ (\boldsymbol{G}_T^k(\hat{v}_T), \boldsymbol{G}_T^k(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\hat{v}_T), S_{\partial T}(\hat{w}_T))_{\partial T} \Big\}. \tag{13}$$

The space semi-discrete wave equation consists of finding $\hat{u}_h := (u_\mathcal{T}, u_\mathcal{F}) \in C^2(\bar{J}; \widehat{\mathcal{U}}_{h,0}^{l,k})$ such that

$$(\partial_t^2 u_\mathcal{T}(t), w_\mathcal{T})_\Omega + a_h(\hat{u}_h(t), \hat{w}_h) = (f(t), w_\mathcal{T})_\Omega, \quad \forall t \in \bar{J},\ \forall \hat{w}_h := (w_\mathcal{T}, w_\mathcal{F}) \in \widehat{\mathcal{U}}_{h,0}^{l,k}. \tag{14}$$

Notice that the homogeneous Dirichlet boundary condition is enforced by the fact that $\hat{u}_h(t) \in \widehat{\mathcal{U}}_{h,0}^{l,k}$ at all times $t \in \bar{J}$. The initial conditions are enforced on the cell degrees of freedom as follows (initial conditions on the face degrees of freedom are not needed):

$$u_\mathcal{T}(0) := \Pi_\mathcal{T}^l(u_0), \quad \partial_t u_\mathcal{T}(0) := \Pi_\mathcal{T}^l(v_0). \tag{15}$$

2.3 Time-discretization with the leapfrog scheme

Let N be the number of discrete time intervals such that $(t^n)_{n\in\{0:N\}}$ are the discrete time nodes with $t^0 = 0$ and $t^N := \mathfrak{T}$. We set $f^n := f(t^n)$ for all $n \in \{0{:}N\}$. For the sake of simplicity, we consider a fixed time step $\Delta t := \frac{\mathfrak{T}}{N}$. The time discrete unknown $\hat{u}_h^n = (u_\mathcal{T}^n, u_\mathcal{F}^n) \in \widehat{\mathcal{U}}_{h,0}^{l,k}$ is meant to be an approximation of the space semi-discrete HHO solution $\hat{u}_h(t^n) \in \widehat{\mathcal{U}}_{h,0}^{l,k}$.

The leapfrog scheme consists of solving

$$\frac{1}{\Delta t^2}(u_\mathcal{T}^{n+1} - 2u_\mathcal{T}^n + u_\mathcal{T}^{n-1}, w_\mathcal{T})_\Omega + a_h(\hat{u}_h^n, \hat{w}_h) = (f^n, w_\mathcal{T})_\Omega, \quad \forall n \in \{1{:}N-1\},\ \forall \hat{w}_h \in \widehat{\mathcal{U}}_{h,0}^{l,k}, \tag{16}$$

where the unknowns are $u_\mathcal{T}^{n+1}$ and $u_\mathcal{F}^n$, whereas $u_\mathcal{T}^n$ and $u_\mathcal{T}^{n-1}$ are known from prior time steps or given by the initial conditions as follows:

$$u_\mathcal{T}^0 = \Pi_\mathcal{T}^l(u_0), \tag{17a}$$

$$a_h(\hat{u}_h^0, (0, w_\mathcal{F})) = 0, \quad \forall w_\mathcal{F} \in \mathcal{U}_{\mathcal{F},0}^k, \tag{17b}$$

$$\big(u_\mathcal{T}^1, w_\mathcal{T}\big)_\Omega = \big(u_\mathcal{T}^0 + \Delta t \Pi_\mathcal{T}^l(v_0), w_\mathcal{T}\big)_\Omega + \frac{\Delta t^2}{2}\big\{(f^0, w_\mathcal{T})_\Omega - a_h(\hat{u}_h^0, (w_\mathcal{T}, 0))\big\}, \quad \forall w_\mathcal{T} \in \mathcal{U}_\mathcal{T}^l. \tag{17c}$$

Notice that we used the initial conditions (15) in (17a) and (17c). At each time step $n \in \{1{:}N-1\}$, the problem (16) is solved by first finding the face unknown $u_\mathcal{F}^n \in \mathcal{U}_{\mathcal{F},0}^k$ from the cell unknown $u_\mathcal{T}^n \in \mathcal{U}_\mathcal{T}^l$ by solving

$$a_h((0, u_\mathcal{F}^n), (0, w_\mathcal{F})) = -a_h((u_\mathcal{T}^n, 0), (0, w_\mathcal{F})), \quad \forall w_\mathcal{F} \in \mathcal{U}_{\mathcal{F},0}^k, \tag{18}$$

and then the cell unknown $u_\mathcal{T}^{n+1} \in \mathcal{U}_\mathcal{T}^l$ is computed by solving

$$\frac{1}{\Delta t^2}\big(u_\mathcal{T}^{n+1} - 2u_\mathcal{T}^n + u_\mathcal{T}^{n-1}, w_\mathcal{T}\big)_\Omega = (f^n, w_\mathcal{T})_\Omega - a_h(\hat{u}_h^n, (w_\mathcal{T}, 0)), \quad \forall w_\mathcal{T} \in \mathcal{U}_\mathcal{T}^l. \tag{19}$$

Owing to the static coupling between face and cell unknowns implied by (18), the scheme is semi-implicit. A computationally effective, iterative algorithm to solve this static coupling based on inverting block-diagonal matrices is introduced and analyzed in [36].

Remark 2.2 (Final step)**.** At the final step $n = N-1$, we compute $u_\mathcal{F}^{N-1}$ from (18) and $u_\mathcal{T}^N$ from (19). Then $u_\mathcal{F}^N$ can be retrieved by solving (18) for $n = N$.

3 Preliminary results for the error analysis

In this section, we collect from the literature various results that are useful to perform the error analysis. In what follows, C denotes a generic positive constant whose value can change at each occurrence as long as it is independent of the mesh size and the time step; the value of C can depend on the mesh shape-regularity and the underlying polynomial degree, and, whenever relevant, on the ratio $\frac{\mu_\sharp}{\mu_\flat}$ related to the contrast in the sound speed. In a few cases, we use a more specific symbol, e.g., for some discrete inverse inequalities, a discrete Poincaré inequality, and the coercivity and boundedness constants of the discrete bilinear form a_h.

Lemma 3.1 (Discrete inverse inequalities)**.** *Let the polynomial degree $m \geq 0$ be fixed. There is $C_{\mathrm{dinv}} > 0$ such that, for all $h > 0$, all $T \in \mathcal{T}_h$ and all $q \in \mathbb{P}_d^m(T)$,*

$$\|\nabla q\|_T \leq C_{\mathrm{dinv}} h_T^{-1} \|q\|_T, \tag{20a}$$

$$\|q\|_{\partial T} \leq C_{\mathrm{dinv}} h_T^{-\frac{1}{2}} \|q\|_T. \tag{20b}$$

Proof. See, e.g., [18, Lem. 1.44 & 1.46]. □

Stability and boundedness. A direct verification shows that the map $\|\cdot\|_{\mathrm{HHO}} : \widehat{\mathcal{U}}_h^{l,k} \to \mathbb{R}_+$ such that

$$\|\hat{v}_h\|_{\mathrm{HHO}}^2 := \sum_{T\in\mathcal{T}_h} \mu_T^2 \big\{ \|\nabla v_T\|_T^2 + h_T^{-1} \|v_{\partial T} - v_T\|_{\partial T}^2 \big\}, \qquad \forall \hat{v}_h \in \widehat{\mathcal{U}}_h^{l,k}, \tag{21}$$

defines a norm on $\widehat{\mathcal{U}}_{h,0}^{l,k}$ (and a seminorm on $\widehat{\mathcal{U}}_h^{l,k}$). We define the seminorm $|\hat{v}_h|_{\mathrm{s}}^2 := \sum_{T\in\mathcal{T}_h} \mu_T^2 h_T^{-1} \|S_{\partial T}(\hat{v}_T)\|_{\partial T}^2$ and the global gradient reconstruction such that $\boldsymbol{G}_{\mathcal{T}}^k(\hat{v}_h)|_T := \boldsymbol{G}_T^k(\hat{v}_T)$ for all $T \in \mathcal{T}_h$.

Lemma 3.2 (Stability and boundedness)**.** *There are $0 < \alpha \leq \varpi < \infty$ such that, for all $h > 0$ and all $\hat{v}_h \in \widehat{\mathcal{U}}_{h,0}^{l,k}$,*

$$\alpha \, \|\hat{v}_h\|_{\mathrm{HHO}}^2 \leq a_h(\hat{v}_h, \hat{w}_h) = \|\mu \boldsymbol{G}_{\mathcal{T}}^k(\hat{v}_h)\|_\Omega^2 + |\hat{v}_h|_{\mathrm{s}}^2 \leq \varpi \, \|\hat{v}_h\|_{\mathrm{HHO}}^2. \tag{22}$$

Proof. See [20, Lem. 4]. □

Lemma 3.3 (Discrete Poincaré inequality)**.** *There is C_{dP} such that for all $h > 0$ and all $\hat{v}_h \in \widehat{\mathcal{U}}_{h,0}^{l,k}$,*

$$\|v_{\mathcal{T}}\|_\Omega \leq C_{\mathrm{dP}} \mu_\sharp^{-1} \ell_\Omega \|\hat{v}_h\|_{\mathrm{HHO}}. \tag{23}$$

Proof. See, e.g., [5, Lem. 6.2]. □

Approximation. We first define the local projection operator $\hat{I}_T^{k,l} : H^1(T) \to \widehat{\mathcal{U}}_T^{l,k}$ for all $T \in \mathcal{T}_h$ such that, for all $v \in H^1(T)$,

$$\hat{I}_T^{k,l}(v) := \big(\Pi_T^l(v), \Pi_{\partial T}^k(v)\big). \tag{24}$$

Then the global projection operator is defined as $\hat{I}_h^{k,l} : H^1(\Omega) \to \widehat{\mathcal{U}}_h^{l,k}$ such that, for all $v \in H^1(\Omega)$,

$$\hat{I}_h^{k,l}(v) := \big((\Pi_T^l(v))_{T\in\mathcal{T}_h}, (\Pi_F^k(v))_{F\in\mathcal{F}_h}\big). \tag{25}$$

Notice that $\hat{I}_h^{k,l}$ maps functions from $H_0^1(\Omega)$ onto $\widehat{\mathcal{U}}_{h,0}^{l,k}$. Moreover, the definition of $\hat{I}_h^{k,l}$ is meaningful since a function $v \in H^1(\Omega)$ does not jump across the mesh interfaces. Let $\mathcal{E}_{\mathcal{T}}^{k+1} : H^1(\Omega) \to \mathcal{U}_{\mathcal{T}}^{k+1}$ denote the broken

elliptic projection onto $\mathcal{U}_{\mathcal{T}}^{k+1}$, such that, for all $v \in H^1(\Omega)$ and all $T \in \mathcal{T}_h$, $\mathcal{E}_{\mathcal{T}}^{k+1}(v)|_T := \mathcal{E}_T^{k+1} \in \mathbb{P}_d^{k+1}(T)$ is uniquely defined by the relations

$$\big(\nabla(\mathcal{E}_T^{k+1}(v) - v), \nabla q\big)_T = 0 \ \forall q \in \mathbb{P}_{d,*}^{k+1}(T), \quad \text{and} \quad \big(\mathcal{E}_T^{k+1}(v) - v, 1\big)_T = 0. \tag{26}$$

Lemma 3.4 (Approximation property for $\boldsymbol{G}_T^k$ and R_T^{k+1})**.** *The following holds true for all $h > 0$ and all $T \in \mathcal{T}_h$:*

$$\boldsymbol{G}_T^k(\hat{I}_T^{k,l}(v)) = \Pi_T^k(\nabla v), \qquad \nabla R_T^{k+1}(\hat{I}_T^{k,l}(v)) = \nabla \mathcal{E}_T^{k+1}(v), \qquad \forall v \in H^1(T). \tag{27}$$

Moreover, there exists a real number $C > 0$ such that, for all $h > 0$, all $T \in \mathcal{T}_h$ and all $v \in H^{k+2}(T)$, we have

$$||v - R_T^{k+1}(\hat{I}_T^{k,l}(v))||_T + h_T^{\frac{1}{2}}||v - R_T^{k+1}(\hat{I}_T^{k,l}(v))||_{\partial T} \le C h_T^{k+2}|v|_{H^{k+2}(T)}, \tag{28}$$

and

$$||\nabla\big(v - R_T^{k+1}(\hat{I}_T^{k,l}(v))\big)||_T + h_T^{\frac{1}{2}}||\nabla\big(v - R_T^{k+1}(\hat{I}_T^{k,l}(v))\big)||_{\partial T} \le C h_T^{k+1}|v|_{H^{k+2}(T)}, \tag{29a}$$

$$||\nabla v - \boldsymbol{G}_T^k(\hat{I}_T^{k,l}(v))||_T + h_T^{\frac{1}{2}}||\nabla v - \boldsymbol{G}_T^k(\hat{I}_T^{k,l}(v))||_{\partial T} \le C h_T^{k+1}|v|_{H^{k+2}(T)}. \tag{29b}$$

Proof. See, e.g., [20, Lem. 3], [10, Lem. 3.1]. □

Lemma 3.5 (Approximation property for $S_{\partial T}$)**.** *There is a constant $C > 0$ such that for all $h > 0$, all $T \in \mathcal{T}_h$, and all $v \in H^1(T)$, we have*

$$\|S_{\partial T}(\hat{I}_T^{k,l}(v))\|_{\partial T} \le C h_T^{\frac{1}{2}}\|\nabla(v - P_T^{k+1}(v))\|_T, \tag{30}$$

where $P_T^{k+1} := \mathcal{E}_T^{k+1}$ if $l = k$ and $P_T^{k+1} := \Pi_T^{k+1}$ if $l = k+1$.

Proof. See, e.g., [20, Eq. (45)], [10, Lem. 2.7]. □

Consistency. For any function $v \in Y := \{v \in H_0^1(\Omega), B(v) := -\nabla \cdot (\mu^2 \nabla v) \in L^2(\Omega)\}$, we consider the linear form $\psi_h(v;\cdot) \in (\widehat{\mathcal{U}}_{h,0}^{l,k})'$ such that

$$\psi_h(v;\hat{w}_h) := (B(v), w_{\mathcal{T}})_\Omega - a_h(\hat{I}_h^{k,l}(v), \hat{w}_h), \qquad \forall \hat{w}_h \in \widehat{\mathcal{U}}_{h,0}^{l,k}. \tag{31}$$

For any function $v \in H^{1+\nu}(\Omega)$, $\nu \in (\frac{1}{2}, 1]$, we consider the seminorm $|v|_{*,h}$ defined by

$$|v|_{*,h}^2 := \sum_{T \in \mathcal{T}_h} \mu_T^2 \big\{ h_T \|\boldsymbol{\gamma}(v) \cdot \mathbf{n}_T\|_{\partial T}^2 + \|\nabla \eta(v)\|_T^2 \big\}, \tag{32}$$

with

$$\boldsymbol{\gamma}(v) := \nabla v - \boldsymbol{G}_{\mathcal{T}}^k(\hat{I}_h^{k,l}(v)), \qquad \eta(v) := v - P_{\mathcal{T}}^{k+1}(v), \tag{33}$$

where P_T^{k+1} is defined in Lemma 3.5 and $P_{\mathcal{T}}^{k+1}$ denotes the broken version of P_T^{k+1} defined elementwise. For a linear form $\phi \in (\widehat{\mathcal{U}}_{h,0}^{l,k})'$, we set

$$\|\phi\|_{(\mathrm{HHO})'} := \sup_{\hat{q}_h \in \widehat{\mathcal{U}}_{h,0}^{l,k}} \frac{|\phi(\hat{q}_h)|}{\|\hat{q}_h\|_{\mathrm{HHO}}},$$

with the norm $\|\cdot\|_{\mathrm{HHO}}$ defined in (21).

Lemma 3.6 (Consistency)**.** *There is a constant c_* such that, for all $h > 0$ and all $v \in Y \cap H^{1+\nu}(\Omega)$, we have*

$$\|\psi_h(v,\cdot)\|_{(\mathrm{HHO})'} \le c_* |v|_{*,h}. \tag{34}$$

Proof. See, e.g., [20, Eq. (43)], [10, Lem. 2.8 & 3.4]. □

4 Energy stability

The notion of discrete energy is central to the proof of convergence of the fully discrete scheme. Let us first introduce the semi-discrete energy balance as in [4, Lem. 3.1]. Recalling that $\hat{u}_h \in C^2(\bar{J};\widehat{\mathcal{U}}_{h,0}^{l,k})$ solves the space semi-discrete problem (14), we set, for all $t \in \bar{J}$,

$$E_h(t) := \frac{1}{2}\|\partial_t u_{\mathcal{T}}(t)\|_\Omega^2 + \frac{1}{2}\hat{a}_h(\hat{u}_h(t),\hat{u}_h(t)). \tag{35}$$

This energy verifies

$$E_h(t) = E_h(0) + \int_0^t (f(s),\partial_t u_{\mathcal{T}}(s))_\Omega \,\mathrm{d}s, \tag{36}$$

as readily follows by testing (14) with $\hat{w}_h := \partial_t \hat{u}_h(t)$ for all $t \in \bar{J}$ and integrating in time.

The first important step is to derive the fully discrete counterpart of the semi-discrete energy identity (36). It is convenient to set, for all $n \in \{0{:}N-1\}$,

$$\hat{u}_h^{n+\frac{1}{2}} := \frac{1}{2}(\hat{u}_h^n + \hat{u}_h^{n+1}), \qquad \delta\hat{u}_h^{n+\frac{1}{2}} := \frac{1}{\Delta t}(\hat{u}_h^{n+1} - \hat{u}_h^n). \tag{37}$$

The fully discrete counterpart of $E_h(t)$ is the discrete energy $E_h^{n+\frac{1}{2}}$ defined as

$$E_h^{n+\frac{1}{2}} := \frac{1}{2}\|\delta u_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega^2 - \frac{\Delta t^2}{8}a_h(\delta\hat{u}_h^{n+\frac{1}{2}},\delta\hat{u}_h^{n+\frac{1}{2}}) + \frac{1}{2}a_h(\hat{u}_h^{n+\frac{1}{2}},\hat{u}_h^{n+\frac{1}{2}}). \tag{38}$$

Lemma 4.1 (Energy balance)**.** *The fully discrete energy verifies, for all $n \in \{1{:}N-1\}$,*

$$E_h^{n+\frac{1}{2}} = E_h^{\frac{1}{2}} + \frac{1}{2}\sum_{m=1}^n \big(f^m, u_{\mathcal{T}}^{m+1} - u_{\mathcal{T}}^{m-1}\big)_\Omega. \tag{39}$$

Proof. Let $n \in \{1{:}N-1\}$. For all $m \in \{1{:}n\}$, we test (16) with $\hat{w}_h := \frac{1}{2}(\hat{u}_h^{m+1} - \hat{u}_h^{m-1})$. We use the identity

$$\begin{aligned}
\frac{1}{2}\big(u_{\mathcal{T}}^{m+1} - 2u_{\mathcal{T}}^m + u_{\mathcal{T}}^{m-1}, u_{\mathcal{T}}^{m+1} - u_{\mathcal{T}}^{m-1}\big)_\Omega &= \frac{1}{2}\big((u_{\mathcal{T}}^{m+1} - u_{\mathcal{T}}^m) - (u_{\mathcal{T}}^m - u_{\mathcal{T}}^{m-1}), (u_{\mathcal{T}}^{m+1} - u_{\mathcal{T}}^m) + (u_{\mathcal{T}}^m - u_{\mathcal{T}}^{m-1})\big)_\Omega \\
&= \frac{1}{2}\|u_{\mathcal{T}}^{m+1} - u_{\mathcal{T}}^m\|_\Omega^2 - \frac{1}{2}\|u_{\mathcal{T}}^m - u_{\mathcal{T}}^{m-1}\|_\Omega^2 \\
&= \frac{\Delta t^2}{2}\|\delta u_{\mathcal{T}}^{m+\frac{1}{2}}\|_\Omega^2 - \frac{\Delta t^2}{2}\|\delta u_{\mathcal{T}}^{m-\frac{1}{2}}\|_\Omega^2.
\end{aligned}$$

We also use the following computation which exploits the symmetry of a_h:

$$\begin{aligned}
a_h(\hat{u}_h^m, \hat{u}_h^{m+1} - \hat{u}_h^{m-1}) &= \frac{1}{4}a_h(\hat{u}_h^{m+1} + 2\hat{u}_h^m + \hat{u}_h^{m-1}, (\hat{u}_h^{m+1} + \hat{u}_h^m) - (\hat{u}_h^m + \hat{u}_h^{m-1})) \\
&\quad - \frac{1}{4}a_h(\hat{u}_h^{m+1} - 2\hat{u}_h^m + \hat{u}_h^{m-1}, (\hat{u}_h^{m+1} - \hat{u}_h^m) + (\hat{u}_h^m - \hat{u}_h^{m-1})) \\
&= a_h(\hat{u}_h^{m+\frac{1}{2}} + \hat{u}_h^{m-\frac{1}{2}}, \hat{u}_h^{m+\frac{1}{2}} - \hat{u}_h^{m-\frac{1}{2}}) - \frac{\Delta t^2}{4}a_h(\delta\hat{u}_h^{m+\frac{1}{2}} + \delta\hat{u}_h^{m-\frac{1}{2}}, \delta\hat{u}_h^{m+\frac{1}{2}} - \delta\hat{u}_h^{m-\frac{1}{2}}) \\
&= a_h(\hat{u}_h^{m+\frac{1}{2}}, \hat{u}_h^{m+\frac{1}{2}}) - a_h(\hat{u}_h^{m-\frac{1}{2}}, \hat{u}_h^{m-\frac{1}{2}}) \\
&\quad - \frac{\Delta t^2}{4}\big(a_h(\delta\hat{u}_h^{m+\frac{1}{2}}, \delta\hat{u}_h^{m+\frac{1}{2}}) - a_h(\delta\hat{u}_h^{m-\frac{1}{2}}, \delta\hat{u}_h^{m-\frac{1}{2}})\big).
\end{aligned}$$

This gives $E_h^{m+\frac{1}{2}} - E_h^{m-\frac{1}{2}} = \frac{1}{2}\big(f^m, u_{\mathcal{T}}^{m+1} - u_{\mathcal{T}}^{m-1}\big)_\Omega$. Summing this identity for $m = 1$ to $m = n$ yields the claim. □

The second important question is whether $E_h^{n+\frac{1}{2}}$ defines a strongly convex functional on $\delta u_{\mathcal{T}}^{n+\frac{1}{2}}$ and $\hat{u}_h^{n+\frac{1}{2}}$. This property can be achieved under a CFL restriction on the time step.

Lemma 4.2 (Strong convexity of $E_h^{n+\frac{1}{2}}$)**.** *Under the following CFL restriction on the time step:*

$$\Delta t \le \eta \mu_\sharp^{-1} h, \qquad \eta := C_{\mathrm{dinv}}^{-1} \varpi^{-\frac{1}{2}}, \tag{40}$$

$E_h^{n+\frac{1}{2}}$ *defines a strongly convex functional on* $\delta u_{\mathcal{T}}^{n+\frac{1}{2}}$ *and* $\hat{u}_h^{n+\frac{1}{2}}$.

Proof. Using the fact that $\hat{u}_h^n$ satisfies (18) and the symmetry of a_h, we infer that, for all $n \in \{1{:}N-1\}$,

$$\begin{aligned}
\Delta t^2 a_h(\delta\hat{u}_h^{n+\frac{1}{2}}, \delta\hat{u}_h^{n+\frac{1}{2}}) &= a_h(\hat{u}_h^{n+1} - \hat{u}_h^n, \hat{u}_h^{n+1} - \hat{u}_h^n) \\
&= a_h(\hat{u}_h^{n+1} - \hat{u}_h^n, (u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0)) \\
&= a_h((u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0), (u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0)) + a_h((0, u_{\mathcal{F}}^{n+1} - u_{\mathcal{F}}^n), (u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0)) \\
&= a_h((u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0), (u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0)) + a_h((u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0), (0, u_{\mathcal{F}}^{n+1} - u_{\mathcal{F}}^n)) \\
&= a_h((u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0), (u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n, 0)) - a_h((0, u_{\mathcal{F}}^{n+1} - u_{\mathcal{F}}^n), (0, u_{\mathcal{F}}^{n+1} - u_{\mathcal{F}}^n)) \\
&= \Delta t^2 a_h((\delta u_{\mathcal{T}}^{n+\frac{1}{2}}, 0), (\delta u_{\mathcal{T}}^{n+\frac{1}{2}}, 0)) - \Delta t^2 a_h((0, \delta u_{\mathcal{F}}^{n+\frac{1}{2}}), (0, \delta u_{\mathcal{F}}^{n+\frac{1}{2}})) \\
&\le \Delta t^2 a_h((\delta u_{\mathcal{T}}^{n+\frac{1}{2}}, 0), (\delta u_{\mathcal{T}}^{n+\frac{1}{2}}, 0)).
\end{aligned}$$

Combining this bound with the coercivity property from Lemma 3.2 gives

$$E_h^{n+\frac{1}{2}} \ge \frac{1}{2}\|\delta u_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega^2 - \frac{\Delta t^2}{8} a_h((\delta u_{\mathcal{T}}^{n+\frac{1}{2}}, 0), (\delta u_{\mathcal{T}}^{n+\frac{1}{2}}, 0)) + \alpha \|\hat{u}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}}^2.$$

Recalling that $\|\cdot\|_{\mathrm{HHO}}$ defines a norm on $\widehat{\mathcal{U}}_{h,0}^{l,k}$ and $\hat{u}_h^{n+\frac{1}{2}} \in \widehat{\mathcal{U}}_{h,0}^{l,k}$, the proof is complete if we show that, under the CFL condition (40), we have, for all $w_{\mathcal{T}} \in \mathcal{U}_{\mathcal{T}}^l$,

$$\frac{1}{2}\|w_{\mathcal{T}}\|_\Omega^2 - \frac{\Delta t^2}{8} a_h((w_{\mathcal{T}}, 0), (w_{\mathcal{T}}, 0)) \ge \frac{1}{4}\|w_{\mathcal{T}}\|_\Omega^2. \tag{41}$$

To this purpose, we observe using the inverse inequalities from Lemma 3.1 and the boundeness property from Lemma 3.2 that

$$\begin{aligned}
a_h((w_{\mathcal{T}}, 0), (w_{\mathcal{T}}, 0)) &\le \varpi \|(w_{\mathcal{T}}, 0)\|_{\mathrm{HHO}}^2 \\
&\le \varpi \mu_\sharp^2 \left\{ \sum_{T \in \mathcal{T}_h} \|\nabla w_T\|_T^2 + h_T^{-1} \|w_T\|_{\partial T}^2 \right\} \\
&\le 2 C_{\mathrm{dinv}}^2 \varpi \mu_\sharp^2 h^{-2} \|w_{\mathcal{T}}\|_\Omega^2.
\end{aligned}$$

Using the CFL condition (40), we infer that

$$\frac{\Delta t^2}{8} a_h((w_{\mathcal{T}}, 0), (w_{\mathcal{T}}, 0)) \le \frac{\Delta t^2}{8} 2 C_{\mathrm{dinv}}^2 \varpi \mu_\sharp^2 h^{-2} \|w_{\mathcal{T}}\|_\Omega^2 \le \frac{1}{4}\|w_{\mathcal{T}}\|_\Omega^2.$$

Re-arranging the terms establishes (41) and completes the proof. □

5 Energy error analysis

In this section, we prove the convergence in energy-norm of the fully discrete wave equation. We start the section by stating our main result, Theorem 5.1, and devote the rest of the section to its proof.

5.1 Main result

Recall that u denotes the solution to the continuous wave equation (1) with the initial conditions (2) and $(\hat{u}_h^n)_{n\in\{0:N\}}$ the solution to the fully discrete wave equation (16) with the initial conditions (17). We define the discrete error

$$\hat{e}_h^n := \hat{u}_h^n - \hat{I}_h^{k,l}(u(t^n)), \quad \forall n \in \{0{:}N\}, \tag{42}$$

which represents the difference between the discrete hybrid solution at time step n and the projection of the continuous solution at the discrete time t^n onto the hybrid space. Recalling the seminorm $|\cdot|_{*,h}$ defined in (32), we set, for all $t \in \bar{J}$,

$$|u|_{C^0(0,t;*,h)} := \sup_{s\in[0,t]} |u(s)|_{*,h}, \quad |\partial_t u|_{L^1(0,t;*,h)} := \int_0^t |\partial_t u(s)|_{*,h}\,\mathrm{d}s. \tag{43}$$

Theorem 5.1 (Energy error estimate)**.** *Assume the CFL condition* (40)*, that $f \in C^1(\bar{J};L^2(\Omega))$, and that $u \in C^4(\bar{J};L^2(\Omega)) \cap W^{5,1}(J;L^2(\Omega))$ and $u \in C^0(\bar{J};H^{1+\nu}(\Omega)) \cap W^{1,1}(J;H^{1+\nu}(\Omega))$, $\nu \in (\frac{1}{2},1]$. The following holds:*

$$\begin{aligned}\max_{n\in\{0:N-1\}} \|\delta e_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega + \max_{n\in\{0:N-1\}} \|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}} &\le C_1\Big\{\|u\|_{C^0([0;t^{N-1}];*,h)} + \|\partial_t u\|_{L^1([0;t^{N-1}];*,h)}\Big\}\\ &+ C_2\Delta t^2\Big\{\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))} + \Theta\big(\|\partial_t^4 u\|_{C^0([0;t^N];L^2(\Omega))} + \|\partial_t^5 u\|_{L^1([0;t^N];L^2(\Omega))}\big)\Big\},\end{aligned} \tag{44}$$

with generic constants C_1 and C_2 and the time scale $\Theta := \mu_\sharp^{-1}\ell_\Omega$.

Remark 5.2 (Regularity assumption)**.** The regularity assumption in space hinging on the shift $\nu \in (\frac{1}{2},1]$ is made to simplify the bound on the consistency error in space. The quasi-minimal setting with $\nu \in (0,1]$ can be handled by using the tools introduced in [23], see also [22, Sec. 41.5].

Remark 5.3 (Convergence order)**.** Under the above regularity assumption on u, we can bound $|u(t)|_{*,h}$ and $|\partial_t u(t)|_{*,h}$, for all $t \in \bar{J}$, using the approximation results from Lemma 3.4. This gives

$$\begin{aligned}\max_{n\in\{0:N-1\}} \|\delta e_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega + \|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}} &\le C_1 h^{k+1}\Big\{|u|_{C^0([0;t^{N-1}];H^{k+2}(\Omega))} + |\partial_t u|_{L^1([t^0,t^{N-1}];H^{k+2}(\Omega))}\Big\}\\ &+ C_2\Delta t^2\Big\{\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))} + \Theta\big(\|\partial_t^4 u\|_{C^0([0;t^N];L^2(\Omega))} + \|\partial_t^5 u\|_{L^1([0;t^N];L^2(\Omega))}\big)\Big\},\end{aligned} \tag{45}$$

which is the optimal convergence order $\mathcal{O}(h^{k+1}+\Delta t^2)$ in the energy norm.

5.2 Consistency errors and error equation

We define the space consistency error, ψ_h^n, at the discrete time t^n, for all $n \in \{0{:}N\}$, as the linear form in $(\widehat{\mathcal{U}}_{h,0}^{l,k})'$ such that, for all $\hat{w}_h \in \widehat{\mathcal{U}}_{h,0}^{l,k}$,

$$\psi_h^n(\hat{w}_h) := \psi_h(u(t^n),\hat{w}_h), \tag{46}$$

where the linear form $\psi_h(v,\cdot)$ is defined in (31) for any function $v \in Y \cap H^{1+\nu}(\Omega)$ (notice that $u(t^n) \in Y \cap H^{1+\nu}(\Omega)$ for all $n \in \{0{:}N\}$ by assumption). We also define the time consistency error, κ^n, to be the error between the centered difference scheme and the exact second-order derivative in time:

$$\kappa^n := \frac{u(t^{n+1}) - 2u(t^n) + u(t^{n-1})}{\Delta t^2} - \partial_t^2 u(t^n), \quad \forall n \in \{1{:}N-1\}, \tag{47}$$

and we set $\kappa^0 := 0$.

Lemma 5.4 (Discrete error equation)**.** *Let $(\hat e_h^n)_{n\in\{0:N\}}$ be the collection of discrete errors defined in* (42)*. The following holds for all $n\in\{1{:}N-1\}$:*

$$\frac{1}{\Delta t^2}\big(e_{\mathcal T}^{n+1}-2e_{\mathcal T}^{n}+e_{\mathcal T}^{n-1},w_{\mathcal T}\big)_\Omega+a_h(\hat e_h^n,\hat w_h)=\psi_h^n(\hat w_h)-(\kappa^n,w_{\mathcal T})_\Omega,\quad\forall\hat w_h\in\widehat{\mathcal U}_{h,0}^{l,k}. \tag{48}$$

Proof. Let us evaluate the left-hand-side of (48) for all $n\in\{1{:}N-1\}$:

$$\begin{aligned}
&\frac{1}{\Delta t^2}(e_{\mathcal T}^{n+1}-2e_{\mathcal T}^{n}+e_{\mathcal T}^{n-1},w_{\mathcal T})_\Omega+a_h(\hat e_h^n,\hat w_h)\\
&\quad=(f^n,w_{\mathcal T})_\Omega-\left(\Pi_{\mathcal T}^l\left[\frac{u(t^{n+1})-2u(t^n)+u(t^{n-1})}{\Delta t^2}\right],w_{\mathcal T}\right)_\Omega-a_h(\hat I_h^{k,l}(u(t^n)),\hat w_h)\\
&\quad=(f^n,w_{\mathcal T})_\Omega-\big(\Pi_{\mathcal T}^l(\partial_t^2u(t^n)+\kappa^n),w_{\mathcal T}\big)_\Omega-a_h(\hat I_h^{k,l}(u(t^n)),\hat w_h)\\
&\quad=\big(\partial_t^2u(t^n),w_{\mathcal T}\big)_\Omega+(B(u(t^n)),w_{\mathcal T})_\Omega-\big(\partial_t^2u(t^n)+\kappa^n,w_{\mathcal T}\big)_\Omega-a_h(\hat I_h^{k,l}(u(t^n)),\hat w_h)\\
&\quad=\psi_h^n(\hat w_h)-(\kappa^n,w_{\mathcal T})_\Omega,
\end{aligned}$$

where we used the discrete scheme (16), the fact that the exact solution satisfies $\partial_t^2u(t^n)+B(u(t^n))=f^n$, the definition of the L^2-projection onto $\mathcal U_{\mathcal T}^l$ and the definition (46) of ψ_h^n. □

5.3 Energy-error identity

It is convenient to define, for all $n\in\{0{:}N-1\}$, the error and velocity error at the half time-steps as follows:

$$\hat e_h^{n+\frac12}:=\frac12(\hat e_h^{n+1}+\hat e_h^n),\quad\delta\hat e_h^{n+\frac12}:=\frac{1}{\Delta t}(\hat e_h^{n+1}-\hat e_h^n). \tag{49}$$

Concerning the consistency errors, we set, for all $n\in\{0{:}N-1\}$,

$$\delta\kappa^{n+\frac12}:=\frac{1}{\Delta t}(\kappa^{n+1}-\kappa^n),\quad\delta\psi_h^{n+\frac12}(\hat w_h):=\frac{1}{\Delta t}(\psi_h^{n+1}(\hat w_h)-\psi_h^n(\hat w_h)),\quad\forall\hat w_h\in\widehat{\mathcal U}_{h,0}^{l,k}. \tag{50}$$

Let us consider the discrete energy (38) evaluated using the error. We set

$$\mathcal E_h^{n+\frac12}:=\frac12\|\delta e_{\mathcal T}^{n+\frac12}\|_\Omega^2-\frac{\Delta t^2}{8}a_h(\delta\hat e_h^{n+\frac12},\delta\hat e_h^{n+\frac12})+\frac12a_h(\hat e_h^{n+\frac12},\hat e_h^{n+\frac12}),\quad\forall n\in\{0{:}N-1\}. \tag{51}$$

Unfortunately, even under a CFL condition, $\mathcal E_h^{n+\frac12}$ does not define a strongly convex functional on $\delta e_{\mathcal T}^{n+\frac12}$ and $\hat e_h^{n+\frac12}$. The reason is that we no longer have $a_h(\hat e_h^n,(0,w_{\mathcal F}))=0$ for all $w_{\mathcal F}\in\mathcal U_{\mathcal F,0}^k$, but only

$$a_h(\hat e_h^n,(0,w_{\mathcal F}))=\psi_h^n((0,w_{\mathcal F})), \tag{52}$$

as a consequence of (48). This leads to the following definition of discrete energy error: For all $n\in\{0{:}N-1\}$,

$$\check{\mathcal E}_h^{n+\frac12}:=\mathcal E_h^{n+\frac12}+\frac{\Delta t^2}{4}\delta\psi_h^{n+\frac12}((0,\delta e_{\mathcal F}^{n+\frac12})). \tag{53}$$

Lemma 5.5 (Discrete energy error)**.** *For all $n\in\{0{:}N-1\}$, we have*

$$\begin{aligned}
\check{\mathcal E}_h^{n+\frac12}=&\frac12\|\delta e_{\mathcal T}^{n+\frac12}\|_\Omega^2+\frac{\Delta t^2}{8}a_h((0,\delta e_{\mathcal F}^{n+\frac12}),(0,\delta e_{\mathcal F}^{n+\frac12}))\\
&-\frac{\Delta t^2}{8}a_h((\delta e_{\mathcal T}^{n+\frac12},0),(\delta e_{\mathcal T}^{n+\frac12},0))+\frac12a_h(\hat e_h^{n+\frac12},\hat e_h^{n+\frac12}).
\end{aligned} \tag{54}$$

Moreover, under the CFL condition (40)*, $\check{\mathcal E}_h^{n+\frac12}$ defines a strongly convex functional on $\delta e_{\mathcal T}^{n+\frac12}$ and $\hat e_h^{n+\frac12}$.*

Proof. Proceeding as in Lemma 4.2, we obtain

$$\begin{aligned}\Delta t^2 a_h(\delta\hat{e}_h^{n+\frac12},\delta\hat{e}_h^{n+\frac12}) &= a_h(\hat{e}_h^{n+1}-\hat{e}_h^n,\hat{e}_h^{n+1}-\hat{e}_h^n)\\ &= a_h(\hat{e}_h^{n+1}-\hat{e}_h^n,(e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0))+a_h(\hat{e}_h^{n+1}-\hat{e}_h^n,(0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n))\\ &= a_h(\hat{e}_h^{n+1}-\hat{e}_h^n,(e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0))+\Delta t\Big\{\psi_h^{n+1}((0,\delta e_{\mathcal{F}}^{n+\frac12}))-\psi_h^n((0,\delta e_{\mathcal{F}}^{n+\frac12}))\Big\}\\ &= a_h((e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0),(e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0))+a_h((0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n),(e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0))\\ &\quad+\Delta t^2\delta\psi_h^{n+\frac12}((0,\delta e_{\mathcal{F}}^{n+\frac12})),\end{aligned}$$

where we used (52) and the notation (49) for $\delta\psi_h^{n+\frac12}$. Owing to the symmetry of a_h and again (52), we infer that

$$\begin{aligned}a_h((0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n),(e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0)) &= a_h((e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0),(0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n))\\ &= -a_h((0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n),(0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n))+\Delta t^2\delta\psi_h^{n+\frac12}((0,\delta e_{\mathcal{F}}^{n+\frac12})).\end{aligned}$$

This gives

$$\begin{aligned}&\Delta t^2 a_h(\delta\hat{e}_h^{n+\frac12},\delta\hat{e}_h^{n+\frac12})\\ &= a_h((e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0),(e_{\mathcal{T}}^{n+1}-e_{\mathcal{T}}^n,0))-a_h((0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n),(0,e_{\mathcal{F}}^{n+1}-e_{\mathcal{F}}^n))+2\Delta t^2\delta\psi_h^{n+\frac12}((0,\delta e_{\mathcal{F}}^{n+\frac12}))\\ &= \Delta t^2 a_h((\delta e_{\mathcal{T}}^{n+\frac12},0),(\delta e_{\mathcal{T}}^{n+\frac12},0))-\Delta t^2 a_h((0,\delta e_{\mathcal{F}}^{n+\frac12}),(0,\delta e_{\mathcal{F}}^{n+\frac12}))+2\Delta t^2\delta\psi_h^{n+\frac12}((0,\delta e_{\mathcal{F}}^{n+\frac12})).\end{aligned}$$

This proves the identity (54). Finally, the strong convexity of $\check{\mathcal{E}}_h^{n+\frac12}$ under the CFL condition (40) is established as in the proof of Lemma 4.2. □

The next step is to write an energy identity mimicking the space semi-discrete case.

Lemma 5.6 (Energy identity)**.** *For all $n\in\{1{:}N-1\}$, the discrete energy error $\check{\mathcal{E}}_h^{n+\frac12}$ verifies*

$$\check{\mathcal{E}}_h^{n+\frac12} = \mathcal{Z}_\psi^n+\mathcal{Z}_\kappa^n+\mathcal{Z}_{\rm IC}^0, \tag{55}$$

with the space consistency error

$$\mathcal{Z}_\psi^n := \psi_h^n(\hat{e}_h^{n+\frac12})-\frac{\Delta t^2}{4}\delta\psi_h^{n+\frac12}((0,\delta e_{\mathcal{F}}^{n+\frac12}))+\Delta t\sum_{m=1}^{n-1}\delta\psi_h^{m+\frac12}(\hat{e}_h^{m+\frac12}), \tag{56a}$$

the time consistency error

$$\mathcal{Z}_\kappa^n := -(\kappa^n,e_{\mathcal{T}}^{n+\frac12})_\Omega+\Delta t\sum_{m=1}^{n-1}(\delta\kappa^{m+\frac12},e_{\mathcal{T}}^{m+\frac12})_\Omega, \tag{56b}$$

and the initial error

$$\mathcal{Z}_{\rm IC}^0 := \mathcal{E}_h^{\frac12}-\psi_h^1(\hat{e}_h^{\frac12})+(\kappa^1,e_{\mathcal{T}}^{\frac12})_\Omega. \tag{56c}$$

Proof. Let us start by remarking that, similarly to the computations in the proof of Lemma 4.1, we have, for all $m\in\{1{:}N-1\}$,

$$\mathcal{E}_h^{m+\frac12}-\mathcal{E}_h^{m-\frac12} = \frac{1}{\Delta t^2}\big(e_{\mathcal{T}}^{m+1}-2e_{\mathcal{T}}^m+e_{\mathcal{T}}^{m-1},e_{\mathcal{T}}^{m+1}-e_{\mathcal{T}}^{m-1}\big)_\Omega+a_h(\hat{e}_h^m,\hat{e}_h^{m+1}-\hat{e}_h^{m-1}).$$

Using Lemma 5.4 with the test function $\hat{w}_h := \frac{1}{2}(\hat{e}_h^{m+1} - \hat{e}_h^{m-1}) = \hat{e}_h^{m+\frac{1}{2}} - \hat{e}_h^{m-\frac{1}{2}}$ yields

$$\mathcal{E}_h^{m+\frac{1}{2}} - \mathcal{E}_h^{m-\frac{1}{2}} = \psi_h^m(\hat{e}_h^{m+\frac{1}{2}} - \hat{e}_h^{m-\frac{1}{2}}) - (\kappa^m, e_{\mathcal{T}}^{m+\frac{1}{2}} - e_{\mathcal{T}}^{m-\frac{1}{2}})_\Omega. \tag{57}$$

Summing (57) from $m = 1$ to $m = n$ yields

$$\mathcal{E}_h^{n+\frac{1}{2}} = \mathcal{E}_h^{\frac{1}{2}} + \sum_{m=1}^{n} \Big\{ \psi_h^m(\hat{e}_h^{m+\frac{1}{2}} - \hat{e}_h^{m-\frac{1}{2}}) - (\kappa^m, e_{\mathcal{T}}^{m+\frac{1}{2}} - e_{\mathcal{T}}^{m-\frac{1}{2}})_\Omega \Big\}. \tag{58}$$

The sums on the right-hand side of (58) can be reordered using the notation (49) to obtain

$$\sum_{m=1}^{n} \psi_h^m(\hat{e}_h^{m+\frac{1}{2}} - \hat{e}_h^{m-\frac{1}{2}}) = \psi_h^n(\hat{e}_h^{n+\frac{1}{2}}) - \psi_h^1(\hat{e}_h^{\frac{1}{2}}) - \Delta t \sum_{m=1}^{n-1} \delta\psi_h^{m+\frac{1}{2}}(\hat{e}_h^{m+\frac{1}{2}}),$$

$$\sum_{m=1}^{n} (\kappa^m, e_{\mathcal{T}}^{m+\frac{1}{2}} - e_{\mathcal{T}}^{m-\frac{1}{2}})_\Omega = (\kappa^n, e_{\mathcal{T}}^{n+\frac{1}{2}})_\Omega - (\kappa^1, e_{\mathcal{T}}^{\frac{1}{2}})_\Omega - \Delta t \sum_{m=1}^{n-1} (\delta\kappa^{m+\frac{1}{2}}, e_{\mathcal{T}}^{m+\frac{1}{2}})_\Omega.$$

This gives

$$\begin{aligned}
\mathcal{E}_h^{n+\frac{1}{2}} &= \mathcal{E}_h^{\frac{1}{2}} - \psi_h^1(\hat{e}_h^{\frac{1}{2}}) + (\kappa^1, e_{\mathcal{T}}^{\frac{1}{2}})_\Omega + \Big\{ \psi_h^n(\hat{e}_h^{n+\frac{1}{2}}) - \Delta t \sum_{m=1}^{n-1} \delta\psi_h^{m+\frac{1}{2}}(\hat{e}_h^{m+\frac{1}{2}}) \Big\} \\
&\quad - (\kappa^n, e_{\mathcal{T}}^{n+\frac{1}{2}})_\Omega + \Delta t \sum_{m=1}^{n-1} (\delta\kappa^{m+\frac{1}{2}}, e_{\mathcal{T}}^{m+\frac{1}{2}})_\Omega \\
&= \psi_h^n(\hat{e}_h^{n+\frac{1}{2}}) - \Delta t \sum_{m=1}^{n-1} \delta\psi_h^{m+\frac{1}{2}}(\hat{e}_h^{m+\frac{1}{2}}) + \mathcal{Z}_\kappa^n + \mathcal{Z}_{\mathrm{IC}}^0,
\end{aligned}$$

owing to the definitions (56b) and (56c). It remains to go from $\mathcal{E}_h^{n+\frac{1}{2}}$ to $\check{\mathcal{E}}_h^{n+\frac{1}{2}}$. Using (53), we have

$$\check{\mathcal{E}}_h^{n+\frac{1}{2}} = \psi_h^n(\hat{e}_h^{n+\frac{1}{2}}) - \Delta t \sum_{m=1}^{n-1} \delta\psi_h^{m+\frac{1}{2}}(\hat{e}_h^{m+\frac{1}{2}}) + \mathcal{Z}_\kappa^n + \mathcal{Z}_{\mathrm{IC}}^0 + \frac{\Delta t^2}{4} \delta\psi_h^{n+\frac{1}{2}}((0, \delta e_{\mathcal{F}}^{n+\frac{1}{2}})).$$

Recalling the definition of $\mathcal{Z}_\psi^n$ proves the claim. □

5.4 Bound on consistency and initial errors

We now bound the three terms on the right-hand side of (55). Each estimate is stated as a separate lemma.

Lemma 5.7 (Estimate on the space consistency error)**.** *Let $\mathcal{Z}_\psi^n$ be defined in* (56a)*. For all $n \in \{1{:}N-1\}$, the following holds:*

$$\begin{aligned}
|\mathcal{Z}_\psi^n| &\le c_* \big(|\partial_t u|_{L^1([t^1,t^n];*,h)} + |u(t^n)|_{*,h} \big) \max_{m\in\{1:n\}} \|\hat{e}_h^{m+\frac{1}{2}}\|_{\mathrm{HHO}} \\
&\quad + \frac{1}{4} c_* \alpha^{-\frac{1}{2}} |\partial_t u|_{L^1([t^n;t^{n+1}];*,h)} \Big[\Delta t^2 a_h((0, \delta e_{\mathcal{F}}^{m+\frac{1}{2}}), (0, \delta e_{\mathcal{F}}^{m+\frac{1}{2}})) \Big]^{\frac{1}{2}},
\end{aligned}$$

where c_* results from (34) *and α is the coercivity constant of the discrete bilinear form a_h.*

Proof. Let $n \in \{1{:}N-1\}$ and let $m \in \{1{:}n\}$. We observe that

$$\Delta t \delta\psi_h^{m+\frac12}(\hat e_h^{m+\frac12}) = \psi_h(u(t^{m+1}),\hat e_h^{m+\frac12}) - \psi_h(u(t^m),\hat e_h^{m+\frac12}) = \int_{t^m}^{t^{m+1}} \psi_h(\partial_t u(s);\hat e_h^{m+\frac12})\,\mathrm{d}s.$$

Since $\partial_t u(t) \in Y \cap H^{1+\nu}(\Omega)$ for all $t \in \bar J$ by assumption (indeed, $B(\partial_t u(t)) = \partial_t f(t) - \partial_t^3 u(t)$, $f \in C^1(\bar J; L^2(\Omega))$, and $u \in C^4(\bar J; L^2(\Omega))$), we can invoke (34) to infer that

$$\|\psi_h(\partial_t u(t);\cdot)\|_{(\mathrm{HHO})'} \le c_*|\partial_t u(t)|_{*,h}.$$

Thus, we obtain

$$\left|\Delta t \sum_{m=1}^{n-1} \delta\psi_h^{m+\frac12}(\hat e_h^{m+\frac12})\right| \le c_* \sum_{m=1}^{n-1} |\partial_t u|_{L^1([t^m,t^{m+1}];*,h)} \|\hat e_h^{m+\frac12}\|_{\mathrm{HHO}}.$$

The same reasoning is used for the second term on the right-hand-side of (56a) which is bounded as follows:

$$\left|\frac{\Delta t^2}{4}\delta\psi_h^{n+\frac12}((0,\delta e_{\mathcal F}^{n+\frac12}))\right| \le \frac14 c_*\alpha^{-\frac12}|\partial_t u|_{L^1([t^n,t^{n+1}];*,h)}\left[\Delta t^2 a_h((0,\delta e_{\mathcal F}^{n+\frac12}),(0,\delta e_{\mathcal F}^{n+\frac12}))\right]^{\frac12},$$

where we used the coercivity of the discrete bilinear form a_h. Finally, the same argument for the first term on the right-hand side of (56a) yields

$$\left|\psi_h^n(\hat e_h^{n+\frac12})\right| \le c_*|u(t^n)|_{*,h}\|\hat e_h^{n+\frac12}\|_{\mathrm{HHO}}. \tag{59}$$

This yields the claim. □

Lemma 5.8 (Estimate on the time consistency error)**.** *Let $\mathcal Z_\kappa^n$ be defined in* (56b)*. For all $n \in \{1{:}N-1\}$, the following holds:*

$$|\mathcal Z_\kappa^n| \le C\Theta\Delta t^2\big(\|\partial_t^5 u\|_{L^1([0;t^{n+1}];L^2(\Omega))} + \|\partial_t^4 u\|_{C^0([t^{n-1};t^{n+1}];L^2(\Omega))}\big)\max_{m\in\{1:n\}}\|\hat e_h^{m+\frac12}\|_{\mathrm{HHO}}.$$

Proof. Let $n \in \{1{:}N-1\}$. A straightforward calculation using fourth-order Taylor expansions with integral remainder shows that

$$\kappa^n = \frac{1}{6\Delta t^2}\left\{\int_{t^n}^{t^{n+1}} (t^{n+1}-t)^3\partial_t^4 u(t)\,\mathrm{d}t - \int_{t^{n-1}}^{t^n}(t^{n-1}-t)^3\partial_t^4 u(t)\,\mathrm{d}t\right\}.$$

Rewriting $\delta\kappa^{m+\frac12}$, for all $m \in \{1{:}n-1\}$, using this identity gives

$$\begin{aligned}
\delta\kappa^{m+\frac12} &= \frac{1}{\Delta t}(\kappa^{m+1}-\kappa^m)\\
&= \frac{1}{6\Delta t^3}\left(\int_{t^{m+1}}^{t^{m+2}}(t^{m+2}-t)^3\partial_t^4 u(t)\,\mathrm{d}t - \int_{t^m}^{t^{m+1}}(t^m-t)^3\partial_t^4 u(t)\,\mathrm{d}t\right.\\
&\quad\left. - \int_{t^m}^{t^{m+1}}(t^{m+1}-t)^3\partial_t^4 u(t)\,\mathrm{d}t + \int_{t^{m-1}}^{t^m}(t^{m-1}-t)^3\partial_t^4 u(t)\,\mathrm{d}t\right)\\
&= \frac{1}{6\Delta t^3}\left(\int_{t^m}^{t^{m+1}}(t^{m+1}-t)^3\int_t^{t+\Delta t}\partial_t^5 u(s)\,\mathrm{d}s\,\mathrm{d}t - \int_{t^{m-1}}^{t^m}(t^{m-1}-t)^3\int_t^{t+\Delta t}\partial_t^5 u(s)\,\mathrm{d}s\,\mathrm{d}t\right).
\end{aligned}$$

Let us now use the following estimate on the first integral:

$$\left\|\int_{t^m}^{t^{m+1}}(t^{m+1}-t)^3\int_t^{t+\Delta t}\partial_t^5 u(s)\,\mathrm{d}s\,\mathrm{d}t\right\|_\Omega \le \Delta t^4\|\partial_t^5 u\|_{L^1([t^m;t^{m+2}];L^2(\Omega))}, \tag{60}$$

and a similar estimate on the second integral. Then, we invoke the discrete Poincaré inequality (23) to obtain

$$\begin{aligned}\left|\Delta t\sum_{m=1}^{n-1}(\delta\kappa^{m+\frac12},e_{\mathcal T}^{m+\frac12})_\Omega\right| &\le C\mu_\sharp^{-1}\ell_\Omega\Delta t^2\sum_{m=1}^{n-1}\|\partial_t^5 u\|_{L^1([t^{m-1};t^{m+2}];L^2(\Omega))}\|\hat e_h^{m+\frac12}\|_{\mathrm{HHO}}\\ &= C\Theta\Delta t^2\sum_{m=1}^{n-1}\|\partial_t^5 u\|_{L^1([t^{m-1};t^{m+2}];L^2(\Omega))}\|\hat e_h^{m+\frac12}\|_{\mathrm{HHO}}.\end{aligned} \tag{61}$$

The first term on the right-hand-side of (56b) can be bounded in the same way, yielding

$$\left|(\kappa^n,e_{\mathcal T}^{n+\frac12})_\Omega\right| \le C\Theta\Delta t^2\|\partial_t^4 u\|_{C^0([t^{n-1};t^{n+1}];L^2(\Omega))}\|\hat e_h^{n+\frac12}\|_{\mathrm{HHO}}. \tag{62}$$

Taking the maximum over $m\in\{1{:}n-1\}$ on the right-hand-side of (61) and summing to (62) yields the expected result. □

Lemma 5.9 (Estimate on the initial error)**.** *Let $\mathcal Z_{\mathrm{IC}}^0$ be defined in* (56c)*. Assume the CFL condition* (40)*. There is a constant C_{IC} such that*

$$\left|\mathcal Z_{\mathrm{IC}}^0\right| \le C_{\mathrm{IC}}\Big\{|u_0|_{*,h}^2+\left|u(t^1)\right|_{*,h}^2+\Delta t^4\big(\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}^2+\Theta^2\|\partial_t^4 u\|_{C^0([t^0;t^2];L^2(\Omega))}^2\big)\Big\}.$$

Proof. Recall that

$$\begin{aligned}\mathcal Z_{\mathrm{IC}}^0 &= \mathcal E_h^{\frac12}-\psi_h^1(\hat e_h^{\frac12})+(\kappa^1,e_{\mathcal T}^{\frac12})_\Omega\\ &= \frac{1}{2\Delta t^2}\|e_{\mathcal T}^1-e_{\mathcal T}^0\|_\Omega^2-\frac18 a_h(\hat e_h^1-\hat e_h^0,\hat e_h^1-\hat e_h^0)+\frac12 a_h(\hat e_h^{\frac12},\hat e_h^{\frac12})-\psi_h^1(\hat e_h^{\frac12})+(\kappa^1,e_{\mathcal T}^{\frac12})_\Omega.\end{aligned} \tag{63}$$

Since the second term is negative, we infer that $\mathcal Z_{\mathrm{IC}}^0\le A_1+A_2+A_3$ with

$$A_1:=\frac{1}{2\Delta t^2}\|e_{\mathcal T}^1-e_{\mathcal T}^0\|_\Omega^2,\qquad A_2:=\frac12 a_h(\hat e_h^{\frac12},\hat e_h^{\frac12}),\qquad A_3:=(\kappa^1,e_{\mathcal T}^{\frac12})_\Omega-\psi_h^1(\hat e_h^{\frac12}).$$

Step 1: Bound on A_1. We use a first-order Taylor expansion of $u(t^1)$ with integral remainder to obtain

$$u(t^1)=u_0+\Delta t v_0+\frac{\Delta t^2}{2}\partial_t^2 u(0)+\frac12\int_0^{t^1}(t^1-s)^2\partial_t^3 u(s)\,\mathrm{d}s.$$

The definition of $u_{\mathcal T}^1$ via the initial condition (17c) then gives, for all $w_{\mathcal T}\in\mathcal U_{\mathcal T}^l$,

$$\begin{aligned}\frac{1}{\Delta t}(e_{\mathcal T}^1,w_{\mathcal T})_\Omega &= \frac{1}{\Delta t}(u_{\mathcal T}^1-u(t^1),w_{\mathcal T})_\Omega\\ &= \frac{\Delta t}{2}(f^0-\partial_t^2 u(0),w_{\mathcal T})_\Omega-\frac{1}{2\Delta t}\int_0^{t^1}(t^1-s)^2(\partial_t^3 u(s),w_{\mathcal T})_\Omega\,\mathrm{d}s-\frac{\Delta t}{2}a_h(\hat u_h^0,(w_{\mathcal T},0))\\ &= \frac{\Delta t}{2}\Big\{(B(u^0),w_{\mathcal T})_\Omega-a_h(\hat u_h^0,(w_{\mathcal T},0))\Big\}-\frac{1}{2\Delta t}\int_0^{t^1}(t^1-s)^2(\partial_t^3 u(s),w_{\mathcal T})_\Omega\,\mathrm{d}s\\ &= \frac{\Delta t}{2}\psi_h^0((w_{\mathcal T},0))-\frac{1}{2\Delta t}\int_0^{t^1}(t^1-s)^2(\partial_t^3 u(s),w_{\mathcal T})_\Omega\,\mathrm{d}s.\end{aligned}$$

We use both discrete inverse inequalities from Lemma 3.1 to obtain

$$\begin{aligned}\|(w_{\mathcal{T}},0)\|_{\mathrm{HHO}} &= \Big(\sum_{T\in\mathcal{T}_h}\mu_T^2\big\{\|\nabla w_T\|_T^2 + h_T^{-1}\|w_T\|_{\partial T}^2\big\}\Big)^{\frac12}\\ &\le C\mu_\sharp h^{-1}\|w_{\mathcal{T}}\|_\Omega.\end{aligned}$$

Using this inequality, the CFL condition (40) and the inequality from (34) to bound $\|\psi_h^0\|_{(\mathrm{HHO})'}$, we obtain

$$\begin{aligned}\frac{1}{\Delta t}|(e_{\mathcal{T}}^1,w_{\mathcal{T}})_\Omega| &\le \frac{\Delta t}{2}\|\psi_h^0\|_{(\mathrm{HHO})'}\|(w_{\mathcal{T}},0)\|_{\mathrm{HHO}} + \frac{\Delta t^2}{2}\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}\|w_{\mathcal{T}}\|_\Omega\\ &\le C\big(|u_0|_{*,h} + \Delta t^2\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}\big)\|w_{\mathcal{T}}\|_\Omega.\end{aligned}$$

Since $\|e_{\mathcal{T}}^1\|_\Omega \le \sup_{w_{\mathcal{T}}\in\mathcal{U}_{\mathcal{T}}^l}\frac{(e_{\mathcal{T}}^1,w_{\mathcal{T}})_\Omega}{\|w_{\mathcal{T}}\|_\Omega}$, we conclude that $\frac{1}{\Delta t}\|e_{\mathcal{T}}^1\|_\Omega \le C\big(|u_0|_{*,h} + \Delta t^2\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}\big)$. Since $e_{\mathcal{T}}^0 = 0$, this finally gives

$$A_1 = \frac{1}{2\Delta t^2}\|e_{\mathcal{T}}^1 - e_{\mathcal{T}}^0\|_\Omega^2 \le C\big(|u_0|_{*,h}^2 + \Delta t^4\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}^2\big). \tag{64}$$

Step 2: Bound on $\|(0,e_{\mathcal{F}}^0)\|_{\mathrm{HHO}}$ *and* $\|(0,e_{\mathcal{F}}^1)\|_{\mathrm{HHO}}$. Since $e_{\mathcal{T}}^0 = 0$, the coercivity of the discrete bilinear form a_h implies that

$$\alpha\|\hat{e}_h^0\|_{\mathrm{HHO}}^2 = \alpha\|(0,e_{\mathcal{F}}^0)\|_{\mathrm{HHO}}^2 \le a_h((0,e_{\mathcal{F}}^0),(0,e_{\mathcal{F}}^0)),$$

with $e_{\mathcal{F}}^0 = u_{\mathcal{F}}^0 - \Pi_{\mathcal{F}}^k(u_0)$. Equation (17b) and the linearity of a_h with respect to its first argument imply that

$$a_h((0,u_{\mathcal{F}}^0),(0,e_{\mathcal{F}}^0)) = -a_h((u_{\mathcal{T}}^0,0),(0,e_{\mathcal{F}}^0)) = -a_h((\Pi_{\mathcal{T}}^l(u_0),0),(0,e_{\mathcal{F}}^0)).$$

where we used (15) in the second equality. Hence, we have

$$\begin{aligned}\alpha\|\hat{e}_h^0\|_{\mathrm{HHO}}^2 = \alpha\|(0,e_{\mathcal{F}}^0)\|_{\mathrm{HHO}}^2 &\le a_h((0,u_{\mathcal{F}}^0 - \Pi_{\mathcal{F}}^k(u_0)),(0,e_{\mathcal{F}}^0))\\ &= -a_h((\Pi_{\mathcal{T}}^l(u_0),0),(0,e_{\mathcal{F}}^0)) - a_h((0,\Pi_{\mathcal{F}}^k(u_0)),(0,e_{\mathcal{F}}^0))\\ &= -a_h(\hat{I}_h^{k,l}(u_0),(0,e_{\mathcal{F}}^0))\\ &= \psi_h(u_0,(0,e_{\mathcal{F}}^0)) \le c_*|u_0|_{*,h}\|(0,e_{\mathcal{F}}^0)\|_{\mathrm{HHO}},\end{aligned}$$

where we used the consistency bound (34) (notice that $u_0 \in Y\cap H^{1+\nu}(\Omega)$ by assumption). This implies that $\|(0,e_{\mathcal{F}}^0)\|_{\mathrm{HHO}} \le C|u_0|_{*,h}$.

Moreover, to bound $e_{\mathcal{F}}^1$, we use the following equality (see (52) for $n=1$ and $w_{\mathcal{F}} = e_{\mathcal{F}}^1$):

$$a_h((0,e_{\mathcal{F}}^1),(0,e_{\mathcal{F}}^1)) = \psi_h^1((0,e_{\mathcal{F}}^1)) - a_h((e_{\mathcal{T}}^1,0),(0,e_{\mathcal{F}}^1)).$$

Using the coercivity and boundedness of the bilinear form a_h, we obtain

$$\|(0,e_{\mathcal{F}}^1)\|_{\mathrm{HHO}}^2 \le \frac{1}{\alpha}a_h((0,e_{\mathcal{F}}^1),(0,e_{\mathcal{F}}^1)) \le \frac{1}{\alpha}\Big(\|\psi_h^1\|_{(\mathrm{HHO})'} + \varpi\|(e_{\mathcal{T}}^1,0)\|_{\mathrm{HHO}}\Big)\|(0,e_{\mathcal{F}}^1)\|_{\mathrm{HHO}},$$

so that

$$\|(0,e_{\mathcal{F}}^1)\|_{\mathrm{HHO}} \le C\big(\|\psi_h^1\|_{(\mathrm{HHO})'} + \|(e_{\mathcal{T}}^1,0)\|_{\mathrm{HHO}}\big). \tag{65}$$

To bound the second term on the right-hand-side, we invoke again the coercivity and boundedness of a_h together with the discrete inverse inequality from Lemma 3.1 and the CFL condition (40), to obtain (since $e_{\mathcal{T}}^0 = 0$)

$$\begin{aligned}\alpha\|(e_{\mathcal{T}}^1,0)\|_{\mathrm{HHO}}^2 \le a_h((e_{\mathcal{T}}^1-e_{\mathcal{T}}^0,0),(e_{\mathcal{T}}^1-e_{\mathcal{T}}^0,0)) &\le \varpi\|(e_{\mathcal{T}}^1-e_{\mathcal{T}}^0,0)\|_{\mathrm{HHO}}^2\\ &\le C\mu_\sharp^2 h^{-2}\|e_{\mathcal{T}}^1-e_{\mathcal{T}}^0\|_\Omega^2\\ &\le C'\frac{1}{\Delta t^2}\|e_{\mathcal{T}}^1-e_{\mathcal{T}}^0\|_\Omega^2,\end{aligned} \tag{66}$$

which can be bounded using (64). Since $\|\psi_h^1\|_{\mathrm{HHO}'} \le c_*|u(t^1)|_{*,h}$, this gives altogether

$$\|(0,e_{\mathcal{F}}^1)\|_{\mathrm{HHO}} \le C\Big(|u_0|_{*,h} + |u(t^1)|_{*,h} + \Delta t^2\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}\Big). \tag{67}$$

Step 3: Bound on $\|\hat{e}_h^{\frac12}\|_{\mathrm{HHO}}$. The symmetry of a_h, the fact that $e_{\mathcal{T}}^0 = 0$ and the above arguments give

$$\begin{aligned}a_h(\hat{e}_h^1+\hat{e}_h^0,\hat{e}_h^1+\hat{e}_h^0) &\le 2a_h((e_{\mathcal{T}}^1+e_{\mathcal{T}}^0,0),(e_{\mathcal{T}}^1+e_{\mathcal{T}}^0,0)) + 2a_h((0,e_{\mathcal{F}}^1+e_{\mathcal{F}}^0),(0,e_{\mathcal{F}}^1+e_{\mathcal{F}}^0))\\ &\le 2a_h((e_{\mathcal{T}}^1,0),(e_{\mathcal{T}}^1,0)) + 4a_h((0,e_{\mathcal{F}}^1),(0,e_{\mathcal{F}}^1)) + 4a_h((0,e_{\mathcal{F}}^0),(0,e_{\mathcal{F}}^0))\\ &= 2a_h((e_{\mathcal{T}}^1-e_{\mathcal{T}}^0,0),(e_{\mathcal{T}}^1-e_{\mathcal{T}}^0,0)) + 4a_h((0,e_{\mathcal{F}}^1),(0,e_{\mathcal{F}}^1)) + 4a_h((0,e_{\mathcal{F}}^0),(0,e_{\mathcal{F}}^0))\\ &\le C\big(\|(e_{\mathcal{T}}^1-e_{\mathcal{T}}^0,0)\|_{\mathrm{HHO}}^2 + \|(0,e_{\mathcal{F}}^1)\|_{\mathrm{HHO}}^2 + \|(0,e_{\mathcal{F}}^0)\|_{\mathrm{HHO}}^2\big)\\ &\le C'\big(\frac{1}{\Delta t^2}\|(e_{\mathcal{T}}^1-e_{\mathcal{T}}^0,0)\|_\Omega^2 + \|(0,e_{\mathcal{F}}^1)\|_{\mathrm{HHO}}^2 + \|(0,e_{\mathcal{F}}^0)\|_{\mathrm{HHO}}^2\big)\\ &\le C''\big(|u_0|_{*,h}^2 + |u(t^1)|_{*,h}^2 + \Delta t^4\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}^2\big),\end{aligned}$$

owing to (64), (65) and (67). Since $\|\hat{e}_h^{\frac12}\|_{\mathrm{HHO}}^2 \le \frac{1}{\alpha}a_h(\hat{e}_h^{\frac12},\hat{e}_h^{\frac12})$, we conclude that

$$\|\hat{e}_h^{\frac12}\|_{\mathrm{HHO}} \le C\Big(|u_0|_{*,h} + |u(t^1)|_{*,h} + \Delta t^2\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}\Big). \tag{68}$$

Step 4: Bound on A_2 and A_3. The bound (68) directly yields

$$|A_2| \le C\Big(|u_0|_{*,h}^2 + |u(t^1)|_{*,h}^2 + \Delta t^4\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}^2\Big).$$

Finally, recalling the arguments from the proofs of Lemmas 5.7 and 5.8, we have

$$\begin{aligned}\Big|(\kappa^1,e_{\mathcal{T}}^{\frac12})_\Omega\Big| &\le C\Theta\Delta t^2\|\partial_t^4 u\|_{C^0([t^0;t^2];L^2(\Omega))}\|\hat{e}_h^{\frac12}\|_{\mathrm{HHO}},\\ \Big|\psi_h^1(\hat{e}_h^{\frac12})\Big| &\le C|u(t^1)|_{*,h}\|\hat{e}_h^{\frac12}\|_{\mathrm{HHO}}.\end{aligned}$$

The above estimate of $\|\hat{e}_h^{\frac12}\|_{\mathrm{HHO}}$ then gives

$$\begin{aligned}|A_3| &\le \Big|(\kappa^1,e_{\mathcal{T}}^{\frac12})_\Omega\Big| + \Big|\psi_h^1(\hat{e}_h^{\frac12})\Big|\\ &\le C\big(|u(t^1)|_{*,h} + \Theta\Delta t^2\|\partial_t^4 u\|_{C^0([t^0;t^2];L^2(\Omega))}\big)\big(|u_0|_{*,h} + |u(t^1)|_{*,h} + \Delta t^2\|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}\big)\\ &\le C'\Big\{|u_0|_{*,h}^2 + |u(t^1)|_{*,h}^2 + \Delta t^4\big(\Theta^2\|\partial_t^4 u\|_{C^0([t^0;t^2];L^2(\Omega))}^2 + \|\partial_t^3 u\|_{C^0([0;t^1];L^2(\Omega))}^2\big)\Big\},\end{aligned}$$

where the last bound follows from Young's inequality. Gathering the two estimates from Step 4 with the one on A_1 established in Step 1 proves the claim. □

5.5 Proof of Theorem 5.1

We are now ready to complete the proof of Theorem 5.1. On the one hand, using the convexity result of Lemma 5.5, we have, for all $n \in \{1{:}N-1\}$,

$$\frac{1}{4}\|\delta e_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega^2 + \alpha\|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}}^2 + \frac{\Delta t^2}{8}a_h((0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}}),(0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}})) \le \check{\mathcal{E}}_h^{n+\frac{1}{2}}. \tag{69}$$

On the other hand, Lemma 5.6 gives, for all $n \in \{1{:}N-1\}$,

$$\begin{aligned}\check{\mathcal{E}}_h^{n+\frac{1}{2}} &\le |\mathcal{Z}_\psi^n| + |\mathcal{Z}_\kappa^n| + |\mathcal{Z}_{\mathrm{IC}}^0| \\ &\le \max_{n\in\{1:N-1\}}|\mathcal{Z}_\psi^n| + \max_{n\in\{1:N-1\}}|\mathcal{Z}_\kappa^n| + |\mathcal{Z}_{\mathrm{IC}}^0|.\end{aligned}$$

Combining the above two inequalities yields, for all $n \in \{1{:}N-1\}$,

$$\frac{1}{4}\|\delta e_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega^2 + \alpha\|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}}^2 + \frac{\Delta t^2}{8}a_h((0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}}),(0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}})) \le \max_{n\in\{1:N-1\}}|\mathcal{Z}_\psi^n| + \max_{n\in\{1:N-1\}}|\mathcal{Z}_\kappa^n| + |\mathcal{Z}_{\mathrm{IC}}^0|.$$

Since the right-hand-side of this inequality does not depend on n and since each term on the left-hand-side is positive, we infer that

$$\begin{aligned}\frac{1}{4}\max_{n\in\{1:N-1\}}\|\delta e_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega^2 &\le \max_{n\in\{1:N-1\}}|\mathcal{Z}_\psi^n| + \max_{n\in\{1:N-1\}}|\mathcal{Z}_\kappa^n| + |\mathcal{Z}_{\mathrm{IC}}^0|,\\ \alpha\max_{n\in\{1:N-1\}}\|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}}^2 &\le \max_{n\in\{1:N-1\}}|\mathcal{Z}_\psi^n| + \max_{n\in\{1:N-1\}}|\mathcal{Z}_\kappa^n| + |\mathcal{Z}_{\mathrm{IC}}^0|,\\ \frac{\Delta t^2}{8}\max_{n\in\{1:N-1\}}a_h((0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}}),(0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}})) &\le \max_{n\in\{1:N-1\}}|\mathcal{Z}_\psi^n| + \max_{n\in\{1:N-1\}}|\mathcal{Z}_\kappa^n| + |\mathcal{Z}_{\mathrm{IC}}^0|.\end{aligned}$$

This gives

$$\begin{aligned}\frac{1}{4}\max_{n\in\{1:N-1\}}\|\delta e_{\mathcal{T}}^{n+\frac{1}{2}}\|_\Omega^2 + \alpha\max_{n\in\{1:N-1\}}\|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}}^2 + \frac{\Delta t^2}{8}\max_{n\in\{1:N-1\}}a_h((0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}}),(0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}}))&\\ \le 3\Big\{\max_{n\in\{1:N-1\}}|\mathcal{Z}_\psi^n| + \max_{n\in\{1:N-1\}}|\mathcal{Z}_\kappa^n| + |\mathcal{Z}_{\mathrm{IC}}^0|\Big\}.&\end{aligned} \tag{70}$$

Let us collect the results of Lemmas 5.7 and 5.8 and apply Young's inequality to the upper bounds on $\mathcal{Z}_\psi^n$ and $\mathcal{Z}_\kappa^n$. We obtain

$$\begin{aligned}\max_{n\in\{1:N-1\}}|\mathcal{Z}_\psi^n| \le C_\psi\Big(|\partial_t u|^2_{L^1([t^1,t^N];*,h)} + |u|^2_{C^0([t^1;t^{N-1}];*,h)}\Big) + \frac{\alpha}{8}\max_{n\in\{1:N-1\}}\|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}}^2&\\ + \frac{\Delta t^2}{24}\max_{n\in\{1:N-1\}}a_h((0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}}),(0,\delta e_{\mathcal{F}}^{n+\frac{1}{2}})),&\end{aligned}$$

for the space consistency error, and

$$\max_{n\in\{1:N-1\}}|\mathcal{Z}_\kappa^n| \le C_\kappa\Theta^2\Delta t^4\Big(\|\partial_t^5 u\|^2_{L^1([0;t^N];L^2(\Omega))} + \|\partial_t^4 u\|^2_{C^0([0;t^N];L^2(\Omega))}\Big) + \frac{\alpha}{8}\max_{n\in\{1:N-1\}}\|\hat{e}_h^{n+\frac{1}{2}}\|_{\mathrm{HHO}}^2,$$

for the time consistency error, with constants C_ψ and C_κ. Using these two bounds and the bound on $\mathcal{Z}^0_{\mathrm{IC}}$ from Lemma 5.9 in (70) and rearranging the terms involving $\max_{n\in\{1:N-1\}}\|\hat{e}_h^{n+\frac12}\|^2_{\mathrm{HHO}}$, we obtain

$$\begin{aligned}\frac14\max_{n\in\{1:N-1\}}\|\delta e_{\mathcal{T}}^{n+\frac12}\|^2_\Omega+\frac14\alpha\max_{n\in\{1:N-1\}}\|\hat{e}_h^{n+\frac12}\|^2_{\mathrm{HHO}}&\le C_\psi\Big(|\partial_t u|^2_{L^1([t^1,t^N];*,h)}+|u|^2_{C^0([t^1;t^{N-1}];*,h)}\Big)\\&+C_\kappa\Theta^2\Delta t^4\Big(\|\partial_t^5u\|^2_{L^1([0;t^N];L^2(\Omega))}+\|\partial_t^4u\|^2_{C^0([0;t^N];L^2(\Omega))}\Big)\\&+C_{\mathrm{IC}}\Big(|u_0|^2_{*,h}+\big|u(t^1)\big|^2_{*,h}+\Delta t^4\big(\|\partial_t^3u\|^2_{C^0([0;t^1];L^2(\Omega))}+\Theta^2\|\partial_t^4u\|^2_{C^0([t^0;t^2];L^2(\Omega))}\big)\Big)\end{aligned}$$

Taking the square root and reorganizing the terms gives

$$\begin{aligned}\max_{n\in\{1:N-1\}}\|\delta e_{\mathcal{T}}^{n+\frac12}\|_\Omega+\max_{n\in\{1:N-1\}}\|\hat{e}_h^{n+\frac12}\|_{\mathrm{HHO}}&\le C_1\Big\{|\partial_t u|_{L^1([t^1,t^N];*,h)}+|u|_{C^0([0;t^{N-1}];*,h)}\Big\}\\&+C_2\Delta t^2\Big\{\|\partial_t^3u\|_{C^0([0;t^1];L^2(\Omega))}+\Theta\|\partial_t^4u\|_{C^0([0;t^N];L^2(\Omega))}+\Theta\|\partial_t^5u\|_{L^1([0;t^N];L^2(\Omega))}\Big\},\end{aligned}$$

which yields the claim.

6 Numerical results

In this section, we present the algebraic formulation of the fully discrete scheme (16) together with the initial conditions (17), and we discuss some numerical results illustrating the theoretical findings on a test case with analytical solution.

6.1 Algebraic formulation

Let $N_{\mathcal{T}}:=\dim(\mathcal{U}^l_{\mathcal{T}})$ and $N_{\mathcal{F}}:=\dim(\mathcal{U}^k_{\mathcal{F},0})$ and let $\{\phi_i\}_{1\le i\le N_{\mathcal{T}}},\{\psi_i\}_{1\le i\le N_{\mathcal{F}}}$ be chosen bases of $\mathcal{U}^l_{\mathcal{T}}$ and $\mathcal{U}^k_{\mathcal{F},0}$, respectively. For all $n\in\{0{:}N\}$, let $(\mathsf{U}^n_{\mathcal{T}},\mathsf{U}^n_{\mathcal{F}})\in\mathbb{R}^{N_{\mathcal{T}}}\times\mathbb{R}^{N_{\mathcal{F}}}$ be the vector of degrees of freedom of the fully discrete solution $\hat{u}_h^n$ on the chosen bases of $\mathcal{U}^l_{\mathcal{T}}$ and $\mathcal{U}^k_{\mathcal{F},0}$, and $\mathsf{F}^n_{\mathcal{T}}\in\mathbb{R}^{N_{\mathcal{T}}}$ the vector having components $((f(t^n),\phi_i)_\Omega)_{1\le i\le N_{\mathcal{T}}}$. The algebraic realization of (16) amounts to

$$\frac{1}{\Delta t^2}\begin{bmatrix}\mathcal{M}&0\\0&0\end{bmatrix}\begin{pmatrix}\mathsf{U}^{n+1}_{\mathcal{T}}-2\mathsf{U}^n_{\mathcal{T}}+\mathsf{U}^{n-1}_{\mathcal{T}}\\\cdot\end{pmatrix}+\begin{bmatrix}\mathcal{A}_{\mathcal{T}\mathcal{T}}&\mathcal{A}_{\mathcal{T}\mathcal{F}}\\\mathcal{A}_{\mathcal{F}\mathcal{T}}&\mathcal{A}_{\mathcal{F}\mathcal{F}}\end{bmatrix}\begin{pmatrix}\mathsf{U}^n_{\mathcal{T}}\\\mathsf{U}^n_{\mathcal{F}}\end{pmatrix}=\begin{bmatrix}\mathsf{F}^n_{\mathcal{T}}\\0\end{bmatrix},\tag{71}$$

with $\mathcal{A}$ the symmetric definite positive stiffness matrix associated with the bilinear form a_h and $\mathcal{M}$ the (block-diagonal) cell mass matrix. As the structure of the global mass matrix makes its definition irrelevant, $\mathsf{U}^n_{\mathcal{F}}$ is replaced by a "$\cdot$" in the acceleration term. The resolution of (71) proceeds in two steps for all $n\in\{1{:}N-1\}$:

1. Compute $\mathsf{U}^n_{\mathcal{F}}$ by solving the linear system $\mathcal{A}_{\mathcal{F}\mathcal{F}}\mathsf{U}^n_{\mathcal{F}}=-\mathcal{A}_{\mathcal{F}\mathcal{T}}\mathsf{U}^n_{\mathcal{T}}$, where $\mathsf{U}^n_{\mathcal{T}}$ is the data and $\mathsf{U}^n_{\mathcal{F}}$ the unknown.
2. Compute $\mathsf{U}^{n+1}_{\mathcal{T}}$ by solving $\mathcal{M}\mathsf{U}^{n+1}_{\mathcal{T}}=\mathcal{M}(2\mathsf{U}^n_{\mathcal{T}}-\mathsf{U}^{n-1}_{\mathcal{T}}+\Delta t^2\mathsf{F}^n_{\mathcal{T}})-\Delta t^2(\mathcal{A}_{\mathcal{T}\mathcal{T}}\mathsf{U}^n_{\mathcal{T}}+\mathcal{A}_{\mathcal{T}\mathcal{F}}\mathsf{U}^n_{\mathcal{F}})$. Since the mass matrix $\mathcal{M}$ is block-diagonal, this linear system is very easy to solve.

We refer the reader to [36] for a computationally effective approach to perform the above two steps. The idea is to perform an iterative process that converges rapidly while mildly affecting the CFL stability condition.

For the first time step $n = 0$, the initial cell unknown U_T^0 is the vector of degrees of freedom of u_T^0 (see (17a)), the initial face unknown $\mathsf{U}_\mathcal{F}^0$ is computed via the algebraic counterpart of (17b), i.e., by solving

$$\mathcal{A}_{\mathcal{F}\mathcal{F}}\mathsf{U}_\mathcal{F}^0 = -\mathcal{A}_{\mathcal{F}\mathcal{T}}\mathsf{U}_\mathcal{T}^0, \tag{72}$$

and the cell unknown at the first time step, $\mathsf{U}_\mathcal{T}^1$, solves

$$\mathcal{M}U_\mathcal{T}^1 = \mathcal{M}(\mathsf{U}_\mathcal{T}^0 + \Delta t\mathsf{V}_\mathcal{T}^0 + \frac{\Delta t^2}{2}\mathsf{F}_\mathcal{T}^0) - \frac{\Delta t^2}{2}(\mathcal{A}_{\mathcal{T}\mathcal{T}}\mathsf{U}_\mathcal{T}^0 + \mathcal{A}_{\mathcal{T}\mathcal{F}}\mathsf{U}_\mathcal{F}^0), \tag{73}$$

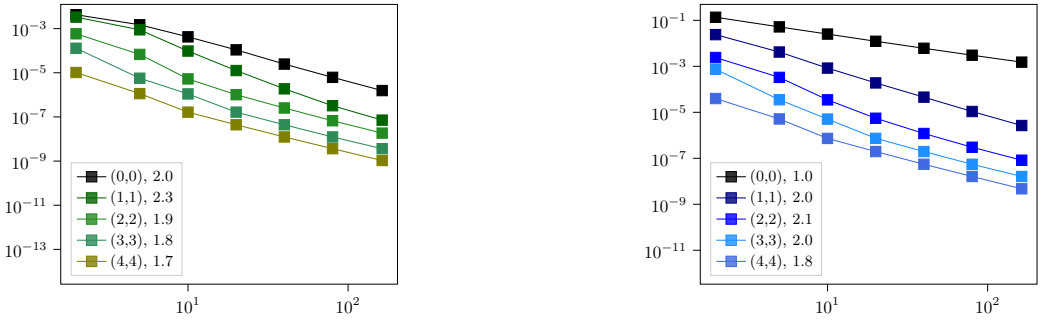
with $\mathsf{V}_\mathcal{T}^0$ the vector of degrees of freedom of v_T^0 (see (15)).

6.2 Verification of the space-time convergence order

Let us set $\Omega := (0,1)^2$, the source term $f(x,y,t) := 0$, the initial conditions $u_0(x,y) := 0$ and $v_0(x,y) := \sin(\pi x)\sin(\pi y)$, and a constant speed of sound $\mu := 1$. The solution to (1) is

$$u(x,y,t) = \frac{1}{\sqrt{2}\pi}\sin(\sqrt{2}\pi t)\sin(\pi x)\sin(\pi y). \tag{74}$$

The expected orders of convergence are Δt^2 in time, h^{k+1} in space for the H^1-error, and h^{k+2} in space for the L^2-error (assuming that full elliptic regularity pickup is available). If the time step is taken so that it respects the CFL condition, and the ratio $\frac{\Delta t}{h}$ remains constant, the time discretization error dominates over the space discretization error on the finest meshes, except for the H^1-error in the lowest-order settings ($k \in \{0,1\}$) and the L^2-error for $k = 0$. This phenomenon is illustrated in Figure 1, where the time step verifies $\Delta t := 0.8\eta\mu^{-1}h$ (see the CFL condition (40)). The convergence rate of the L^2-error for polynomial orders $(k,l) = (0,0)$ and the H^1-error for the polynomial orders $(k,l) = (0,0)$ and $(k,l) = (1,1)$ are unaffected. But for higher orders, the optimal space convergence rate is observed on the coarse meshes, whereas a second-order decay rate dictated by the time discretization error is observed on the finest meshes. The transition between the two regimes is smooth.

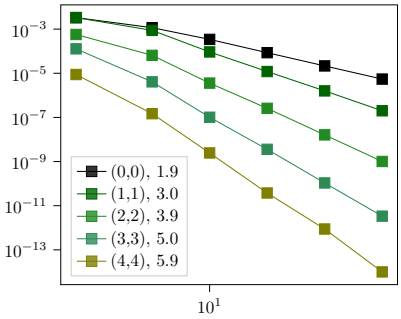


(a) L^2-error w.r.t the mesh size

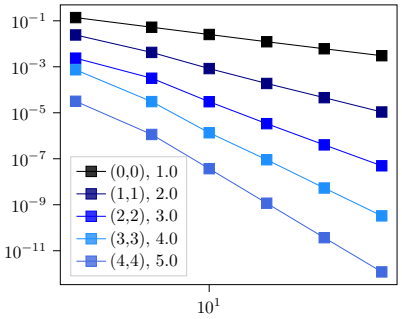
(b) H^1-error w.r.t the mesh size

Figure 1: Convergence of the L^2- and H^1-errors at the final time as a function of the mesh size, polynomial orders $k \in \{0,1,2,3,4\}$, equal-order setting, $\mathfrak{T} = 0.2$, $\Delta t = 0.8\eta\mu^{-1}h$.

If the time step is taken to be constant for all the meshes in the sequence, but much smaller than the critical time step, the time discretization error becomes negligible w.r.t the space discretization error and higher convergence rates in space are recovered, see Figure 2. This shows (as expected) that, to balance space and time discretization errors, the time step is typically set at a (much) smaller value than the one imposed by the CFL stability condition.



(a) L^2-error w.r.t the mesh size



(b) H^1-error w.r.t the mesh size

Figure 2: Convergence of the L^2- and H^1-errors at the final time as a function of the mesh size, polynomial orders $k \in \{0, 1, 2, 3, 4\}$, equal-order setting, $\mathfrak{T} = 0.2$, $\Delta t \ll h$.

Declarations

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Data availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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