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Invited Review

Fuzzy sets, probability and measurement *

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Abstract: In recent years, the problem of uncertainty modeling has received much attention from scholars in artificial intelligence and decision theory. Various formal settings, including but not restricted to fuzzy sets and possibility measures, have been proposed, based on different intuitions, and dealing with various kinds of uncertain data. The two main research directions are upper and lower probabilities which convey the idea of imprecisely estimated probability measures, and distorted probabilities for the descriptive assessment of partial belief. Possibility measures, and thereby fuzzy sets, stand at the crossroads of these new approaches. Traditional views, interpretive settings and canonical experiments for the measurement of probability such as frequentist approaches, betting theories, comparative uncertainty relations are currently extended to the generalized uncertainty measures. These works shed new light on various interpretations of fuzzy sets and clarify their links with probability theory; conversely Zadeh's logical point of view on fuzzy sets suggests a set-theoretic perspective on uncertainty measures, that brings together numerical quantification and logic.

Keywords: Fuzzy sets, possibility theory, probability theory, statistics, random set, measurement

Introduction

The question of uncertainty modeling has become a major issue in several fields such as artificial intelligence, decision theory, and the social sciences; in the last thirty years many works have focused on the limitations of probability theory as a tool for representing uncertainty. Among them, fuzzy set theory has emerged as one of the earliest and most seminal attempts; its originality lies in its being the first systematic and computationally reasonable treatment of vagueness in human-originated information. Contrasting with its intu-

tive appeal, the foundational aspects of fuzzy set theory have been much controversial for many years; from a mathematical point of view, a rigorous formulation of basic axioms, and the link between membership functions and probabilities were not available from the first. On a practical level, the question of membership function elicitation has bothered many a scholar, and the lack of simple convincing techniques has sometimes raised many a criticism (e.g. French [27]). This question is still pending, but this state of facts might rapidly change. Indeed, recent progress in fuzzy set and possibility theory and related fields (e.g. Shafer's evidence theory [56]) has clarified many issues that were not well understood ten years ago and especially the links between fuzzy sets and probability theory. This question has created many a misunderstanding in the past, and many scholars, especially in probability-related fields are still unaware of recent findings.

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This paper surveys foundational aspects of fuzzy set theory, from the point of view of applied mathematics and tries to explain the links between fuzzy sets and probability measures by considering so-called random sets. First, fuzzy sets are cast into a more general framework for the representation of non-additive probabilities, via the concept of a possibility measure. The strength of fuzzy sets lies in their dual nature of generalized sets, and non-additive partial belief models. This is the topic of the second section that offers a brief review of fuzzy set theoretic operations, points out their links with random set operations and surveys recent advances in non-standard approaches to information theory. Section 3 is devoted to interpretive issues. The first part considers measurement problems with a subjectivist point of view. Then we argue that frequentist views of fuzzy sets make sense, exploiting the formal relationships with random sets. Fuzzy sets can then be viewed as approximations of random, possibly imprecise data, in the sense of a generalized concept of inclusion. The links between fuzzy sets and confidence intervals are explained. The last section is a survey of joint applications of fuzzy sets and probability to statistics and decision analysis.

Other applications of fuzzy sets to operations research (e.g. optimization) have been reviewed in the recent past by Zimmermann in this journal [80] and in two books [81, Ch. 13-82, Ch. 2-5]; the reader may find interesting material in the volume edited by Kaoprzyk and Orlovsky [34], as a follow-up of Zimmermann's review. Besides, fuzzy set people are very active in the field of knowledge-based systems, from a theoretical point of view (see Dubois and Pradé [21, Ch. 4, 5, 6]) as well as in applications (e.g. Sugeno [62]). However the scope of this paper is essentially at the basic level.

It is believed that many people are still reluctant to consider fuzzy sets as a mathematical tool just like others, because they have not clarified the issue of distinguishing between a fuzzy set and a probability measure. Hence apart from displaying applications that work it seems worthwhile to explain what the concept of a fuzzy set is, in terms understandable by probability theory scholars. This would avoid such amazing claims made by experts in probability theory commenting on the success of fuzzy controllers in applications (such as the Sendai metro system in Japan): "we

know that probabilistic controllers would perform better". One answer to such claims is to show that the laws of fuzzy set theory are not in contradiction with the laws of probability theory.

1. Fuzzy sets, possibility measures, random sets

Zadeh's [74] original idea of a fuzzy set F is to consider a referential set Ω and a function from Ω to the unit interval, say μ_F , that extends the notion of set-characteristic function to intermediary degrees of membership. The next breakthrough consisted in acknowledging that membership functions could serve as distributions of possibility weights, viewing a fuzzy set as the set of (more or less) possible values of some variable; such distributions characterize set-functions called possibility measures (Zadeh [77]). The basic step for relating possibility (hence fuzzy sets) to probability was taken realizing that possibility measures were a special kind of Shafer's [56] evidence measures (Dubois and Pradé [9]) or equivalently random sets (Nguyen [45]); as Kampé de Féniet [35] points it out, a degree of membership can be considered as a degree of plausibility in the sense of Shafer. These three statements form the core of present-day fuzzy set theory, and are now further commented upon. See Klir and Folger [39], Dubois and Pradé [21] for monographs displaying this type of concern.

1.1. Three representations of a fuzzy set

A fuzzy set F on a referential set Ω can be viewed as a mapping μ_F from Ω to $[0, 1]$: each element ω belongs to F to degree $\mu_F(\omega)$, where $\mu_F(\omega) = 1$ (resp. 0) means total membership (resp. non-membership). This is the 'vertical' representation of a fuzzy set F , viewing Ω as an horizontal axis or plane, and putting the unit interval on a vertical axis (cf. Figure 1).

Contrastively, a fuzzy set F can be represented in terms of its level-cuts

$$\{F_\alpha \mid \alpha \in [0, 1]\}$$

where $F_\alpha = \{\omega \mid \mu_F(\omega) \geq \alpha\}$. Namely (Zadeh [75]),

$$\mu_F(\omega) = \sup\{\alpha \in [0, 1] \mid \omega \in F_\alpha\}. \quad (1)$$

Hence given a family of sets $\{F_\alpha \mid \alpha \in [0, 1]\}$ such

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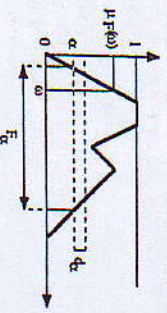


Figure 1. Vertical and horizontal views of fuzzy sets

that $\alpha \geq \alpha' \Leftrightarrow F_\alpha \subseteq F_{\alpha'}$ and $F_\alpha = \lim_{\beta \uparrow \alpha} F_\beta$, then it can be considered as the level-cuts of a fuzzy set F . This is the 'horizontal' representation of a fuzzy set as a nested family of level-cuts.

The membership degree can also be expressed as an integral:

$$\mu_F(\omega) = \int_0^1 \mu_{F_\alpha}(\omega) d\alpha \quad (2)$$

where $\mu_{F_\alpha}(\omega) = 1$ if $\omega \in F_\alpha$ and 0 otherwise. In that case $\{F_\alpha | \alpha \in (0, 1)\}$ is viewed as a uniformly distributed random set, consisting of the Lebesgue measure on $[0, 1]$ and the set-valued mapping $\alpha \rightarrow F_\alpha$ from $(0, 1)$ to \mathcal{Q} . If $\{F_\alpha | \alpha \in (0, 1)\}$ is finite, then F is called a finitely discrete fuzzy set; then let

$$M(F) = \mu_F(\mathcal{Q}) - \{0\} = \{\alpha_1 > \alpha_2 > \dots > \alpha_n\}$$

be the set of positive membership values, and let $m_i = \alpha_i - \alpha_{i+1}$, with $\alpha_{n+1} = 0$ by convention; then (2) also writes (Dubois and Prade [11])

$$\mu_F(\omega) = \sum_{\omega \in F_{\alpha_i}} m_i. \quad (3)$$

Clearly $\sum_{i=1}^n m_i = \alpha_1 = 1$ as soon as $\exists \omega, \mu_F(\omega) = 1$; F is then said to be normal. With this third representation, we come closer to probability theory, since any finite nested family of sets $\{A_i \subseteq \dots \subseteq A_n\}$ together with positive weights m_1, \dots, m_n summing to 1 characterize a finitely discrete, normal fuzzy set, and conversely, m_i can be viewed as the probability that F_{α_i} stands as a crisp (non-fuzzy) representative of F .

$\{(F_{\alpha_i}, m_i) | i = 1, \dots, n\}$ is usually called a 'random set' (Kendall [37], Matheron [42]) and more specifically, a finitely discrete, *consonant* random set. Generally, a finitely discrete random set is a set

$$\mathcal{R} = \{(A_i, m_i), i = 1, \dots, n\}$$

where the A_i are subsets of \mathcal{Q} called focal ele-

ments by Shafer [56], the m_i are positive and sum to 1. The consonance property ($A_i \subseteq A_j \dots \subseteq A_n$) is generally not required. Moreover the A_i 's need not be distinct. More generally, a random set is a set-valued mapping from a probability space to \mathcal{R}^* , that possesses measurability properties. A fuzzy set can always be described as a consonant random set. Note that the word 'random' in random sets is misleading in that respect. A random set is nothing but a convex combination of sets, and a fuzzy set can be viewed as such. 'Randomness' as a physical phenomenon has nothing to do here; the m_i 's are just weights in a convex sum.

Conversely, given a finitely discrete random set $\mathcal{R} = \{(A_i, m_i) | i = 1, \dots, n\}$ where $\sum_{i=1}^n m_i = 1$, its one-point coverage function (also called contour function by Shafer [56])

$$\mu(\omega) = \sum_{\omega \in A_i} m_i, \quad (4)$$

defines the membership function of a fuzzy set. However, this fuzzy set is normal only if $\bigcap_{i=1}^n A_i$ is not empty. Moreover when the A_i 's are not nested, the knowledge of the membership function is not enough to reconstruct the random set \mathcal{R} . In fact each fuzzy set F defines an equivalence class of random sets whose contour function is the membership function μ_F (Goodman and Nguyen [31], Wang and Sanchez [64]).

1.2. Possibility theory

Let S be a subset of \mathcal{Q} , viewed as the set of admissible, mutually exclusive values of a variable x . Let A be another subset of \mathcal{Q} ; if we ask whether or not A contains x there can be three answers: if A contains S then there is total certainty that $x \in A$; if $A \cap S = \emptyset$, then it is impossible that $x \in A$; otherwise it is possible that $x \in A$. If some values of x are more possible than others, S contains a fuzzy set F of possible values, and the answers pertaining to any subset A become graded: one may evaluate to what extent A intersects F (possibility of event A) and to what extent A contains F (certainty of event A). In the spirit of (1),

$$\Pi(A) = \sup \{\alpha | A \cap F_\alpha \neq \emptyset\}, \quad (5)$$

$$N(A) = 1 - \sup \{\alpha | \bar{A} \cap F_\alpha \neq \emptyset\}. \quad (6)$$

are respectively the degree of possibility and the

necessity (certainty) of event A derived from F . Equation (6) means that $N(A) = 1 - \Pi(\bar{A})$, i.e. the certainty of A reflects the impossibility of its complement \bar{A} . When μ_F is continuous, (6) also writes $N(A) = \sup\{1 - \alpha | F_\alpha \subseteq A\} = \inf\{1 - \alpha | F_\alpha \subseteq A\}$. F is supposedly normal so that $\Pi(\mathcal{Q}) = 1$. It means that the support of F , i.e.

$$\text{supp}(F) = \{\omega | \mu_F(\omega) > 0\}$$

contains x with certainty. Moreover $\Pi(\emptyset) = 0$ (impossible event), so that $N(\mathcal{Q}) = 1$, $N(\emptyset) = 0$. Axioms that characterize set functions Π and N are, for any indexed family $\{A_i | i \in I\}$ of subsets of \mathcal{Q} ,

$$\Pi\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \Pi(A_i), \quad (7)$$

$$N\left(\bigcap_{i \in I} A_i\right) = \inf_{i \in I} N(A_i). \quad (8)$$

Taking $I = A$, and choosing the singletons of A for the indexed family leads to

$$\Pi(A) = \sup\{\mu_F(\omega) | \omega \in A\}. \quad (9)$$

Note that generally,

$$N(A \cup B) \geq \max(N(A), N(B))$$

and

$$\Pi(A \cap B) \leq \min(\Pi(A), \Pi(B))$$

and no equality holds, unless A and B refer to variables that are not linked (Dubois and Prade [21]). When F is finitely discrete, (5) and (6) also write, using the random set representation (3):

$$\Pi(A) = \sum_{F_{\alpha_i} \cap A \neq \emptyset} m_i, \quad (10)$$

$$N(A) = \sum_{F_{\alpha_i} \subseteq A} m_i. \quad (11)$$

Π is called a possibility measure (Zadeh [77]) and N a necessity measure (Dubois and Prade [9,21]).

Formulas (10)–(11) clearly indicate that N and Π are particular cases of belief and plausibility functions, respectively, in the sense of Shafer [56], where the set of level-cuts is taken as the set of focal elements and the m_i 's play the role of a basic probability assignment m with $m(F_{\alpha_i}) = m_i$. In the general case nothing is assumed about the focal elements B with weights $m(B) > 0$. Namely,

(10) and (11) make sense even when the F_{α_i} 's are not nested. If these sets are all singletons (i.e. $m(B) > 0 \Leftrightarrow \exists \omega, B = \{\omega\}$), then

$$\sum_{A \cap B \neq \emptyset} m(B) = \sum_{B \subseteq A} m(B) = P(A).$$

is the probability of A ;

$$\mathcal{R} = \{(B, m(B)) | m(B) > 0\}$$

can also be viewed as a random set as shown earlier. Imbedding possibility measures into belief function theory thus clarifies the link between fuzzy sets and probability: fuzzy sets correspond to consonant imprecise evidence while probability correspond to dissonant evidence, with precise observations. See Dubois and Prade [15] for discussions on this point.

1.3. Projections of random sets and fuzzy sets

Consider a bi-dimensional universe $\Omega_1 \times \Omega_2$. A random relation is a random set \mathcal{R} on $\Omega_1 \times \Omega_2$, and a fuzzy relation R corresponds to a consonant random relation on $\Omega_1 \times \Omega_2$. The projection of \mathcal{R} on Ω_1 , denoted $\Omega_1(\mathcal{R})$ is a random set \mathcal{R}_1 with basic probability assignment m_1 such that [57]

$$\forall A_1 \subseteq \Omega_1,$$

$$m_1(A_1) = \sum (m(C) | A_1 = \Omega_1(C)), \quad (12)$$

where m is the basic probability assignment of \mathcal{R} and A_1 the projection of C on Ω_1 . The projection of a fuzzy relation \mathcal{R} on Ω_1 is the fuzzy set $F_1 = \Omega_1(\mathcal{R})$ defined by (Zadeh [76])

$$\mu_{F_1}(\omega_1) = \sup_{\omega_2 \in \Omega_2} \mu_R(\omega_1, \omega_2). \quad (13)$$

The two definitions are in perfect agreement: when \mathcal{R} is a consonant random set with contour function μ_R , then the membership function μ_{F_1} of the projection of R is the contour function of $\Omega_1(\mathcal{R})$ (Shafer [57]). Fuzzy sets and random sets agree with respect to projection operations from multiple-dimension scales.

2. Non-additive probabilities versus generalized sets

In his book, Shafer [56] introduces belief functions as generalized probabilities. In contrast, the

point of view of random sets, mathematically equivalent to belief functions (Nguyen [45]), lays bare the dual nature of non-additive set-functions as being also generalized sets. Fuzzy set theory adopted the logical point of view first, developing the algebra of fuzzy sets, and the measure-theoretic approach through possibility measures came later. On such a basis it is interesting to study the algebra of random sets [31,16] as well as possibilistic expectation [13]. However there are two ways of deriving possibility measures in a probabilistic perspective: one is to introduce imprecision in the additive model; the other is to distort additivity by some homomorphism that accounts for subjectivity. This is the topic of the first section. Then random set-theoretic operations are compared with fuzzy set operations. It is shown that basic fuzzy set operations can be appropriately justified in a random set setting. Lastly recent advances in the theory of information outside probability theory are reported. It sheds light on the different facets of uncertainty.

2.1. Distorted versus upper and lower probabilities

Let us assume as in classical decision theory that an individual's state of uncertainty should be expressed by means of a probability measure on some description of the world, say a set Ω . One may admit that the individual is not able to precisely describe this probability, and that he can only point out some subset \mathcal{P} of probability measures which contains his actual probability. Then only upper and lower probabilities of events are available, namely

$$P_*(A) = \inf\{P(A) \mid P \in \mathcal{P}\}, \quad (14)$$

$$P^*(A) = \sup\{P(A) \mid P \in \mathcal{P}\}. \quad (15)$$

It can be checked that (14) and (15) define set-functions more general than belief and plausibility measures (e.g. Walley and Fine [63]); hence possibility measures are a special kind of upper probabilities and, because $P_*(A) = 1 - P^*(A)$, necessity measures are a special kind of lower probabilities in the sense of (14). In fact possibility theory proposes the simplest system of upper and lower probabilities, as advocated in a previous paper [22] that surveys upper and lower probability systems.

Instead of assuming that an individual's state of uncertainty is described by an unreachable probability measure, one may as well reject this assumption, and admit that a degree of confidence is described by a non-additive set-function g that is monotonic under set-inclusion, i.e. $A \subseteq B \Rightarrow g(A) \leq g(B)$. In order to be computationally simple, one may naturally assume that g is decomposable as in (Dubois and Prade [10], Weber [67]); there is some operation $*$ on the unit interval such that

$$A \cap B = g \Rightarrow g(A \cup B) = g(A) * g(B). \quad (16)$$

Assuming $*$ is continuous, the properties of the Boolean algebra force $*$ to be associative, commutative, monotonic in the wide sense, and such that $a * 0 = a$, $\forall a \in [0, 1]$, i.e. to be a triangular co-norm (Schweizer and Sklar [54]). Namely $a * b \geq \max(a, b)$ and the main solutions for $*$, different from $*$ = max are

$$a * b = \min(1, a + b) \quad (\text{bounded sum})$$

and

$$a * b = a + b - ab \quad (\text{probabilistic sum}).$$

Decomposable measures are simple because they are distributional, i.e. if $\Omega = \{\omega_1, \dots, \omega_n\}$, g is characterized by the set of weights $\{g_1, \dots, g_n\}$ with $g_i = g(\{\omega_i\})$, satisfying the normalization condition $g_1 * g_2 * \dots * g_n = 1$, and if $A = \{\omega_{i_1}, \dots, \omega_{i_j}\}$,

$$g(A) = g_{i_1} * g_{i_2} * \dots * g_{i_j}.$$

With $*$ = bounded sum, g is a probability measure as soon as $\sum_{i=1}^n g_i = 1$. A wide class of candidates for $*$ are such that $f(a * b) = f(a) + f(b)$ where f is a strictly increasing mapping from $[0, 1]$ to $[0, +\infty)$, so that g is often a distorted probability (Chateauneuf [2]). The limit case $*$ = max corresponds to extreme distortion. Hence possibility measures are also extreme cases of distorted probabilities.

These two links between possibility and probability suggest two extensions of expectation in the possibilistic setting: upper and lower expectations, deriving from (14) and (15), changing $P(A)$ into classical expectation (Dempster [5]), and distorted expectation changing sums and products into triangular norm-based operations (Weber [67]). Let f be a function from Ω to \mathbb{R} ,

and F be a fuzzy set on Ω . The upper and lower expectation of f deriving from the possibility distribution μ_F are, consistently with (14) and (15) (e.g. Dubois and Prade [13]),

$$E_+(f) = \int_0^1 \inf\{f(\omega) \mid \omega \in F_\alpha\} d\alpha \\ = \sum_{i=1}^n m_i \cdot \min\{f(\omega) \mid \omega \in F_{\alpha_i}\}, \quad (17)$$

$$E_-(f) = \int_0^1 \sup\{f(\omega) \mid \omega \in F_\alpha\} d\alpha \\ = \sum_{i=1}^n m_i \cdot \max\{f(\omega) \mid \omega \in F_{\alpha_i}\}. \quad (18)$$

Particularly, when F is a fuzzy interval, i.e. a fuzzy set of the real line with a unimodal, upper-semi continuous membership function, then (17) and (18) define the mean value of the fuzzy interval F (Dubois and Prade [18]); it is an interval $[E_-, E^*]$ where

$$E_- = \int_0^1 \inf F_\alpha d\alpha, \quad E^* = \int_0^1 \sup F_\alpha d\alpha.$$

This is not surprising. Expectation summarizes a random number by means of a precise number, thus eliminating uncertainty. A fuzzy interval combines imprecision and uncertainty. When uncertainty is eliminated, imprecision remains.

The distorted expectation for possibility measures is Sugeno's [61] integral; it makes sense when f maps into $[0, 1]$, i.e. is the membership function of a fuzzy set G . Sugeno's integral of μ_G in the sense of μ_F coincides with Zadeh's [77] definition of the possibility of a fuzzy event G , i.e. the extension of (9) to

$$\Pi(G) = \sup_{\omega} \min(\mu_F(\omega), \mu_G(\omega)). \quad (19)$$

2.2. Generalized set-theoretic operations

Random sets and fuzzy sets (as well as upper-lower probabilities) are generalized sets. Hence set-theoretic operations should be extended to generalized sets. Let S_1 and S_2 be generalized sets belonging to a given theory \mathcal{G} (fuzzy sets or random sets). Let \square be a set-theoretic operation on $2^{\mathcal{G}}$, and \square its extension. The definition of set-theoretic operations should obey the following requirements:

(i) *Closure*: $S_1 \in \mathcal{G}$ and $S_2 \in \mathcal{G}$ implies $S_1 \square S_2 \in \mathcal{G}$.

(ii) *Reduction to usual set-theoretic connectives*: $S_1 = A \subseteq \Omega$, $S_2 = B \subseteq \Omega$ implies $S_1 \square S_2 = A \square B$.

(i) means that combining fuzzy sets should give fuzzy sets and combining random sets should give random sets. In the following it is noticed that fuzzy set connectives are consistent with random set connectives with respect to the identity between membership functions and one-point coverage functions. But random set operations may fail to preserve consonance. We shall consider complementation, intersection and inclusion.

2.2.1. Complementation

Complementation of a fuzzy set F is usually defined as a fuzzy set \bar{F} such that $\mu_{\bar{F}}(\omega) = 1 - \mu_F(\omega)$. It is a solution of the set of axioms:

CI. *Pointwiseness*:

$$\exists C: \forall \omega, \mu_{\bar{F}}(\omega) = C(\mu_F(\omega)).$$

C2. *Strict-decreasingness*:

$$\mu_F(\omega) > \mu_{\bar{F}}(\omega) \Rightarrow \mu_{\bar{F}}(\omega) < \mu_F(\omega).$$

C3. *Involutiveness*: $\bar{\bar{F}} = F$.

However any function $C(x) = f^{-1}(1 - f(x))$ where f is an increasing bijection of $[0, 1]$ on itself is a solution of CI-C3 as well [1].

The complement of a random set [16],

$$\bar{\mathcal{Q}} = \{(A_i, m_i), i = 1, \dots, n\}$$

is

$$\bar{\mathcal{Q}} = \{(\bar{A}_i, m_i), i = 1, \dots, n\}.$$

If $\forall i = 1, \dots, n$, $A_i = F_{\alpha_i}$ for some α_i , then $\bar{\mathcal{Q}}$ reduces to F via (3), but $\bar{\mathcal{Q}}$ reduces to \bar{F} if only if $C(x) = 1 - x$.

2.2.2. Intersection and union

Intersection of fuzzy sets is defined via the following axioms:

II. *pointwiseness*:

$$\mu_{F \cap G}(\omega) = \mu_F(\omega) \wedge \mu_G(\omega).$$

12. associativity of \cap :

$$a \cap (b \cap c) = (a \cap b) \cap c, \quad \forall (a, b, c) \in [0, 1]^3.$$

13. commutativity of \cap :

$$a \cap b = b \cap a, \quad \forall (a, b) \in [0, 1]^2.$$

14. non-decreasingness:

$$\begin{aligned} \mu_F(\omega) \geq \mu_F(\omega') \text{ and } \mu_G(\omega) \geq \mu_G(\omega') \\ \Rightarrow \mu_{F \cap G}(\omega) \geq \mu_{F \cap G}(\omega'). \end{aligned}$$

$$15. F \cap G = G; F \cap G = F.$$

It follows that $\mu_{F \cap G}(\omega) = \mu_F(\omega) \cap \mu_G(\omega)$ where \cap must be a triangular norm (Schweizer and Sklar [54]), i.e. the dual of a triangular co-norm * introduced in the previous section, via $a \cap b = 1 - (1 - a) * (1 - b)$. Namely $a \cap b \leq \min(a, b)$, and usual candidates for \cap are $a \cap b = a \cdot b$ (product) and $a \cap b = \max(0, a + b - 1)$ (linear intersection), as well as $a \cap b = \min(a, b)$.

Random set intersection can be defined by considering the two random sets

$$\mathcal{A} = \{(A_i, m_i), i \in I\} \text{ and } \mathcal{B} = \{(B_j, p_j), j \in J\}$$

as the marginals of a random set

$$\mathcal{C} = \{(C_{ij}, q_{ij}), (i, j) \in I \times J\}$$

such that

$$C_{ij} = A_i \times B_j, \quad m_i = \sum_j q_{ij}, \quad p_j = \sum_i q_{ij}.$$

Then

$$\mathcal{A} \cap \mathcal{B} = \{(A_i \cap B_j, q_{ij}) | i \in I, j \in J\}. \quad (20)$$

The main problem is to define q_{ij} in terms of $\{m_i | i \in I\}$ and $\{p_j | j \in J\}$. Two assumptions can be set forth:

- stochastic independence: $q_{ij} = m_i \cdot p_j, \forall i, j$;
- total dependence: \mathcal{A} and \mathcal{B} are always observed conjointly as pairs (A_i, B_j) so that $I = J = m_i = p_j, \forall i$. Then $q_{ij} = 0$ if $i \neq j$ and $q_{ii} = m_i = p_j$. Note that the condition $I = J$ is not really restrictive since the A_i 's (resp. B_j 's) need not be distinct.

The first assumption leads to the unnormalized version of Dempster rule of combination (e.g. Shafer [56]). If \mathcal{A} and \mathcal{B} are consonant random sets associated to fuzzy sets F and G , then

$$\forall \omega, \quad \sum_{i,j | \omega \in A_i \cap B_j} m_i \cdot p_j = \mu_F(\omega) \cdot \mu_G(\omega). \quad (21)$$

In other words, (21) justifies the product rule of fuzzy set intersection when random sets are represented via their contour function. But the lack of consonance of $\mathcal{A} \cap \mathcal{B}$ indicates that (21) is only an approximation of the actual result. For instance the upper probabilities derived from $\mathcal{A} \cap \mathcal{B}$ are usually greater than the degrees of possibilities derived using $\mu_F \cdot \mu_G$, i.e.

$$\forall A \subseteq \Omega,$$

$$\sum_{A \cap A_i \cap B_j \neq \emptyset} m_i \cdot p_j \geq \sup_{\omega \in A} \mu_F(\omega) \cdot \mu_G(\omega). \quad (22)$$

It is possible to justify the two other intersection rules of fuzzy set theory (minimum and linear intersection) by means of random set arguments. Ohlsvang [50] gave a sufficient condition to justify the minimum rule, and Goodman and Nguyen [31] give the general setting. Here we shall describe the results in the finite case, for simplicity.

Let us denote

$$\mu_F(\omega) = \sum_{\omega \in A_i} m_i, \quad \mu_G(\omega) = \sum_{\omega \in B_j} p_j$$

and

$$\mu_{F \cap G}(\omega) = \sum_{\omega \in A_i \cap B_j} q_{ij}$$

where the q_{ij} 's are supposedly known. The following quantities can then be obtained:

$$\begin{aligned} \sum_{\omega \in A_i \cap \bar{B}_j} q_{ij} &= \sum_{\omega \in A_i} p_j - \sum_{\omega \in A_i \cap B_j} q_{ij} \\ &= \mu_F(\omega) - \mu_{F \cap G}(\omega), \end{aligned}$$

$$\sum_{\omega \in \bar{A}_i \cap B_j} q_{ij} = \mu_G(\omega) - \mu_{F \cap G}(\omega),$$

$$\sum_{\omega \in \bar{A}_i \cap \bar{B}_j} q_{ij} = 1 - \mu_G(\omega) - \mu_F(\omega) + \mu_{F \cap G}(\omega).$$

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As a consequence we can state:

Proposition. (a) $\mu_{F \cap G} = \min(\mu_F, \mu_G)$ if and only if for each ω , one of the two following conditions hold:

$$q_{ij} = 0 \text{ for all } i, j \text{ such that } \omega \in A_i, \omega \in B_j, \quad (23)$$

$$q_{ij} = 0 \text{ for all } i, j \text{ such that } \omega \in B_j, \omega \in A_i, \quad (24)$$

(b) $\mu_{F \cap G} = \max(0, \mu_F + \mu_G - 1)$ if and only if for each ω , one of the following conditions hold:

$$q_{ij} = 0 \text{ for all } i, j \text{ such that } \omega \in A_i \cap B_j, \quad (23')$$

$$q_{ij} = 0 \text{ for all } i, j \text{ such that } \omega \in \bar{A}_i \cup B_j. \quad (24')$$

Proof. We shall use the three identities just stated above.

$$(a) \quad \min(\mu_F(\omega), \mu_G(\omega))$$

$$= \mu_{F \cap G}(\omega) + \min \left(\sum_{\omega \in A_i \cap \bar{B}_j} q_{ij}, \sum_{\omega \in \bar{A}_i \cap B_j} q_{ij} \right);$$

one of the terms inside the min in the last expression must be zero to get the minimum rule between the contour functions. It gives one of the conditions (23) or (24) since $q_{ij} \geq 0$.

$$(b) \quad \max(0, \mu_F(\omega) + \mu_G(\omega) - 1)$$

$$= \max \left(0, \mu_{F \cap G}(\omega) - \sum_{\omega \in \bar{A}_i \cap \bar{B}_j} q_{ij} \right)$$

$$= \mu_{F \cap G}(\omega) - \min \left(\sum_{\omega \in \bar{A}_i \cap \bar{B}_j} q_{ij}, \sum_{\omega \in \bar{A}_i \cup B_j} q_{ij} \right).$$

Hence conditions (23') and (24'). For instance the linear intersection applies when $q_{ij} = 0$ for all i, j such that $A_i \cup B_j \neq \Omega$.

Example. $\Omega = \{a, b, c\}$.

$$\mathcal{A} = \{(b, 0.4), (c, 0.05), (ab, 0.2), (ac, 0.05),$$

$$(bc, 0.15), (abc, 0.15)\},$$

$$\mathcal{B} = \{(a, 0.2), (ab, 0.1), (ac, 0.4), (bc, 0.1),$$

$$(abc, 0.2)\}$$

where ab stands for $\{a, b\}$ etc. The q_{ij} 's are given in Table 1.

Note that $\mu_{F \cap G} = \min(\mu_F, \mu_G)$. Indeed the following conditions are satisfied: zero mass for all entries (A, B) such that $a \in A, a \notin B$ (marked 0_a), zero mass for all entries (A, B) such that $b \notin A, b \in B$ (marked 0_b), zero mass for all entries

Table 1
 $F = \{(a, 0.4), (b, 0.9), (c, 0.4)\}$
 $G = \{(a, 0.2), (b, 0.4), (c, 0.7)\}$

F	G					
	a	ab	ac	bc	abc	
b	0.15	0.05	0.1	0.05	0.05	
c	0 _a	0 _a	0.05	0 _a	0 _a	
ab	0.05	0.05	0.05	0 _a	0.05	
ac	0 _a	0 _a	0.05	0.05	0 _a	
bc	0 _a	0 _a	0.05	0.05	0.05	
abc	0 _a	0 _a	0.1	0 _a	0.05	

$$F \cap G = \{(a, 0.4), (b, 0.4), (c, 0.4)\}.$$

(A, B) such that $c \in A, c \notin B$ (marked 0_c). In this example, the condition that holds for $\omega = b$ is not the same as the one for $\omega \neq b$.

When \mathcal{A} and \mathcal{B} are consonant, it is possible to require that $\mathcal{A} \cap \mathcal{B}$ be also consonant and that $\mu_{F \cap G} = \min(\mu_F, \mu_G)$. This was done previously by the authors [17]: it is enough to build the set $M(F) \cup M(G) = \{\gamma_i = 1 \geq \gamma_2 \geq \dots \geq \gamma_n > 0\}$ and to define the weights q_{ij} as

$$q_{ij} = \begin{cases} \gamma_n - \gamma_{n+1} & \text{if } A_i = F_n, B_j = G_n, \\ 0 & \text{otherwise.} \end{cases}$$

This is the case of total dependence, where the A_i 's and B_j 's are observed by pairs. So, all fuzzy set theoretic operations can be justified from the standpoint of random sets.

The union rule of fuzzy sets can be studied the same way as intersection, changing triangular norms into triangular co-norms. See Alsina et al. [11], Chapter 3 of [21], or its long version [14] for instance, for details about fuzzy set-theoretic operations. Random set union is obtained by changing intersection into union in (20). See e.g. [16].

2.2.3. Inclusion

Inclusion of fuzzy sets is simply defined by Zadeh [74] as an inequality between membership functions, i.e. $F \subseteq G \Rightarrow \mu_F \leq \mu_G$. When F and G are normal, and $F \subseteq G$, F is said to be more specific than G . Random set inclusion is a slightly more difficult issue. A definition has been proposed by several authors (Yager [71], Dubois and Prade [16], Delgado and Moral [4]). Namely $\mathcal{A} \subseteq \mathcal{B}$ if and only if there are non-negative coefficients w_{ij} , $i \in I, j \in J$, such that $\forall i, m_i = \sum_j w_{ij} \cdot p_j$, $p_j = \sum_i w_{ij}$, and $w_{ij} > 0 \Rightarrow A_i \subseteq B_j$. Hence $\mathcal{A} \subseteq \mathcal{B}$

corresponds to a set of triples $(A, \subseteq B, w_{ij})_{i,j \in I}$ forming a weighted inclusion such that $\sum_{i,j} w_{ij} = 1$.

This definition of inclusion coincides with Zadeh's inclusion on consonant random sets. See Dubois and Prade [16] for other, weaker, notions of random set inclusions. Particularly, if

$$P^*(C) = \sum_{A \subseteq C} m_i,$$

and

$$P^{**}(C) = \sum_{B \subseteq C} p_i,$$

are the lower probabilities defined from \mathcal{Q} and \mathcal{S} (i.e. belief functions in the sense of Shafer), the inequality $P^* \geq P^{**}$ also extends fuzzy set inclusion to random sets. However it is weaker than the former definition in the sense that $\mathcal{Q} \subseteq \mathcal{S}$ implies $P^* \geq P^{**}$ but not conversely. This property is called the bracketing property and points out the fact that when $\mathcal{Q} \subseteq \mathcal{S}$, \mathcal{S} is a bracketing approximation of \mathcal{Q} . Particularly, intervals of probability $[P^*(A), P^{**}(A)]$ derived from \mathcal{Q} are always included into intervals of probability derived from \mathcal{S} . Moreover if the weights w_{ij} describe the joint random set underlying \mathcal{Q} and \mathcal{S} , then it is easy to check that $\mathcal{Q} \subseteq \mathcal{S} \iff \mathcal{Q} \cap \mathcal{S} = \mathcal{Q}$, and that if $\mu_F, \mu_G, \mu_{F \cap G}$ are the contour functions of \mathcal{Q} , \mathcal{S} and $\mathcal{Q} \cap \mathcal{S}$, we have $\mu_{F \cap G} = \min(\mu_F, \mu_G)$. Indeed, when $\mathcal{Q} \subseteq \mathcal{S}$, the following condition is fulfilled:

$$\text{if } \exists \omega \in A_i, \omega \in B_j \text{ then } w_{ij} = 0. \quad (25)$$

Note that $\mu_{F \cap G} = \min(\mu_F, \mu_G)$ does not imply $\mathcal{Q} \subseteq \mathcal{S}$ since the above condition corresponds to applying (23) for all ω . The example given in Section 2.2.2 is a case where the minimum rule is used while $\mathcal{Q} \not\subseteq \mathcal{S}$, because property (23) holds for some elements ω , and (24) for others.

2.3. Fuzzy sets and information measures

Shannon's information theory is now playing an important role in probability applications (Shore and Johnson [58]): namely the entropy measure is useful for the elicitation of missing probabilities and relative entropy measures are useful to justify inductive inference patterns such as Bayes' theorem and extend them to more gen-

eral probability kinematics problems (Domorot [6]).

In possibility theory and evidence theory the situation, although not so advanced, has recently improved in a dramatic way. Several kinds of information measures have been defined and studied for random sets. See Dubois and Prade [19] and the issue containing this article. See also Klir and Folger [39]. A random set being a generalized set as well as a generalized probability, cardinality and entropy are liable of being extended in the random set setting.

Let $\mathcal{Q} = \{(A_i, m_i) | i = 1, \dots, n\}$ be a random set on a finite universe Ω . The imprecision of \mathcal{Q} is measured by index $I(\mathcal{Q})$ defined by

$$I(\mathcal{Q}) = \sum_{i=1}^n m_i \cdot f(|A_i|) \quad (26)$$

where $|A_i|$ is the cardinality of A_i , f is any monotonically increasing function. Usually f is the identity or is equal to Log_2 (in order to equip I with additivity properties). $I(\mathcal{Q})$ is maximal when $\mathcal{Q} = \Omega$ (it expresses total ignorance) and minimal when $|A_i| = 1, \forall i$, i.e. \mathcal{Q} is a probability measure. Then $I(\mathcal{Q}) = f(1)$, while $I(\Omega) = f(|\Omega|)$. When f is monotonically decreasing the index is called specificity index (Yager [70]).

The dissonance of \mathcal{Q} , i.e. to what extent the A_i 's are scattered, is evaluated by index $D(\mathcal{Q})$ defined by Yager [70]:

$$D(\mathcal{Q}) = - \sum_{i=1}^n m_i \cdot \text{Log } P^*(A_i) \quad (27)$$

where P^* is the upper probability function associated to \mathcal{Q} , i.e.

$$P^*(A) = \sum \{m_i | A \cap A_i \neq \emptyset\}.$$

$D(\mathcal{Q}) = 0$ and is minimum as soon as \mathcal{Q} is consistent, i.e. $\bigcap_{i=1}^n A_i \neq \emptyset$. Particularly $D(\mathcal{Q}) = 0$ if \mathcal{Q} is consonant, i.e. corresponds to a fuzzy set.

$$D(\mathcal{Q}) = \frac{\text{Log } |\Omega|}{|\Omega|}$$

and is maximal when \mathcal{Q} is the uniformly distributed probability on Ω . Index $D(\mathcal{Q})$ is thus a generalized Shannon entropy. See [19,39] for the dual measure

$$C(\mathcal{Q}) = - \sum_{i=1}^n m_i \cdot \text{Log } P_*(A_i),$$

called confusion measure, with $P_*(A) = 1 - P^*(\bar{A})$. The latter evaluates to what extent \mathcal{Q} is identical to its complement $\bar{\mathcal{Q}} = \{(\bar{A}_i, m_i), i = 1, \dots, n\}$.

In the case of fuzzy sets, measures of imprecision play a role analogous to the one of measures of entropy for probability measures. Note that when $f = \text{identity}$, the index of imprecision of the consonant random set \mathcal{Q} that represents a fuzzy set F is the scalar cardinality of the fuzzy set. Namely if

$$\mathcal{Q} = \{(F_i, m_i), i = 1, \dots, n\}$$

with

$$m_i = \alpha_i - \alpha_{i+1}, \quad F_i = \{\omega | \mu_F(\omega) \geq \alpha_i\},$$

then

$$I(\mathcal{Q}) = \sum_{i=1}^n m_i \cdot |F_i| = \sum_{\omega \in \Omega} \mu_F(\omega) \triangleq |F|. \quad (28)$$

A principle, similar to the principle of maximum entropy, has been set forth for fuzzy sets, viewed as possibility distributions: the principle of minimal specificity. Namely a fuzzy set F is said to be more specific than F' if $\exists \omega, \mu_F(\omega) = \mu_{F'}(\omega) = 1$ and F is included in F' [70]. In other words F encompasses less alternatives than F' . As a consequence $|F| \leq |F'|$. The principle of minimum specificity states that if the set $\{\mu_F(\omega) | \omega \in \Omega\}$ of membership values is constrained to remain in some area $\mathcal{Q} \subseteq [0, 1]^{|\Omega|}$, then the least arbitrary representation F of this fuzzy set is such that F is not included in any other admissible fuzzy set. If F is not uniquely defined then the fuzzy cardinality index can discriminate among the ties, using its maximization. Note that the imprecision index $I(\mathcal{Q})$ is consistent with the ordering of random sets induced by the strong inclusion concept introduced above; see [19]. Imprecision measures should not be confused with measures of fuzziness which evaluate to what extent F is fuzzy, i.e. its membership values differ from 0 and 1 (see Dubois and Prade [9], Ch. 1 of Part II, and Klir and Folger [39], Ch. 5 for a review on this topic).

3. Interpretation of possibility and membership

Beside sound mathematical foundations, is the problem of providing fuzzy sets with some in-

terpretive settings that suggest practical methods for elicitation. At this point, the example of probability teaches us that uncertainty can be envisaged from a subjectivist or an objectivist point of view (although the latter is not always void of any subjectivity). Subjective probability expresses subjective degrees of belief while objective probability reflects frequency of objectively observed events. Measurement of subjective probability can be stated as a standard measurement problem (e.g. Krantz et al. [40]), or related to a betting behavior paradigm as suggested by De Finetti and its followers. For fuzzy sets, the same points of views can be adopted and existing results from frequentist, or betting probabilities, as well as measurement theory can be very useful. See Giles [30] for non-additive betting probability functions that include possibility measures. Here only measurement-theoretic approaches and frequentist views are discussed.

3.1. The meaning of possibility

Before proceeding further it may be useful to discuss the notions captured by fuzzy sets, and especially the notion of possibility. The reader is referred to a previous paper [15] in this journal. A fuzzy set restricts to the set of possible values of a variable. 'Possible' has two meanings, mainly, in natural language:

- *physical*: possible means 'easy to achieve', as in the sentence "it is possible for Hans to eat 6 eggs for breakfast".

- *logical*: possible means 'compatible with available information' as in the sentence "it is possible that it rains to-morrow".

Physical possibility has been advocated by Zadeh [77] so as to justify the axiomatic rule of possibility measures,

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

In order to perform $A \cup B$, the easiest thing is to perform the most possible among A and B . Logical possibility corresponds to what is described in Section 1.2. If the available knowledge about x is that $x \in S$, then $x \in A$ is possible when $A \cap S \neq \emptyset$. Logical possibility may be objective, when knowledge about S derive from statistical experiments. It may be subjective when S derive from linguistic information modelled as a fuzzy set. This logical view of possibility has been advocated by Prade

[51], Yager [68] and is also present in Shackle's [55] works, as well in Shafer [56] concept of plausibility.

3.2. Measurement of possibility

It is natural, when describing an individual's state of uncertainty, to start with a weak ordering of events, rather than to assign numbers to them right away. Denote by \succsim a relation on a Boolean algebra of events \mathcal{E} represented by subsets A, B, C, \dots of Ω . \succsim is supposed to be complete ($\forall A, B, A \succsim B$ or $B \succsim A$) and transitive. $A \succ B$ means that A is at least as likely to occur as B . The following axioms are intuitively satisfactory for \succsim :

- $\Omega \succ \emptyset$; $A \succ \emptyset$ ($A \succ B$ means $A \succsim B$ and not $B \succ A$);
- $A \cap (B \cup C) = \emptyset$ and $B \succsim C$ imply $A \cup B \succsim A \cup C$ (monotonicity).

A pair (\mathcal{E}, \succsim) satisfying these axioms is called a monotonic belief structure (M.B.S.). A set function $g: \mathcal{E} \rightarrow [0, 1]$ is said to be compatible with \succsim if and only if $\forall A, B, g(A) \geq g(B) \Leftrightarrow A \succsim B$.

It can be proved that the only set-functions compatible with a finite M.B.S. are decomposable measures (i.e. satisfy axiom (16) (Chateaufort [2], Dubois [7])). The monotonicity axiom can be specialized in two ways.

- $A \cap (B \cup C) = \emptyset$ implies that $B \succsim C$ is equivalent to $A \cup B \succsim A \cup C$ (additivity). Probability measures induce orderings that obey this axiom. Additive belief structures are known in the literature as 'qualitative probability' [40,26].
- Dropping the condition $A \cap (B \cup C) = \emptyset$. Then the only compatible set-functions are possibility measures [7], i.e. set-functions Π such that $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$.

However the underlying possibility distribution is defined only up to a strictly increasing transformation f such that $f(0) = 0$ and $f(1) = 1$. Hence we are led to the problem of eliciting the membership function.

3.3. Measurement of membership and fuzzy set-theoretic operations

Let Ω be a set, and F be the name of a fuzzy set. A qualitative view of gradual set-membership function can be formalized by means of a relation \succsim_F over Ω ; $\omega \succsim_F \omega'$ means that ω belongs to F

more than ω' . This is a classical problem in measurement theory [40]. \succsim_F is complete and transitive, in order to be compatible with the unit interval. However, due to the poorness of the structure, the membership will be defined only up to a strictly increasing transformation!

A richer structure can be obtained by considering membership elicitation as a difference-measurement problem [40]. Namely \succsim_F is viewed as a binary relation on Ω^2 . $(\omega_1, \omega_2) \succsim_F (\omega'_1, \omega'_2)$ means that " ω_1 is more F with respect to ω_2 than ω'_1 is F with respect to ω'_2 ". It seems reasonable to request this type of information from individuals. On continuous sets, theoretical results exist which ensure the unicity of a membership function μ_F compatible with \succsim_F in the sense that

$$\begin{aligned} (\omega_1, \omega_2) \succsim_F (\omega'_1, \omega'_2) \\ \Leftrightarrow \mu_F(\omega_1) - \mu_F(\omega_2) \geq \mu_F(\omega'_1) - \mu_F(\omega'_2). \end{aligned} \quad (29)$$

This approach has been followed by Norwich and Turkmen [46].

They have indicated that for fuzzy categories S defined on continuous scales Ω (height, age, any real-valued parameter), the structure (Ω^2, \succsim_F) is an algebraic difference structure, i.e.

- (Ω^2, \succsim_F) is a weak order (complete and transitive).
- $(\omega_1, \omega_2) \succsim_F (\omega_3, \omega_4) \Leftrightarrow (\omega_4, \omega_3) \succsim_F (\omega_2, \omega_1)$ (sign-reversal).
- If $(\omega_1, \omega_2) \succsim_F (\omega'_1, \omega'_2)$ and $(\omega_2, \omega_3) \succsim_F (\omega'_2, \omega'_3)$ then $(\omega_1, \omega_3) \succsim_F (\omega'_1, \omega'_3)$.
- If $(\omega_1, \omega_2) \succsim_F (\omega_3, \omega_4) \succsim_F (\omega, \omega)$ then $3\omega'_1, \omega'_2$ such that $(\omega_1, \omega'_2) \sim_F (\omega_3, \omega_4) \sim_F (\omega'_1, \omega'_2)$ where \sim_F is the symmetric part of \succsim_F .
- Any standard sequence that is strictly bounded is finite (Archimedean assumption); i.e. $\omega_1, \omega_2, \dots$ such that

$$\forall i, (\omega_{i+1}, \omega_i) \sim_F (\omega_2, \omega_1) \succ_F (\omega_1, \omega_i)$$

and

$$3\omega_1, \omega', (\omega_i, \omega') \succ_F (\omega_i, \omega_1) \succ_F (\omega', \omega), \forall i.$$

D1–D3 reflect the 'rationality' of behavior of the individual who describes his linguistic category F . D4 and D5 require that, in practice, Ω be the real line. Under such conditions the membership function μ_F exists and is an interval scale, i.e. μ_F and $\mu_{F'}$ are compatible with \succsim_F if and only if

$\mu_{F'} = a\mu_F + b$. Note that \succsim_F induces a weak ordering on Ω letting $\omega_1 \succsim_F \omega_2$ if and only if $(\omega_1, \omega_2) \succsim_F (\omega, \omega)$ for any $\omega \in \Omega$. If $\exists \omega, \bar{\omega}$ such that

$$\forall \omega \in \Omega, \bar{\omega} \succsim_F \omega \succsim_F \bar{\omega}.$$

it is natural to let $\mu_F(\bar{\omega}) = 0$, $\mu_F(\bar{\omega}) = 1$, and μ_F is then uniquely defined by the properties of (Ω^2, \succsim_F) . See Norwich and Turkmen [46] for an experimental setting dedicated to this approach. Saaty's [53] pairwise comparison method for membership is more demanding: it assumes that \succsim_F defines a ratio-scale and requires data that represent the values of $\mu_F(\omega)/\mu_F(\omega')$ for various pairs of objects (ω, ω') .

This approach can be extended to the problem of determining fuzzy set-theoretic connectives, when stated in terms of conjoint measurement (Krantz et al. [40], Ch. 6 and 7). For instance Ω is a set of objects, a_1 and a_2 functions from Ω to sets A_1 and A_2 respectively; a_1 and a_2 describe properties of Ω . If objects ω and ω' can be compared in terms of their membership in a fuzzy set F in Ω , this fuzzy set can be described by means of two fuzzy sets F_1 and F_2 on A_1 and A_2 and a fuzzy set-theoretic operation f such that

$$\mu_F(x_1, x_2) = f(\mu_{F_1}(x_1), \mu_{F_2}(x_2)) \quad (30)$$

where $x_i = a_i(\omega)$, $i = 1, 2$, for some $\omega \in \Omega$.

If \succsim_F defines a complete transitive relation on $A_1 \times A_2$, let \sim_F denote the symmetric part of \succsim_F . The following conditions ensure the existence of μ_{F_1} , μ_{F_2} and a strictly monotonic function f such that (30) holds:

- the quotient set A/\sim_F has an order-dense subset B (i.e. if $x \succ_F y$ are in A/\sim_F , $\exists z \in B$, $x \succ_F z \succ_F y$);
- $\mu_{F_1}(x_1, x_2) = \mu_{F_2}(y_1, y_2) \Leftrightarrow x_1 = y_1$ (and the same for the other argument).

Results in Krantz et al. [40] and followers could certainly be adapted to membership measurement with multi-dimensional scales. However in [47], Norwich and Turkmen try to justify the minimum and the maximum rule of fuzzy sets using a scale invariance argument which is debatable [28]. Namely admissible fuzzy set-theoretic operations * should be such that if μ_{F_1} and μ_{F_2} define interval scales then $\mu_{F_1} * \mu_{F_2}$ should define another interval scale. This condition looks very drastic. It sounds more reasonable to address the

problem of defining simultaneously *, μ_{F_1} and μ_{F_2} as a conjoint measurement problem. Papers by French [27,28] although sometimes aggressively against the idea of fuzzy sets, contain interesting preliminary considerations for a measurement-theoretic approach to fuzzy set connectives. See Zimmermann [81, Ch. 14: 82, Ch. 5] for practical aspects of membership and fuzzy connectives measurement.

3.4. Frequentist views of possibility distributions

A random set $\mathcal{Q} = \{A_i, m_i\}$, $i = 1, \dots, n$ on Ω has an obvious frequentist interpretation in terms of set-valued statistics: A_i is a set-valued observation and m_i is the frequency of observation of A_i . For instance, A_i is an imprecise observation: the outcome of a random trial is in A_i , but we do not know which element. See e.g. Wang et al. [65,66]. A fuzzy set can be defined as an approximation of such a random set. The simplest definition of such a fuzzy set is, in the spirit of (3), F_* such that

$$\mu_{F_*}(\omega) = \sum_{i \in A_i} m_i. \quad (31)$$

This is the contour function [56] of the belief function defined by \mathcal{Q} also called the falling shadow of \mathcal{Q} [64,65]. It can be proved that $\Pi_* \leq P^*$ and that, if \mathcal{Q} is consistent $N_* \geq P_*$ where N_* is the necessity measure associated to F_* .

$$N_*(A) = \inf_{x \in A} 1 - \mu_{F_*}(\omega) = 1 - \Pi_*(\bar{A})$$

and P_* derives from \mathcal{Q} , i.e.

$$P_*(A) = \sum_{i \in A_i} m_i = 1 - P^*(\bar{A}).$$

F_* is the greatest fuzzy set such that $\Pi_* \leq P^*$ [16]. F_* is called an inner approximation: it is not necessarily normal. But $F_* \subseteq \mathcal{Q}$ does not hold, generally, in the sense of the strong inclusion of Section 2.2.3.

Example. Set-valued statistics look sometimes more robust and more natural than ordinary point-valued statistics. Consider an opinion poll carried out two months before an election takes place. A sample of individuals is asked about the preferred choice among the set of candidates $C = \{a, b, c, d, e, f, g\}$. Instead of forcing people to

give an explicit name, it is possible for individuals to point out a subset A of C that contains their favorite candidate, in case they have not made up their minds yet.

Let $(A_i, i = 1, \dots, n)$ the subsets that have been considered relevant by the voters. $A_i \subseteq C$ can model such entities as 'a leftist' ($A_i = \{a, b, c\}$), 'a rightist' ($A_i = \{e, f, g\}$), 'no opinion' ($A_i = C$), etc. Let m_i be the proportion of voters that selected A_i , as containing their best choice. Then the following quantities have operational meaning:

(a) $P^*(A) = \sum_{i: A_i \supseteq A} m_i$, is the proportion of people who might vote for a candidate in A .

(b) $P_*(A) = \sum_{i: A_i \subseteq A} m_i$, is the proportion of people who for sure will vote for candidate in A .

(c) F_* is the fuzzy set of preferred candidates,

$$\mu_{F_*}(\omega) = \sum_{\omega \in A_i} m_i,$$

being the proportion of people that do not exclude ω from their final decision.

Using such an opinion poll analysis, one would get a bracketing on the actual percentages of ballots for each candidate, $\mu_{F_*}(\omega)$ being the most optimistic score for candidate ω . The expected cardinality $J(\mathcal{A}) = \sum m_i |A_i|$ would be a good measure of how hesitant the voters are in proposing a final choice.

This procedure can be applied to model linguistic categories for groups of people. Namely, consider a linguistic term F referring to a well-defined scale for instance the real line. Assume that the meaning of F is something like "between x and y , approximately". Then F can be viewed as a fuzzy interval, and modeled as a random set. Let m_i be the proportion of individuals in the group that declared that $A_i = [a_i, b_i]$ is their best representative for F ; then (31) can be used to define the membership function of F for the group. See Wang et al. [64,65,66]. Contrary to the opinion poll example, the case of group-modelling of a linguistic term usually gives a consistent random set.

For some authors, the membership function μ_F can be viewed as a likelihood function in probability theory, i.e.

$$\mu_{F^*}(\omega) = P(F|\omega).$$

This proposal is made by Hisdal [32] in the setting

of experimental work, where $P(F|\omega)$ denotes a proportion of experiments in which object ω has been found to be compatible with label F . In that case experiments are simply of the Yes-No type, while the random set setting requires that the individual proposes an explicit range for the attribute values that represent F . If we admit that, when submitted to a series of stimuli $\omega_1, \dots, \omega_k$, an individual i uses a thresholding procedure for decision of the form

$$\omega \text{ is } F \text{ if } \omega \in A_i,$$

$$\omega \text{ is not } F \text{ if } \omega \notin A_i,$$

then

$$\mu_{F^*}(\omega) = P(F|\omega) = \sum_{\omega \in A_i} m_i,$$

since if an individual answers yes to the stimulus, it means that he would have proposed a set A_i containing ω as a proper representative of F in the random set experiment. Note that $P(F|\cdot)$ is not a probability measure on Ω , generally!

3.5. Outer approximations of random sets

The above elicitation procedure for membership function has the drawback of not leading to normal membership functions. This is especially true if the measured set is likely to be very much fluctuating, e.g. the opinion poll example. Subnormalization occurs when $\bigcap_{i=1}^m A_i = \emptyset$. When eliciting the membership function of a linguistic category F for a group of people, there is usually some consensus about what is a typical value for F , and $\bigcap_{i=1}^m A_i$ is likely to contain this typical value. However it would be useful to have another procedure that generates a normal fuzzy set from any random set \mathcal{A} , be it a probability measure.

Another type of approximation is a fuzzy set F^* such that $\mathcal{A} \subseteq F^*$; F^* is called an *outer* approximation. Given a set of m nested sets $B_1 \subset B_2 \subset \dots \subset B_m$ of the m -element set Ω ($B_1 = \{\omega_1\}$, $B_m = \Omega$), an outer approximation of \mathcal{A} can be built as follows [15]:

(i) $\forall A_i$, define $k(i) = \inf\{j | A_i \subseteq B_j\}$;

(ii) let $m_j^* = \sum_{i \in K^{-1}(j)} m_i$;

then the outer approximation F^* is defined by

$$\mu_{F^*}(\omega) = \sum_{\omega \in B_j} m_j^*. \quad (32)$$

(i) assigns A_i to the most specific B_j that contains it; (ii) allocates to each B_j the weights of the A_i 's that are assigned to B_j . (32) defines the most specific outer approximation based on the B_j 's. By construction, the inclusion $\mathcal{A} \subseteq F^*$ holds.

Especially the set $\{B_1, \dots, B_m\}$ can be chosen as the set of level-cuts of the inner approximation F_* [16]. Note that if \mathcal{A} is consonant, then the inner and the outer approximation coincide. Moreover the inner approximation of a probability distribution is the probability distribution itself. In that case the outer approximation is more interesting than the inner approximation. Especially an outer approximation is *always* normal, while F_* is normal only if $\bigcap_{i=1}^m A_i \neq \emptyset$.

An outer approximation of a probability distribution $\mathcal{A} = \{\omega_i, p_i\}$, $i = 1, \dots, m$, can be defined for any substitution σ of $\{1, 2, \dots, m\}$ by

$$\forall i, \quad \mu_{F^*}(\omega_i) = \sum_{j=1}^m p_{\sigma(j)}. \quad (33)$$

See e.g. Delgado and Moral [4]. Letting

$$B_j = \{\omega_{\sigma(1)}, \dots, \omega_{\sigma(j)}\},$$

(33) is clearly a particular case of (32). The case when $\sigma = \bar{\sigma}$ such that $p_{\sigma(1)} \geq \dots \geq p_{\sigma(m)}$ is studied by Dubois and Prade [11]. Then the B_j 's are the level-cuts of the probability assignment. The inclusion $\mathcal{A} \subseteq F^*$ expresses a formulation of the following probability/possibility consistency principle [9], that specializes the weak random set inclusion:

$$\forall A, \quad N^*(A) \leq P(A) \leq \Pi^*(A) \quad (34)$$

where Π^* and N^* are based on μ_{F^*} .

Given statistical data under the form of a probability distribution one may look for a fuzzy set satisfying (34) such that F is as specific as possible in the sense of fuzzy set inclusion. The solutions are the m fuzzy sets defined by (33). Given a fuzzy set F , the quantity $P(F) = \sum_{i=1}^m \mu_{F^*}(\omega_i) p_i$ evaluates a degree of consistency between F and P , that may help discriminate among the F^* defined in (33) [77,4]. One might require that $P(F^*)$ be as close as possible to 1, as proposed by Ciavarlar and Trussell [3]. However it is easy to check that

$$\forall \sigma, \quad P(F^*) = \frac{1}{2} \left(1 + \sum_{i=1}^m p_i^2 \right).$$

i.e. this criterion is useless under constraint (34) and maximum specificity in the sense of inclusion. Another natural specificity criterion may be the minimization of cardinality

$$|F^*| = \sum_{i=1}^m \mu_{F^*}(\omega_i).$$

It can be checked that any substitution σ such that $|F^*|$ is minimum, is defined by ordering the probability numbers so that $p_{\sigma(1)} \geq p_{\sigma(2)} \geq \dots \geq p_{\sigma(m)}$, i.e. the transformation first proposed by the authors [11], that uses the level-cuts of the inner approximation and the substitution $\bar{\sigma}$. Extending these results to the outer approximation of general random sets is a more tricky problem.

This transformation from a probability distribution to a possibility distribution is closely related to the notion of confidence interval in statistics (e.g. Kendall [36] chap. 19). Namely given any substitution σ let

$$B_j^* = \{\omega_{\sigma(1)}, \dots, \omega_{\sigma(j)}\}.$$

Then

$$\mu_{F^*}(\omega_{\sigma(1)}) = 1 - P(B_{m-1}^*),$$

i.e. the probability that the variable v , whose value is described by the probability assignment P , lies outside of B_{m-1}^* . In other words, any substitution σ defines a nested family of confidence intervals $\{B_i^*, i = 1, \dots, m\}$ and the induced membership grades are calculated from the confidence levels $P(B_i^*)$ attached to these sets. Choosing $\bar{\sigma}$ such that $P_{\bar{\sigma}(1)} \geq \dots \geq P_{\bar{\sigma}(m)}$ comes down to find for all $k = 1, \dots, m$ the k -element subset of Ω with the greatest level of confidence, i.e. find $A \subseteq \Omega$, $|A| = k$ and $P(A)$ is maximal; clearly this subset is

$$A = \{\omega_{\sigma(1)}, \omega_{\sigma(2)}, \dots, \omega_{\sigma(k)}\}.$$

In the continuous case, a nested family of confidence intervals associated to a probability density p on \mathbb{R} can be defined by means of any value x_0 in the support $[\underline{x}, \bar{x}]$ of p , and any function

$$f: [\underline{x}, x_0] \rightarrow [x_0, \bar{x}]$$

such that

$$x \leq x' \Rightarrow f(x) \geq f(x'), \quad f(\underline{x}) = \bar{x}, \quad f(x_0) = x_0.$$

The family

$$\{([x, f(x)], \alpha(x)), x \in [\underline{x}, x_0]\}$$

is a nested family of confidence intervals with

$$\alpha(x) = \int_{-\infty}^x p(u) du + \int_{f(x)}^{+\infty} p(u) du \\ = 1 - P([x, f(x)]); \quad (35)$$

the corresponding fuzzy interval is defined by $\mu_F(x) = \alpha(x) = \mu_F(f(x))$ and it verifies $N(A) \leq P(A) \leq P(A)$, $\forall A$. For instance x_0 can be the median of p and $f(x) = x^*$ such that

$$\int_{-\infty}^x p(u) du = \int_x^{+\infty} p(u) du,$$

i.e. using the quantiles of p . Rather than choosing an arbitrary threshold (e.g. 5%) to define a confidence interval, it is more satisfactory to use the whole family of confidence intervals as a fuzzy confidence interval. This result comes close to a proposal made by McCain [44]: a fuzzy interval is a family of confidence intervals; here confidence levels are given an operational meaning.

The converse problem, i.e. given a fuzzy set F , find a probability measure consistent with (34) was considered as the particular case of a more general problem [15]: finding a probabilistic approximation to a random set

$$\mathcal{R} = \{(A_i, m_i) | i = 1, \dots, n\}.$$

The insufficient reason principle leads us to turn each focal set A_i into a uniformly distributed probability measure with support A_i , and consider the resulting mixture with weights m_i , i.e.

$$\forall i, \quad p_i = \sum_{\omega_i \in A_i} \frac{m_i}{|A_i|}. \quad (36)$$

In the case of a fuzzy set F , we obtain

$$p_i = \sum_{j=1}^n \frac{1}{j} (\mu_F(\omega_j) - \mu_F(\omega_{j+1})),$$

where the $\mu_F(\omega_j)$ are decreasingly ordered and using $\mu_F(\omega_{n+1}) = 0$. Note that this transformation between probability measures and fuzzy sets is bijective, but its converse is not (33). It corresponds to the following simulation process based on a fuzzy set F : select $\alpha \in [0, 1]$ at random, and an outcome $\omega \in F_\alpha$ at random (e.g. Yager [69]).

3.6. Membership function elicitation using extremal information principles

In practice it is not always possible to set up a measurement experiment and come up with a

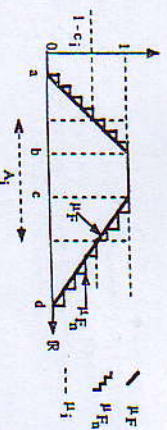


Figure 2. Trapezoidal fuzzy interval

membership function. Moreover, the most commonly encountered fuzzy sets are the fuzzy intervals, i.e. fuzzy sets of real numbers whose level-cuts are closed intervals. They often represent the value of some parameter as described (in a linguistic form, sometimes) by an individual.

The simplest approximation of a fuzzy interval F is a trapezoidal membership function (cf. Figure 2 and Dubois and Prade [21] for instance).

It is completely characterized by its support $[a, d] = \text{supp}(F)$ and its core $F^* = [b, c]$. Basically $[a, d]$ is a pessimistic bracketing of the value of a parameter, while $[b, c]$ is the most optimistic bracketing. These rough pieces of knowledge are rather easy to obtain.

F is then viewed as an ill-known interval $[x, y]$ where $x \in [a, b]$ and $y \in [c, d]$. If we admit that $[x, y]$ is a random interval, let $p(x, y)$ be a density function that describes it. Following Wang and Sanchez [64],

$$\mu_F(r) = \int_r^{-\infty} \left(\int_{-\infty}^r p(x, y) dx \right) dy \\ = \int_{r \in [x, y]} p(x, y) dx dy. \quad (37)$$

All we know about p is that $p(x, y) > 0$ only if $a < x < b$ and $c < y < d$. In the absence of further knowledge, we can model p as a uniform density on $[a, b] \times [c, d]$, with value $1/(b-a)(d-c)$. In that case, it is easy to verify that μ_F has the shape of the trapezoid on Figure 2.

More generally the information about F may be available as a nested sequence $A_0 \subseteq \dots \subseteq A_n$ of confidence intervals with levels of confidence $0 = c_0 \leq \dots \leq c_n = 1$ where c_i is interpreted as the lower bound on the probability $P(A_i)$ that a variable v belongs to A_i . The constraint $P(A_i) > c_i$ can be interpreted without any approximation as $N(A_i) = c_i = \inf\{1 - \mu_i(\omega) | \omega \notin A_i\}$

where μ_i is a possibility distribution that represents the set of probability measures $\{P | P(A_i) \geq c_i\}$. Using the principle of minimum specificity, we must look for the greatest membership grades $\mu_i(u)$, $\forall u \in \mathbb{R}$, such that the constraint

$$\sup_{u \in A_i} \mu_i(u) = 1 - c_i \quad (38)$$

holds. Denoting μ_{A_i} the characteristic function of A_i , the solution is

$$\mu_i = \max(\mu_{A_i}, 1 - c_i), \quad \forall i.$$

The solution to the set of inequalities

$$\{N(A_i) = c_i | i = 1, \dots, n\}$$

is given by

$$\mu_{F_n}(\omega) = \min_{i=1, \dots, n} \max(\mu_{A_i}(\omega), 1 - c_i) \quad (39)$$

which gives the staircase-like fuzzy interval on Figure 2. Letting

$$A_n = [a, d], \quad c_i = i/n,$$

$$A_i = [b - (b-a)i/n, c + (d-c)i/n],$$

we get a uniformly distributed random set F_n (i.e. $m(A_i) = 1/n$, $i = 1, \dots, n$) that, when n goes to infinity comes close to the trapezoidal fuzzy number F .

4. Conjoint uses of fuzzy sets and probability theory

Fuzzy set and probability theory are often used together in problems where uncertainty appears under several forms: random appearance in observed phenomena is often captured by probabilities while imprecision (measurement inaccuracies, lexical vagueness) is more naturally modelled by fuzzy set and possibility theory. In this section we survey various classes of situations where a probabilistic model can be pervaded with possibilistic imprecision: the set of considered events might include fuzzy (ill-defined) events; the set of observations that defines a statistic contain ill-observed data; subjective probabilities may be linguistically assessed. See Kacprzyk and Fedzizi [33] for a collection of papers where probability and fuzzy sets are used conjointly.

4.1. Fuzzy events

When events whose occurrence is inquired about vaguely described, for instance using verbal imprecise statements, one may call them 'fuzzy events' [79]. A fuzzy event can be modeled by a fuzzy set G over some universe, or frame of discernment Ω , with a membership function $\mu_G: \Omega \rightarrow [0, 1]$. Zadeh has defined the probability of a fuzzy event, in the spirit of the traditional view of probability, as the expectation of its membership function, i.e. in a finite setting

$$P(G) = \sum_{\omega \in \Omega} m(\omega) \cdot \mu_G(\omega) \quad (40)$$

where m is a basic probability assignment focusing on singletons. Defining the union and intersection of fuzzy sets in the usual way, i.e.

$$\mu_{A \cup B} = \max(\mu_A, \mu_B)$$

and

$$\mu_{A \cap B} = \min(\mu_A, \mu_B),$$

it is easy to check that P possesses the usual additivity property, i.e. at the order 2,

$$P(A \cup B) + P(A \cap B) = P(A) + P(B).$$

This notion has been investigated with great care in a more general setting (e.g. [38,60], among others). Definition (40) can be extended to plausibility and credibility measures. Smets [59] has naturally defined the plausibility (resp. credibility) of a fuzzy event as the upper (resp. lower) expectation of its membership function, i.e. consistently with the crisp event case

$$P(G) = E^*(\mu_G), \quad C(G) = E_*(\mu_G) \quad (41)$$

where upper and lower expectations are defined in (17) and (18). More generally the scalar evaluation of a set can be extended to the evaluation of fuzzy set consistently with the above definitions, taking a fuzzy set as a constant random set. If $f(A) \in \mathbb{R}$ evaluates the set A , then $f(F)$ is defined by [25]:

$$f(F) = \int_0^1 f(F_\alpha) d\alpha. \quad (42)$$

It is the expectation of the evaluation of a random set. Yager's [70] measure of specificity is of this form, with $f(F_\alpha) = 1/|F_\alpha|$. But (42) may be useful in many purposes. For instance if Ω is a set of

workers, F is the set of young workers (a fuzzy set defined on an age scale) and $\bar{s}(A)$ denotes the average salary of workers in the set A , i.e.

$$\bar{s}(A) = \sum_{\omega \in A} s(\omega) / |A|$$

where $s(\omega)$ is the salary of worker ω , then the average salary of young workers is defined by

$$\bar{s}(F) = \int_0^1 \sum_{\omega \in F_\alpha} s(\omega) / |F_\alpha| d\alpha. \quad (43)$$

Probabilities of fuzzy events are applied to problems of inference and decision-making in Bayesian statistics where updating information appears in the form of fuzzy events (e.g. [49,29]). More generally, evaluation of fuzzy sets via random set expectations have been used in image processing [8] and query processing in information systems [25].

4.2. Fuzzy random variables

There are situations where a random process is not accurately observed. This is the case with self-valued statistics explained in a previous section, where measurement devices are inaccurate. When the information about outcomes is given by individuals which express themselves with linguistic terms, then we obtain fuzzy set-valued statistics. This is especially the case with opinion polls where responses may be different from the standard yes-no type, and where individuals are asked to report estimated values on numerical scales. In that case the random process consists of a random variable which is fuzzy-valued, i.e. a set $((F_i, p_i), i = 1, \dots, n)$ where F_i is a fuzzy set of the real line and p_i is the probability of observing an outcome labeled F_i . This is a (discrete) fuzzy random variable. It is a generalized random set, since the random set is obtained when the F_i 's are simple subsets.

There is a whole literature on this topic. The reader is referred to the book by Kruse and Meyer [41] where many concepts of classical statistics are extended to the case of fuzzy random variables, such as expected value, variance, distribution functions. Limit theorems have been proved such as the strong law of large (fuzzy) numbers, and non-Bayesian statistical methods for parameter estimation and hypothesis testing have been developed. See Yager [73] for an application to deci-

sion-making in which classical criteria (minimax, maximax, Hurwitz, minimal regret) are generalized to fuzzy random sets, that describe uncertainty about the world. Note that these techniques heavily rely on fuzzy interval analysis [9,20,21]. For instance the expectation of a fuzzy random variable $v = ((F_i, p_i) | i = 1, \dots, n)$ is a fuzzy quantity

$$E(v) = \sum_{i=1}^n p_i F_i \quad (44)$$

such that

$$\mu_{E(v)}(x) = \sup \left\{ \min_i \mu_{F_i}(x_i) \mid \sum p_i x_i = x \right\}.$$

In practice the membership function of $E(v)$ is very easy to compute.

4.3. Linguistic probabilities

Much of quantitative expert knowledge in some domains is under the form of ill-defined proportions, i.e. such as "Most of the A 's and B 's", where 'most' refers to some ill-defined proportion and A and B can be fuzzy sets. Formally, it expresses that the conditional probability $P(A|B)$ is fuzzy restricted by some possibility distribution μ_Q on the unit interval. Practical experiments in the estimation of membership function of probability phrases have been carried out recently (e.g. [52]). It seems that individuals often have fuzzy rather than point probabilities in their minds.

Probabilistic methods straightforwardly extend to accommodate linguistic probabilities. Zadeh [78] and the authors [23] handle fuzzy probabilities in syllogisms involving fuzzy quantifiers: it is a step towards handling fuzziness in probabilistic diagnosis networks. Decision evaluation methods using fuzzy-valued probabilities are surveyed in Dubois and Prade [24] from the algorithmic point of view. One of the main problems is to compute expected utilities in the presence of fuzzy probabilities, i.e. compute $\bar{u} = \sum_{i=1}^n u_i p_i$, where u_i is the utility of some decision in state i and p_i is an ill-known probability value that lies in a fuzzy interval $Q_i \subseteq [0, 1]$. The calculation of \bar{u} comes down to the following optimization problem:

$$\begin{aligned} \forall \alpha \in (0, 1], \\ e_\alpha(\bar{u}) &= \inf \left\{ \sum u_i \cdot p_i, p_i \in Q_i, \sum p_i = 1 \right\}, \\ \bar{e}_\alpha(\bar{u}) &= \sup \left\{ \sum u_i \cdot p_i, p_i \in Q_i, \sum p_i = 1 \right\}. \end{aligned}$$

where Q_i is the α -cut of Q_i and the α -cut of \bar{u} is $[e_\alpha(\bar{u}), \bar{e}_\alpha(\bar{u})]$. This problem, including the case when elementary utilities u_i are only fuzzily known, has been solved in Dubois and Prade [12]. Yager [72] solves a similar problem, i.e. find the expected value of a variable v on which a collection of fuzzy statements $P(v \in F_i) \in Q_i$ is available. Nau [43] has studied fuzzy probabilities as a generalization of upper and lower probabilities and has tried to provide an operational basis on such 'confidence-weighted' probabilities in terms of betting behavior.

Conclusion

This review paper has tried to point out that fuzzy sets should play a central role among the mathematical models of uncertainty which have been recently developed and which generalize probabilities. First, it has been shown that probability theory and fuzzy set theory are not to be seen as rival approaches, but rather as complementary ones that are encompassed by a more general setting, namely the so-called random sets. Possibility measures introduce set-theory within the measure-theoretic framework under the form of fuzzy sets, and mathematical models of uncertainty extending probability measures also generalize possibility measures. Moreover, it has been indicated that measurement techniques introduced in the past could serve in elicitation problems, and that frequentist views of fuzzy set theory naturally derive from the formal links existing between fuzzy sets and random sets. Some of the fuzzy operations that naturally appear with axiomatic and measurement-theoretic considerations can be justified with the random set paradigm (e.g. minimum, product and linear conjunction for intersection). Looking at fuzzy sets from the standpoint of general models of uncertainty enables a clear assessment of their role in the representation of imprecision and vagueness, and their complementarity with respect to classical probability.

The close interaction between set theory and measure theory as exhibited in the fuzzy set framework as well as more general models such as random sets and upper/lower probabilities suggests that fuzzy sets, due to their conceptual and computational simplicity, stand as a useful tool to develop new uncertainty representation and

processing methods that are more respectful of information as it appears and introduce sophisticated sensitivity analysis techniques in decision analysis and optimization.

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