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A LAW OF LARGE NUMBERS FOR NON CONVEX FUZZY  
SETS OF THE REAL LINE

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CERT/DERA  
TOULOUSE

Recently there have been a few papers trying to establish laws of large numbers in the framework of fuzzy set theory ; namely Stein and Talati [6] consider a sum  $\frac{(X_1 + X_2 \dots + X_n)}{n}$  where the  $X_i$ 's are possibilistic variables with the same convex distribution  $\pi$ ,  $\pi$  is thus the membership function of a fuzzy number [2]. It is proved that under this convexity assumption, the above sum is a possibilistic variable with distribution  $\pi$ . These authors then take advantage of this result in deriving similar properties for fuzzy random variables as stated by Nahmias [5] .. Another paper by Badard [1] also mentions the same results and moreover deals with the case of sup product extension principle (see also [3])

In this short note, we address the case of  $X_i$ 's represented by a non convex possibility distribution on the real line  $\mathbb{R}$ , in the framework of sup min extension principle First let  $M$  be the fuzzy set of the real line such that  $\pi \triangleq \mu_M$ . We have the first preliminary result, which invalidates a theorem by Stein and Talatti [6] : Let  $p_1, \dots, p_n$  be  $n$  strictly positive numbers summing to 1

PROPOSITION 1 if  $M$  is a convex fuzzy set, then  $p_1 M \oplus \dots \oplus p_n M = M$  ;  
the converse does not hold.

Proof : the membership of  $p_1 M \oplus \dots \oplus p_n M$  is  $\mu$  such that,  $\forall z$

$$\mu(z) \triangleq \sup_{x_1, \dots, x_n} : \sum_{i=1}^n p_i x_i = z \quad \min_{i=1, \dots, n} \mu_M(x_i) \geq \mu_M(z) \quad (\text{Choose } x_i = z, \forall i)$$

but  $\mu_M(z) \geq \min_{i=1, \dots, n} \mu_M(x_i) \quad \forall (x_1, \dots, x_n) \text{ s.t. } \sum_{i=1}^n p_i x_i = z$ , due to convexity of  $M$ .  
Q.E.D.

That the converse does not hold can be checked on the following counter-example.

Let  $M = [0,1] \cap \mathbb{Q}$ , i.e. the set of rational numbers in the unit interval. Then  $M$  is obviously not convex. Assume  $p_1 \dots p_n$  are rational.

Then  $\forall (x_1 \dots x_n) \in M^n \quad \sum_{i=1}^n p_i x_i \in [0,1]$  and is rational, thus belongs to  $M$  i.e.

$$p_1 M \oplus \dots \oplus p_n M \subseteq M$$

Since the converse inequality always holds, we have an example of non convex distribution such that  $p_1 M \oplus \dots \oplus p_n M = M$

However the converse of proposition 1 does hold for a broad class of fuzzy sets of  $\mathbb{R}$

PROPOSITION 2 if  $M$  is the union of a finite number of convex fuzzy sets of  $\mathbb{R}$  then  $M$  is a convex if and only if  $p_1 M \oplus \dots \oplus p_n M = M$

Proof: it is sufficient to show that when  $M$  is not convex then

$$p_1 M \oplus \dots \oplus p_n M \neq M$$

Assume  $M$  is not convex.  $M \triangleq \bigcup_{i=1}^k M_i$

where  $M_i$  is convex  $\forall i$ . It is not restrictive to assume :

$\rightarrow \forall i \neq j, M_i \cap M_j = \emptyset$  and  $M_i \cup M_j$  is not convex

$\rightarrow i < j \Rightarrow \forall x \in S(M_i), \forall y \in S(M_j), x < y$

where  $S(M_i)$  denotes the support of  $M_i$ , i.e.  $\{x \mid \mu_{M_i}(x) > 0\}$

In other words the  $M_i$ 's are disjoint, ranked in increasing order, and in representing  $M$  as

$$\bigcup_{i=1}^k M_i$$

$k$  is minimum, i.e. we have a minimal representation of  $M$ .

Now denote

$$\alpha_i = \sup \mu_{M_i}$$

$$\alpha_{i+1} = \sup \mu_{M_{i+1}}$$

for two consecutive  $M_i$ 's.

Let  $x = \lim_{\alpha \uparrow \alpha_i} \sup (M_i)_\alpha$

$y = \lim_{\alpha \uparrow \alpha_{i+1}} \inf (M_{i+1})_\alpha$

where  $(M)_\alpha$  denotes the  $\alpha$ -cut of  $M$ , i.e.  $\{x \mid \mu_M(x) \geq \alpha\}$

Assume  $x \neq y$

Let  $z = (p_1 + p_2 + \dots + p_k) x + (p_{k+1} + \dots + p_n) y$  ( $k < n$ ),  $z \in ]x, y[$

for  $k < n$ . Then if  $\mu$  is the membership function of  $p_1 M \oplus \dots \oplus p_n M$

$$\mu(z) \geq \min(\mu_M(x), \mu_M(y)) = \min(\alpha_i, \alpha_{i+1}) > \mu_M(z)$$

Hence

$$p_1 M \oplus \dots \oplus p_n M \neq M$$

Q.E.D.

NB : If  $x = y$  it can be proved that  $\mu_M(x) = 0$  ;  $\mu(x) > 0$  and thus the result still holds

Note that if we change proposition 2 from "finite unions" to "countable unions" then this proposition becomes false :

$[0,1] \cap \mathbb{Q}$  is made of such a countable union of convex "fuzzy sets" (singletons actually)

Anyway , when  $M$  is not convex there are many instances when the quantity

$$M \oplus \dots \oplus M \quad (n \text{ times})$$

is not the same as the fuzzy set  $nM$ , such that  $\mu_{nM}(z) = \mu_M(z/n)$

However when  $\mu_M$  is continuous, then  $\frac{M \oplus \dots \oplus M}{n}$  has a limit when  $n$  becomes large. This is the topic of the following proposition which requires the preliminary definition (Lowen [4] )

Convex hull: let  $M$  a fuzzy set of  $\mathbb{R}$ ; the convex hull  $\hat{M}$  of  $M$  is defined by :

$$\mu_{\hat{M}}(z) = \sup_{x,y : x \leq z \leq y} \min(\mu_M(x), \mu_M(y))$$

PROPOSITION 3 Let  $X_1 \dots X_n$  be possibilistic variables with a continuous non convex distribution  $\pi \triangleq \mu_M$  then

$$\lim_{n \rightarrow +\infty} \frac{X_1 \dots + X_n}{n}$$

has distribution  $\mu_{\hat{M}}$ , i.e. the convex hull of  $M$ . (Here we prove pointwise convergence).

Proof : Let  $\mu_n$  be the membership function associated with

$$\frac{X_1 + X_2 \dots + X_n}{n}$$

i.e

$$\forall t, \mu_n(t) = \sup_{x_1, \dots, x_n} \min_{i=1, n} \mu_M(x_i) \quad (1)$$

$$x_1 + \dots + x_n = nt$$

a) -  $\forall t, \mu_n(t) \geq \mu_M(t)$

Indeed  $\mu_M(t)$  is obtained from (1) by adding the constraint  $x_i = t, \forall i = 1, n$

b) -  $\forall t, \mu_n(t) \leq \widehat{\mu}_M(t)$

To see this first note the following inclusion

$$\{(x_1, \dots, x_n) \mid x_1 + \dots + x_n = nt\} \subseteq \{(x_1, \dots, x_n) \mid \min_i x_i \leq t \leq \max_i x_i\}$$

Hence we have

$$\mu_n(t) \leq \sup_{x_1, \dots, x_n: \min_i x_i \leq t \leq \max_i x_i} \min_i \mu_M(x_i)$$

The expression on the right hand side is easily recognized as being  $\widehat{\mu}_M(t)$  since the supremum is always reached for some  $(x_1, \dots, x_n)$  such that  $x_1 = x^*$  and  $x_2 = \dots = x_n = y^*$  with  $x^* \leq t \leq y^*$  owing to the symmetry of the expression. Hence we have proved

$$M \subseteq \frac{M \oplus \dots \oplus M}{n} \subseteq \widehat{M} \quad \text{so far.}$$

c) -  $\forall t, \exists$  a sequence  $\{u_n\}_m$  such that  $\lim_{n \rightarrow \infty} u_n = t$  and

$$\mu_n(u_n) = \widehat{\mu}_M(t), \quad \forall n > 1.$$

Let  $x^*$  and  $y^*$  such that  $x^* \neq y^*$  and:

$$\widehat{\mu}_M(t) = \min(\mu_M(x^*), \mu_M(y^*))$$

We know that, say,  $x^* \leq t \leq y^*$ , and  $x^* \neq y^*$  for some  $t$ , otherwise  $\widehat{M} = M$ . Let us share  $[x^*, y^*]$  into  $n$  equal parts ( $n > 1$ )

$$\left[ \frac{j x^* + (n-j) y^*}{n}, \frac{(j-1) x^* + (n-j+1) y^*}{n} \right]$$

and assume  $t$  belongs to the  $\bar{j}(n)$ <sup>th</sup> part,  $\forall n$ .

Let

$$u_n^- = \frac{\bar{j}(n)x^* + (n - \bar{j}(n))y^*}{n} \quad u_n^+ = \frac{(\bar{j}(n)-1)x^* + (n - \bar{j}(n)+1)y^*}{n}$$

Clearly  $\lim_{n \rightarrow +\infty} u_n^+ - u_n^- = \lim_{n \rightarrow +\infty} \frac{y^* - x^*}{n} = 0$

Hence since

$$u_n^- \leq t \leq u_n^+, \quad \lim_{n \rightarrow +\infty} u_n^- = \lim_{n \rightarrow +\infty} u_n^+ = t$$

Now

$$\mu_n(u_n^-) \geq \min(\mu_n(x^*), \mu_n(y^*)) \triangleq \mu_{\hat{M}}(u_n^-) \text{ since we can choose}$$

$$x_i = x^* \quad \forall i=1, \bar{j}(n) \quad ; \quad x_i = y^*, \quad \forall i = \bar{j}(n)+1, n$$

From b) we conclude

$$\mu_n(u_n^-) = \mu_{\hat{M}}(u_n^-), \quad \forall n$$

The same holds for  $u_n^+$ .

$$\text{d) - } \lim_{n \rightarrow +\infty} \mu_n(t) = \mu_{\hat{M}}(t)$$

From c) we have

$$\lim_{n \rightarrow +\infty} \mu_n(u_n^-) = \lim_{n \rightarrow +\infty} \mu_{\hat{M}}(u_n^-) = \mu_{\hat{M}}(\lim_{n \rightarrow +\infty} u_n^-) = \mu_{\hat{M}}(t) \quad (2)$$

due to the continuity of  $\mu_{\hat{M}}$ , hence that of  $\mu_{\hat{M}}$

Now since  $\mu_{\hat{M}}$  is continuous,  $\mu_n$  is too; more-over

$$|\mu_n(t) - \mu_{\hat{M}}(t)| \leq |\mu_n(t) - \mu_n(u_n^-)| + |\mu_n(u_n^-) - \mu_{\hat{M}}(t)| \quad (3)$$

it is patent that because  $\lim_{n \rightarrow +\infty} u_n^- = t$  and  $\mu_{\hat{M}}$  is continuous we have

$$\lim_{n \rightarrow +\infty} |\mu_n(t) - \mu_n(u_n^-)| = 0$$

Clearly, the other quantity in the right-hand side of (3) has a limit 0, from (2). Hence the quantity on the left-hand side of (3) is bounded by arbitrary small numbers as  $n$  increases; hence

$$\lim_{n \rightarrow +\infty} \mu_n(t) = \mu_{\widehat{M}}(t)$$

with establishes the pointwise convergence of the possibility distribution of

$$\frac{X_1 + \dots + X_n}{n}$$

toward the convex hull of the possibility distribution of the  $X_i$ 's  
QED

For fuzzy sets of the real line satisfying assumptions of proposition 2, i.e. finite union of convex fuzzy sets, without continuity proposition 3 does not hold. For instance, consider the fuzzy set made of the union of two singletons, i.e.

$$M = \{0, 1\}, \text{ and } \widehat{M} = [0, 1]$$

then

$$\frac{M \oplus M}{2} = \{0, \frac{1}{2}, 1\}$$

$$\frac{M \oplus M \oplus M}{3} = \{0, \frac{1}{3}, 1\} \dots \text{etc...}$$

Let  $\frac{p}{q}$  be a rational number lying in  $[0, 1]$ . It is clear that if  $n = kq$ ,  $\mu_n(\frac{p}{q}) = 1$ ,  $\forall k \geq 1$ ; and  $\mu_n(\frac{p}{q}) = 0 \forall n \neq kq$  for all positive integers  $k$ ; hence :

$$\lim_{n \rightarrow +\infty} \mu_n(\frac{p}{q}) \text{ does not exist}$$

If  $t$  is irrational then  $\mu_n(t) = 0 \neq \mu_{\widehat{M}}(t)$

Hence  $\lim_{n \rightarrow +\infty} \mu_n(t) = 0$ . In other words  $\lim_{n \rightarrow +\infty} \frac{X_1 + \dots + X_n}{n}$  has no limiting distribution



In the proof of proposition 3, it is clear that using

$$M = \{0,1\}$$

$\mu_n$  is not continuous and  $|\mu_n(u_n^-) - \mu_n(t)| = 1$  in eqn (3) whenever  $t$  is irrational.

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