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QUANTITATIVE OBSERVABILITY FOR ONE-DIMENSIONAL SCHRÖDINGER EQUATIONS WITH POTENTIALS

PEI SU, CHENMIN SUN, AND XU YUAN

ABSTRACT. In this note, we prove the quantitative observability with an explicit control cost for the 1D Schrödinger equation over \mathbb{R} with real-valued, bounded continuous potential on thick sets. Our proof relies on different techniques for low-frequency and high-frequency estimates. In particular, we extend the large time observability result for the 1D free Schrödinger equation in Theorem 1.1 of Huang-Wang-Wang [20] to any short time. As another byproduct, we extend the spectral inequality of Lebeau-Moyano [27] for real-analytic potentials to bounded continuous potentials in the one-dimensional case.

1. INTRODUCTION

1.1. **Main result.** Consider the 1D Schrödinger equation,

$$i\partial_t u - \partial_x^2 u + V(x)u = 0, \quad u|_{t=0} = u_0 \in L^2(\mathbb{R}), \quad (1.1)$$

where the potential V is a real-valued, continuous and bounded function. We are interested in the observability of the Schrödinger equation (1.1) on rough sets, which concerns the following inequality,

$$\|u_0\|_{L^2(\mathbb{R})}^2 \leq C(T, V, \Omega) \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt, \quad (1.2)$$

for all solutions $u(t)$ of (1.1) where $\Omega \subset \mathbb{R}$ is a measurable subset.

The inequality (1.2) measures how solutions of Schrödinger equations can concentrate on subsets of the domain. Such a property is linked to the high-frequency wave propagation phenomenon and to concentration properties for quasimodes of the Schrödinger operator. The results are sensitive for different underlying manifolds and the corresponding Schrödinger operators. Another motivation for establishing the observability estimate (1.2) is to prove the exact controllability for the associated control system. See Corollary 1.4 for the precise statement.

In the general framework, there are three parameters that affect the observability estimates for Schrödinger type equations. These are the underlying geometry (the background manifold on which the equation is posed and the associated Schrödinger operator), the control region Ω , and the time $T > 0$ to achieve the observability. When observability holds for any time $T > 0$, the control cost, i.e. the blow-up rate of the optimal constant $C(T, V, \Omega)$ is also an object of study.

In this note, we address the observability problem for the 1D Schrödinger equation on an unbounded set by measurable control regions. To the best of our knowledge, this setup is much less studied in the literature. To state the main result, we recall the thickness condition for the control region.

Definition 1.1. Let $0 < \zeta < 1$ and $0 < L < \infty$. We say that $\Omega \subset \mathbb{R}$ is a (L, ζ) -thick set if Ω is a measurable set and satisfies

$$|\Omega \cap [x, x + L]| \geq \zeta L, \quad \text{for any } x \in \mathbb{R}.$$

The main result of this note is the following quantitative observability for (1.1).

Theorem 1.2. *Let Ω be a (L, ζ) -thick set and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists a constant $C = C(V, L, \zeta) > 0$, depending only on V , L and ζ , such that for any $T > 0$ and any solution u of (1.1) we have*

$$\|u_0\|_{L^2(\mathbb{R})}^2 \leq C e^{\frac{C}{T^2}} \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt. \quad (1.3)$$

Remark 1.3. In [20, Theorem 1.1], the authors considered the 1D free Schrödinger equation. They showed that (1.2) holding for some time $T > 0$ is equivalent to the control region Ω being thick. Theorem 1.2 extends their results to any observability time $T > 0$ for general 1D Schrödinger operators with bounded and continuous potential. In particular, we answer the question raised in Remark (a1) below [20, Theorem 1.1] concerning the short time observability for the 1D Schrödinger equation. Moreover, the proof of Theorem 1.2 is purely quantitative which provides the upper bound for the control cost in terms of $T > 0$ and the parameter defining the thickness. See Section 5 for more precise comments on the proof.

As a consequence of the classical HUM method (see [29]), we have the following exact controllability result for 1D Schrödinger equations.

Corollary 1.4. *Let Ω be a (L, ζ) -thick set and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For any $T > 0$ and any $(u_0, u_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, there exists a control $f \in L^2((0, T) \times \mathbb{R})$ such that the unique solution of the 1D inhomogeneous Schrödinger equation*

$$i\partial_t u - \partial_x^2 u + V(x)u = \mathbf{1}_\Omega f$$

with initial data $u(0) = u_0$ satisfies $u(T) = u_1$.

1.2. Previous results. In the existing literature, observability for Schrödinger equations on compact manifolds and bounded domains has been extensively studied. Sufficient geometric conditions are often imposed on the control region Ω . When Ω is an open set satisfying the geometric control condition (GCC), Lebeau [26] proved that observability is true for an arbitrarily short time $T > 0$. The essential behind Lebeau's theorem is the infinite speed of propagation of singularities for high-frequency wave-packets of solutions to the Schrödinger equation.

The condition GCC is in general not necessary. The stability (or instability) property of the geodesic flow of the underlying manifold plays a decisive role. Specifically, it has been shown (though this list is not exhaustive) that observability is valid for any $T > 0$ and for any non-empty open control region if the underlying manifold is a torus [2, 8, 13, 21], or compact hyperbolic surfaces [22] (see also [4] for negatively curved manifolds), or the disk [3], provided the control region encompasses a neighborhood of some portion of the boundary. We also mention here that for certain subelliptic Schrödinger equations, the minimal observability time is strictly positive (see [11, 16]).

In recent years, there has been a growing focus on observability for Schrödinger equations by rough control regions or in non-compact settings [14, 20, 25, 32, 35]. For the Schrödinger equation on two-dimensional tori, [14] showed that the short time observability result holds for any measurable control region with a positive measure. Very recently, this result has been extended to Schrödinger equations (with periodic potential) on \mathbb{R}^2 having a periodic control region. It is worth mentioning that the thickness condition (see Definition 1.1) in the one-dimensional context is a version of GCC on rough sets. The interest of the result in [25] lies in the fact that the control region, though $2\pi\mathbb{Z}^2$ -periodic, might not satisfy GCC. Lastly, we refer to [20, 32] for discussions on the observability of Schrödinger operators on \mathbb{R}^d with confined potentials.

1.3. Comments on the proof. Our proof of Theorem 1.2 relies on three steps: spectral inequality for low-frequency part, high-frequency part observability via resolvent estimate (of Hautus type inequality), and a quantitative gluing argument. The first part of our analysis is the proof of the low-frequency observability by establishing the spectral inequality for 1D Schrödinger operators with continuous bounded potentials (see Lemma 3.8). This particularly generalizes the spectral inequality in [27] for real-analytic potentials to bounded continuous potentials in the one-dimensional context. Our proof of the spectral inequality leverages a very recent work [36] on the propagation of smallness for elliptic equations on the plane. Second, the resolvent estimate for the high-frequency part is essentially not new and has already been obtained in [19, Proposition 1] for the 1D free Schrödinger operator (i.e. the case of $V = 0$). The proof in [19] utilizes the Logvinenko-Sereda uncertainty principle (see [24, Theorem 2]), which exploits the straightforward geometric feature of the 1D Fourier space. More precisely, in the 1D case, the measure of ξ such that $|\xi| - \lambda \leq O(1)$ is bounded by $O(1)$, independent of the position of λ . Instead, we provide a different, more elementary proof (see Lemma 4.1) of the high-frequency resolvent estimate for 1D Schrödinger operators.

Another delicate issue in the analysis is gluing low-frequency and high-frequency estimates to obtain the full observability estimate. We briefly revisit the classical compactness-uniqueness method that is extensively used in the literature in the compact setting (in the context where the Schrödinger operator has a compact resolvent). The strategy is due to Bardos-Lebeau-Rauch [7] (see also [34, Chapter 6] for a similar abstract framework). Given a general Schrödinger type equation associated with a self-adjoint operator \mathcal{A} on a Hilbert space \mathcal{H} and a control operator \mathcal{C} :

$$i\partial_t u = \mathcal{A}u, \quad u|_{t=0} = u_0 \in \mathcal{H}.$$

We are concerned with the following observability inequality:

$$\|u_0\|_{\mathcal{H}}^2 \leq C \int_0^T \|\mathcal{C}u(t)\|_{\mathcal{H}}^2 dt. \quad (1.4)$$

The compactness-uniqueness strategy can be summarized as follows. Let us first assume that we have already established the high-frequency observability estimate,

$$\|u_0\|_{\mathcal{H}}^2 \leq C \int_0^T \|\mathcal{C}u(t)\|_{\mathcal{H}}^2 dt + C \|\mathcal{A}^{-1}u_0\|_{\mathcal{H}}^2. \quad (1.5)$$

Here, for the sake of clarity, we assume that $0 \notin \text{Spec}(\mathcal{A})$. Given $T > 0$, we define

$$\mathcal{N}_T := \{u_0 \in \mathcal{H} : \mathcal{C}u(t) \equiv 0 \text{ in } L^2((0, T); \mathcal{H})\}.$$

When \mathcal{A} has a compact resolvent (thus $\text{Spec}(\mathcal{A})$ is discrete), the compact embedding and (1.5) ensures that \mathcal{N}_T exists as a finite-dimensional linear subspace of \mathcal{H} . Furthermore, it can be deduced that the restriction of \mathcal{A} on \mathcal{N}_T is a well-defined linear operator. By taking an eigenfunction $\phi \neq 0$ of \mathcal{A} on \mathcal{N}_T , we find that $\mathcal{C}\phi \equiv 0$. Thus, the observability (1.4) results from (1.5) coupled with the unique continuation property of eigenfunctions:

$$\mathcal{A}\phi = \lambda\phi \quad \text{and} \quad \mathcal{C}\phi \equiv 0 \implies \phi \equiv 0.$$

However, when considering the non-compact setting where \mathcal{A} does not have a compact resolvent, we are not aware of any compactness-uniqueness type argument to deduce (1.4) from (1.5). Actually, in [25], this issue does not exist as the problem can be reduced to the compact setting \mathbb{T}^2 , thanks to the periodicity of both the control region and the potential. Here, we prove a quantitative unique continuation property for Schrödinger equations, which facilitates to glue the high-frequency observability with low-frequency estimates. This idea was introduced

in Phung [31] for the Schrödinger equation on bounded domain over \mathbb{R}^d . Following this strategy, we begin by establishing the backward-in-time observability for the associated heat semigroup (see Proposition 3.10). This allows us to deduce a quantitative unique continuation estimate for Schrödinger equations by taking the Fourier–Bros–Iagolnitzer (FBI) transformation in time. We refer to [8, Appendix A] where another quantitative uniqueness-compactness argument was introduced in the compact setting.

Though the passage from the observability of heat-semigroup and the high-frequency observability to the full observability is quite robust in both the compact and non-compact settings, it is worth mentioning that this zigzag path exceeds what is essentially required to ensure the Schrödinger observability estimate. For instance, we consider the half Laplacian $\mathcal{A} = |\nabla|$ on the 1D torus \mathbb{T} . It is a known fact that for any non-empty open set $\omega \subset \mathbb{T}$, the associated heat semigroup is not observable by control operator $\mathcal{C} = \mathbf{1}_\omega$ (see [23, Subsection 2.1]). However, the Schrödinger semigroup $e^{\pm it\mathcal{A}}$ (which is the half-wave operator) is observable by control operator $\mathcal{C} = \mathbf{1}_\omega$ for a certain $T > 0$ (this is a very special case of the exact controllability for the wave equation under the geometric control condition). This discussion raises a natural question of extending the compactness-uniqueness approach, especially concerning the observability estimates of Schrödinger type equations in non-compact domains. We expect that such an extension would be potentially useful when addressing observability estimates for many other Schrödinger type equations on non-compact domains.

The note is organized as follows: First, in Section 2, we state notation and conventions. Second, in Section 3, we derive the propagation of smallness for the 2D elliptic equation and then establish the spectral inequality for the Schrödinger operator H as a consequence. Third, in Section 4, we deduce the high-frequency observability based on the resolvent estimate. Then, in Section 5, we complete the proof of Theorem 1.2 based on the gluing of the low-frequency and high-frequency estimates. Last, for self-contained reason, we give the details of proof for Theorem 3.3 and Proposition 3.10 in Appendix A and B, respectively.

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2. NOTATION AND CONVENTIONS

The Fourier transform of a function $f \in L^1(\mathbb{R})$, denoted by \widehat{f} , is defined as:

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \quad \text{for any } \xi \in \mathbb{R}.$$

Recall that, the Fourier transform defines a linear bounded operator from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Moreover, this operator is an isometry and so there is a unique bounded extension \mathcal{F} defined in all $L^2(\mathbb{R})$. The operator \mathcal{F} is called the Fourier transform in $L^2(\mathbb{R})$. To shorten notation, we denote $\widehat{f} = \mathcal{F}f$ for $f \in L^2(\mathbb{R})$.

Note that, if $u(t)$ is a solution for (1.1), then for any $\theta > 0$, $U(t) = e^{i\theta t}u(t)$ is a solution for

$$i\partial_t U - \partial_x^2 U + V(x)U + \theta U = 0, \quad U|_{t=0} = u_0 \in L^2(\mathbb{R}).$$

This gauge transformation leaves the observability inequality (1.3) invariant. Therefore, without loss of generality, we can assume the potential V is a continuous

bounded function with $V \geq 1$ throughout the article. To shorten notation, for a given potential $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$, we denote

$$\|V\|_\infty = \|V\|_{L^\infty(\mathbb{R})} \quad \text{and} \quad H = -\partial_x^2 + V \quad \text{with domain } H^2(\mathbb{R}).$$

Note also that, the operator H on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ is self-adjoint with non-negative continuous spectrum. Based on the spectral theorem (see for instance [15, Section 2.5]), there exists a spectral measure dm_λ such that

$$F(\sqrt{H}) = \int_0^\infty F(\lambda) dm_\lambda, \quad \text{for any bounded function } F.$$

Moreover, for any bounded functions F and G , we have

$$\left(F(\sqrt{H})f, G(\sqrt{H})f \right)_{L^2(\mathbb{R})} = \int_0^\infty F(\lambda) \overline{G(\lambda)} (dm_\lambda f, f)_{L^2(\mathbb{R})}, \quad \text{for } f \in L^2(\mathbb{R}).$$

Here, dm_λ is the spectral measure of the operator \sqrt{H} . In particular, for $\mu > 0$, the spectral projector Π_μ , associate with the function $F(\lambda) = \mathbf{1}_{\{\lambda \leq \mu\}}$, is defined by

$$\Pi_\mu = \mathbf{1}_{\{\sqrt{H} \leq \mu\}} = \int_0^\mu dm_\lambda.$$

On the other hand, the operator iH generates a unitary group e^{itH} and so the solution for (1.1) can be written as $u(t) = e^{itH} u_0 \in L^2(\mathbb{R})$.

For future reference, we define

$$D_1 = \mathbb{R} \times \left[-\frac{1}{2}, \frac{1}{2} \right], \quad D_2 = \mathbb{R} \times \left[-\frac{3}{2}, \frac{3}{2} \right] \quad \text{and} \quad D_3 = \mathbb{R} \times \left[-\frac{5}{2}, \frac{5}{2} \right].$$

Next, for any $\ell \in \mathbb{Z}$, we define

$$I_{1\ell} = [\ell, \ell + 1], \quad I_{2\ell} = [\ell - 1, \ell + 2] \quad \text{and} \quad I_{3\ell} = [\ell - 2, \ell + 3].$$

Moreover, for any $\ell \in \mathbb{Z}$, we set

$$D_{1\ell} = I_{1\ell} \times \left[-\frac{1}{2}, \frac{1}{2} \right], \quad D_{2\ell} = I_{2\ell} \times \left[-\frac{3}{2}, \frac{3}{2} \right] \quad \text{and} \quad D_{3\ell} = I_{3\ell} \times \left[-\frac{5}{2}, \frac{5}{2} \right].$$

Without loss of generality, we can assume for a given thick set $\Omega \subset \mathbb{R}$, there exists a constant $0 < \zeta < 1$ such that

$$|\Omega \cap [x, x + 1]| \geq \zeta \quad \text{for any } x \in \mathbb{R},$$

that is, the constant $L = 1$ in the Definition 1.1. For a given thick set Ω , we set

$$\Omega_\ell = \Omega \cap I_{1\ell} = \Omega \cap [\ell, \ell + 1], \quad \text{for any } \ell \in \mathbb{Z}.$$

By abuse of notation, in this article, we use the same letters α and C for some small or large constants and state their dependency on other parameters.

For $a \in \mathbb{R}^2$ and $r > 0$, we denote by $B_r(a)$ the ball of \mathbb{R}^2 of center a and of radius r . To simplify notation, we also denote by B_r the ball in \mathbb{R}^2 centred at the origin with radius $r > 0$.

For $\delta > 0$, we denote by \mathcal{H}_δ the δ -dimensional Hausdorff content, that is, for a subset $E \subset \mathbb{R}^2$, we define

$$\mathcal{H}_\delta(E) = \inf \left\{ \sum_{n=1}^\infty r_n^\delta : E \subset \bigcup_{n=1}^\infty B_{r_n}(a_n), \quad a_n \in \mathbb{R}^2 \right\}.$$

Let $\omega \subset B_1 \cap \ell_0$ satisfy $|\omega| > 0$ for some line ℓ_0 in \mathbb{R}^2 . From the definition of 1D Lebesgue measure and δ -Hausdorff content, we have

$$|\omega| = \inf \left\{ \sum_{n=1}^\infty |I_n| : \omega \subset \bigcup_{n=1}^\infty I_n, \quad I_n \subset \ell_0 \right\} = 2\mathcal{H}_1(\omega). \quad (2.1)$$

3. SPECTRAL INEQUALITY

3.1. Propagation of smallness. In this subsection, we introduce the propagation of smallness for solutions of elliptic equations in \mathbb{R}^2 . We start with the following technical lemma for the second-order ODE

$$-\varphi''(x) + V(x)\varphi(x) = 0. \quad (3.1)$$

Lemma 3.1. *Let $I = [a, b]$ be a finite interval and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. There exists a C^2 positive solution φ of (3.1) on the interval I such that*

$$1 \leq \varphi(x) \leq e^{(b-a)^2 \|V\|_\infty}, \quad \text{for any } x \in I.$$

Remark 3.2. The continuity of the potential V is to ensure the existence and regularity of the solution for the second-order ODE (3.1). Such a regular solution can help us to reduce the elliptic equation (3.6) of non-divergence form to divergence form (3.3). This is the only reason why we assume $V \in C(\mathbb{R})$. Actually, we expect Theorem 1.2 still hold true for any potential $V \in L^\infty(\mathbb{R})$.

Proof of Lemma 3.1. Consider the following initial-value problem

$$-\varphi''(x) + V(x)\varphi(x) = 0 \quad \text{with } (\varphi(a), \varphi'(a)) = (1, 0). \quad (3.2)$$

Based on Picard's existence theorem and $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique C^2 solution φ to the initial-value problem (3.2). We now establish the lower and upper bounded estimates for φ on the interval I . First, from (3.2) and $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$, we see that $\varphi(x) \geq 0$ on I . Therefore, for any $x \in I$, we have

$$\varphi'(x) = \int_a^x V(s)\varphi(s)ds \geq 0 \implies \varphi(x) \geq \varphi(a) = 1.$$

Second, using again (3.2) and $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$,

$$\varphi(x) \leq 1 + (b-a)\|V\|_\infty \int_a^x \varphi(s)ds, \quad \text{for any } x \in I.$$

It follows from Grönwall's inequality that

$$\varphi(x) \leq e^{\|V\|_\infty \int_a^x (b-a)ds} \leq e^{(b-a)^2 \|V\|_\infty}, \quad \text{for any } x \in I.$$

Combining the above inequalities, we complete the proof of Lemma 3.1. \square

Second, we consider the 2D elliptic equation in divergence form

$$\nabla \cdot (A(z)\nabla\phi(z)) = 0 \quad \text{in } B_4. \quad (3.3)$$

Here $z = (x, y) \in \mathbb{R}^2$, and the real symmetric matrix $A(z) = (a_{jk}(z))_{2 \times 2}$ is elliptic, that is, there exists a constant $\Lambda > 1$ such that

$$\Lambda^{-1} \leq \xi^T A(z)\xi \leq \Lambda, \quad \text{for any } \xi \in B_1 \text{ and } z \in B_4. \quad (3.4)$$

We recall the following propagation of smallness for solutions to (3.3) from [36].

Theorem 3.3 ([36]). *Let $\omega \subset B_1 \cap \ell_0$ satisfy $|\omega| > 0$ for some line ℓ_0 in \mathbb{R}^2 with the normal vector \mathbf{e}_0 . There exist some constants $\alpha = \alpha(\Lambda, |\omega|) \in (0, 1)$ and $C = C(\Lambda, |\omega|) > 0$, depending only on Λ and $|\omega|$, such that for any real-valued H_{loc}^2 solution ϕ of (3.3) with $A\nabla\phi \cdot \mathbf{e}_0 = 0$ on $B_1 \cap \ell_0$, we have*

$$\sup_{B_1} |\phi| \leq C \left(\sup_{\omega} |\phi|^\alpha \right) \left(\sup_{B_2} |\phi|^{1-\alpha} \right). \quad (3.5)$$

Remark 3.4. By scaling and translation, the interpolation inequality (3.5) remains true if we replace B_1, B_2 by balls $B_r(a), B_{2r}(a)$ for H_{loc}^2 solutions to (3.3) in $B_{4r}(a)$, where $a \in \mathbb{R}^2$ and $r > 0$, and the constant C depends only on $\Lambda, |\omega|$ and $r > 0$.

Remark 3.5. We mention here that Theorem 3.3 is just a special version of [36, Theorem 1.1]. Actually, [36, Theorem 1.1] shows the propagation of smallness for solutions from any $\omega \subset B_1$ lying on a line with $\mathcal{H}_\delta(\omega) > 0$. Here $\mathcal{H}_\delta(\omega)$ means δ -dimensional Hausdorff content of ω . For the sake of completeness and the readers' convenience, the sketch of the proof for Theorem 3.3 is given in Appendix A.

Last, we introduce the L^2 -propagation of smallness for H_{loc}^2 solution of the following 2D elliptic equation in nondivergence form

$$-\Delta\phi(z) + V(x)\phi(z) = 0 \quad \text{with} \quad \partial_y\phi|_{y=0} = 0. \quad (3.6)$$

Following [30, Section 2], we see that the equation (3.6) in nondivergence form can be reduced to the divergence form (3.3). More precisely, for the given potential $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$, we first consider the positive C^2 solution φ constructed in Lemma 3.1 for the second-order ODE (3.1). Then, by an elementary computation, on $[a, b] \times \mathbb{R}$, we deduce that

$$\begin{cases} \partial_y\phi|_{y=0} = 0 \implies \partial_y \left(\frac{\phi(z)}{\varphi(x)} \right) \Big|_{y=0} = 0, \\ -\Delta\phi(z) + V(x)\phi(z) = 0 \implies \nabla \cdot \left(\varphi^2(x) \nabla \left(\frac{\phi(z)}{\varphi(x)} \right) \right) = 0. \end{cases} \quad (3.7)$$

Combining the above reduction with Theorem 3.3, we now establish the following L^2 -propagation of smallness for the H_{loc}^2 solution of 2D elliptic equation (3.6). The proof is inspired by the techniques developed in Burq-Moyano [10, Section 2].

Proposition 3.6. *Let $\ell \in \mathbb{Z}$ and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. Then for any measurable set $\omega \subset I_{1\ell}$ with $|\omega| > 0$ and any real-valued H_{loc}^2 solution ϕ of (3.6), we have*

$$\|\phi\|_{L^2(D_{1\ell})} \leq C \|\phi\|_{L^2(\omega)}^\alpha \left(\sup_{D_{2\ell}} |\phi|^{1-\alpha} \right),$$

where $\alpha = \alpha(V, |\omega|) \in (0, 1)$ and $C = C(V, |\omega|) > 0$ depend only on V and $|\omega|$.

Remark 3.7. In the spirit of Burq-Moyano [10] (see also [9] in a similar context), the L^2 -propagation of smallness can be used to establish the spectral inequality which can help us to obtain the observability estimate for 1D homogenous heat equation with a potential (see more details in §3.2).

Proof of Proposition 3.6. Step 1. L^∞ -propagation of smallness. We claim that, for any measurable set $\omega \subset I_{1\ell}$ with $|\omega| > 0$ and any H_{loc}^2 solution ϕ of (3.6),

$$\sup_{D_{1\ell}} |\phi| \leq C \left(\sup_{\omega} |\phi|^\alpha \right) \left(\sup_{D_{2\ell}} |\phi|^{1-\alpha} \right), \quad (3.8)$$

where $\alpha = \alpha(V, |\omega|) \in (0, 1)$ and $C = C(V, |\omega|) > 0$ depend only on V and $|\omega|$. Indeed, we first consider the positive C^2 solution φ constructed in Lemma 3.1 for the second-order ODE (3.1) on $[\ell - 5, \ell + 5]$. Then, from the reduction (3.7), we can apply Theorem 3.3 to (ϕ/φ) with $\ell_0 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. Then, according to scaling and translation, there exist some constants $\alpha = \alpha(\varphi, |\omega|) \in (0, 1)$ and $C = C(\varphi, |\omega|) > 0$, depending only on $|\omega|$ and the lower and upper bounds of φ on $[\ell - 5, \ell + 5]$, such that for any H_{loc}^2 solution ϕ of (3.6), we have

$$\sup_{B_{1\ell}} \left| \frac{\phi}{\varphi} \right| \leq C \left(\sup_{\omega} \left| \frac{\phi}{\varphi} \right|^\alpha \right) \left(\sup_{B_{2\ell}} \left| \frac{\phi}{\varphi} \right|^{1-\alpha} \right),$$

where $B_{1\ell}$ and $B_{2\ell}$ are defined by

$$B_{1\ell} = B_{\frac{\sqrt{2}}{2}} \left(\left(\ell + \frac{1}{2}, 0 \right) \right) \quad \text{and} \quad B_{2\ell} = B_{\sqrt{2}} \left(\left(\ell + \frac{1}{2}, 0 \right) \right).$$

It follows directly from Lemma 3.1 that

$$\sup_{B_{1\ell}} \left| \frac{\phi}{\varphi} \right| \leq C_1 \left(\sup_{\omega} |\phi|^\alpha \right) \left(\sup_{B_{2\ell}} |\phi|^{1-\alpha} \right).$$

where $C_1 = C(V, |\omega|) > 0$ is a constant depending only on V and $|\omega|$.

On the other hand, from the definition of $D_{1\ell}$, $D_{2\ell}$, $B_{1\ell}$ and $B_{2\ell}$, we observe that

$$D_{1\ell} \subset B_{1\ell} \subset B_{2\ell} \subset D_{2\ell}, \quad \text{for all } \ell \in \mathbb{Z}.$$

Therefore, using again Lemma 3.1, we conclude that

$$\begin{aligned} \sup_{D_{1\ell}} |\phi| &\leq \left(\sup_{I_{1\ell}} |\varphi| \right) \left(\sup_{B_{1\ell}} \left| \frac{\phi}{\varphi} \right| \right) \\ &\leq C_1 e^{100\|V\|_\infty} \left(\sup_{\omega} |\phi|^\alpha \right) \left(\sup_{D_{2\ell}} |\phi|^{1-\alpha} \right). \end{aligned}$$

This completes the proof of (3.8).

Step 2. Replacing L^∞ norm with L^2 norm. From (3.8), there exists some constants $\alpha_1 = \alpha_1(\Lambda, |\omega|) \in (0, 1)$ and $C_2 = C_2(\Lambda, |\omega|) > 0$, depending only on Λ and $|\omega|$, such that for any $\tilde{\omega} \subset \omega$ with $\frac{1}{2}|\omega| \leq |\tilde{\omega}| \leq |\omega|$, we have

$$\sup_{D_{1\ell}} |\phi| \leq C_2 \left(\sup_{\tilde{\omega}} |\phi|^{\alpha_1} \right) \left(\sup_{D_{2\ell}} |\phi|^{1-\alpha_1} \right). \quad (3.9)$$

Assume $\phi \not\equiv 0$ on $D_{1\ell}$. Let $0 < \varepsilon < 1$ be a small constant to be chosen later. Define

$$\delta = \varepsilon \left(\left(\sup_{D_{1\ell}} |\phi|^{\frac{1}{\alpha_1}} \right) / \left(\sup_{D_{2\ell}} |\phi|^{\frac{1}{\alpha_1}-1} \right) \right) \quad \text{and} \quad \omega_\delta = \{x \in \omega : |\phi(x, 0)| \leq \delta\} \subset \omega.$$

We claim that, there exists a constant $\varepsilon = \varepsilon(V, |\omega|)$, depending only on V and $|\omega|$, such that $|\omega_\delta| \leq \frac{1}{2}|\omega|$. Indeed, otherwise, the inequality (3.9) would hold with $\tilde{\omega}$ replaced by ω_δ (with same constants α_1 and C_2). Hence, from the definition of ω_δ ,

$$\sup_{D_{1\ell}} |\phi| \leq C_2 \left(\sup_{\omega_\delta} |\phi|^{\alpha_1} \right) \left(\sup_{D_{2\ell}} |\phi|^{1-\alpha_1} \right) \leq C_2 \delta^{\alpha_1} \left(\sup_{D_{2\ell}} |\phi|^{1-\alpha_1} \right) \leq C_2 \varepsilon^{\alpha_1} \sup_{D_{1\ell}} |\phi|.$$

This is a contradiction with $\phi \not\equiv 0$ on $D_{1\ell}$ for ε small enough such that $C_2 \varepsilon^{\alpha_1} < 1$, and so we have $|\omega_\delta| \leq \frac{1}{2}|\omega|$ for a constant $\varepsilon = \varepsilon(V, |\omega|) \in (0, 1)$ which depends only on V and $|\omega|$. In conclusion, using again the definition of ω_δ , we obtain

$$\|\phi\|_{L^2(\omega)}^2 \geq \|\phi\|_{L^2(\omega \setminus \omega_\delta)}^2 \geq \frac{\delta^2}{2} |\omega| \geq \frac{1}{2} \varepsilon^2 |\omega| \left(\left(\sup_{D_{1\ell}} |\phi|^{\frac{2}{\alpha_1}} \right) / \left(\sup_{D_{2\ell}} |\phi|^{\frac{2}{\alpha_1}-2} \right) \right),$$

which implies

$$\|\phi\|_{L^2(D_{1\ell})} \leq \sup_{D_{1\ell}} |\phi| \leq 2(\varepsilon^2 |\omega|)^{-\frac{\alpha_1}{2}} \|\phi\|_{L^2(\omega)}^{\alpha_1} \left(\sup_{D_{2\ell}} |\phi|^{1-\alpha_1} \right).$$

The proof of Proposition 3.6 is complete. \square

3.2. Spectral inequality. In this subsection, we deduce the spectral inequality and then establish the observability estimate for the 1D heat equation as a consequence. Following Burq-Moyano [10], we first establish the spectral inequality for the low-frequency part from L^2 -propagation of smallness.

Lemma 3.8. *Let Ω be a $(1, \zeta)$ -thick set and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. Then there exists a constant $C = C(V, \zeta) > 0$, depending only on V and ζ , such that for any $\mu > 0$ and any $f \in L^2(\mathbb{R})$, we have*

$$\|\Pi_\mu f\|_{L^2(\mathbb{R})} \leq C e^{3\mu} \|\Pi_\mu f\|_{L^2(\Omega)}.$$

Proof. Without loss of generality, we assume that $f \in L^2(\mathbb{R})$ is real-valued. For $\mu > 0$, $(x, y) \in D_3$ and $f \in L^2(\mathbb{R})$, we set

$$F_\mu(x, y) = \int_0^\mu \cosh(y\lambda) dm_\lambda f(x).$$

Using the fact that $(\cosh s)'' = \cosh s$ and the definition of dm_λ ,

$$\partial_y^2 F_\mu = H F_\mu = \int_0^\mu \lambda^2 \cosh(y\lambda) dm_\lambda f.$$

It follows that

$$-\Delta F_\mu + V(x)F_\mu = -\partial_y^2 F_\mu + H F_\mu = 0.$$

On the other hand, for the case of $y = 0$, using the fact that $(\cosh s)' = \sinh s$,

$$\partial_y F_\mu|_{y=0} = \int_0^\mu \lambda \sinh(0\lambda) dm_\lambda f = 0.$$

Hence, the function F_μ is a H_{loc}^2 solution for (3.6) with $F_\mu(x, 0) = \Pi_\mu f(x)$ on \mathbb{R} . We now apply Proposition 3.6 to F_μ , and thus, for any $\ell \in \mathbb{Z}$, we obtain

$$\|F_\mu\|_{L^2(D_{1\ell})} \leq C \|\Pi_\mu f\|_{L^2(\Omega_\ell)}^\alpha \left(\sup_{D_{2\ell}} |F_\mu|^{1-\alpha} \right).$$

where $C = (V, \zeta) > 0$ is a constant depending only on V and ζ . Therefore, from Young's inequality for products, for any ε small enough, we have

$$\|F_\mu\|_{L^2(D_{1\ell})}^2 \leq \frac{C_1}{\varepsilon} \|\Pi_\mu f\|_{L^2(\Omega_\ell)}^2 + C_1 \varepsilon \|F_\mu\|_{L^\infty(D_{2\ell})}^2,$$

where $C_1 = C_1(V, \zeta) > 0$ depends only on V and ζ . Summing over $\ell \in \mathbb{Z}$, we find,

$$\|F_\mu\|_{L^2(D_1)}^2 \leq \frac{C_1}{\varepsilon} \|\Pi_\mu f\|_{L^2(\Omega)}^2 + C_1 \varepsilon \sum_{\ell \in \mathbb{Z}} \|F_\mu\|_{L^\infty(D_{2\ell})}^2. \quad (3.10)$$

Then we fix suitable cut-off functions. Let $\chi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^2 function such that

$$\chi \equiv 1 \quad \text{on} \quad \left[-\frac{3}{2}, \frac{3}{2}\right]^2 \quad \text{and} \quad \text{supp} \chi \subset \left[-\frac{5}{2}, \frac{5}{2}\right]^2.$$

For any $\ell \in \mathbb{Z}$, we set

$$\chi_\ell(x, y) = \chi\left(x - \ell - \frac{1}{2}, y\right) \implies \chi_\ell \equiv 1 \quad \text{on} \quad D_{2\ell} \quad \text{and} \quad \text{supp} \chi_\ell \subset D_{3\ell}.$$

Therefore, using the 2D Sobolev embedding theorem, we deduce that

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}} \|F_\mu\|_{L^\infty(D_{2\ell})}^2 &\leq \pi \sum_{\ell \in \mathbb{Z}} \|\chi_\ell F_\mu\|_{H^2(\mathbb{R}^2)}^2 \\ &\leq C_2 \|F_\mu\|_{H^2(D_3)}^2 \leq C_3 (1 + \mu^4) \|F_\mu\|_{L^2(D_3)}^2, \end{aligned} \quad (3.11)$$

where $C_2 > 0$ and $C_3 > 0$ depend only on V and the choice of χ . Combining (3.10) and (3.11), for any ε small enough, we have

$$\|F_\mu\|_{L^2(D_1)}^2 \leq \frac{C_1}{\varepsilon} \|\Pi_\mu f\|_{L^2(\Omega)}^2 + C_1 C_3 \varepsilon (1 + \mu^4) \|F_\mu\|_{L^2(D_3)}^2.$$

On the other hand, from the definition of the spectral projector Π_μ ,

$$\begin{aligned} \|F_\mu\|_{L^2(D_1)}^2 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^\mu \cosh^2(y\lambda) (dm_\lambda f, f)_{L^2(\mathbb{R})} dy \geq \|\Pi_\mu f\|_{L^2(\mathbb{R})}^2, \\ \|F_\mu\|_{L^2(D_3)}^2 &= \int_{-\frac{5}{2}}^{\frac{5}{2}} \int_0^\mu \cosh^2(y\lambda) (dm_\lambda f, f)_{L^2(\mathbb{R})} dy \leq 5e^{5\mu} \|\Pi_\mu f\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Gathering the above three inequalities, we obtain

$$\|\Pi_\mu f\|_{L^2(\mathbb{R})}^2 \leq \frac{C_1}{\varepsilon} \|\Pi_\mu f\|_{L^2(\Omega)}^2 + 5C_1 C_3 \varepsilon e^{5\mu} (1 + \mu^4) \|\Pi_\mu f\|_{L^2(\mathbb{R})}^2,$$

which completes the proof of Lemma 3.8 for taking ε small enough. \square

Remark 3.9. The assumption $V \geq 1$ can be removed in Lemma 3.8, thus extending the result of Lebeau-Moyano [27] to bounded potentials. Here, we consider the spectral measure for H instead of \sqrt{H} . The additional argument can be sketched as follows. Although $V(x)$ may take negative values, by Sturm-Liouville theory, we are still able to construct $\varphi(x)$, bounded from below and above as in Lemma 3.1 by restricting the size of interval I such that $|I| \leq 2\sigma_0$, where $\sigma_0 = 1/4\pi\sqrt{\|V\|_{L^\infty}}$. Consequently, the analogue of Proposition 3.6 remains true if we replace $D_{1\ell}, D_{2\ell}$ with boxes of the form $I_1 \times [-\frac{1}{2}, \frac{1}{2}]$, $I_2 \times [-\frac{3}{2}, \frac{3}{2}]$ such that $I_1 = [x_0, x_0 + \sigma_0]$, $I_2 = [x_0 - \sigma_0, x_0 + 2\sigma_0]$. By dividing the interval $[\ell, \ell + 1]$ into at most $\lfloor 1/\sigma_0 \rfloor + 1$ intervals of size σ_0 and using the pigeonhole principle, finally we are able to obtain the same statement as Proposition 3.6.

Note that, from Lemma 3.8 and a standard argument, we directly have the observability estimate for 1D homogeneous heat equation (not necessarily real-valued)

$$\partial_t u - \partial_x^2 u + V(x)u = 0, \quad u|_{t=0} = u_0 \in L^2(\mathbb{R}). \quad (3.12)$$

Recall that, the operator $-H$ generates a semigroup e^{-tH} and so the solution for (3.12) can be written as $u(t) = e^{-tH}u_0 \in L^2(\mathbb{R})$.

Proposition 3.10. *Let Ω be a $(1, \zeta)$ -thick set and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. Then there exists a constant $C = C(V, \zeta) > 0$, depending only on V and ζ , such that for any $T > 0$ and any solution u of (3.12) we have*

$$\|u(T)\|_{L^2(\mathbb{R})}^2 \leq C e^{\frac{C}{T}} \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt.$$

Proof. For the sake of completeness and the readers' convenience, the details of the proof for Proposition 3.10 is given in Appendix B. \square

Thanks to the above Proposition, we obtain the observability estimate for the 1D inhomogeneous heat equation

$$\partial_t u - \partial_x^2 u + V(x)u = F \in L^2((0, \infty) : L^2(\mathbb{R})), \quad u|_{t=0} = u_0 \in H^2(\mathbb{R}). \quad (3.13)$$

Corollary 3.11. *Let Ω be a $(1, \zeta)$ -thick set and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. Then there exists a constant $C = C(V, \zeta) > 0$, depending only on V and ζ , such that for any $T > 0$ and any solution u of (3.13) we have*

$$\|u(T)\|_{L^2(\mathbb{R})}^2 \leq C e^{\frac{C}{T}} \int_0^T \left(\|Hu(t)\|_{L^2(\Omega)}^2 + \|F(t)\|_{L^2(\mathbb{R})}^2 \right) dt.$$

Proof. We decompose the solution u as

$$u(t, x) = u_1(t, x) + u_2(t, x), \quad \text{on } [0, T] \times \mathbb{R},$$

where u_1 and u_2 are the solutions for the following 1D homogeneous or inhomogeneous heat equations

$$\begin{cases} \partial_t u_1 - \partial_x^2 u_1 + V(x)u_1 = 0, & u_1|_{t=0} = u_0, \\ \partial_t u_2 - \partial_x^2 u_2 + V(x)u_2 = F, & u_2|_{t=0} = 0. \end{cases} \quad (3.14)$$

First, from $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$, we have

$$H \geq \text{Id} \implies \|Hu_1\|_{L^2(\mathbb{R})} \geq \|u_1\|_{L^2(\mathbb{R})}.$$

It follows from Proposition 3.10 that

$$\|u_1(T)\|_{L^2(\mathbb{R})}^2 \leq \|Hu_1(T)\|_{L^2(\mathbb{R})}^2 \leq Ce^{\frac{C}{T}} \int_0^T \|Hu_1(t)\|_{L^2(\Omega)}^2 dt,$$

where $C = C(V, \zeta) > 0$ depends only on V and ζ . Note that, from (3.14), the term Hu_1 can be rewritten as $Hu_1 = Hu + \partial_t u_2 - F$ and so we obtain

$$\begin{aligned} \|u_1(T)\|_{L^2(\mathbb{R})}^2 &\leq 3Ce^{\frac{C}{T}} \int_0^T \|Hu(t)\|_{L^2(\Omega)}^2 dt \\ &\quad + 3Ce^{\frac{C}{T}} \int_0^T \left(\|\partial_t u_2(t)\|_{L^2(\mathbb{R})}^2 + \|F(t)\|_{L^2(\mathbb{R})}^2 \right) dt. \end{aligned} \quad (3.15)$$

Second, using again (3.14), we directly have

$$(\partial_t u_2)^2 + \frac{1}{2} \partial_t \left((\partial_x u_2)^2 + V(x)u_2^2 \right) - \partial_x \left((\partial_t u_2)(\partial_x u_2) \right) = F \partial_t u_2.$$

Integrating the above identities over $[0, T] \times \mathbb{R}$, and then using Cauchy-Schwarz inequality, we see that

$$\|u_2(T)\|_{L^2(\mathbb{R})}^2 + \int_0^T \|\partial_t u_2(t)\|_{L^2(\mathbb{R})}^2 dt \leq \int_0^T \|F(t)\|_{L^2(\mathbb{R})}^2 dt. \quad (3.16)$$

Here, we used the fact that $H \geq \text{Id}$ and the zero initial condition of u_2 in $H^2(\mathbb{R})$. Combining (3.15) and (3.16) with Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|u_1(T)\|_{L^2(\mathbb{R})}^2 &\leq 2 \left(\|u_1(T)\|_{L^2(\mathbb{R})}^2 + \|u_2(T)\|_{L^2(\mathbb{R})}^2 \right) \\ &\leq (6C + 1)e^{\frac{C}{T}} \int_0^T \left(\|Hu(t)\|_{L^2(\Omega)}^2 + 2\|F(t)\|_{L^2(\mathbb{R})}^2 \right) dt. \end{aligned}$$

The proof of Proposition 3.10 is complete. \square

For the notational convenience of introducing the FBI transformation later, we can reverse the time t to $T - t$ to obtain the following observability estimate for the 1D inhomogeneous backward heat equation

$$\partial_t u + \partial_x^2 u - V(x)u = F \in L^2((0, \infty) : L^2(\mathbb{R})), \quad u|_{t=T} = u_T \in H^2(\mathbb{R}). \quad (3.17)$$

Corollary 3.12. *Let Ω be a $(1, \zeta)$ -thick set and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. Then there exists a constant $C = C(V, \zeta) > 0$, depending only on V and ζ , such that for any $T > 0$ and any solution u of (3.17) we have*

$$\|u(0)\|_{L^2(\mathbb{R})}^2 \leq Ce^{\frac{C}{T}} \int_0^T \left(\|Hu(t)\|_{L^2(\Omega)}^2 + \|F(t)\|_{L^2(\mathbb{R})}^2 \right) dt.$$

4. RESOLVENT ESTIMATE

In this section, we recall the resolvent estimate for 1D Schrödinger operator H and then deduce the observability estimate for the high-frequency part. Recall that, the resolvent estimate for operator $H = -\partial_x^2$ was first given in [19, Proposition 1].

Lemma 4.1 ([19]). *Let Ω be a $(1, \zeta)$ -thick set and let $V \in L^\infty(\mathbb{R})$. Then there exist some constants $\mu_0 = \mu_0(V, \zeta)$ and $C = C(V, \zeta)$, depending only on V and ζ , such that for any $\mu > \mu_0$ and any $f \in H^2(\mathbb{R})$, we have*

$$\|f\|_{L^2(\mathbb{R})}^2 \leq \frac{C}{\mu} \|(H - \mu)f\|_{L^2(\mathbb{R})}^2 + C\|f\|_{L^2(\Omega)}^2.$$

For the reader's convenience, we will provide a proof of Lemma 4.1, different from the one given in [19]. We first need the following technical estimate.

Lemma 4.2. *Let $0 < \zeta < 1$. Then there exists a constant $c = c(\zeta) > 0$, depending only on ζ , such that for any measurable set $\omega \subset (0, 1)$ with $\zeta \leq |\omega| \leq 1$ and any $\lambda > 1$, we have*

$$\inf_{x_0 \in \mathbb{R}} \int_{\omega} \cos^2(\lambda(x - x_0)) dx \geq c(\zeta). \quad (4.1)$$

Remark 4.3. We mention here that Lemma 4.2 is not a direct consequence of the Riemann-Lebesgue lemma and the proof will require more quantitative analysis, since the lower bound in estimate (4.1) only depends on the size of the measurable set ω which is different from the statement of Riemann-Lebesgue lemma.

Proof of Lemma 4.2. Using the structure of a measurable set in \mathbb{R} , for any $0 < \varepsilon < 1$, there exists a finite union of disjoint open intervals

$$U = \bigcup_{n=1}^N I_n \quad \text{with} \quad I_n = (a_n, b_n) \subset (0, 1) \quad \text{for any } n \in \{1, \dots, N\},$$

such that $|\omega \setminus U| + |U \setminus \omega| < \varepsilon$ and $|U| \geq \zeta - \varepsilon$. It follows that

$$\int_{\omega} \cos^2(\lambda(x - x_0)) dx \geq \sum_{n=1}^N \int_{a_n}^{b_n} \cos^2(\lambda(x - x_0)) dx - \varepsilon. \quad (4.2)$$

By an elementary computation, on any finite interval $I_n = (a_n, b_n)$, we have

$$\begin{aligned} & \int_{a_n}^{b_n} \cos^2(\lambda(x - x_0)) dx \\ &= \frac{1}{2} \int_{a_n}^{b_n} (1 + \cos(2\lambda(x - x_0))) dx \\ &= \frac{1}{2} \left((b_n - a_n) + \frac{1}{\lambda} \sin(\lambda(b_n - a_n)) \cos(\lambda(b_n + a_n - 2x_0)) \right). \end{aligned} \quad (4.3)$$

Let $0 < \delta \ll 1$ be a small constant to be chosen later. For any $m \in \mathbb{Z}$, we set

$$J_{m,\delta} = \left(x_0 + \frac{m\pi}{2\lambda} - \frac{\delta}{2\lambda}, x_0 + \frac{m\pi}{2\lambda} + \frac{\delta}{2\lambda} \right) \cap (0, 1) \quad \text{and} \quad S = \bigcup_{m \in \mathbb{Z}} J_{m,\delta}.$$

Note that, $\{J_{m,\delta}\}_{m \in \mathbb{Z}}$ are disjoint sets. For further reference, we consider

$$\mathcal{A} = \{m \in \mathbb{Z} : J_{m,\delta} \neq \emptyset\}, \quad \text{and thus we have} \quad \#\mathcal{A} \leq 3\lambda.$$

We split the index set $\{1, \dots, N\}$ to the following three cases according to the size and position of the finite interval I_n and then establish estimate for each case.

Case 1. Let $(b_n - a_n) \geq \frac{\delta}{\lambda}$. From the fact that $\frac{\sin x}{x}$ is decreasing on $[\delta, \pi)$, for $0 < \delta \ll 1$,

$$\sup_{[\delta, \infty)} \left| \frac{\sin x}{x} \right| \leq \frac{\sin \delta}{\delta} \implies \frac{1}{\lambda} |\sin(\lambda(b_n - a_n))| \leq \frac{\sin \delta}{\delta} (b_n - a_n).$$

Based on the above estimate and (4.3), we obtain

$$\int_{a_n}^{b_n} \cos^2(\lambda(x - x_0)) dx \geq \frac{1}{2} \left(1 - \frac{\sin \delta}{\delta} \right) (b_n - a_n). \quad (4.4)$$

Case 2. Let $0 < (b_n - a_n) < \frac{\delta}{\lambda}$ with $S \cap I_n = \emptyset$. First, from the definition of S and $S \cap I_n = \emptyset$, there exists $m \in \mathbb{Z}$ such that

$$I_n \subset \left(x_0 + \frac{m\pi}{2\lambda} + \frac{\delta}{2\lambda}, x_0 + \frac{(m+1)\pi}{2\lambda} - \frac{\delta}{2\lambda} \right),$$

and thus, from $\sin^2 x + \cos^2 x = 1$, we find

$$\text{dist}(\lambda(b_n + a_n - 2x_0), \pi\mathbb{Z}) \geq \delta \implies |\cos(\lambda(b_n + a_n - 2x_0))| \leq \sqrt{1 - \sin^2 \delta}.$$

Therefore, using again (4.3) and $|\sin x| \leq |x|$, we obtain

$$\int_{a_n}^{b_n} \cos^2(\lambda(x-x_0))dx \geq \frac{1}{2} \left(1 - \sqrt{1 - \sin^2 \delta}\right) (b_n - a_n). \quad (4.5)$$

Case 3. We now consider the last case, that is, the case of $0 < (b_n - a_n) < \frac{\delta}{\lambda}$ with $S \cap I_n \neq \emptyset$. To simplify notation, we denote

$$\mathcal{B} = \left\{ n \in \{1, \dots, N\} : 0 < (b_n - a_n) < \frac{\delta}{\lambda} \text{ with } S \cap I_n \neq \emptyset \right\}.$$

Note that, for any $n \in \mathcal{B}$, there exists $m \in \mathcal{A}$ such that

$$I_n \subset J_{m,3\delta} \quad \text{where} \quad J_{m,3\delta} = \left(x_0 + \frac{m\pi}{2\lambda} - \frac{3\delta}{2\lambda}, x_0 + \frac{m\pi}{2\lambda} + \frac{3\delta}{2\lambda} \right) \cap (0, 1).$$

Next, from $0 < \delta \ll 1$, for any $(m, m') \in \mathbb{Z}^2$ with $m \neq m'$, we find $J_{m,3\delta} \cap J_{m',3\delta} = \emptyset$. Therefore, using the fact that $\{I_n\}_{n=1}^N$ are disjoint intervals and $\#\mathcal{A} \leq 3\lambda$, we obtain

$$\sum_{n \in \mathcal{B}} (b_n - a_n) = \left| \bigcup_{n \in \mathcal{B}} I_n \right| \leq \left| \bigcup_{m \in \mathcal{A}} J_{m,3\delta} \right| \leq \frac{3\delta}{\lambda} \#\mathcal{A} \leq 9\delta. \quad (4.6)$$

Combining (4.2), (4.4), (4.5), (4.6) with $|U| \geq \zeta - \varepsilon$, we conclude that

$$\inf_{x_0 \in \mathbb{R}} \int_{\omega} \cos^2(\lambda(x-x_0))dx \geq \frac{1}{2} \left(1 - \max\left(\frac{\sin \delta}{\delta}, \sqrt{1 - \sin^2 \delta}\right)\right) (\zeta - \varepsilon - 9\delta) - \varepsilon.$$

We see that (4.1) follows from the above estimate for ε and δ small enough. \square

We now give an alternative proof of Lemma 4.1 for the reader's convenience.

Proof of Lemma 4.1. Step 1. Estimate for the flat case. Let $V = 0$ and $\ell \in \mathbb{Z}$. For $\mu > 1$, we denote

$$F = -\partial_x^2 f - \mu f \quad \text{on } \mathbb{R}.$$

For any $s \in I_{1\ell}$, the function f can be expressed by

$$f(x) = \cos(\sqrt{\mu}(x-s))f(s) + \frac{\sin(\sqrt{\mu}(x-s))}{\sqrt{\mu}}f'(s) - \int_s^x \frac{\sin(\sqrt{\mu}(x-y))}{\sqrt{\mu}}F(y)dy.$$

Note that, in the above identity, the sum of the first two terms can be rewritten as

$$\cos(\sqrt{\mu}(x-s))f(s) + \frac{\sin(\sqrt{\mu}(x-s))}{\sqrt{\mu}}f'(s) = r \cos\left(\sqrt{\mu}\left(x-s - \frac{\theta}{\sqrt{\mu}}\right)\right),$$

where

$$\theta \in [0, 2\pi) \quad \text{and} \quad r = \sqrt{|f(s)|^2 + \mu^{-1}|f'(s)|^2}.$$

Therefore, from Lemma 4.2 and $\zeta \leq |\Omega_\ell| \leq 1$, there exists a constant $c = c(\zeta) > 0$, depending only on ζ , such that

$$c|f(s)|^2 \leq \left\| \cos(\sqrt{\mu}(x-s))f(s) + \frac{\sin(\sqrt{\mu}(x-s))}{\sqrt{\mu}}f'(s) \right\|_{L^2(\Omega_\ell)}^2.$$

Combining the above estimate with the explanation of $f(x)$, we find

$$\begin{aligned} c|f(s)|^2 &\leq \left\| f(x) + \int_s^x \frac{\sin(\sqrt{\mu}(x-y))}{\sqrt{\mu}}F(y)dy \right\|_{L^2(\Omega_\ell)}^2 \\ &\leq 2\|f\|_{L^2(\Omega_\ell)}^2 + \frac{2}{\mu} \left\| \int_s^x |F(y)|dy \right\|_{L^2(\Omega_\ell)}^2 \\ &\leq 2\|f\|_{L^2(\Omega_\ell)}^2 + \frac{2}{\mu} \|F\|_{L^2(I_{1\ell})}^2. \end{aligned}$$

Integrating the above estimate with respect to variable s over $I_{1\ell}$ and then summing over $\ell \in \mathbb{Z}$, we conclude that

$$\|f\|_{L^2(\mathbb{R})}^2 \leq \frac{C}{\mu} \|(-\partial_x^2 - \mu)f\|_{L^2(\mathbb{R})}^2 + C\|f\|_{L^2(\Omega)}^2, \quad (4.7)$$

where $C = C(\zeta) > 0$ is a constant depending only on ζ .

Step 2. Conclusion. Note that, from (4.7) and $V \in L^\infty(\mathbb{R})$, we obtain

$$\begin{aligned} \|f\|_{L^2(\mathbb{R})}^2 &\leq \frac{C}{\mu} \|(-\partial_x^2 - \mu)f\|_{L^2(\mathbb{R})}^2 + C\|f\|_{L^2(\Omega)}^2 \\ &\leq \frac{C}{\mu} \|(H - \mu)f\|_{L^2(\mathbb{R})}^2 + \frac{C}{\mu} \|V\|_\infty^2 \|f\|_{L^2(\mathbb{R})}^2 + C\|f\|_{L^2(\Omega)}^2, \end{aligned}$$

which completes the proof of Lemma 4.1 by taking μ large enough. \square

Combining the above resolvent estimate with an argument in [12, Section 3], we obtain the following observability inequality for the high-frequency part.

Corollary 4.4. *Let Ω be a $(1, \zeta)$ -thick set and let $V \in L^\infty(\mathbb{R})$ with $V \geq 1$. Then there exist some constants $\mu_1 = \mu_1(V, \zeta)$ and $C = C(V, \zeta)$, depending only on V and ζ , such that for any $T > 0$, $\mu > \mu_1(1 + T^{-\frac{1}{2}})$ and $f \in L^2(\mathbb{R})$, we have*

$$\|(1 - \Pi_\mu)f\|_{L^2(\mathbb{R})}^2 \leq \frac{C}{T} \int_0^T \|e^{itH}(1 - \Pi_\mu)f\|_{L^2(\Omega)}^2 dt.$$

Proof. Let $f \in H^2(\mathbb{R})$ and let F be the solution of the 1D Schrödinger equation

$$i\partial_t F - \partial_x^2 F + V(x)F = 0 \quad \text{with } F|_{t=0} = (1 - \Pi_\mu)f \in H^2(\mathbb{R}).$$

We fix a cut-off C^2 function $\chi : \mathbb{R} \rightarrow [0, 1]$ satisfying

$$\chi \equiv 1 \text{ on } \left[\frac{1}{4}, \frac{3}{4}\right], \quad \text{supp}\chi \subset [0, 1] \quad \text{and} \quad \chi' \in [-5, 5].$$

For time $T > 0$, we consider a new function

$$\Psi(t, x) = \chi\left(\frac{t}{T}\right) F(t, x) \implies i\partial_t \Psi - \partial_x^2 \Psi + V(x)\Psi = \frac{i}{T} \chi'\left(\frac{t}{T}\right) F.$$

Taking the Fourier transform in the above equation with respect to t ,

$$(H - \xi)\widehat{\Psi}(\xi, x) = \frac{i}{T} \mathcal{F}_{t \rightarrow \xi} \left(\chi'\left(\frac{t}{T}\right) F \right) (\xi, x). \quad (4.8)$$

Let $\mu > \mu_0 + 2$ where μ_0 is the parameter appearing in Lemma 4.1. On the one hand, we apply Lemma 4.1 to $\widehat{\Psi}$. Hence, based on the identity (4.8), for $\xi > \mu > \mu_0 + 2$, we directly have

$$\left\| \widehat{\Psi}(\xi, x) \right\|_{L_x^2(\mathbb{R})}^2 \leq \frac{C}{\mu} \left\| \mathcal{F}_{t \rightarrow \xi} \left(\chi'\left(\frac{t}{T}\right) F \right) (\xi, x) \right\|_{L_x^2(\mathbb{R})}^2 + C \left\| \widehat{\Psi}(\xi, x) \right\|_{L_x^2(\Omega)}^2, \quad (4.9)$$

where $C = C(V, \zeta)$ is a constant depending only on V and ζ .

On the other hand, for $\xi \leq \mu$, we estimate (4.8) directly using the fact that $(H - \xi)(\text{Id} - \Pi_\mu)$ is invertible. More precisely, from the definition of dm_λ and Ψ , we have

$$(H - \xi)\widehat{\Psi}(\xi, x) = \int_\mu^\infty (\lambda^2 - \xi) dm_\lambda \widehat{\Psi}(\xi, x).$$

Observe that $\frac{\mu^2}{2} < \lambda^2 - \mu$ for $\lambda \geq \mu$, since $\mu > \mu_0 + 2 > 2$. Therefore, for any $\xi \leq \mu$, we see that

$$\begin{aligned} \frac{1}{2}\mu^2 \left\| \widehat{\Psi}(\xi, x) \right\|_{L_x^2(\mathbb{R})}^2 &\leq \int_{\mu}^{\infty} (\lambda^2 - \mu) \left(dm_{\lambda} \widehat{\Psi}(\xi, x), \widehat{\Psi}(\xi, x) \right)_{L_x^2(\mathbb{R})} \\ &\leq \left((H - \xi) \widehat{\Psi}(\xi, x), \widehat{\Psi}(\xi, x) \right)_{L_x^2(\mathbb{R})}. \end{aligned}$$

Combining the above inequality with (4.8), for $\xi \leq \mu$, we obtain

$$\left\| \widehat{\Psi}(\xi, x) \right\|_{L_x^2(\mathbb{R})}^2 \leq \frac{4}{\mu^4 T^2} \left\| \mathcal{F}_{t \rightarrow \xi} \left(\chi' \left(\frac{t}{T} \right) F \right) (\xi, x) \right\|_{L_x^2(\mathbb{R})}^2. \quad (4.10)$$

Gathering (4.9) and (4.10), and then integrating over \mathbb{R} for the variable ξ , we find

$$\begin{aligned} \left\| \widehat{\Psi}(\xi, x) \right\|_{L^2(\mathbb{R}^2)}^2 &\leq \frac{C}{\mu} \left\| \mathcal{F}_{t \rightarrow \xi} \left(\chi' \left(\frac{t}{T} \right) F \right) (\xi, x) \right\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad + \frac{4}{\mu^4 T^2} \left\| \mathcal{F}_{t \rightarrow \xi} \left(\chi' \left(\frac{t}{T} \right) F \right) (\xi, x) \right\|_{L^2(\mathbb{R}^2)}^2 + C \left\| \widehat{\Psi}(\xi, x) \right\|_{L^2(\mathbb{R} \times \Omega)}^2. \end{aligned}$$

Then, using the Plancherel theorem for the variables t and ξ ,

$$\left\| \Psi(t, x) \right\|_{L^2(\mathbb{R}^2)}^2 \leq \left(\frac{C}{\mu} + \frac{4}{\mu^4 T^2} \right) \left\| \chi' \left(\frac{t}{T} \right) F(t, x) \right\|_{L^2(\mathbb{R}^2)}^2 + C \left\| \Psi(t, x) \right\|_{L^2(\mathbb{R} \times \Omega)}^2.$$

Therefore, by the conservation of L_x^2 for F and the definition of χ and Ψ , we obtain

$$\begin{aligned} \left\| (1 - \Pi_{\mu}) f \right\|_{L^2(\mathbb{R})}^2 &\leq 50 \left(\frac{C}{\mu} + \frac{4}{\mu^4 T^2} \right) \left\| (1 - \Pi_{\mu}) f \right\|_{L^2(\mathbb{R})}^2 \\ &\quad + \frac{2C}{T} \int_0^T \left\| e^{itH} (1 - \Pi_{\mu}) f \right\|_{L^2(\Omega)}^2 dt, \end{aligned}$$

which completes the proof of $f \in H^2(\mathbb{R})$ for taking μ large enough. Last, using a density argument, we complete the proof of Corollary 4.4 for any $f \in L^2(\mathbb{R})$. \square

5. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. The proof is based on the general strategy introduced in Phung [31] (inspired by Lebeau-Robbiano [28]) for the Schrödinger equation in a similar context. We start with the following quantitative unique continuation estimate for the 1D Schrödinger equation which plays a crucial role in our proof for Theorem 1.2. The key idea of the proof is to take an FBI transformation that transfers the 1D Schrödinger equation to the 1D heat equation.

Proposition 5.1. *Let Ω be a $(1, \zeta)$ -thick set and let $V \in C(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ with $V \geq 1$. Then there exist some constants $h_0 = h_0(V, \zeta) \in (0, 1)$ and $C = C(V, \zeta) > 0$, depending only on V and ζ , such that for any $T > 0$, $0 < h < h_0 (1 + T^{-3})^{-1}$ and $f \in H^2(\mathbb{R})$, we have*

$$\begin{aligned} \|f\|_{L^2(\mathbb{R})}^2 &\leq Ch \left(1 + \frac{1}{T^2} \right) \|Hf\|_{L^2(\mathbb{R})}^2 \\ &\quad + C e^{\frac{2T^2}{h} + \frac{C}{T}} \int_0^T \|H e^{itH} f\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (5.1)$$

Proof. Step 1. FBI transformation. Following Zworski [37], we introduce the definition of FBI transformation. For $0 < h < 1$, $z = \tau + is \in \mathbb{C}$ and L^2 -valued regular function $\Gamma(t)$, we define

$$\mathcal{T}_h \Gamma(z) = \frac{2^{\frac{1}{4}}}{(2\pi h)^{\frac{3}{4}}} \int_{\mathbb{R}} e^{-\frac{(z+t)^2}{2h}} \Gamma(t) dt.$$

By an elementary computation and integration by parts, we directly have

$$(\partial_s + \partial_x^2 - V(x)) \mathcal{T}_h \Gamma(z) = -\mathcal{T}_h ((i\partial_t - \partial_x^2 + V(x)) \Gamma)(z). \quad (5.2)$$

This is the key point to transfer the observability estimate for 1D Schrödinger equation to the observability estimate for the 1D heat equation.

Fix $T > 0$. We define a cut-off C^1 function $\chi : \mathbb{R} \rightarrow [0, 1]$ satisfying

$$\chi \equiv 1 \text{ on } [2T, 8T], \quad \text{supp} \chi \subset [0, 10T] \quad \text{and} \quad \chi' \in \left[-\frac{2}{T}, \frac{2}{T} \right]. \quad (5.3)$$

To simplify notation, we will prove (5.1) is true for $10T$ and then based on the arbitrary choice of T , we can complete the proof of (5.1) for any $T > 0$. For any $f \in H^2(\mathbb{R})$, we denote $F = e^{itH} f$ and $\tilde{F} = \chi F$. We directly have

$$i\partial_t \tilde{F} - \partial_x^2 \tilde{F} + V(x) \tilde{F} = i\chi'(t)F.$$

Taking the FBI transformation on both sides of the above identity and then using (5.2), we obtain

$$\partial_s W + \partial_x^2 W - V(x)W = G,$$

where $W = \mathcal{T}_h \tilde{F}$ and $G = -\mathcal{T}_h(i\chi'F)$. From Corollary 3.12, there exists a constant $C = C(V, \zeta)$, depending only on V and ζ such that for any $\tau \in \mathbb{R}$, we have

$$\begin{aligned} \|W(\tau)\|_{L^2(\mathbb{R})}^2 &\leq C e^{\frac{C}{T}} \int_0^T \|HW(\tau + is)\|_{L^2(\Omega)}^2 ds \\ &\quad + C e^{\frac{C}{T}} \int_0^T \|G(\tau + is)\|_{L^2(\mathbb{R})}^2 ds. \end{aligned} \quad (5.4)$$

Step 2. $L_s^1 L_x^2$ estimates on HW and G . First, from the definition of FBI transformation and $W(\tau + is)$, we have

$$HW(\tau + is) = \frac{2^{\frac{1}{4}}}{(2\pi h)^{\frac{3}{4}}} e^{\frac{s^2}{2h}} \int_{\mathbb{R}} e^{-\frac{(\tau+t)^2}{2h}} e^{-i\frac{s(\tau+t)}{h}} \chi(t) HF(t) dt.$$

It follows from Cauchy-Schwarz inequality that

$$\|HW(\tau + is)\|_{L^2(\Omega)}^2 \leq \frac{20T}{(2\pi h)^{\frac{3}{2}}} e^{\frac{s^2}{h}} \int_0^{10T} \|HF(t)\|_{L^2(\Omega)}^2 dt.$$

Integrating the above inequality on $[0, T]$, we see that

$$\sup_{\tau \in \mathbb{R}} \int_0^T \|HW(\tau + is)\|_{L^2(\Omega)}^2 ds \leq \frac{20T^2}{(2\pi h)^{\frac{3}{2}}} e^{\frac{T^2}{h}} \int_0^{10T} \|HF(t)\|_{L^2(\Omega)}^2 dt. \quad (5.5)$$

Second, using again the definition of FBI transformation,

$$G(\tau + is) = -i \frac{2^{\frac{1}{4}}}{(2\pi h)^{\frac{3}{4}}} e^{\frac{s^2}{2h}} \int_{\mathbb{R}} e^{-\frac{(\tau+t)^2}{2h}} e^{-i\frac{s(\tau+t)}{h}} \chi'(t) F(t) dt.$$

Note that, from the definition of χ in (5.3), we infer that

$$|\tau + t| \geq 2T, \quad \text{for any } (\tau, t) \in [-6T, -4T] \times \text{supp} \chi'.$$

It follows from $\|F(t)\|_{L^2(\mathbb{R})} = \|e^{itH} f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ that

$$\max_{\tau \in [-6T, -4T]} \|G(\tau + is)\|_{L^2(\mathbb{R})}^2 \leq \frac{128}{(2\pi h)^{\frac{3}{2}}} e^{\frac{s^2}{h}} e^{-\frac{4T^2}{h}} \|f\|_{L^2(\mathbb{R})}^2.$$

Integrating the above inequality on $[0, T]$, we see that

$$\max_{\tau \in [-6T, -4T]} \int_0^T \|G(\tau + is)\|_{L^2(\mathbb{R})}^2 ds \leq \frac{128T}{(2\pi h)^{\frac{3}{2}}} e^{-\frac{3T^2}{h}} \|f\|_{L^2(\mathbb{R})}^2. \quad (5.6)$$

Step 3. Conclusion. Combining (5.4) and (5.6) with (5.5), we obtain

$$\begin{aligned} \max_{\tau \in [-6T, -4T]} \|W(\tau)\|_{L^2(\mathbb{R})}^2 &\leq \frac{128CT}{(2\pi h)^{\frac{3}{2}}} e^{-\frac{3T^2}{h} + \frac{C}{T}} \|f\|_{L^2(\mathbb{R})}^2 \\ &+ \frac{20CT^2}{(2\pi h)^{\frac{3}{2}}} e^{\frac{T^2}{h} + \frac{C}{T}} \int_0^{10T} \|HF(t)\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (5.7)$$

On the other hand, using again $\|F(t)\|_{L^2(\mathbb{R})} = \|e^{itH}f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$,

$$\|f\|_{L^2(\mathbb{R})}^2 = \frac{1}{2T} \int_{4T}^{6T} \|F(\tau)\|_{L^2(\mathbb{R})}^2 d\tau \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \frac{1}{T} \int_{4T}^{6T} \left\| \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{(\tau-t)^2}{2h}} \tilde{F}(t) dt \right\|_{L^2(\mathbb{R})}^2 d\tau, \\ I_2 &= \frac{1}{T} \int_{4T}^{6T} \left\| F(\tau) - \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{(\tau-t)^2}{2h}} \tilde{F}(t) dt \right\|_{L^2(\mathbb{R})}^2 d\tau. \end{aligned}$$

Based on the definition of FBI transformation and $W(\tau)$, we have

$$I_1 = \frac{1}{T} \int_{-6T}^{-4T} \left\| \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{(\tau+t)^2}{2h}} \tilde{F}(t) dt \right\|_{L^2(\mathbb{R})}^2 d\tau = \frac{\sqrt{\pi h}}{T} \int_{-6T}^{-4T} \|W(\tau)\|_{L^2(\mathbb{R})}^2 d\tau.$$

It follows from (5.7) that

$$\begin{aligned} I_1 &\leq \frac{128CT}{\pi h} e^{-\frac{3T^2}{h} + \frac{C}{T}} \|f\|_{L^2(\mathbb{R})}^2 \\ &+ \frac{20CT^2}{\pi h} e^{\frac{T^2}{h} + \frac{C}{T}} \int_0^{10T} \|HF(t)\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (5.8)$$

Next, using the fact that $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$ and the definition of χ , we rewrite I_2 as

$$I_2 = \frac{1}{T} \int_{4T}^{6T} \left\| \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{t^2}{2h}} (F(\tau) - \chi(\tau-t)F(\tau-t)) dt \right\|_{L^2(\mathbb{R})}^2 d\tau = I_{21} + I_{22},$$

where

$$\begin{aligned} I_{21} &= \frac{1}{T} \int_{4T}^{6T} \left\| \frac{1}{\sqrt{2\pi h}} \int_{|t| \geq 6T} e^{-\frac{t^2}{2h}} F(\tau) dt \right\|_{L^2(\mathbb{R})}^2 d\tau, \\ I_{22} &= \frac{1}{T} \int_{4T}^{6T} \left\| \frac{1}{\sqrt{2\pi h}} \int_{-6T}^{6T} e^{-\frac{t^2}{2h}} (F(\tau) - \chi(\tau-t)F(\tau-t)) dt \right\|_{L^2(\mathbb{R})}^2 d\tau. \end{aligned}$$

Using again $\|F(t)\|_{L^2(\mathbb{R})} = \|e^{itH}f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$, we have

$$I_{21} \leq \frac{1}{\pi h} \left(\int_{|t| \geq 6T} e^{-\frac{t^2}{2h}} dt \right)^2 \|f\|_{L^2(\mathbb{R})}^2 \leq 2e^{-\frac{18T^2}{h}} \|f\|_{L^2(\mathbb{R})}^2.$$

Note that, from $f \in H^2(\mathbb{R})$ and $H \geq \text{Id}$, we have

$$\begin{aligned} \|F(t)\|_{L^2(\mathbb{R})} &\leq \|HF(t)\|_{L^2(\mathbb{R})} = \|Hf\|_{L^2(\mathbb{R})}, \\ \|\partial_t F(t)\|_{L^2(\mathbb{R})} &= \|HF(t)\|_{L^2(\mathbb{R})} = \|Hf\|_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore, from the mean-value theorem, for any $\tau \in (4T, 6T)$, we have

$$\begin{aligned} & \|F(\tau) - \chi(\tau - t)F(\tau - t)\|_{L^2(\mathbb{R})} \\ & \leq \|F(\tau) - F(\tau - t)\|_{L^2(\mathbb{R})} + |\chi(\tau) - \chi(\tau - t)|\|F(\tau - t)\|_{L^2(\mathbb{R})} \\ & \leq |t| (\|Hf\|_{L^2(\mathbb{R})} + \|\chi'\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}) \leq |t| \left(1 + \frac{2}{T}\right) \|Hf\|_{L^2(\mathbb{R})}. \end{aligned}$$

Based on the above inequality and Minkowski inequality, we directly have

$$I_{22} \leq \frac{4}{\pi h} \left(1 + \frac{2}{T}\right)^2 \left(\int_0^{6T} e^{-\frac{t^2}{2h}} dt\right)^2 \|Hf\|_{L^2(\mathbb{R})}^2 \leq \frac{32h}{\pi} \left(1 + \frac{1}{T^2}\right) \|Hf\|_{L^2(\mathbb{R})}^2.$$

By the above estimates for I_{21} and I_{22} , we see that

$$I_2 \leq I_{21} + I_{22} \leq 2e^{-\frac{18T^2}{h}} \|f\|_{L^2(\mathbb{R})}^2 + \frac{32h}{\pi} \left(1 + \frac{1}{T^2}\right) \|Hf\|_{L^2(\mathbb{R})}^2. \quad (5.9)$$

Gathering (5.8) and (5.9) together, we conclude that

$$\begin{aligned} \|f\|_{L^2(\mathbb{R})}^2 & \leq \frac{32h}{\pi} \left(1 + \frac{1}{T^2}\right) \|Hf\|_{L^2(\mathbb{R})}^2 \\ & \quad + \frac{20CT^2}{\pi h} e^{\frac{T^2}{h} + \frac{C}{T}} \int_0^{10T} \|HF(t)\|_{L^2(\Omega)}^2 dt \\ & \quad + \left(\frac{128CT}{\pi h} e^{-\frac{3T^2}{h} + \frac{C}{T}} + 2e^{-\frac{18T^2}{h}}\right) \|f\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

This completes the proof of Proposition 5.1 for taking h_0 small enough where h satisfies $0 < h < h_0(1 + T^{-3})^{-1}$. \square

We are in a position to complete the proof of Theorem 1.2.

End of the proof of Theorem 1.2. Recall that, without loss of generality, we assume $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. It follows that the operator $H \geq \text{Id}$ and thus H is invertible and is a bijection from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$. For any $u_0 \in L^2(\mathbb{R})$, we denote $U_0 = H^{-1}u_0 \in H^2(\mathbb{R})$ and $u(t) = e^{itH}u_0$ be the corresponding solution of (1.1).

First, from Proposition 5.1, there exist some constants $h_0 = h_0(V, \zeta) \in (0, 1)$ and $C = C(V, \zeta) > 0$ depending only on V and ζ , such that for any $T > 0$ and any $0 < h < h_0(1 + T^{-3})^{-1}$, we have

$$\begin{aligned} \|U_0\|_{L^2(\mathbb{R})}^2 & \leq Ch \left(1 + \frac{1}{T^2}\right) \|u_0\|_{L^2(\mathbb{R})}^2 \\ & \quad + Ce^{\frac{2T^2}{h} + \frac{C}{T}} \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (5.10)$$

Second, using Corollary 4.4 and the triangle inequality, there exists $\mu_1 = \mu_1(V, \zeta) > 0$ and $C_1 = C_1(V, \zeta) > 0$, depending only on V and ζ , such that for any $T > 0$ and $\mu > \mu_1(1 + T^{-\frac{1}{2}})$, we have

$$\|u_0\|_{L^2(\mathbb{R})}^2 \leq \frac{C_1}{T} \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt + (C_1 + 1) \|\Pi_\mu u_0\|_{L^2(\mathbb{R})}^2,$$

Note that, from the definition of U_0 and Π_μ , we find

$$\|\Pi_\mu u_0\|_{L^2(\mathbb{R})}^2 = \int_0^\mu (dm_\lambda u_0, u_0)_{L^2(\mathbb{R})} = \int_0^\mu \lambda^4 (dm_\lambda U_0, U_0)_{L^2(\mathbb{R})} \leq \mu^4 \|U_0\|_{L^2(\mathbb{R})}^2,$$

which implies

$$\|u_0\|_{L^2(\mathbb{R})}^2 \leq \frac{C_1}{T} \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt + (C_1 + 1) \mu^4 \|U_0\|_{L^2(\mathbb{R})}^2. \quad (5.11)$$

Combining (5.10) and (5.11), we conclude that

$$\begin{aligned} \|u_0\|_{L^2(\mathbb{R})}^2 &\leq C(C_1 + 1)\mu^4 h \left(1 + \frac{1}{T^2}\right) \|u_0\|_{L^2(\mathbb{R})}^2 \\ &\quad + \left(C(C_1 + 1)\mu^4 e^{\frac{2T^2}{h} + \frac{C}{T}} + \frac{C_1}{T}\right) \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

This completes the proof of Theorem 1.2 for taking $\mu = 2\mu_1(1 + T^{-\frac{1}{2}})$ and $h = \varepsilon(1 + T^{-4})^{-1}$ where ε small enough. The proof of Theorem 1.2 is complete. \square

APPENDIX A. PROOF OF THEOREM 3.3

In this appendix, we repeat the proof of Theorem 3.3 in [36] based on complex analysis. First, we recall the following Jensen's formula from [33, Theorem 15.18].

Lemma A.1 (Jensen's formula). *Let $0 < r_1 < r_2 < \infty$. Let f be a holomorphic function in B_{r_2} with $f(0) \neq 0$ and a_1, a_2, \dots, a_N are the zeros of f in B_{r_1} (repeated according to their respective multiplicity), then we have*

$$\log |f(0)| = \sum_{n=1}^N \log \left(\frac{|a_n|}{r_1} \right) + \frac{1}{2\pi} \int_0^{2\pi} |f(r_1 e^{i\theta})| d\theta.$$

Note that, Jensen's formula can be used to estimate the number of zeros of the holomorphic function in an open disk.

Corollary A.2. *Let $0 < r_1 < r_2 < r_3 < \infty$ and $a \in \mathbb{C}$. Let f be a holomorphic function in $B_{r_3}(a)$ with $f(a) \neq 0$ and a_1, a_2, \dots, a_N are the zeros of f in $B_{r_1}(a)$ (repeated according to their respective multiplicity), then we have*

$$N \leq \frac{\log M - \log |f(a)|}{\log r_2 - \log r_1}, \quad \text{where } M = \max_{|z|=r_2} |f(z)|.$$

Second, we recall Hadamard's three-circle theorem from [33, Page 264].

Lemma A.3. *Let f be a holomorphic function in B_R for $0 < R < \infty$. Let $M(r)$ be the maximum of $|f(z)|$ on the circle $|z| = r$ for $0 < r < R$. Then we have*

$$\log \left(\frac{r_3}{r_1} \right) \log M(r_2) \leq \log \left(\frac{r_3}{r_2} \right) \log M(r_1) + \log \left(\frac{r_2}{r_1} \right) \log M(r_3),$$

for any three concentric circles of radii $0 < r_1 < r_2 < r_3 < R$.

Third, following from [17, Theorem 4.3], we introduce the Remez-type inequality for holomorphic polynomials for further reference.

Lemma A.4 ([17]). *Let $P(z)$ be a holomorphic polynomial of degree N . Let $E \subset B_1$. Then for any $\delta > 0$, we have*

$$\sup_{B_1} |P(z)| \leq \left(\frac{6e}{\mathcal{H}_\delta(E)} \right)^{\frac{N}{\delta}} \sup_E |P(z)|.$$

Here, $\mathcal{H}_\delta(E)$ means the δ -dimensional Hausdorff content of E .

Now we recall some elementary properties of quasiconformal mapping and the presentation is close to [6]. For complex function f , we write the derivative as

$$Df(z)h = \frac{\partial f}{\partial z}(z)h + \frac{\partial f}{\partial \bar{z}}(z)\bar{h}, \quad \text{for } h \in \mathbb{C}.$$

The norm of the derivative and Jacobian can be explained as

$$|Df(z)| = \left| \frac{\partial f}{\partial z}(z) \right| + \left| \frac{\partial f}{\partial \bar{z}}(z) \right| \quad \text{and} \quad Jf(z) = \left| \frac{\partial f}{\partial z}(z) \right|^2 - \left| \frac{\partial f}{\partial \bar{z}}(z) \right|^2.$$

Definition A.5. Let U and V be open sets of \mathbb{C} and take $K \geq 1$.

(i) An orientation-preserving mapping $f : U \rightarrow V$ is K -quasiregular mapping if

$$f \in W_{loc}^{1,2}(U) \quad \text{and} \quad |Df(z)|^2 \leq KJf(z), \quad \text{for almost every } z \in U.$$

(ii) An orientation-preserving homeomorphism $f : U \rightarrow V$ is K -quasiconformal if

$$f \in W_{loc}^{1,2}(U) \quad \text{and} \quad |Df(z)|^2 \leq KJf(z), \quad \text{for almost every } z \in U.$$

Following [6, Chapter 16], we denote

$$* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{and} \quad ** = -\text{Id}.$$

For any solution ϕ of (3.3), we find the field $(*A\nabla\phi)$ is curl-free, and thus from Poincaré lemma, there exists a Sobolev function $\psi \in W_{loc}^{1,2}(B_4)$ such that

$$\nabla\psi = *A\nabla\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A\nabla\phi.$$

Note that the function ψ is unique up to an additive constant and it is called the A -harmonic conjugate of ϕ . For any $a \in B_1$, we consider the complex function $f_a = \phi + i\psi_a$ where ψ_a is A -harmonic conjugate with $\psi_a(a) = 0$. By an elementary computation and the definition of ψ_a

$$|Df_a|^2 \leq |\nabla\phi|^2 + |\nabla\psi_a|^2 \leq (\Lambda + \Lambda^{-1}) \nabla\phi \cdot A\nabla\phi \leq (\Lambda + \Lambda^{-1}) \nabla\phi \cdot (-* \nabla\psi_a).$$

Note that, from the definition of $*$, we find

$$\nabla\phi \cdot (-* \nabla\psi_a) = (\partial_x\phi)(\partial_y\psi_a) - (\partial_y\phi)(\partial_x\psi_a) = Jf_a.$$

Based on the above inequalities, we obtain

$$|Df_a(z)|^2 \leq (\Lambda + \Lambda^{-1})Jf_a(z), \quad \text{for almost every } z \in B_4,$$

which means that f_a is a $(\Lambda + \Lambda^{-1})$ -quasiregular mapping. It then follows from the representation theorem (see [1, Section 2] and [6, Corollary 5.5.3]) that f_a can be written as

$$f_a = F \circ G, \quad \text{on } B_2,$$

where F is holomorphic in B_2 , and G is a $(\Lambda + \Lambda^{-1})$ -quasiconformal homeomorphism from B_2 onto B_2 which verifies $G(0) = 0$ and

$$C^{-1}|z_2 - z_1|^{\frac{1}{\alpha}} \leq |G(z_2) - G(z_1)| \leq C|z_2 - z_1|^\alpha, \quad \text{when } (z_1, z_2) \in B_2 \times B_2, \quad (\text{A.1})$$

for some constants $\alpha = \alpha(\Lambda) \in (0, 1)$ and $C = C(\Lambda) > 0$ depending only on Λ . Moreover, from [6, Corollary 5.9.2], we know that G has a continuous homeomorphic extension to the boundary ∂B_2 . Therefore, from (A.1), there exists a constant $r = r(\Lambda) \in (0, 2)$, depending only Λ , such that

$$0 < 2 - r \leq \text{dist}(G(B_1), \partial B_2) \implies G(B_1) \subset B_r. \quad (\text{A.2})$$

For any $a \in B_1$, we consider the holomorphic self-homeomorphism of B_2 to itself:

$$R_a(z) = 4 \frac{z - G(a)}{4 - G(a)z} : B_2 \rightarrow B_2.$$

Based on [6, Theorem 3.1.2], we know that f_a can be rewritten as

$$f_a = (F \circ R_a^{-1}) \circ (R_a \circ G) := F_a \circ G_a, \quad \text{on } B_2, \quad (\text{A.3})$$

where F_a is holomorphic in B_2 , and G_a is a $(\Lambda + \Lambda^{-1})$ -quasiconformal homeomorphism from B_2 onto B_2 which verifies $G_a(a) = 0$. Moreover, from (A.1) and (A.2),

$$C^{-1}|z_2 - z_1|^{\frac{1}{\alpha}} \leq |G_a(z_2) - G_a(z_1)| \leq C|z_2 - z_1|^\alpha, \quad \text{when } (z_1, z_2) \in B_2 \times B_2, \quad (\text{A.4})$$

where $\alpha = \alpha(\Lambda) \in (0, 1)$ and $C = C(\Lambda) > 0$ are constants depending only on Λ . For any $a \in B_1$, using (A.4), we know that (A.2) is also true for G_a with probably different constant $r = r(\Lambda)$ but it still depends only on Λ .

The following proposition plays a crucial role in our proof of Theorem 3.3.

Proposition A.6. *Let $\omega \subset B_1 \cap \ell_0$ satisfy $|\omega| > 0$ for some line $\ell_0 \in \mathbb{R}^2$. Then there exist $z_0 \in \omega$ and some constants $\alpha = \alpha(\Lambda, |\omega|) \in (0, 1)$ and $C = C(\Lambda, |\omega|) > 0$, depending only on Λ and $|\omega|$, such that for any H_{loc}^1 solution ϕ of (3.3) with its A -harmonic conjugate satisfying $\psi_{z_0}(z_0) = 0$, we have*

$$\sup_{B_1} |\phi| \leq C \left(\sup_{\omega} |\phi + i\psi_{z_0}|^\alpha \right) \left(\sup_{B_2} |\phi|^{1-\alpha} \right).$$

Proof. From $\omega \subset B_1 \cap \ell_0$ and (A.4), there exist $r_1 = r_1(|\omega|) \in (0, 1)$ depending only on $|\omega|$ and $r_2 = r_2(\Lambda, |\omega|) \in (0, \frac{1}{6})$ depending only on Λ and $|\omega|$ such that

$$\frac{1}{2}|\omega| \leq |\omega \cap B_{r_1}| \quad \text{and} \quad B_{6r_2} \subset \bigcap_{a \in B_{r_1}} G_a(B_1). \quad (\text{A.5})$$

Moreover, from (A.4) and (A.5), there exist $z_0 \in \omega \cap B_{r_1}$ and $r_3 = r_3(\Lambda, |\omega|) \in (0, 1)$ depending only on Λ and $|\omega|$ such that

$$c|\omega| \leq |\omega \cap B_{r_3}(z_0)| \quad \text{and} \quad G_{z_0}(B_{r_3}(z_0)) \subset B_{r_2}, \quad (\text{A.6})$$

where $c = c(\Lambda, |\omega|)$ depends only on Λ and $|\omega|$. We denote $\tilde{\omega} = \omega \cap B_{r_3}(z_0)$.

Let a_1, a_2, \dots, a_N be the zeros of F_{z_0} in B_{2r_2} (repeated according to their respective multiplicity). Let $a_0 \in \partial B_{r_2}$ be such that $|F_{z_0}(a_0)|$ is the maximum of $|F_{z_0}(z)|$ on B_{r_2} . We consider the following complex polynomial $P_{z_0}(z)$ and holomorphic non-vanishing function $h_{z_0}(z)$ on B_{6r_2} ,

$$P_{z_0}(z) = \prod_{n=1}^N (z - a_n) \quad \text{and} \quad h_{z_0}(z) = \frac{F_{z_0}(z)}{P_{z_0}(z)}.$$

On the one hand, we denote by \tilde{N} the number of zeros of F_{z_0} in $B_{3r_2}(a_0)$. From $B_{2r_2} \subset B_{3r_2}(a_0)$, $B_{\frac{7}{2}r_2}(a_0) \subset B_{5r_2}$ and Corollary A.2,

$$\begin{aligned} N \leq \tilde{N} &\leq \frac{1}{\log \frac{7}{6}} \left(\log \sup_{B_{\frac{7}{2}r_2}(a_0)} |F_{z_0}| - \log |F_{z_0}(a_0)| \right) \\ &\leq \frac{1}{\log \frac{7}{6}} \left(\log \sup_{B_{5r_2}} |F_{z_0}| - \log \sup_{B_{r_2}} |F_{z_0}| \right). \end{aligned} \quad (\text{A.7})$$

Then, using the definition of h_{z_0} and P_{z_0} , we see that

$$\sup_{B_{r_2}} |F_{z_0}| = |F_{z_0}(a_0)| \leq |h_{z_0}(a_0)| |P_{z_0}(a_0)| \leq (3r_2)^N |h_{z_0}(a_0)|. \quad (\text{A.8})$$

Using the maximum modulus principle on B_{5r_2} , we find

$$\sup_{B_{5r_2}} |h_{z_0}| \leq \left(\sup_{\partial B_{5r_2}} |F_{z_0}| \right) \left(\sup_{\partial B_{5r_2}} |P_{z_0}^{-1}| \right) \leq \left(\frac{1}{3r_2} \right)^N \sup_{B_{5r_2}} |F_{z_0}|. \quad (\text{A.9})$$

Since h_{z_0} is a holomorphic non-vanishing function, $\log |h_{z_0}|$ is a harmonic function on B_{2r_2} . From Harnack's inequality for harmonic function, there exists a constant $C = C(\Lambda) > 1$, depending only on Λ , such that

$$\sup_{B_{r_2}} \left(\sup_{B_{5r_2}} \log |h_{z_0}| - \log |h_{z_0}(z)| \right) \leq C \inf_{B_{r_2}} \left(\sup_{B_{5r_2}} \log |h_{z_0}| - \log |h_{z_0}(z)| \right).$$

It follows that

$$|h_{z_0}(a_0)|^C \sup_{B_{5r_2}} |h_{z_0}| \leq \left(\sup_{B_{5r_2}} |h_{z_0}|^C \right) \left(\inf_{B_{r_2}} |h_{z_0}| \right).$$

Combining (A.8) and (A.9) with the above inequality, we obtain

$$\left(\sup_{B_{r_2}} |F_{z_0}|^C \right) \left(\sup_{B_{5r_2}} |h_{z_0}| \right) \leq \left(\sup_{B_{5r_2}} |F_{z_0}|^C \right) \left(\inf_{B_{r_2}} |h_{z_0}| \right). \quad (\text{A.10})$$

On the other hand, using Lemma A.4, we find

$$\sup_{B_{r_2}} |P_{z_0}| \leq \left(\frac{6e}{\mathcal{H}_\alpha(G_{z_0}(\tilde{\omega}))} \right)^{\frac{N}{\alpha}} \sup_{G_{z_0}(\tilde{\omega})} |P_{z_0}|.$$

From (2.1), (A.4) and (A.6), there exists a constant $C_1 = C_1(\Lambda) > 0$, depending only on Λ , such that

$$c|\omega| \leq |\tilde{\omega}| \leq C_1 \mathcal{H}_\alpha(G_{z_0}(\tilde{\omega})) \implies \sup_{B_{r_2}} |P_{z_0}| \leq \left(\frac{6C_1 e}{c|\omega|} \right)^{\frac{N}{\alpha}} \sup_{G_{z_0}(\tilde{\omega})} |P_{z_0}|. \quad (\text{A.11})$$

Thanks to (A.6), (A.10) and (A.11), we find

$$\begin{aligned} \sup_{B_{r_2}} |F_{z_0}|^{1+C} &\leq \left(\sup_{B_{r_2}} |F_{z_0}|^C \right) \left(\sup_{B_{5r_2}} |h_{z_0}| \right) \left(\sup_{B_{r_2}} |P_{z_0}| \right) \\ &\leq C_2^N \left(\sup_{B_{5r_2}} |F_{z_0}|^C \right) \left(\sup_{G_{z_0}(\tilde{\omega})} |P_{z_0}| \right) \left(\inf_{B_{r_2}} |h_{z_0}| \right) \\ &\leq C_2^N \left(\sup_{B_{5r_2}} |F_{z_0}|^C \right) \left(\sup_{G_{z_0}(\tilde{\omega})} |F_{z_0}| \right), \end{aligned}$$

where $C_2 = C_2(\Lambda, |\omega|) > 1$ is a constant depending only on Λ and $|\omega|$. Combining the above inequality with (A.7), there exists a constant $C_3 = C_3(\Lambda, |\omega|) > 0$, depending only on Λ and $|\omega|$, such that

$$\sup_{B_{r_2}} |F_{z_0}|^{1+C+C_3} \leq \left(\sup_{B_{5r_2}} |F_{z_0}|^{C+C_3} \right) \left(\sup_{G_{z_0}(\tilde{\omega})} |F_{z_0}| \right),$$

and thus we have

$$\sup_{B_{r_2}} |F_{z_0}| \leq \left(\sup_{B_{5r_2}} |F_{z_0}|^{1-\alpha} \right) \left(\sup_{G_{z_0}(\tilde{\omega})} |F_{z_0}|^\alpha \right), \quad \text{where } \alpha = \frac{1}{1+C+C_3} \in (0, 1).$$

Therefore, from (A.2) and Lemma A.3, there exists a constant $r = r(\Lambda) \in (0, 2)$, depending only on Λ , such that

$$\begin{aligned} \sup_{G_{z_0}(B_1)} |F_{z_0}| &\leq \sup_{B_r} |F_{z_0}| \leq \left(\sup_{B_{r_2}} |F_{z_0}|^{\alpha_1} \right) \left(\sup_{B_{\frac{2+r}{2}}} |F_{z_0}|^{1-\alpha_1} \right) \\ &\leq \left(\sup_{G_{z_0}(\tilde{\omega})} |F_{z_0}|^{\alpha_1} \right) \left(\sup_{B_{\frac{2+r}{2}}} |F_{z_0}|^{1-\alpha_1} \right), \end{aligned} \quad (\text{A.12})$$

where $\alpha_1 = \alpha_1(\Lambda, |\omega|) \in (0, 1)$ is a constant depending only on Λ and $|\omega|$.

From the definition of F_{z_0} , the holomorphic function F_{z_0} can be written as

$$F_{z_0} = f_{z_0} \circ G_{z_0}^{-1} = \phi \circ G_{z_0}^{-1} + i\psi_{z_0} \circ G_{z_0}^{-1}. \quad (\text{A.13})$$

Therefore, using the Cauchy-Riemann equation and $\psi_{z_0} \circ G_{z_0}^{-1}(0) = 0$,

$$\psi_{z_0} \circ G_{z_0}^{-1}(x, y) = \int_0^1 (y, -x) \cdot \nabla(\phi \circ G_{z_0}^{-1})(x\sigma, y\sigma) d\sigma, \text{ on } B_2,$$

and thus, from interior estimates of derivatives of harmonic function (see for instance [18, Theorem 2.10]), there exists a constant $C_4 = C_4(\Lambda)$, depending only on Λ , such that

$$\sup_{B_{\frac{2+r}{2}}} |\psi_{z_0} \circ G_{z_0}^{-1}| \leq 2 \sup_{B_{\frac{2+r}{2}}} |\nabla(\phi \circ G_{z_0}^{-1})| \leq C_4 \sup_{B_{\frac{6+r}{4}}} |\phi \circ G_{z_0}^{-1}|.$$

Hence, using again (A.13), we find

$$\begin{aligned} \sup_{B_{\frac{2+r}{2}}} |F_{z_0}| &\leq \sup_{B_{\frac{2+r}{2}}} |\phi \circ G_{z_0}^{-1}| + \sup_{B_{\frac{2+r}{2}}} |\psi_{z_0} \circ G_{z_0}^{-1}| \\ &\leq (1 + C_4) \sup_{B_{\frac{6+r}{4}}} |\phi \circ G_{z_0}^{-1}| \leq (1 + C_4) \sup_{B_2} |\phi|. \end{aligned} \quad (\text{A.14})$$

Combining (A.12) and (A.14) with $\tilde{\omega} \subset \omega$, we conclude that

$$\begin{aligned} \sup_{G_{z_0}(B_1)} |F_{z_0}| &\leq (1 + C_4)^{1-\alpha\alpha_1} \left(\sup_{G_{z_0}(\tilde{\omega})} |F_{z_0}|^{\alpha\alpha_1} \right) \left(\sup_{B_2} |\phi|^{1-\alpha\alpha_1} \right) \\ &\leq (1 + C_4)^{1-\alpha\alpha_1} \left(\sup_{G_{z_0}(\omega)} |F_{z_0}|^{\alpha\alpha_1} \right) \left(\sup_{B_2} |\phi|^{1-\alpha\alpha_1} \right), \end{aligned}$$

which completes the proof of Proposition A.6 \square

We are in a position to complete the proof of Theorem 3.3.

End of the proof of Theorem 3.3. Let $z_0 \in \omega$ be the point appearing in Proposition A.6 and ψ_{z_0} be the A -harmonic conjugate of ϕ satisfying $\psi_{z_0}(z_0) = 0$. Note that, from $A\nabla\phi \cdot \mathbf{e}_0 = 0$ and the definition of ψ_{z_0} , we obtain

$$\nabla\psi_{z_0} \cdot \mathbf{e}_0^\perp = *A\nabla\phi \cdot \mathbf{e}_0^\perp = A\nabla\phi \cdot (-*\mathbf{e}_0^\perp) = A\nabla\phi \cdot \mathbf{e}_0 = 0.$$

It follows that $\psi_{z_0}(z) = \psi_{z_0}(z_0) = 0$ on ℓ_0 . Therefore, from $\omega \subset B_1 \cap \ell_0$ and Proposition A.6, we conclude that, there exist some constants $\alpha = \alpha(\Lambda, |\omega|) \in (0, 1)$ and $C = C(\Lambda, |\omega|) > 0$, depending only on Λ and $|\omega|$, such that

$$\sup_{B_1} |\phi| \leq C \left(\sup_{\omega} |\phi + i\psi_{z_0}|^\alpha \right) \left(\sup_{B_2} |\phi|^{1-\alpha} \right) \leq C \left(\sup_{\omega} |\phi|^\alpha \right) \left(\sup_{B_2} |\phi|^{1-\alpha} \right).$$

The proof of Theorem 3.3 is complete. \square

APPENDIX B. PROOF OF PROPOSITION 3.10

The implication of backward observability for the heat equation from the spectral inequality is standard. Here we closely follow the strategy of [9, Theorem 8 and Corollary 4.1] which builds upon the early work in [5]. The key point is the following interpolation estimates for solutions of the heat equation.

Lemma B.1. *Let Ω be a $(1, \zeta)$ -thick set and let $V \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V \geq 1$. Then there exists a constant $C = C(V, \zeta) > 0$, depending only on V and ζ , such that for any $\alpha \in (0, 1)$, $0 \leq s < t < \infty$ and $f \in L^2(\mathbb{R})$, we have*

$$\|e^{-tH} f\|_{L^2(\mathbb{R})} \leq C e^{\frac{3}{\alpha(t-s)}} \|e^{-tH} f\|_{L^2(\Omega)}^{1-\alpha} \|e^{-sH} f\|_{L^2(\mathbb{R})}^\alpha.$$

Proof. First, for any $\mu > 0$ and $0 \leq s < t < \infty$, we have

$$\begin{aligned} \|e^{-tH}(1 - \Pi_\mu)f\|_{L^2(\mathbb{R})}^2 &= \int_\mu^\infty e^{-2t\lambda^2} (dm_\lambda f, f)_{L^2(\mathbb{R})} \\ &\leq e^{-2(t-s)\mu^2} \|e^{-sH}f\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (\text{B.1})$$

It follows from Lemma 3.8 that

$$\begin{aligned} \|e^{-tH}\Pi_\mu f\|_{L^2(\mathbb{R})} &\leq Ce^{3\mu} (\|e^{-tH}f\|_{L^2(\Omega)} + \|e^{-tH}(1 - \Pi_\mu)f\|_{L^2(\mathbb{R})}) \\ &\leq Ce^{3\mu} (\|e^{-tH}f\|_{L^2(\Omega)} + e^{-(t-s)\mu^2} \|e^{-sH}f\|_{L^2(\mathbb{R})}). \end{aligned}$$

From AM-GM inequality, for any $\mu > 0$, $\alpha \in (0, 1)$ and $0 \leq s < t < \infty$, we obtain

$$3\mu \leq \alpha(t-s)\mu^2 + \frac{3}{\alpha(t-s)} \implies 3\mu - \alpha(t-s)\mu^2 \leq \frac{3}{\alpha(t-s)},$$

which implies

$$\begin{aligned} \|e^{-tH}\Pi_\mu f\|_{L^2(\mathbb{R})} &\leq Ce^{\frac{3}{\alpha(t-s)}} e^{\alpha(t-s)\mu^2} \|e^{-tH}f\|_{L^2(\Omega)} \\ &\quad + Ce^{\frac{3}{\alpha(t-s)}} e^{-(1-\alpha)(t-s)\mu^2} \|e^{-sH}f\|_{L^2(\mathbb{R})}. \end{aligned} \quad (\text{B.2})$$

Now we make the choice of $\mu \in (0, \infty)$ such that

$$e^{(t-s)\mu^2} = \frac{\|e^{-sH}f\|_{L^2(\mathbb{R})}}{\|e^{-tH}f\|_{L^2(\Omega)}} \in (1, \infty).$$

This is always possible since for $f \neq 0$, we find

$$0 < \|e^{-tH}f\|_{L^2(\Omega)} \leq \|e^{-tH}f\|_{L^2(\mathbb{R})} \leq \|e^{-sH}f\|_{L^2(\mathbb{R})}.$$

Therefore, from (B.1) and (B.2), we conclude that

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq (2C + 1)e^{\frac{3}{\alpha(t-s)}} \|e^{-tH}f\|_{L^2(\Omega)}^{1-\alpha} \|e^{-sH}f\|_{L^2(\mathbb{R})}^\alpha.$$

The proof of Lemma B.1 is complete. \square

We are in a position to complete the proof of Proposition 3.10.

End of the proof of Proposition 3.10. We split the proof into the following two steps.

Step 1. Recursive estimates. First, from Lemma B.1, for $0 < t_1 < t_2 \leq T < \infty$, $t \in (\frac{t_1+t_2}{2}, t_2)$ and $\alpha = \frac{1}{4}$, we have

$$\begin{aligned} \|u(t_2)\|_{L^2(\mathbb{R})} &\leq \|u(t)\|_{L^2(\mathbb{R})} \leq Ce^{\frac{12}{t-t_1}} \|u(t)\|_{L^2(\Omega)}^{\frac{3}{4}} \|u(t_1)\|_{L^2(\mathbb{R})}^{\frac{1}{4}} \\ &\leq Ce^{\frac{24}{t_2-t_1}} \|u(t)\|_{L^2(\Omega)}^{\frac{3}{4}} \|u(t_1)\|_{L^2(\mathbb{R})}^{\frac{1}{4}}, \end{aligned}$$

where $C = C(V, \zeta) > 2$ is a constant depending only on V and ζ . Integrating the above inequality over $(\frac{t_1+t_2}{2}, t_2)$ and then using Hölder inequality, we see that

$$\|u(t_2)\|_{L^2(\mathbb{R})}^2 \leq C^2 e^{\frac{48}{t_2-t_1}} \left(\frac{t_2-t_1}{2}\right)^{-\frac{3}{4}} \|u(t_1)\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|u(t)\|_{L^2(\Omega)}^2 dt\right)^{\frac{3}{4}}.$$

It follows that

$$\begin{aligned} \|u(t_2)\|_{L^2(\mathbb{R})}^2 e^{-\frac{99}{t_2-t_1}} &\leq C^3 (t_2-t_1)^{-\frac{3}{4}} \left(\|u(t_1)\|_{L^2(\mathbb{R})}^2 e^{-\frac{198}{t_2-t_1}}\right)^{\frac{1}{4}} \\ &\quad \times e^{-\frac{3}{2(t_2-t_1)}} \left(\int_{t_1}^{t_2} \|u(t)\|_{L^2(\Omega)}^2 dt\right)^{\frac{3}{4}} \\ &\leq \left(\|u(t_1)\|_{L^2(\mathbb{R})}^2 e^{-\frac{198}{t_2-t_1}}\right)^{\frac{1}{4}} \left(C^4 (t_2-t_1) \int_{t_1}^{t_2} \|u(t)\|_{L^2(\Omega)}^2 dt\right)^{\frac{3}{4}}. \end{aligned}$$

Based on the above inequality and Young's inequality, we conclude that

$$\|u(t_2)\|_{L^2(\mathbb{R})}^2 e^{-\frac{99}{t_2-t_1}} \leq \frac{1}{4} \|u(t_1)\|_{L^2(\mathbb{R})}^2 e^{-\frac{198}{t_2-t_1}} + C^4(t_2-t_1) \int_{t_1}^{t_2} \|u(t)\|_{L^2(\Omega)}^2 dt. \quad (\text{B.3})$$

Step 2. Conclusion. For $T > 0$ and $n \in \mathbb{N}$, we set

$$S_n = \frac{T}{2^n} \quad \text{and} \quad a_n = \|u(S_n)\|_{L^2(\mathbb{R})}^2 e^{-\frac{99}{S_n-S_{n+1}}}.$$

By an elementary computation,

$$S_{n+1} - S_{n+2} = \frac{S_n - S_{n+1}}{2} \implies a_{n+1} = \|u(S_{n+1})\|_{L^2(\mathbb{R})}^2 e^{-\frac{198}{S_n-S_{n+1}}}.$$

Therefore, from (B.3), for any $n \in \mathbb{N}$, we have

$$a_n \leq \frac{1}{4} a_{n+1} + C^4 T \int_{\frac{T}{2^{n+1}}}^{\frac{T}{2^n}} \|u(t)\|_{L^2(\Omega)}^2 dt,$$

which implies

$$\|u(T)\|_{L^2(\mathbb{R})}^2 e^{-\frac{198}{T}} = a_0 \leq \frac{1}{4} a_n + C^4 T \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt.$$

Using the fact that $a_n \rightarrow 0$ as $n \rightarrow \infty$, we complete the proof of Proposition 3.10. \square

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