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► **To cite this version:**

Didier Dubois. A law for large numbers for non convex fuzzy sets fo the real line. Bulletin pour les sous-ensembles flous et leurs applications, 1982, 9, pp.31-38. hal-04199816v1

HAL Id: hal-04199816

<https://hal.science/hal-04199816v1>

Submitted on 8 Sep 2023 (v1), last revised 29 Apr 2024 (v2)

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LANGAGES ET SYSTEMES INFORMATIQUES

Equipe de Recherche Associee au C.N.R.S. 298

Université Paul Sabatier, 118 route de Narbonne - 31077 Toulouse Cedex - Tél. (61) 63.11.20



A LAW OF LARGE NUMBERS FOR NON CONVEX FUZZY SETS OF THE REAL LINE

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Recently there have been a few papers trying to establish laws of large numbers in the framework of fuzzy set theory; namely Stein and Talatti [6] consider a sum $\frac{1}{n} \sum_{i=1}^n (X_i + X_i \dots + X_i)$ where the X_i 's are possibilistic variables with the same convex distribution \mathcal{T} , \mathcal{T} is thus the membership function of a fuzzy number [2]. It is proved that under this convexity assumption, the above sum is a possibilistic variable with distribution \mathcal{T} . These authors then take advantage of this result in deriving similar properties for fuzzy random variables as stated by Nahmias [5]. Another paper by Badard [1] also mentions the same results and moreover deals with the case of sup product extension principle (see also [3]).

In this short note, we address the case of X_i 's represented by a non convex possibility distribution on the real line \mathbb{R} , in the framework of sup min extension principle. First let M be the fuzzy set of the real line such that $\mathcal{T} \triangleq \mu_M$. We have the first preliminary result, which invalidates a theorem by Stein and Talatti [6]: Let p_1, \dots, p_n be n strictly positive numbers summing to 1

PROPOSITION 1 If M is a convex fuzzy set, then $p_1 M \odot \dots \odot p_n M = M$; the converse does not hold.

Proof: the membership of $p_1 M \odot \dots \odot p_n M$ is μ such that, $\forall \bar{z}$

$$\mu(\bar{z}) \triangleq \sup_{x_1, \dots, x_n} \min_{i=1, \dots, n} p_i \mu(x_i) \quad \text{with } \sum_{i=1}^n p_i x_i = \bar{z} \quad \text{and } \min_{i=1, \dots, n} \mu(x_i) \geq \mu(\bar{z}) \quad (\text{choose } x_i = \bar{z}, \forall i)$$

but $\mu(\bar{z}) \geq \min_{i=1, \dots, n} \mu(x_i) \quad \forall (x_1, \dots, x_n) \text{ s.t. } \sum_{i=1}^n p_i x_i = \bar{z}$, due to convexity of M .
Q.E.D.

That the converse does not hold can be checked on the following counter-example.

Let $M = [0,1] \cap \mathbb{Q}$, i.e. the set of rational numbers in the unit interval. Then M is obviously not convex. Assume $P_1 \dots P_n$ are rational.

Then $\forall (x_1 \dots x_n) \in M^n \quad \sum_{i=1}^n P_i x_i \in [0,1]$ and is rational, thus belongs to M . i.e.

$$P_1 M \oplus \dots \oplus P_n M \subseteq M$$

Since the converse inequality always holds, we have an example of non convex distribution such that $P_1 M \oplus \dots \oplus P_n M = M$

However the converse of proposition 1 does hold for a broad class of fuzzy sets of \mathbb{R}

PROPOSITION 2 if M is the union of a finite number of convex fuzzy sets of \mathbb{R} then M is a convex if and only if $P_1 M \oplus \dots \oplus P_n M = M$

PROOF: it is sufficient to show that when M is not convex then

$$P_1 M \oplus \dots \oplus P_n M \neq M$$

Assume M is not convex. $M \triangleq \bigcup_{i=1}^k M_i$

where M_i is convex $\forall i$, It is not restrictive to assume :

$\rightarrow \forall i \neq j, M_i \cap M_j = \emptyset$ and $M_i \cup M_j$ is not convex

$\rightarrow 1 < j \Rightarrow \forall x \in S(M_i), \forall y \in S(M_j), x < y$

where $S(M_i)$ denotes the support of M_i , i.e. $\{x \mid \mu_{M_i}(x) > 0\}$

In other words the M_i 's are disjoint, ranked in increasing order, and in representing M as

$$\bigcup_{i=1}^k M_i$$

k is minimum, i.e. we have a minimal representation of M .

Now denote

$$\alpha_1 = \sup \mu_{M_1}$$

$$\alpha_{i+1} = \sup \mu_{M_{i+1}}$$

for two consecutive M_i 's.

$$\text{Let } x = \lim_{\alpha \uparrow \alpha_k} \sup (M_i)_\alpha \quad y = \lim_{\alpha \uparrow \alpha_{k+1}} \inf (M_{i+1})_\alpha$$

where $(M)_\alpha$ denotes the α -cut of M , i.e. $\{x \mid \mu_M(x) \geq \alpha\}$

Assume $x \neq y$

Let $z = (P_1 + P_2 + \dots + P_k) x + (P_{k+1} + \dots + P_n) y$ ($k < n$), $z \in]x, y[$

For $k < n$, then if μ is the membership function of $P_1 M \oplus \dots \oplus P_n M$

$$\mu(z) \geq \min(\mu_M(x), \mu_M(y)) = \min(\alpha_1, \alpha_{i+1}) > \mu_M(z)$$

Hence

$$P_1 M \oplus \dots \oplus P_n M \neq M$$

Q.E.D.

NB : If $x = y$ it can be proved that $\mu_M(x) = 0$; $\mu(x) > 0$ and thus the result still holds

Note that if we change proposition 2 from "finite unions" to "countable unions" then this proposition becomes false :

[0,1] ∩ Q is made of such a countable union of convex "fuzzy sets (singletons actually)

Anyway, when M is not convex there are many instances when the quantity

$$M \oplus \dots \oplus M \quad (n \text{ times})$$

is not the same as the fuzzy set nM, such that $\mu_M(z) = \mu_M(z/n)$

However when μ_M is continuous, then $\frac{M \oplus \dots \oplus M}{n}$ has a limit when n becomes large. This is the topic of the following proposition which requires the preliminary definition (Lowen [4])

Convex hull: let M a fuzzy set of R, the convex hull \hat{M} of M is defined by :

$$\mu_{\hat{M}}(z) = \sup_{x, y : x \leq z \leq y} \min(\mu_M(x), \mu_M(y))$$

PROPOSITION 3 Let X_1, \dots, X_n be possibilistic variables with a continuous non convex distribution $\pi \in \mu_M$ then

$$\lim_{n \rightarrow +\infty} \frac{X_1 + \dots + X_n}{n}$$

has distribution $\mu_{\hat{M}}$, i.e. the convex hull of M. (Here we prove pointwise convergence).

Proof : Let μ be the membership function associated with

$$\frac{X_1 + X_2 + \dots + X_n}{n}$$

i.e

$$\forall t, \mu_n(t) = \sup_{x_1, \dots, x_n} \min_{i=1, \dots, n} \mu_M(x_i) \quad (1)$$

$$x_1 + \dots + x_n = nt$$

a) - $\forall t, \mu_n(t) \geq \mu_M(t)$

Indeed $\mu_M(t)$ is obtained from (1) by adding the constraint $x_1 = t, \forall i = 1, n$

b) - $\forall t, \mu_n(t) \leq \mu_{\hat{M}}(t)$

To see this first note the following inclusion

$$\{(x_1, \dots, x_n) \mid x_1 + \dots + x_n = nt\} \subseteq \left\{ (x_1, \dots, x_n) \mid \min_{i=1} x_i \leq t \leq \max_{i=1} x_i \right\}$$

Hence we have

$$\mu_n(t) \leq \sup_{x_1, \dots, x_n} \min_{i=1} x_i \leq t \leq \max_{i=1} x_i \implies \min_{i=1} \mu_M(x_i)$$

The expression on the right hand side is easily recognized as being $\mu_{\hat{M}}(t)$ since the supremum is always reached for some (x_1, \dots, x_n) such that $x_1 = x^*$ and $x_2 = \dots = x_n = y^*$ with $x^* \leq t \leq y^*$ owing to the symmetry of the expression. Hence we have proved

$$M \subseteq \frac{M \oplus \dots \oplus M}{n} \subseteq \hat{M} \quad \text{so far.}$$

c) - $\forall t, \exists$ a sequence $\{u_n\}$ such that $\lim_{n \rightarrow \infty} u_n = t$ and $\frac{\mu(u_n)}{n} = \mu_M(t), \forall n > 1$.

Let x^* and y^* such that $x^* \neq y^*$ and :

$$\mu_{\hat{M}}(t) = \min(\mu_M(x^*), \mu_M(y^*))$$

We know that, say, $x^* \leq t \leq y^*$, and $x^* \neq y^*$ for some t, otherwise $\hat{M} = M$. Let us share $[x^*, y^*]$ into n equal parts (n > 1)

$$\left[\frac{j x^* + (n-j) y^*}{n}, \frac{(j-1) x^* + (n-j+1) y^*}{n} \right]$$

Let

$$u_n^- = \frac{\bar{j}(n)x^* + (n-\bar{j}(n))y^*}{n} \quad u_n^+ = \frac{(\bar{j}(n)-1)x^* + (n-\bar{j}(n)+1)y^*}{n}$$

Clearly $\lim_{n \rightarrow +\infty} u_n^+ - u_n^- = \lim_{n \rightarrow +\infty} \frac{y^* - x^*}{n} = 0$

Hence since

$$u_n^- \leq t \leq u_n^+, \quad \lim_{n \rightarrow +\infty} u_n^- = \lim_{n \rightarrow +\infty} u_n^+ = t$$

Now

$$\mu(u_n^-) \geq \min(\mu(x^*), \mu(y^*)) \triangleq \mu_{\bar{j}(n)}(u_n^-) \quad \text{since we can choose}$$

$$x_i = x^* \quad \forall i=1, \bar{j}(n) \quad ; \quad x_i = y^* \quad \forall i = \bar{j}(n)+1, n$$

From b) we conclude

$$\mu_{\bar{j}(n)}(u_n^-) = \mu_{\bar{j}(n)}(u_n^-), \quad \forall n$$

The same holds for u_n^+ .

$$d) - \lim_{n \rightarrow +\infty} \mu_{\bar{j}(n)}(t) = \mu_{\bar{j}(n)}(t)$$

From c) we have

$$\lim_{n \rightarrow +\infty} \mu(u_n^-) = \lim_{n \rightarrow +\infty} \mu_{\bar{j}(n)}(u_n^-) = \mu_{\bar{j}(n)}(\lim_{n \rightarrow +\infty} u_n^-) = \mu_{\bar{j}(n)}(t) \quad (2)$$

due to the continuity of μ , hence that of $\mu_{\bar{j}(n)}$

Now since $\mu_{\bar{j}(n)}$ is continuous, $\mu_{\bar{j}(n)}$ is too ; more-over

$$|\mu_{\bar{j}(n)}(t) - \mu_{\bar{j}(n)}(t)| \leq |\mu_{\bar{j}(n)}(t) - \mu_{\bar{j}(n)}(u_n^-)| + |\mu_{\bar{j}(n)}(u_n^-) - \mu_{\bar{j}(n)}(t)|$$

It is patent that because $\lim_{n \rightarrow +\infty} u_n^- = t$ and $\mu_{\bar{j}(n)}$ is continuous we have

$$\lim_{n \rightarrow +\infty} |\mu_{\bar{j}(n)}(t) - \mu_{\bar{j}(n)}(u_n^-)| = 0$$

Clearly, the other quantity in the right-hand side of (3) has a limit 0, from (2). Hence the quantity on the left-hand side of (3) is bounded by arbitrary small numbers as n increases; hence

$$\lim_{n \rightarrow +\infty} \mu(t) = \mu_{\bar{j}(n)}(t)$$

with establishes the pointwise convergence of the possibility distribution of

$$\frac{X_1 + \dots + X_n}{n}$$

toward the convex hull of the possibility distribution of the X_i 's

Q.E.D

For fuzzy sets of the real line satisfying assumptions

of proposition 2, i.e. finite union of convex fuzzy sets, without continuity proposition 3 does not hold. For instance, consider the fuzzy set made of the union of two singletons, i.e.

$$M = \{0, 1\}, \quad \text{and} \quad \hat{M} = [0, 1]$$

then

$$\frac{M \oplus M}{2} = \{0, \frac{1}{2}, 1\}$$

$$\frac{M \oplus M \oplus M}{3} = \{0, \frac{1}{3}, 1\} \dots \text{etc.}$$

Let $\frac{k}{q}$ be a rational number lying in $[0, 1]$. It is clear that

$$\text{if } n = kq, \quad \mu_{\bar{j}(n)}(\frac{k}{q}) = 1, \quad \forall k \geq 1 \text{ and } \mu_{\bar{j}(n)}(\frac{k}{q}) = 0 \quad \forall n \neq kq$$

for all positive integers k; hence :

$$\lim_{n \rightarrow +\infty} \mu_{\bar{j}(n)}(\frac{k}{q}) \quad \text{does not exist}$$

If t is irrational then $\mu_{\bar{j}(n)}(t) = 0 \neq \mu_{\bar{j}(n)}(t)$

Hence $\lim_{n \rightarrow +\infty} \mu_{\bar{j}(n)}(t) = 0$. In otherwords

$$\lim_{n \rightarrow +\infty} \frac{X_1 + \dots + X_n}{n}$$

has no limiting distribution

In the proof of proposition 3, it is clear that using

$$M = \{0,1\}$$

μ_n is not continuous and $|\mu_n(x_n) - \mu_n(t)| = 1$ in eqn (3) whenever t is irrational.

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